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The crossing model for regular A_n-crystals ☆

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ABSTRACT

A regular A_n-crystal is an edge-colored directed graph, with n colors, related to an irreducible highest weight integrable module over $U_q(sl_{n+1})$. Based on Stembridge's local axioms for regular simply-laced crystals and a structural characterization of regular A_2-crystals in [V.I. Danilov, A.V. Karzanov, G.A. Koshevoy, Combinatorics of regular A_2-crystals, J. Algebra 310 (2007) 218–234], we present a new combinatorial construction, the so-called *crossing model*, and prove that this model generates precisely the set of regular A_n-crystals.

Using the model, we obtain a series of results on the combinatorial structure of such crystals and properties of their subcrystals.

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1. Introduction

The notion of a *crystal* introduced by Kashiwara [7,8] has proved its importance in representation theory. This is an edge-colored directed graph, with n colors, in which each connected monochromatic subgraph is a finite path, and there are certain interrelations on the lengths of such paths, described via coefficients of an $n \times n$ Cartan matrix M (this matrix characterizes the *type* of a crystal). The central role in the theory of Kashiwara is played by crystals of representations, or *regular* crystals; these are associated to irreducible highest weight integrable modules (representations) over the quantum enveloping algebra related to M. There are several global models to characterize the regular crystals for a variety of types; e.g., via generalized Young tableaux [11], Lusztig's canonical bases [15], Littelmann's path model [12,14].

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Stembridge [16] pointed out a list of "local" graph-theoretic defining axioms for the regular *simply-laced* crystals. These concern simply-laced Cartan matrices M, i.e., those having coefficients $m_{ii} = 2$ and $m_{ij} = m_{ji} \in \{0, -1\}$ for $i \neq j$. He showed that if M has full rank, then for each n-tuple $\mathbf{c} = (c_1, \dots, c_n)$ of nonnegative integers, there is precisely one graph K satisfying his axioms and such that: K is acyclic and has a unique minimal vertex (*source*) s, and the lengths of maximal monochromatic paths with colors $1, \dots, n$ beginning at s are equal to $c_1, \dots, c_n$, respectively. Moreover, K is a regular crystal related to M (it is the crystal graph of the integrable module of highest weight $\sum_i c_i \omega_i$ over the corresponding quantum enveloping algebra, where ω_i is ith fundamental weight). So in this case (and when M is fixed) $\mathbf{c}$ may be regarded as the *parameter* of K, and we may denote K by $K(\mathbf{c})$.

This paper is devoted to a combinatorial study of regular simply-laced crystals of A_n-type, or *regular A_n-crystals*; for brevity we throughout call them *RAN-crystals*. They are related to the algebra $U_q(sl_{n+1})$, and the off-diagonal coefficients m_{ij} of the Cartan matrix (which is of full rank) are equal to -1 if $|i - j| = 1$, and 0 otherwise.

In our previous paper [3] we described the combinatorial structure of regular A_2-crystals K and demonstrated additional combinatorial and polyhedral properties of these crystals and their extensions. The structure turns out to be rather transparent: K always has a (unique) source, and therefore, $K = K(c_1, c_2)$ for some $c_1, c_2 \in \mathbb{Z}_+$, and it can be produced by a certain operation of replicating and gluing together from the crystals $K(c_1, 0)$ and $K(0, c_2)$. The latter crystals are of simple form and are viewed as triangle-shaped parts of square grids. (In fact, K is the largest component of the tensor product of $K(c_1, 0)$ and $K(0, c_2)$.)

When $n > 2$, the structure of an RAN-crystal becomes much more sophisticated, even for $n = 3$. To explore this structure, in this paper we introduce a certain combinatorial construction, called the *crossing model*. This model consists of three ingredients: (i) a finite directed graph G, called the *supporting graph*, depending only on the number n of colors; (ii) a set $\mathcal{F}$ of integer-valued *feasible functions* on the vertices of G, depending on a parameter $\mathbf{c} \in \mathbb{Z}_+^n$; and (iii) n sets $\mathcal{E}_1, \dots, \mathcal{E}_n$, each consisting of *transformations* $f \mapsto f'$ of feasible functions. (In fact, the crossing model is a sort of decomposition of the Gelfand–Tsetlin pattern model [6].)

Our main working theorem asserts that the n-colored directed graph formed by $\mathcal{F}$ as the vertex set and by $\mathcal{E}_1, \dots, \mathcal{E}_n$ as the edge sets of colors $1, \dots, n$, respectively, is isomorphic to the RAN-crystal $K(\mathbf{c})$. In addition, we explain that *any* finite graph satisfying Stembridge's axioms for the A_n case has a source. Therefore, the crossing model produces precisely the set of crystals of representations for $U_q(sl_{n+1})$. Our construction and proofs rely merely on Stembridge's axiomatics and combinatorial arguments and do not appeal explicitly to powerful tools, such as the Path Model, or so.

Then we take advantages from the description of RAN-crystals via the crossing model. The supporting graph G consists of n pairwise disjoint subgraphs $G^1, \dots, G^n$, and given a parameter $\mathbf{c}$, the values of any feasible function to each G^i $(i = 1, \dots, n)$ are between 0 and c_i. The feasible functions that are constant within each G^i are of especial interest to us. We refer to the vertices of the crystal $K(\mathbf{c})$ corresponding to such functions as *principal* ones, and to the set Π of these vertices as the *principal lattice*. So there are $(c_1 + 1) \times \cdots \times (c_n + 1)$ principal vertices, each corresponding to an n-tuple $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}_+^n$ with $\mathbf{a} \leqslant \mathbf{c}$, being denoted as $v[\mathbf{a}]$. The principal lattice Π is proved to have the following properties:

(i) for any $\mathbf{a}, \mathbf{b} \in \mathbb{Z}_+^n$ with $\mathbf{a} \leqslant \mathbf{b} \leqslant \mathbf{c}$, the interval of $K(\mathbf{c})$ between the principal vertices $v[\mathbf{a}]$ and $v[\mathbf{b}]$ is isomorphic to the RAN-crystal $K(\mathbf{b} - \mathbf{a})$;
(ii) there are exactly $|\Pi|$ maximal (connected) subcrystal of $K(\mathbf{c})$ with colors $1, \dots, n-1$ and each of them contains exactly one principal vertex; a similar property takes place for the maximal subcrystals with colors $2, \dots, n$.

We also establish other features of maximal subcrystals K' with colors $1, \dots, n-1$ (or $2, \dots, n$). In particular, the parameter of K' is expressed by an explicit linear function of $\mathbf{c}$ and $\mathbf{a}$, where $v[\mathbf{a}]$ is the principal vertex in K'.

The crossing model enables us to reveal one more interesting object in the crystal $K(\mathbf{c})$. When a feasible function varies within some subgraph G^i and is constant within each of the other subgraphs G^j of the supporting graph G, we obtain an n-colored subcrystal of $K(\mathbf{c})$ having the parameter $\mathbf{c}'$ such that $c'_i = c_i$ and $c'_j = 0$ for $j \neq i$. (This is the crystal graph of the integrable module of $U_q(sl_{n+1})$ with the highest weight $c_i\omega_i$.) The union of these subcrystals (for all i) forms a canonical subgraph that we call the *skeleton* of $K(\mathbf{c})$. It coincides with the whole crystal $K(\mathbf{c})$ when $n = 2$, and is typically smaller when $n \geqslant 3$.

By use of the crossing model, we also can derive natural infinite analogs of RAN-crystals, in which some or all maximal monochromatic paths are infinite (this generalizes the construction of infinite A_2-crystals in [3]).

This paper is organized as follows. Section 2 states Stembridge's axioms for RAN-crystals, recalls some basic properties of crystals, and briefly reviews results on A_2-crystals from [3]. Also, relying on a structural characterization of regular A_2-crystals, we explain in this section that any RAN-crystal has a source (Corollary 2.5). The crossing model is described throughout Section 3 (concerning the supporting graph and feasible functions) and Section 4 (concerning transformations of feasible functions). The equivalence between the objects generated by the crossing model and the RAN-crystals is proved in Section 5 (Theorem 5.2). Section 6 introduces the principal lattice, principal intervals and the skeleton of an RAN-crystal and explains relations between these objects. Also infinite analogs of RAN-crystals and their properties are discussed in this section. Section 7 is devoted to a study of maximal $(n-1)$-colored subcrystals; here we prove the above-mentioned relation between these subcrystals and the principal lattice, compute their parameters and multiplicities, and discuss additional issues.

Our study of RAN-crystals by use of the crossing model will be continued in the forthcoming paper [5] where we characterize the pairwise intersections of maximal subcrystals with colors $1, \ldots, n-1$ and colors $2, \ldots, n$ and, as a result, obtain a recursive description of the combinatorial structure and an algorithm of assembling of the RAN-crystal for a given parameter. (Also, using results on B_2-crystals from [4], we discuss there a relation between RAN-crystals and regular crystals of types B and C.)

2. Axioms of RAN-crystals and backgrounds

Throughout, by an n-colored digraph we mean a (finite or infinite) directed graph $K = (V(K), E(K))$ with vertex set $V(K)$ and with edge set $E(K)$ partitioned into n subsets $E_1, \ldots, E_n$. We say that an edge in E_i has *color* i and for brevity call it an *i-edge*.

2.1. Axioms

Stembridge [16] pointed out local graph-theoretic axioms that precisely characterize the set of regular simply-laced crystals. The RAN-crystals (which form a subclass of regular simply-laced crystals) are defined by axioms (A1)–(A5) below; we give axiomatics in a slightly different, but equivalent, form compared with [16]. In what follows an n-colored digraph K is assumed to be a (weakly) *connected*, i.e., it is not representable as the disjoint union of two nonempty digraphs.

The first axiom concerns the structure of monochromatic subgraphs (V, E_i).

(A1) For $i = 1, \ldots, n$, each maximal connected subgraph (component) of (V, E_i) is a simple *finite* path, i.e., a sequence of the form $(v_0, e_1, v_1, \ldots, e_k, v_k)$, where $v_0, v_1, \ldots, v_k$ are distinct vertices and each e_i is an edge from v_{i-1} to v_i.

In particular, for each i, each vertex has at most one incoming i-edge and at most one outgoing i-edge, and therefore, one can associate to the set E_i partial invertible operator F_i acting on vertices: (u, v) is an i-edge if and only if F_i is applicable to u and $F_i(u) = v$. Since K is connected, one can use the operator notation to express any vertex via another one. For example, the expression $F_1^{-1}F_3^2F_2(v)$ (where F_p^{-1} stands for the partial operator inverse to F_p) determines the vertex w obtained from a vertex v by traversing 2-edge (v, v'), followed by traversing 3-edges (v', u) and (u, u'), followed by

traversing 1-edge (w, u') in backward direction. Emphasize that every time we use such an operator expression in what follows, this automatically indicates that all involved edges do exist in K.

We refer to a *maximal* monochromatic path with color i on the edges as an *i-line*. The i-line passing a vertex v (possibly consisting of the only vertex v) is denoted by $P_i(v)$, its part from the first vertex to v by $P_i^{\text{in}}(v)$, and its part from v to the last vertex by $P_i^{\text{out}}(v)$. The lengths of $P_i^{\text{in}}(v)$ and $P_i^{\text{out}}(v)$ (i.e., the numbers of edges in these paths) are denoted by $t_i(v)$ and $h_i(v)$, respectively.

Axioms (A2)–(A5) tell us about interrelations of different colors i, j. Taken together, they are equivalent to saying that each component of the digraph $(V(K), E_i \cup E_j)$ forms a regular A_2-crystal when colors i, j are *neighboring*, i.e., $|i-j|=1$, and forms a regular $A_1 \times A_1$-crystal (the Cartesian product of two paths) otherwise.

The second axiom indicates possible changes of the head and tail part lengths of j-lines when one traverses an edge of another color i; these changes depend on the Cartan matrix.

(A2) For any two colors $i \neq j$ and for any edge (u, v) with the color i, one holds $t_j(v) \leqslant t_j(u)$ and $h_j(v) \geqslant h_j(u)$. The value $(t_j(v) - t_j(u)) + (h_j(u) - h_j(v))$ is equal to the coefficient m_{ij} in the Cartan matrix M. Furthermore, h_j is convex on each i-path, in the sense that if (u, v), (v, w) are consecutive i-edges, then $h_j(u) + h_j(w) \geqslant 2h_j(v)$.

This can be rewritten as follows.

(2.1) When $|i-j|=1$, each i-line P contains a vertex r such that: for any edge (u, v) in $P_i^{\text{in}}(r)$, one holds $t_j(v) = t_j(u) - 1$ and $h_j(v) = h_j(u)$, and for any edge (u', v') in $P_i^{\text{out}}(r)$, one holds $t_j(v') = t_j(u')$ and $h_j(v') = h_j(u') + 1$. When $|i-j| \geqslant 2$, any i-edge (u, v) satisfies $t_j(v) = t_j(u)$ and $h_j(v) = h_j(u)$.

Such a vertex r (which is unique) is called the *critical* vertex for P, i, j. It is convenient to assign to each i-edge e *label* $\ell_j(e)$ taking value 0 if e occurs in the corresponding i-line *before* the critical vertex, and 1 otherwise. Emphasize that the critical vertex (and therefore, edge labels) on an i-line P depends on j: the critical vertices on P with respect to the neighboring colors $j = i-1$ and $j = i+1$ may be different.

Two operators $F = F_i^{\alpha}$ and $F' = F_j^{\beta}$, where $\alpha, \beta \in \{1, -1\}$, are said to *commute* at a vertex v if each of F, F' acts at v and $FF'(v) = F'F(v)$. The third axiom points out the situations when operators commute for neighboring colors i, j.

(A3) Let $|i-j|=1$. (a) If a vertex u has outgoing i-edge (u, v) and outgoing j-edge (u, v') and if $\ell_j(u, v) = 0$, then $\ell_i(u, v') = 1$ and $F_i F_j(u) = F_j F_i(u)$. Symmetrically: (b) if a vertex v has incoming i-edge (u, v) and incoming j-edge (u', v) and if $\ell_j(u, v) = 1$, then $\ell_i(u', v) = 0$ and $F_i^{-1} F_j^{-1}(v) = F_j^{-1} F_i^{-1}(v)$. (See the picture.)



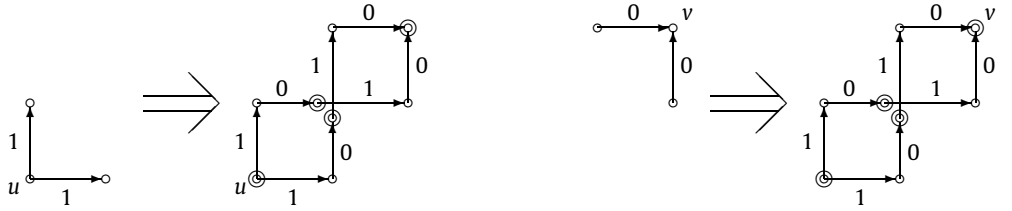
Note that for each "square" u, v, v', w, where $v = F_i(u)$, $v' = F_j(u)$ and $w = F_j(v) = F_i(v')$, the trivial relations $h_j(u) = h_j(v') + 1$ and $h_j(v) = h_j(w) + 1$ imply that the opposite i-edges (u, v) and (v', w) have equal labels ℓ_j; similarly $\ell_i(u, v') = \ell_i(v, w)$. Another important consequence of (A3) is that

(2.2) for $|i-j|=1$, if v is the critical vertex on an i-line with respect to the color j, then v is the critical vertex on the j-line passing v with respect to the color i,

i.e., we can speak of common critical vertices for the pair $\{i, j\}$. Indeed, if a vertex v has incoming i-edge (u, v) with $\ell_j(u, v) = 0$ and outgoing j-edge (v, w), then $h_j(u) = h_j(v) \geqslant 1$, and hence u has outgoing j-edge (u, v'). By (A3), $w = F_i(v')$ and $\ell_i(u, v') = 1$; the latter implies $\ell_i(v, w) = 1$. Symmetrically, if v has outgoing i-edge e with $\ell_j(e) = 1$ and incoming j-edge e', then $\ell_i(e') = 0$.

The fourth axiom points out the situations when for neighboring i, j, the operators F_i, F_j and their inverse ones "remotely commute" (they are said to satisfy the "Verma relation of degree 4").

(A4) Let $|i - j| = 1$. (i) If a vertex u has outgoing edges with the colors i and j and if each edge is labeled 1 (with respect to the other color), then $F_i F_j^2 F_i(u) = F_j F_i^2 F_j(u)$. Symmetrically: (ii) if v has incoming edges with the color i and j and if both are labeled 0, then $F_i^{-1}(F_j^{-1})^2 F_i^{-1}(v) = F_j^{-1}(F_i^{-1})^2 F_j^{-1}(v)$. (See the picture.)



One can show that the labels with respect to i or j of all involved edges are determined uniquely, just as indicated in the above picture (where the circles indicate the critical vertices).

The final axiom concerns nonneighboring colors.

(A5) Let $|i - j| \geqslant 2$. Then for any $F \in \{F_i, F_i^{-1}\}$ and $F' \in \{F_j, F_j^{-1}\}$, the operators F, F' commute at each vertex where both act.

This is equivalent to saying that for $|i - j| \geqslant 2$, each component of the 2-colored subgraph $(V(K), E_i \cup E_j)$ is the Cartesian product of a path with the color i and a path with the color j, i.e., it is an $A_1 \times A_1$-*crystal* (viewed as a rectangular grid).

2.2. Some properties of RAN-crystals

We review some known properties of RAN-crystals that will be used later.

We say that a vertex v of a finite or infinite digraph G is the *source* (resp. *sink*) if any inclusion-wise maximal path begins (resp. ends) at v; in particular, v has zero indegree (resp. zero outdegree). When such a vertex exists, we say that G *has source* (resp. *has sink*). The importance of simply-laced crystals with source is emphasized by a result of Stembridge in [16]; in the A_n case it reads as follows:

(2.3) For any n-tuple $\mathbf{c} = (c_1, \ldots, c_n)$ of nonnegative integers, there exists precisely one RAN-crystal K with source s such that $h_i(s) = c_i$ for $i = 1, \ldots, n$. This K is the crystal graph of the integrable $U_q(sl_{n+1})$-module of highest weight $\mathbf{c}$.

(Hereinafter we usually denote n-tuples in bold.) We say that $\mathbf{c}$ is the *parameter* (tuple) of such a K and denote K by $K(\mathbf{c})$. If we reverse the edges of K while preserving their colors, we again obtain an RAN-crystal (since (A1)–(A5) remain valid for it). It is called the *dual* of K and denoted by K^*.

Another property, indicated in [16] for simply-laced crystals with a nonsingular Cartan matrix, is easy.

(2.4) An RAN-crystal K is *graded* for each color i, which means that for any cycle ignoring the orientation of edges, the number of i-edges in one direction is equal to the number of i-edges in the other direction. (One also says that K admits a weight mapping.) In particular, K is acyclic and has no parallel edges.

(Indeed, associate to each vertex v the n-vector $wt(v)$ whose jth entry is equal to $h_j(v) - t_j(v)$, $j = 1, \dots, n$. Then for each i-edge (u, v), the difference $wt(u) - wt(v)$ coincides with the ith row vector m_i of the Cartan matrix M, in view of axiom (A2) and the obvious equality $h_i(u) - t_i(u) = h_i(v) - t_i(v) + 2$. So under the map $wt: V(K) \to \mathbb{R}^n$, the edges of each color i correspond to parallel translations of one and the same vector m_i, and now (2.4) follows from the fact that the vectors $m_1, \dots, m_n$ are linearly independent.)

In general a regular simply-laced crystal need not have source and/or sink; it may be infinite and may contain directed cycles. One simple result on regular simply-laced crystals in [16] remains valid for more general digraphs, in particular, for a larger class of crystals of representations.

Proposition 2.1. *Let G be an (uncolored) connected and graded digraph with the following property $(*)$: for any vertex v and any edges e, e' entering v, there exist two paths from some vertex w to v such that one path contains e and the other contains e'. Then either G has source or all maximal paths in G are infinite in backward direction.*

(A similar assertion concerns sinks and infinite paths in forward direction. For any RAN-crystal, condition $(*)$ in the proposition is provided by axioms (A3)–(A5).)

Proof. Suppose this is not so. Then, since G is connected and acyclic (as it is graded), there exists a vertex v and two paths P, P' ending at v such that P begins at a zero-indegree vertex s, while P' either is infinite in backward direction or begins at a zero-indegree vertex different from s. Let such v, P, P' be chosen so that the length $|P|$ of P is minimum. Then the last edges $e = (u, v)$ and $e' = (u', v)$ of P and P', respectively, are different. By $(*)$, there is a vertex w, a path Q from w to v containing e and a path Q' from w to v containing e'. Extend Q to a maximal path $\bar{Q}$ ending at v. Three cases are possible: (i) $\bar{Q}$ is infinite in backward direction; (ii) $\bar{Q}$ begins at a (zero-indegree) vertex different from s; and (iii) $\bar{Q}$ begins at s. In cases (i), (ii), we come to a contradiction with the minimality of P by taking the vertex u and the part of P from s to u. And in case (iii), there is a path $\bar{Q}'$ from s to v that contains e'. Since G is graded, $|\bar{Q}'| = |P|$. Then we again get a contradiction with the minimality of P by taking u', the part of $\bar{Q}'$ from s to u', and the part of P' ending at u'. □

(The fact that G is graded is important. Indeed, take G with the vertices s and u_i, v_i for all $i \in \mathbb{Z}_+$, and the edges (s, u_0) and (u_i, u_{i+1}), (v_{i+1}, v_i), (u_i, v_i) for all i. This G satisfies $(*)$, the vertex s has zero indegree, and the path on the vertices v_i is infinite in backward direction. One can also construct a locally finite graph satisfying $(*)$ and having many zero-indegree vertices.)

Our crossing model will generate n-colored graphs satisfying axioms (A1)–(A5); moreover, it generates one RAN-crystal with source for each parameter tuple $\mathbf{c} \in \mathbb{Z}_+^n$. In light of (2.3) and Proposition 2.1, a reasonable question is whether *every* RAN-crystal has source and sink (or, equivalently, is finite). The question will be answered affirmatively in the next subsection, thus implying that the crossing model gives the whole set of RAN-crystals.

As a consequence of the crossing model, we also will observe the following anti-symmetric property of an RAN-crystal K: if we reverse the numeration of colors (regarding each color i as $n-i+1$) in the dual crystal K^*, then the resulting crystal is isomorphic to K. In other words, $h_i(s_K) = t_{n-i+1}(\bar{s}_K)$ for $i = 1, \dots, n$, where s_K and $\bar{s}_K$ are the source and sink of K, respectively.

Finally, recall that a Gelfand–Tsetlin pattern [6], or a *GT-pattern* for short, is a triangular array $X = (x_{ij})_{1 \leqslant j \leqslant i \leqslant n}$ of integers satisfying $x_{ij} \geqslant x_{i-1,j}, x_{i+1,j+1}$ for all i, j. Given a weakly decreasing n-tuple $\mathbf{a} = (a_1 \geqslant \cdots \geqslant a_n)$ of nonnegative integers, one says that X *is bounded* by $\mathbf{a}$ if $a_j \geqslant x_{n,j} \geqslant a_{j+1}$ for $j = 1, \dots, n$, letting $a_{n+1} := 0$. It is known that GT-patterns, as well as the corresponding semi-standard Young tableaux, are closely related to crystals of representations for $U_q(sl_{n+1})$ (cf. [2,9,11,13]). More precisely,

(2.5) for any $\mathbf{c} \in \mathbb{Z}_+^n$, there is a bijection between the vertex set of the RAN-crystal $K(\mathbf{c})$ and the set of GT-patterns bounded by the n-tuple $\mathbf{c}^\Sigma = (c_1^\Sigma, \dots, c_n^\Sigma)$, defined by $c_j^\Sigma := c_1 + \cdots + c_{n-j+1}$ for $j = 1, \dots, n$.

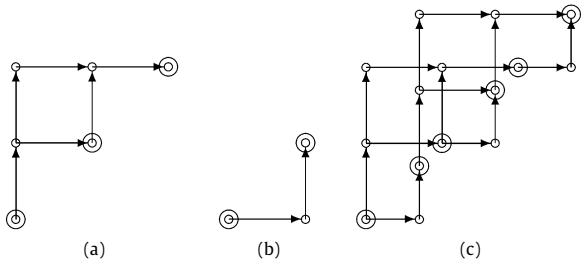


Fig. 1. (a) $K(0,2)$, (b) $K(1,0)$, (c) $K(1,0) \bowtie K(0,2)$.

As mentioned in the Introduction, there is a correspondence between GT-patterns and feasible functions in the crossing model; it will be exposed in Proposition 3.1.

2.3. Properties of A_2-crystals

In this subsection we give a brief review of certain results from [3] for the simplest case $n = 2$, namely, for regular A_2-crystals, or *RA2-crystals* for short. They describe the combinatorial structure of such crystals and demonstrate some additional properties.

An RA2-crystal K is defined by axioms (A1)–(A4) with $\{i, j\} = \{1, 2\}$ (since (A5) becomes redundant). It turns out that these crystals can be produced from elementary 2-colored crystals by use of a certain operation of replicating and gluing together. This operation can be introduced for a pair of arbitrary finite or infinite graphs as follows. (In Section 6 the construction is generalized to n graphs, in connection with the so-called skeleton of an RAN-crystal.)

Consider graphs $G = (V, E)$ and $H = (V', E')$ with distinguished vertex subsets $S \subseteq V$ and $T \subseteq V'$. Take $|T|$ disjoint copies of G, denoted as G_t ($t \in T$), and $|S|$ disjoint copies of H, denoted as H_s ($s \in S$). We glue these copies together in the following way: for each $s \in S$ and each $t \in T$, the vertex s in G_t is identified with the vertex t in H_s. The resulting graph, consisting of $|V||T| + |V'||S| - |S||T|$ vertices and $|E||T| + |E'||S|$ edges, is denoted as $(G, S) \bowtie (H, T)$.

In our special case the role of G and H is played by 2-colored digraphs R and L viewed as triangular parts of square grids. More precisely, R depends on a parameter $c_1 \in \mathbb{Z}_+$ and its vertices v correspond to the integer points (i, j) in the plane such that $0 \leqslant j \leqslant i \leqslant c_1$. The vertices v of L, depending on a parameter $c_2 \in \mathbb{Z}_+$, correspond to the integer points (i, j) such that $0 \leqslant i \leqslant j \leqslant c_2$. We say that v has the *coordinates* (i, j) in the sail. The edges with the color 1 in these digraphs correspond to all possible pairs $((i, j), (i+1, j))$, and the edges with the color 2 to the pairs $((i, j), (i, j+1))$. We call R the *right sail* of size c_1, and L the *left sail* of size c_2.

It is easy to check that R satisfies axioms (A1)–(A4) and is just the crystal $K(c_1, 0)$, and that the set of critical vertices in R coincides with the *diagonal* $D_R = \{(i, i)\colon\ i = 0, \ldots, c_1\}$. Similarly, $L = K(0, c_2)$, and the set of critical vertices in it coincides with the diagonal $D_L = \{(i, i)\colon\ i = 0, \ldots, c_2\}$. These diagonals are just taken as the distinguished subsets in these digraphs. The vertices in D_R (D_L) are ordered in a natural way, according to which (i, i) is referred as the ith critical vertex in R (L).

We refer to the digraph obtained by use of operation $\bowtie$ in this case as the *diagonal-product* of R and L, and for brevity write $R \bowtie L$, omitting the distinguished subsets. The edge colors in the resulting graph are inherited from R and L. Using the above ordering in the diagonals, we may speak of pth right sail in $R \bowtie L$, denoted by R_p. Here $0 \leqslant p \leqslant c_2$, and R_p is the copy of R corresponding to the vertex (p, p) of L. In a similar way, one defines qth left sail L_q in $R \bowtie L$ for $q = 0, \ldots, c_1$. The common vertex of R_p and L_q is denoted by $v_{p,q}$.

One checks that $R \bowtie L$ has source and sink and satisfies axioms (A1)–(A4). Moreover, it is exactly the RA2-crystal $K(c_1, c_2)$. The critical vertices in it are just $v_{p,q}$ for all p, q, the source is $v_{0,0}$ and the sink is v_{c_1,c_2}. The case $c_1 = 1$ and $c_2 = 2$ is illustrated in Fig. 1; here the critical vertices are indicated by circles, 1-edges by horizontal arrows, and 2-edges by vertical arrows.

Theorem 2.2. *(See [3].) Any RA2-crystal K is representable as $K(a, 0) \bowtie K(0, b)$ for some $a, b \in \mathbb{Z}_+$ (in particular, K is finite). The set of RA2-crystals is exactly $\{K(\mathbf{c})\colon\ \mathbf{c} \in \mathbb{Z}_+^2\}$.*

A useful consequence of the above construction is that the vertices v of K one-to-one correspond to the quadruples $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ of integers such that

(2.6) (i) $0 \leqslant \alpha_2 \leqslant \alpha_1 \leqslant c_1$, (ii) $0 \leqslant \beta_1 \leqslant \beta_2 \leqslant c_2$, and (iii) at least one of the equalities $\alpha_2 = \alpha_1$ and $\beta_1 = \beta_2$ takes place,

and each i-edge ($i = 1, 2$) corresponds to the increase by 1 of one of α_i, β_i, subject to maintaining (2.6).

Under this correspondence, if $\beta_1 = \beta_2$ then v occurs in the right sail with the number β_1 and has the coordinates (α_1, α_2) in it, while if $\alpha_2 = \alpha_1$ then v occurs in the left sail with the number α_1 and has the coordinates (β_1, β_2). In particular, a critical vertex $v_{p,q}$ corresponds to (q, q, p, p).

Remark 1. The representation of the vertices of K as the above quadruples satisfying (2.6) gives rise to constructing the crossing model for the simplest case $n = 2$, as we explain in the next section. A more general numerical representation (which is beyond our consideration in this paper) does not impose condition (iii) in (2.6). In this case the admissible transformations of quadruples $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ (giving the edges of a digraph on the quadruples) are assigned as follows. For $\Delta := \min\{\alpha_1 - \alpha_2, \beta_2 - \beta_1\}$, we choose one of $\alpha_1, \alpha_2, \beta_1, \beta_2$ and increase it by 1 unless this increase violates (i) or (ii) in (2.6) or changes Δ. One can see that the resulting digraph Q is the disjoint union of $1 + \min\{c_1, c_2\}$ RA2-crystals, namely, $K(c_1 - \Delta, c_2 - \Delta)$ for $\Delta = 0, \ldots, \min\{c_1, c_2\}$. (This Q is the tensor product of crystals (sails) $K(c_1, 0)$ and $K(0, c_2)$.)

One more useful result in [3] is as follows.

Proposition 2.3. *Part* (ii) *of axiom* (A4) *for RAN-crystals is redundant. Furthermore, axiom* (A4) *itself follows from* (A1)–(A3) *if we add the condition that each component of* $(V, E_i \cup E_j)$ *with* $|i - j| = 1$ *has exactly one zero-indegree* (*or exactly one zero-outdegree*) *vertex.*

In conclusion of this section, return to an arbitrary RAN-crystal K. For a color i, let H_i denote the operator on $V(K)$ that brings a vertex v to the end vertex of the path $P_i(v)$, i.e., $H_i(v) = F_i^{h_i(v)}(v)$ (letting $F_i^0 = \mathrm{id}$). We observe that

(2.7) for neighboring colors i, j and a vertex v, if $h_i(v) = 0$ then the vertex $w = H_i H_j(v)$ satisfies $h_i(w) = h_j(w) = 0$.

Indeed, the RA2-subcrystal with the colors i, j in K that contains v is $K(c_i, c_j)$ for some $c_i, c_j \in \mathbb{Z}_+$. Represent v as quadruple $q = (\alpha_i, \alpha_j, \beta_i, \beta_j)$ in (2.6) (with i, j in place of 1, 2). Then $h_i(q) = 0$ implies $\alpha_i = c_i$ and $\beta_i = \beta_j$. One can see that applying H_j to q results in the quadruple $q' = (c_i, c_i, \beta_i, c_j)$ and applying H_i to q' results in (c_i, c_i, c_j, c_j). This gives (2.7).

Using (2.7), we can show the following important property of RAN-crystals.

Proposition 2.4. *Any RAN-crystal K has a zero-outdegree vertex.*

Proof. For a vertex u, let $p(u)$ be the maximum integer p such that $h_i(u) = 0$ for $i = 1, \ldots, p - 1$. Assuming $p(u) < n + 1$, we claim that the vertex $w = H_1 H_2 \cdots H_{p(u)}(u)$ satisfies $p(w) > p(u)$, whence the result will immediatelly follow. (In other words, by applying the operator $\bar{H}_n \bar{H}_{n-1} \cdots \bar{H}_1$ to an arbitrary vertex, we get a zero-outdegree vertex, where $\bar{H}_i$ stands for $H_1 H_2 \cdots H_i$.)

Indeed, let $p = p(u)$. For the vertex $v_p := H_p(u)$, we have $h_p(v_p) = 0$ and $h_i(v_p) = h_i(u)$ for all $i \neq p - 1, p + 1$ (since the colors p, i commute), while $h_{p-1}(v_p)$ may differ from $h_{p-1}(u)$. So $h_i(v_p) = 0$ for $i = 1, \ldots, p - 2, p$. Similarly, the vertex $v_{p-1} := H_{p-1}(v_p)$ satisfies $h_{p-1}(v_{p-1}) = 0$ and $h_i(v_{p-1}) = h_i(v_p)$ for all $i \neq p - 2, p$. Moreover, applying (2.7) to $v = u$, $i = p - 1$ and $j = p$, we obtain $h_p(v_{p-1}) = 0$. So $h_i(v_{p-1}) = 0$ for $i = 1, \ldots, p - 3, p - 1, p$. On the next step, in a similar fashion one shows that $v_{p-2} := H_{p-2}(v_{p-1})$ satisfies $h_i(v_{p-2}) = 0$ for all $i \in \{1, \ldots, p\} \setminus \{p - 3\}$, and so on. Then

the final vertex $v_1 := H_1 \cdots H_p(u)$ in the process has the property $h_i(v_1) = 0$ for $i = 1, \ldots, p$, as required in the claim. □

Also K has a zero-indegree vertex (since Proposition 2.4 can be applied to the dual crystal K^*). This together with (2.3) and Proposition 2.1 gives the following.

Corollary 2.5. *Every RAN-crystal K is finite and has source and sink. Therefore, $K = K(\mathbf{c})$ for some $\mathbf{c} \in \mathbb{Z}_+^n$.*

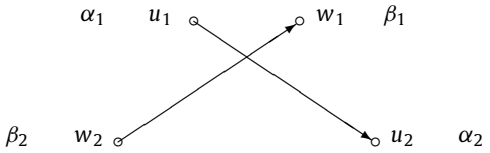
3. Description of the crossing model

As mentioned in the Introduction, the *crossing model* $\mathcal{M}_n$ for RAN-crystals consists of three ingredients:

(i) a certain digraph $G = (V(G), E(G))$ depending only on the number n of colors, called the *supporting graph* (the *structural* part of $\mathcal{M}_n$);
(ii) a certain set $\mathcal{F} = \mathcal{F}(\mathbf{c})$ of nonnegative integer-valued functions on $V(G)$, called *feasible functions*, depending on an n-tuple of parameters $\mathbf{c} \in \mathbb{Z}_+^n$ (the *numerical* part);
(iii) n partial operators acting on $\mathcal{F}$, called *moves* (the *operator* part).

(The feasible functions will correspond to the vertices of the crystal with the parameter $\mathbf{c}$, and the moves to the edges of this crystal.) Parts (i) and (ii) are described in this section, and part (iii) in the next one. To avoid a possible mess when both a crystal and the supporting graph are considered simultaneously, we will refer to a vertex of the latter graph as a *node*.

To explain the idea, we first consider the simplest case $n = 2$ and a 2-colored crystal $K = K(c_1, c_2)$. The model $\mathcal{M}_2$ is constructed by relying on encoding (2.6) of the vertices of K. The supporting graph G is formed by two disjoint edges (u_1, u_2) and (w_2, w_1) (which are related to the elementary crystals, or sails, $K(c_1, 0)$ and $K(0, c_2)$). A feasible function f on $V(G)$ takes values $f(u_1) = \alpha_1$, $f(u_2) = \alpha_2$, $f(w_1) = \beta_1$, $f(w_2) = \beta_2$ for α, β as in (2.6). So the direction of each edge e of G indicates the corresponding inequality to be imposed on the values of any feasible function f on the end nodes of e, and each f one-to-one corresponds to a vertex of K. The graph G is illustrated on the picture:



Note that each admissible quadruple $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ generates the GT-pattern X of size 2 (see Subsection 2.3), defined by $x_{11} := \alpha_1 + \beta_1$, $x_{21} := \beta_2 + c_1$ and $x_{22} := \alpha_2$ (see the diagram below). This pattern is bounded by $\mathbf{c}^\Sigma = (c_1 + c_2, c_1)$.

$$\alpha_1 + \beta_1$$

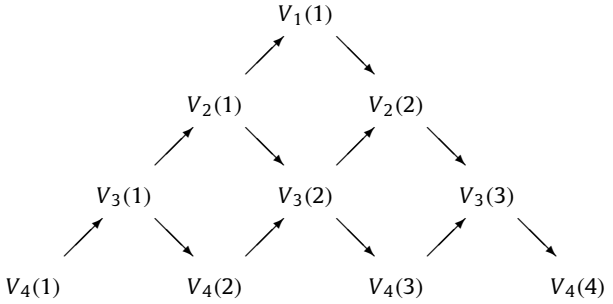
$$c_1 + \beta_2 \qquad\qquad \alpha_2$$

Next we start describing the model for an arbitrary n. The "simplest" case of an n-colored graph $K = K(\mathbf{c})$ arises when all entries in $\mathbf{c} = (c_1, \ldots, c_n)$ are zero except for one entry c_k. In this case we say that K is the kth *base crystal* of size c_k and denote it by $K_n^k(c_k)$.

3.1. The supporting graph of $\mathcal{M}_n$

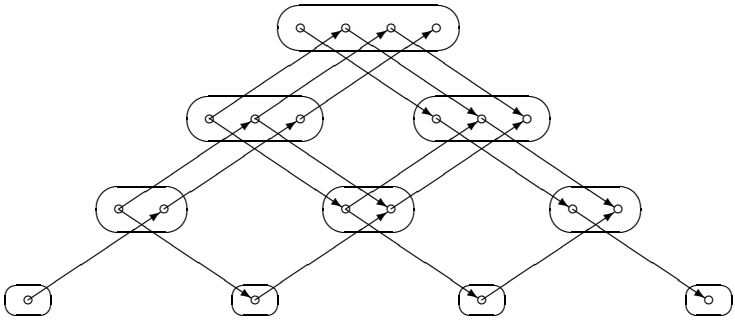
To facilitate understanding the construction of the supporting graph G, we first introduce an auxiliary digraph $\mathcal{G} = \mathcal{G}_n$, called the *proto-graph* of G. Its node set consists of elements $V_i(j)$ for all

$1 \leqslant j \leqslant i \leqslant n$. Its edge set consists of all possible pairs of the form $(V_i(j), V_{i-1}(j))$ (*ascending* edges) or $(V_i(j), V_{i+1}(j+1))$ (*descending* edges). We say that the nodes $V_i(1), \dots, V_i(i)$ form the ith *level* of $\mathcal{G}$ and order them as indicated (by increasing j). We visualize $\mathcal{G}$ by drawing it on the plane so that the nodes of the same level lie on a horizontal line, the edges have equal lengths, the ascending edges point North–East, and the descending edges point South–East. See the picture for $n = 4$.

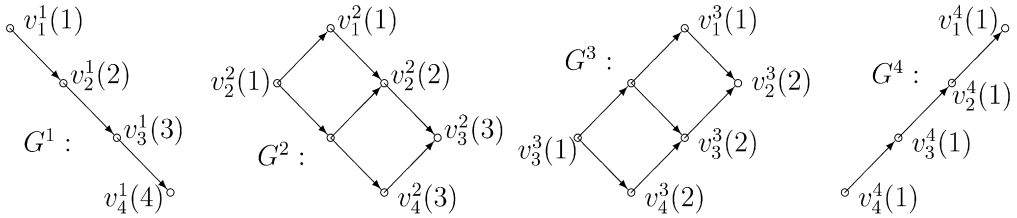


The supporting graph G is formed by replicating elements of $\mathcal{G}$ as follows. Each node $V_i(j)$ generates $n - i + 1$ nodes of G, denoted as $v_i^k(j)$ for $k = i - j + 1, \dots, n - j + 1$, which are ordered by increasing k (and accordingly follow from left to right in the visualization). We identify $V_i(j)$ with the set of these nodes and call it a *multinode* of G. Each edge of $\mathcal{G}$ generates a set of edges of G (a *multi-edge*) connecting the elements with equal upper indexes. More precisely, $(V_i(j), V_{i-1}(j))$ gives $n-i+1$ ascending edges $(v_i^k(j), v_{i-1}^k(j))$ for $k = i-j+1, \dots, n-j+1$, and $(V_i(j), V_{i+1}(j+1))$ gives $n - i$ descending edges $(v_i^k(j), v_{i+1}^k(j+1))$ for $k = i - j + 1, \dots, n - j$.

The resulting G is the disjoint union of n digraphs $G^1, \dots, G^n$. Here $G^k = G_n^k$ contains all vertices of the form $v_i^k(j)$ (the indexes i, j range over $1 \leqslant j \leqslant n-k+1$ and $0 \leqslant i-j \leqslant k-1$, and G^k is viewed as a square (or, better to say, rhombic) grid of size $k-1$ by $n-k$; we shall see later that G^k is, in fact, the supporting graph for the base crystal K_n^k). For example: for $n = 4$, the graph G is viewed as



(where the multinodes are surrounded by ovals) and its components G^1, G^2, G^3, G^4, called the *base subgraphs* of G, are viewed as



Thus, each node $v = v_i^k(j)$ of G has at most four incident edges, namely, $(v_{i-1}^k(j-1), v)$, $(v_{i+1}^k(j), v)$, $(v, v_{i-1}^k(j))$, $(v, v_{i+1}^k(j+1))$; we refer to them, when exist, as the *NW-*, *SW-*, *NE-*, and *SE-edges*, and denote by $e^{\mathrm{NW}}(v)$, $e^{\mathrm{SW}}(v)$, $e^{\mathrm{NE}}(v)$, $e^{\mathrm{SE}}(v)$, respectively.

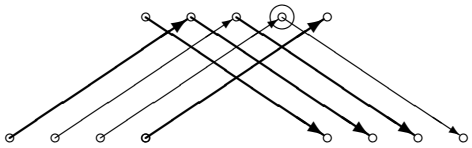
Four nodes of each G^k are distinguished: the leftmost node $v_k^k(1)$, the rightmost node $v_{n-k+1}^k(n-k+1)$, the topmost node $v_1^k(1)$, and the bottommost node $v_n^k(n-k+1)$, denoted by $left^k$, $right^k$, top^k, and $bottom^k$, respectively. Note that $left^k$ is the source and $right^k$ is the sink of G^k.

3.2. Weights of nodes

We consider nonnegative integer-valued functions f on $V(G)$ and refer to the value $f(v)$ as the *weight* of a node v. A function f is called *feasible* if it satisfies the following three conditions. Here for an edge $e = (u, v)$, $\partial f(e)$ denotes the difference $f(u) - f(v)$, and e is called *tight* for f, or f*-tight*, if $\partial f(e) = 0$.

(3.1) (i) f is *monotone* on the edges, in the sense that $\partial f(e) \geqslant 0$ for all $e \in E(G)$;
(ii) $0 \leqslant f(v) \leqslant c_k$ for each $v \in V(G^k)$, $k = 1, \ldots, n$ (or, equivalently, $f(left^k) \leqslant c_k$ and $f(right^k) \geqslant 0$, in view of (i));
(iii) each multinode $V_i(j)$ contains a node v such that: the edge $e^{\mathrm{SE}}(u)$ is tight for each node $u \in V_i(j)$ preceding v, and $e^{\mathrm{SW}}(u')$ is tight for each node $u' \in V_i(j)$ succeeding v.

We say that such a v in (iii) satisfies the *switch* condition. The *first* of such nodes $v = v_i^k(j)$ (i.e., with k minimum) is called the *switch-node* in the multinode $V_i(j)$. It plays an important role in transformations of feasible functions in the model. (We shall see later that the *forward moves*, related to acting operators F_i, handle just switch-nodes, while the *backward moves*, related to acting F_i^{-1}, handle *last* nodes satisfying the switch condition.) See the picture, where tight edges are drawn bold and only one node, marked by a circle, satisfies the switch condition.



The fact that the feasible functions one-to-one correspond to the vertices of the crystal $K(\mathbf{c})$ can be shown by two methods. A direct proof of the assertion that $\mathcal{F}$ along with the moves obeys axioms (A1)–(A5) will be given in Section 5. Another way consists in showing a correspondence to GT-patterns and relies on property (2.5). For $p, q \in \{1, \ldots, n\}$ with $p \leqslant q$, let $c[p:q]$ denote $c_p + \cdots + c_q$. As before, c_j^{Σ} stands for $c[1:n-j+1]$.

Proposition 3.1. *For $1 \leqslant j \leqslant i \leqslant n$, define*

$$x_{i,j} := \bar{f}_i(j) + c[1:i-j], \tag{3.2}$$

where $\bar{f}_i(j)$ denotes the sum of values of f on the nodes in $V_i(j)$. This gives a bijection between the set of feasible functions f and the set of GT-patterns $X = (x_{i,j})$ of size n bounded by $\mathbf{c}^{\Sigma}$.

(Note that this leads to an alternative proof of (2.5), via the crossing model.)

Proof. For a weight function f satisfying (3.1)(i), (ii) (but not necessarily (3.1)(iii)), define X by (3.2). Each multinode $V_n(j)$ in the bottom level consists of the single node $v = v_n^{n-j+1}(j)$, and we have $0 \leqslant f(v) \leqslant c_{n-j+1}$ (since v is in G^{n-j+1}). Therefore, $x_{n,j}$ is between $c[1:n-j]$ and $c[1:n-j+1]$.

The inequality $x_{i,j} \geqslant x_{i+1,j+1}$ is provided by nonincreasing f along the edges from $V_i(j)$ to $V_{i+1}(j+1)$ and by the fact that the term in (3.2) concerning $\mathbf{c}$ is the same for (i,j) and for $(i+1,j+1)$. The inequality $x_{i+1,j} \geqslant x_{i,j}$ follows from nonincreasing f along the edges from $V_{i+1}(j)$ to $V_i(j)$ and from the inequality $c_{i-j+1} \geqslant f(v_i^{i-j+1}(j))$. Thus, X is a GT-pattern bounded by $\mathbf{c}^{\Sigma}$.

Conversely, let X be a GT-pattern bounded by $\mathbf{c}^{\Sigma}$. We construct the desired f step by step, starting from the bottom level. For each node $v = v_n^{n-j+1}(j)$ (forming $V_n(j)$), we define $f(v) := x_{n,j} - c[1 : n-j]$. This value is nonnegative, and (3.2) holds for $i = n$.

Now consider a multinode $V_i(j)$ with $i < n$, assuming that f is already determined for all levels $i' > i$ and satisfies (3.1) and (3.2) for the nodes in these levels and the edges between them. We show that f can be properly extended to the nodes in $V_i(j)$ and that such an extension is unique. Consider an *intermediate* node v in $V_i(j)$ (existing when $i < n-1$). It has both SW- and SE-edges, say, (u,v), (v,w). The weights of u and w (already defined) satisfy $f(u) \geqslant f(w)$ (since for the node v' in the level $i+2$ such that $(u,v') = e^{\mathrm{SE}}(u)$ and $(v',w) = e^{\mathrm{SW}}(w)$, we have $f(u) \geqslant f(v') \geqslant f(w)$). The maximum possible weight of v not violating (3.1)(i) is $f(u)$, while the minimum possible weight is $f(w)$. In its turn, the *first* node v of $V_i(j)$ is connected with the level $i+1$ by the unique edge $e^{\mathrm{SE}}(v)$, say, (v,w), and the maximum possible weight of v is c_{i-j+1} (since v belong to G^{i-j+1}), while the minimum one is $f(w)$. And the *last* node v of $V_i(j)$ is connected with the level $i+1$ by the unique edge $e^{\mathrm{SW}}(v)$, say, (u,v), the maximum possible weight of v is $f(u)$, and the minimum one is zero.

Thus, the maximum assignment of weights for all nodes of $V_i(j)$ would give $\bar{f}_i(j) = \bar{f}_{i+1}(j) + c_{i-j+1}$, implying $x_{i,j} \leqslant \bar{f}_i(j) + c[1 : i-j]$, in view of $x_{i,j} \leqslant x_{i+1,j} = \bar{f}_{i+1}(j) + c[1 : i-j+1]$. And the minimum assignment would give $\underline{f}_i(j) = \underline{f}_{i+1}(j+1)$, implying $x_{i,j} \geqslant \underline{f}_i(j) + c[1 : i-j]$, in view of $x_{i,j} \geqslant x_{i+1,j+1} = \underline{f}_{i+1}(j+1) + c[1 : i-j]$. Therefore, starting with the maximum assignment, scanning the nodes in $V_i(j)$ according to their ordering and decreasing their weights step by step, one can always correct the weights so as to satisfy (3.1)(iii) and (3.2), while maintaining (3.1)(i), (ii). Moreover, (3.1)(iii) guarantees that the weights within $V_i(j)$ are determined uniquely. Eventually, after handling level 1, we obtain the desired function f on $V(G)$. □

4. Moves in the model

So far, we have dealt with the case of nonnegative *upper* bounds (parameters) $c_1, \ldots, c_n$ and zero *lower* bounds, i.e., for any feasible function f, the weight $f(v)$ of each node v of a kth base subgraph lies between 0 and c_k. However, it is useful for us to slightly extend the setting by admitting nonzero lower bounds (in particular, for purposes of Subsection 6.3 where the model is extended to produce crystals with possible infinite monochromatic paths).

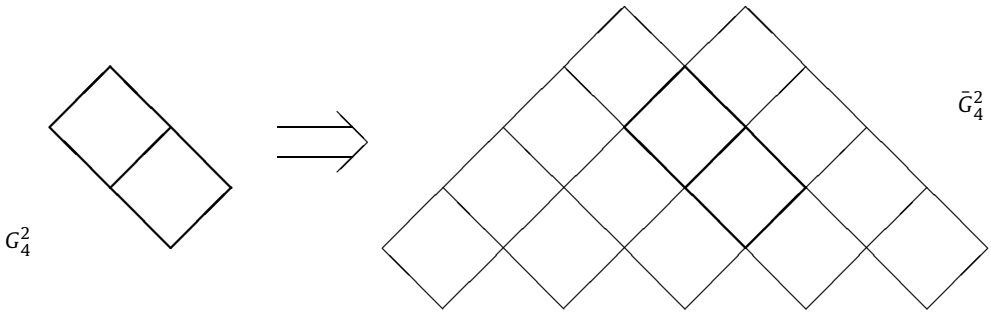
Formally: for $\mathbf{c}, \mathbf{d} \in \mathbb{Z}^n$ with $\mathbf{c} \geqslant \mathbf{d}$, we define a feasible function to be an integer function f on $V(G)$ satisfying (3.1)(i), (iii) and the relation

$$d_k \leqslant f\big(v_i^k(j)\big) \leqslant c_k \quad \text{for all } k, i, j, \tag{4.1}$$

instead of (3.1)(ii). The set of feasible functions for $(\mathbf{c}, \mathbf{d})$ is denoted by $\mathcal{F}(\mathbf{c}, \mathbf{d})$. Clearly the numerical part of the model remains equivalent when for any k, we add a constant to both c_k and d_k and accordingly add this constant to any weight function for G^k. In particular, $\mathcal{F}(\mathbf{c}, \mathbf{d})$ is isomorphic to $\mathcal{F}(\mathbf{c} - \mathbf{d}, \mathbf{0})$, and when $\mathbf{d} = \mathbf{0}$, $\mathcal{F}(\mathbf{c}, \mathbf{d})$ coincides with $\mathcal{F}(\mathbf{c})$ as above.

Now we start describing the desired transformations of functions in $\mathcal{F}(\mathbf{c}, \mathbf{d})$, or moves (that will correspond to edges of the crystal $K(\mathbf{c} - \mathbf{d})$). Each transformation is performed only within one level i, in which case it is called an *i-move*. We need some additional definitions, notation and construction.

First of all, to simplify technical details, we extend each G^k by adding *extra* nodes and edges. More precisely, in the extended digraph $\bar{G}^k$, the node set consists of elements $v_i^k(j)$ for $(i,j) = (0,0)$ and for all i, j such that $0 \leqslant i, j \leqslant n+1$ and $j \leqslant i+1$, except for $(i,j) = (n+1, 0)$. The edge set of $\bar{G}^k$ consists of all possible pairs of the form $(v_i^k(j), v_{i-1}^k(j))$ or $(v_i^k(j), v_{i+1}^k(j+1))$ (as before). An instance is illustrates in the picture; here $n = 4$, $k = 2$, and the thick lines indicate the edges of the original graph G_4^2.



The disjoint union of these $\bar{G}^k$ gives the *extended supporting graph* $\bar{G}$. It possesses the property that the original multinodes become balanced, in the sense that for the set J of index pairs (i, j) satisfying $1 \leqslant j \leqslant i \leqslant n$, the extended multinodes $\bar{V}_i(j)$ contain the same number n of nodes (these are $v_i^1(j), \dots, v_i^n(j)$). Also each node $v = v_i^k(j)$ of $\bar{G}$ with $(i, j) \in J$ has exactly four incident edges, namely, all of $e^{\mathrm{NW}}(v)$, $e^{\mathrm{SW}}(v)$, $e^{\mathrm{NE}}(v)$, and $e^{\mathrm{SE}}(v)$.

Each feasible function on $V(G)$ is extended to the extra nodes $v = v_i^k(j)$ as follows:

(i) put $f(v) := c_k$ if there is a path from v to G^k (equivalently: $j = 0$ or $i - j > k - 1$; one may say that v lies on the left from G^k); and
(ii) put $f(v) := d_k$ otherwise (equivalently: $j > i$ or $j > n - k + 1$, saying that v lies on the right from G^k).

One can see that such an extension maintains conditions (3.1)(i), (ii), (iii) everywhere. Also

(4.2) each edge of $\bar{G}$ with both ends not in G is tight; and for any $(i, j) \in J$, a node $v \in V_i(j)$ satisfies the switch condition in G if and only if it does so in $\bar{G}$.

Given a feasible function f on $V(\bar{G})$, the move from f in a level $i \in \{1, \dots, n\}$ changes f within some multinode in this level. The choice of this multinode depends on so-called residual slacks.

First, for a node $v = v_i^k(j)$, define

$$\epsilon(v) := \partial f\bigl(e^{\mathrm{NW}}(v)\bigr) \quad \text{and} \quad \delta(v) := \partial f\bigl(e^{\mathrm{SE}}(v)\bigr)$$

when the corresponding NW- or SE-edge exists in $\bar{G}$ (i.e., when $i, j \geqslant 1$ in the former case and $i, j \leqslant n$ in the latter case). We call these the *upper slack* and the *lower slack* of f at v, respectively.

Next, define the upper slack $\epsilon_i(j)$ and the lower slack $\delta_i(j)$ at a multinode $\bar{V}_i(j)$ as

$$\epsilon_i(j) := \sum_{k=1}^{n} \epsilon\bigl(v_i^k(j)\bigr) \quad \text{and} \quad \delta_i(j) := \sum_{k=1}^{n} \delta\bigl(v_i^k(j)\bigr) \tag{4.3}$$

(the former when $i, j \geqslant 1$, and the latter when $i, j \leqslant n$). Note that

$$\epsilon_i(1) \geqslant \delta_i(0) \quad \text{for } i = 1, \dots, n \tag{4.4}$$

(as $f(v_{i-1}^k(0)) = f(v_i^k(0)) = c_k$ and $f(v_i^k(1)) \leqslant f(v_{i+1}^k(1))$ for all k). Also $\epsilon_i(i + 1) = 0$.

Finally, we define the *residual upper slack* $\tilde{\epsilon}_i(j)$ and the *residual lower slack* $\tilde{\delta}_i(j)$ by the following rule:

(4.5) (a) for $0 \leqslant p < q \leqslant i+1$, put $\pi(p,q) := \sum_{r=p+1}^{q} \epsilon_i(r) - \sum_{r=p}^{q-1} \delta_i(r)$; (b) for $j = 1, \dots, i+1$, put $\tilde{\epsilon}_i(j) := \max\{0, \min\{\pi(p,j)\colon\ 0 \leqslant p < j\}\}$; and (c) for $j = 0, \dots, i$, put $\tilde{\delta}_i(j) := \max\{0, \min\{-\pi(j,q)\colon\ j < q \leqslant i+1\}\}$.

Remark 2. The residual slacks can also be computed via the following recursive *cancelation process*. Initially, put $\tilde{\epsilon}_i := \epsilon_i$ and $\tilde{\delta}_i := \delta_i$. At each step, choose some pair $j' < j$ such that $\tilde{\epsilon}_i(j) > 0$, $\tilde{\delta}_i(j') > 0$, and $\tilde{\epsilon}_i(q) = \tilde{\delta}_i(q) = 0$ for all $j' < q < j$. Subtract from each of $\tilde{\epsilon}_i(j)$ and $\tilde{\delta}_i(j')$ their minimum. Repeat. Upon termination of this process (when j', j as above no longer exist), we obtain $\tilde{\epsilon}$ and $\tilde{\delta}$ exactly as in (4.5) for all j. This observation will be used in Section 5.

The residual slacks are integers and there exists $j \in \{1, \dots, i\}$ such that

$$\tilde{\delta}_i(0) = \cdots = \tilde{\delta}_i(j-1) = 0 \quad \text{and} \quad \tilde{\epsilon}_i(j+1) = \cdots = \tilde{\epsilon}_i(i+1) = 0. \tag{4.6}$$

(Indeed, suppose $\tilde{\epsilon}_i(j) > 0$ and $\tilde{\delta}_i(j') > 0$ for some $j > j'$. Then, by (4.5)(b), (c), $\pi(j', j) > 0$ and $-\pi(j', j) > 0$; a contradiction.) Take the *minimum* j satisfying (4.6) (if there are many). If $\tilde{\epsilon}_i(j) > 0$, then we say that $\bar{V}_i(j)$ is the *active* multinode in the level i (otherwise $\tilde{\epsilon}_i(1) = \cdots = \tilde{\epsilon}_i(i) = 0$ takes place).

The *moving operator* ϕ_i in level i is applicable when the active multinode $\bar{V}_i(j)$ does exist, and its action is simple: it increases by one the value of f on the switch-node in $\bar{V}_i(j)$, preserving f on all other nodes of $\bar{G}$.

To show that ϕ_i is well defined, we will examine some rhombi of $\bar{G}$, where by a (little) *rhombus* we mean a quadruple ρ of nodes of the form $v_i^k(j)$, $v_{i-1}^k(j)$, $v_i^k(j+1)$, $v_{i+1}^k(j+1)$, called the *left*, *upper*, *right*, and *lower* nodes of ρ, respectively. The following simple observation is useful:

(4.7) for a rhombus ρ, define $\partial f(\rho) := f(u') + f(w') - f(z') - f(v')$, where z', u', v', w' are the left, upper, right and lower nodes of ρ, respectively; then $\partial f(\rho) = \epsilon(v') - \delta(z')$; in particular, $\epsilon(v') - \delta(z')$ is nonnegative if $e^{\mathrm{NE}}(z')$ or $e^{\mathrm{SE}}(z')$ is f-tight, and nonpositive if $e^{\mathrm{NW}}(v')$ or $e^{\mathrm{SW}}(v')$ is f-tight.

It follows that if $\bar{V}_i(j)$ is the active multinode in level i and v is the switch-node in it, then v belongs to G. Indeed, suppose v occurs before the first node of $V_i(j)$. Then, in view of (3.1)(iii) and (4.2), the SW-edges of all nodes of $\bar{V}_i(j)$ are tight, implying $\pi(j-1, j) \leqslant 0$ (by (4.7)), contrary to $\tilde{\epsilon}_i(j) > 0$. The fact that v cannot occur after the last node of $V_i(j)$ is easy as well. So we can speak of active multinodes within G and of the switch-nodes there.

Proposition 4.1. *The function $f' := \phi_i(f)$ is feasible.*

Proof. We have to check validity of (i) and (iii) in (3.1) for f' (then (4.1) for f' will follow automatically). Below, when speaking of switch-nodes or using expressions with ϵ, $\tilde{\epsilon}$, δ, $\tilde{\delta}$, π, we always mean the corresponding objects for f. Let $X := \bar{V}_i(j)$ be the active multinode for f and i. Denote the nodes in X by $v^1, \dots, v^n$ (in this order), and the switch-node by v.

Suppose $\partial f'(e) < 0$ for some edge e. This is possible only if $\partial f(e) = 0$ and e enters v, i.e., e is $e^{\mathrm{NW}}(v)$ or $e^{\mathrm{SW}}(v)$.

(a) Let $e = e^{\mathrm{SW}}(e)$. If $v \neq v^1$, then $\partial f(e) > 0$ (otherwise the switch-node in X would occur before v). So $v = v^1$. Then the SW-edges of all nodes in X are f-tight, by (3.1)(iii). In view of (4.7), this implies $\pi(j-1, j) \leqslant 0$, contrary to $\tilde{\epsilon}_i(j) > 0$.

(b) Now let $e = e^{\mathrm{NW}}(v)$. The beginning node of e belongs to the multinode $V_{i-1}(j-1)$. Consider the rhombi $\rho^1, \dots, \rho^n$ containing $v^1, \dots, v^n$ as right nodes, respectively. Let z^k, u^k, w^k denote, respectively, the left, upper and lower nodes in ρ^k. So $z^1, \dots, z^n$ are the elements of $\bar{V}_i(j-1)$; $u^1, \dots, u^n$ are the elements of $\bar{V}_{i-1}(j-1)$; and $w^1, \dots, w^n$ are the elements of $\bar{V}_{i+1}(j)$ (and the indices grow according to the orderings in these multinodes). Let $v = v^p$, and let u^q be the switch-

node in $\bar{V}_{i-1}(j-1)$. By (3.1)(iii), the edges (w^k, v^k) for $k = p+1, \dots, n$ and the edges $(u^{k'}, v^{k'})$ for $k' = 1, \dots, q-1$ are tight for f. This gives

$$\epsilon\big(v^k\big) \leqslant \delta\big(z^k\big) \quad \text{for } k = 1, \dots, q-1 \text{ and for } k = p+1, \dots, n$$

(in view of (4.7)). Also the tightness of e gives $\epsilon(v^p) \leqslant \delta(z^p)$. Suppose $q < p$. Then u^p occurs in $\bar{V}_{i-1}(j-1)$ after the switch-node u^q, and therefore, (z^p, u^p) is tight for f. We have $f(z^p) = f(u^p) = f(v^p)$, which implies the f-tightness of all edges in ρ^p. Then $\partial f(e^{\mathrm{SW}}(v)) = 0$, contrary to shown in (a). Thus, $q \geqslant p$. This implies $\epsilon(v^k) \leqslant \delta(z^k)$ for all k, and therefore, $\epsilon_i(j) \leqslant \delta_i(j)$; a contradiction.

So, (3.1)(i) for f' is proven. Next, since $\partial f'(e) \leqslant \partial f(e)$ for all SW- and SE-edges e of nodes in $\bar{V}_{i-1}(j-1)$, (3.1)(iii) is valid for f' and this multinode. Also (3.1)(iii) is, obviously, valid for f' and X. It remains to examine the multinode $Y := V_{i-1}(j)$ since for the edge $e = e^{\mathrm{NE}}(v) = (v, u)$, which is the SW-edge for the node u in Y, the value $\partial f'(e)$ becomes greater than $\partial f(e)$. If e is not f-tight or if the *last* node u' in Y satisfying the switch condition for f does not occur before u, then (3.1)(iii) follows automatically.

Suppose $\partial f(e) = 0$ and u' occurs before u. We show that this is not the case by arguing in a way close to (b). For $k = 1, \dots, n$, let z^k, u^k, v^k, w^k denote, respectively, the left, upper, right and lower nodes of the rhombus whose upper node (namely, u^k) is contained in Y. Then the node v (as before) is z^p, and $u' = u^q$ for some p, q with $q < p$. The fact that both v, u' satisfy the switch condition for f (in their multinodes), together with $q < p$, implies that for each $k = 1, \dots, n$, at least one of $\partial f(z^k, u^k)$ and $\partial(z^k, w^k)$ is zero. This gives (cf. (4.7)):

$$\delta\big(z^k\big) \leqslant \epsilon\big(v^k\big) \quad \text{for all } k.$$

Moreover, this inequality is strict for $k = q$. Indeed, we have $\partial f(z^q, w^q) = 0$ and $\partial f(u^q, v^q) > 0$ (otherwise the node in Y next to u would satisfy the switch condition for f as well, but u' is the last of such nodes). So we obtain $\delta_i(j) < \epsilon_i(j+1)$. This implies (in view of $\tilde{\epsilon}_i(j) > 0$ and $\tilde{\delta}_i(j') = 0$ for $j' = 0, \dots, j-1$) that $\tilde{\delta}_i(j) = 0$ and $\tilde{\epsilon}_i(j+1) > 0$, and therefore, the active multinode in level i should occur after $V_i(j)$; a contradiction.

This completes the proof of the proposition. □

In conclusion of this section we discuss one more important aspect.

Backward moves. Besides the above description of partial operators ϕ_i that increase functions in $\mathcal{F}(\mathbf{c}, \mathbf{d})$, we can describe explicitly the corresponding decreasing operators, which make *backward moves*. For $i = 1, \dots, n$, such an operator ψ_i acts on a feasible function f as follows (as before, we prefer to deal with extended functions on $V(\bar{G})$). We take the *first* multinode $V_i(j)$ (with j minimum) in level i for which $\tilde{\delta}_i(j) > 0$; the operator does not act when $\tilde{\delta}_i(j) = 0$ for all j. In view of (4.4), $1 \leqslant j \leqslant n$. In this multinode, called *active in backward direction*, we take the *last* node v possessing the switch condition (3.1)(iii), called the *switch-node in backward direction*. Then the action of ψ_i consists in decreasing the weight $f(v)$ by one, preserving the weights of all other nodes of G.

Proposition 4.2. *The function $f' := \psi_i(f)$ is feasible. Moreover, ϕ_i is applicable to f', and $\phi_i(f') = f$.*

Proof. One can prove this by arguing in a similar spirit as in the proof of Proposition 4.1. Instead, we can directly apply that proposition to a certain reversed model. This is based on a simple observation, as follows.

For a node $v \in V(\bar{G})$, define $\mu(v) := \partial f(e^{\mathrm{NE}}(v))$ and $\nu(v) := \partial f(e^{\mathrm{SW}}(v))$ (when such an NE- or SW-edge exists in $\bar{G}$). The *alternative* upper and lower slacks at a multinode $V_i(j)$ are defined to be, respectively, the sum of numbers $\mu(v)$ and the sum of numbers $\nu(v)$ for the nodes v in this multinode (the former is defined for $j = 0, \dots, i$, and the latter for $j = 1, \dots, i+1$). Compare (4.3).

Considering the little rhombus containing nodes $u = v_i^k(j-1)$ and $v = v_i^k(j)$, we have $\nu(v) - \mu(u) = \epsilon(v) - \delta(u)$ (cf. (4.7)). This gives

$$\nu_i(j) - \mu_i(j-1) = \epsilon_i(j) - \delta_i(j-1). \tag{4.8}$$

The *reversed model* $\mathcal{M}^r$ is obtained by reversing the edges of G, by replacing the upper bound $\mathbf{c}$ by $-\mathbf{d}$, and by replacing the lower bound $\mathbf{d}$ by $-\mathbf{c}$ (one may think that we now read the original model from right to left). Accordingly, a feasible function f in M is replaced by $f^r := -f$. One can see that f^r is feasible for M^r and that the *last* node satisfying the switch condition for f in an original multinode $V_i(j)$ turns into the switch-node for f^r in the corresponding multinode $V_i^r(j')$ in M^r. Also $\epsilon_i^r(j') = \mu_i(j)$ and $\delta_i^r(j') = \nu_i(j)$ (where ϵ^r, δ^r stand for ϵ, δ in the reversed model). In view of (4.8), expressions in (4.5) with f^r in M^r will give $\tilde{\epsilon}_i^r(j') = \tilde{\delta}_i(j)$ and $\tilde{\delta}_i^r(j') = \epsilon_i(j)$ for all j.

These observations enable us to conclude that the function $(f^r)'$ obtained by the forward move from f^r in M^r generates the function $f' = \psi_i(f)$ in M. Therefore, f' is feasible. To see the second part of the proposition, let v be the node of the active in backward direction multinode $V_i(j)$ where f decreases (by one) to produce f'. The edge $e^{\mathrm{SW}}(v)$ is nontight for f', which implies that v is the *unique* node in $V_i(j)$ satisfying the switch condition for f', and therefore, v becomes the switch-node there. Also decreasing f by one at v results in increasing $\epsilon(v)$, and one can see that the residual slack $\tilde{\epsilon}_i(j)$ for f' is greater by one than that for f. This and (4.6) imply that $V_i(j)$ is just the active multinode for f' and i. Hence the forward move from f' increases it by one at v, and we obtain $\phi_i\psi_i(f) = f$, as required. □

By this proposition, the operator ψ_i is injective. The "doubly reversed" model coincides with the original one, and therefore, Proposition 4.2 implies that $\psi_i\phi_i(f) = f$ for each f to which ϕ_i is applicable. So ϕ and ψ are inverse to each other and we may denote ψ_i by ϕ_i^{-1}.

5. The relation of the model to RAN-crystals

We have seen that the feasible functions in the model one-to-one correspond to the vertices of a crystal, by using the GT-pattern model for the latter, see Proposition 3.1. In this section we directly verify that the set $\mathcal{F}$ of these functions and the set of (forward) moves satisfies axioms (A1)–(A5), and therefore, they constitute an RAN-crystal. One may assume that the lower bounds are zero, i.e., $\mathcal{F} = \mathcal{F}(\mathbf{c})$ for $\mathbf{c} \in \mathbb{Z}_+^n$. When the operator ϕ_i is applicable to an $f \in \mathcal{F}$, we say that f and $f' := \phi_i(f)$ are connected by the directed edge (f, f') with the color i; the set of these edges is denoted by $\mathcal{E}_i$. This produces the n-colored digraph $\mathcal{K}(\mathbf{c}) = (\mathcal{F}, \mathcal{E})$ in which $\mathcal{E}$ is partitioned into the color classes $\mathcal{E}_1, \dots, \mathcal{E}_n$. So we are going to show the following.

Theorem 5.1. *$\mathcal{K}(\mathbf{c})$ is an RAN-crystal.*

Proof. As before, it is more convenient to operate with the extended supporting graph $\bar{G}$ and assume that the functions in $\mathcal{F}$ are properly extended to the nodes in $V(\bar{G}) - V(G)$.

Axiom (A1) immediately follows from properties of operators ϕ_i and ψ_i. Next we observe the following. For $f \in \mathcal{F}$ and a color i, if $V_i(j)$ is the active multinode, then the action of ϕ_i decreases $\tilde{\epsilon}_i(j)$ by 1, increases $\tilde{\delta}_i(j)$ by 1, and does not change the residual slacks $\tilde{\epsilon}$ and $\tilde{\delta}$ for the other multinodes in level i. This follows from (4.6) and the fact that under increasing f by 1 at the switch-node v in $V_i(j)$, $\epsilon(v)$ decreases by 1 and $\delta(v)$ increases by 1. Similarly, if $V_i(j')$ is the active multinode in backward direction, then ψ_i decreases $\tilde{\delta}_i(j')$ by 1, increases $\tilde{\epsilon}_i(j')$ by 1, and preserves the residual slacks for the other multinodes in level i. This implies

$$h_i(f) = \sum_{j=1}^{i} \tilde{\epsilon}_i(j) \quad \text{and} \quad t_i(f) = \sum_{j=1}^{i} \tilde{\delta}_i(j), \tag{5.1}$$

regarding f as a vertex of $\mathcal{K}$.

If i, i' are two colors with $|i-i'| \geqslant 2$, then any changes of f in the level i do not affect the numbers $\epsilon(v)$ and $\delta(v)$ for nodes v in the level i'. So $h_{i'}(f) = h_{i'}(f')$ and $t_{i'}(f) = t_{i'}(f')$ for $f' = \phi_i(f)$. This implies validity of axiom (A5).

In order to verify axioms (A2), (A3) and (especially) (A4) for neighboring colors, we need a more careful analysis of the behavior of residual slacks. The following interpretation for the cancelation process (see Remark 2 in Section 4) is of help.

For $f \in \mathcal{F}$ and a fixed level i'', we may think of $V(j)$ as a *box* where $\epsilon(j)$ *white balls* and $\delta(j)$ *black balls* are contained (we omit the subindex i'' hereinafter). Imagine that there is a set C of *couples*, each involving one black ball b from a box $V(j)$ and one white ball w from a box $V(j')$ such that $j < j'$ (each ball occurs in at most one couple). We associate to a couple (b, w) the integer *interval* $[j(b), j(w)]$, where

$$j(q) \text{ denotes the number of the box containing a ball } q.$$

The set $\mathcal{I}$ of these intervals (with possible multiplicities) is required to form an *interval family*, which means that there are no two intervals $[\alpha, \beta]$, $[\alpha', \beta']$ such that $\alpha < \alpha' < \beta < \beta'$ (i.e., no crossing intervals). In particular, the set of maximal intervals in $\mathcal{I}$, not counting multiplicities, forms a linear order in a natural way. Also it is required that: (i) C is maximal, in the sense that there are no uncoupled, or *free*, a black ball b and a white ball w such that $j(b) < j(w)$; and (ii) no free ball lies in the interior of an interval in $\mathcal{I}$.

It is easy to realize that such a C exists and unique, up to recombining couples with equal intervals. We denote the set of free white (free black) balls by W (resp. B) and call (C, W, B) the *arrangement* for the given collection of black and white balls. Furthermore, for each j, the number of free white balls (free black balls) in $V(j)$ is precisely $\tilde{\epsilon}(j)$ (resp. $\tilde{\delta}(j)$).

Let p denote the maximal number $j(w)$ among $w \in W$ (letting $p = -\infty$ if $W = \emptyset$), and q the minimal number $j(b)$ among $b \in B$ (letting $q = \infty$ if $B = \emptyset$). Then $p \leqslant q$. One can see that if some *black* ball b is *removed*, then the arrangement changes as follows (we indicate only the changes important for us).

(5.2) If b is free, it is simply deleted from B. And if b is coupled and occurs in a maximal interval $\sigma = [\alpha, \beta]$, then: (a) if $\beta \leqslant q$ then one of the previously coupled white balls w with $j(w) = \beta$ becomes free (and σ is replaced by a maximal interval $[\alpha, \beta']$ for some $j(b) < \beta' \leqslant \beta$, unless σ vanishes at all); and (b) if $q \leqslant \alpha$, then one free black ball b' whose number $j(b')$ is maximum provided that $j(b') \leqslant \alpha$ becomes coupled and generates the maximal interval $[j(b'), \beta]$.

On the other hand, when a new *white* ball w is *added*, the changes are as follows.

(5.3) In case $j(w) \leqslant q$: (a) if $j(w)$ is in the interior of some maximal interval $[\alpha, \beta]$, then w becomes coupled and one previously coupled white ball w' with $j(w') = \beta$ becomes free; (b) otherwise w is simply added to W. And in case $j(w) > q$: (c) w becomes coupled and one free black ball b with $j(b)$ maximum provided that $j(b) < j(w)$ becomes coupled as well.

Using this interpretation, we now check axioms (A2)–(A4) for neighboring levels (viz. colors) i and $i-1$ in the model. Here for $f \in \mathcal{F}$ in question, the number of the active multinode (the active multinode in backward direction) in the level i is denoted by $p = p(f)$ (resp. $q = q(f)$), and $p' = p'(f)$ and $q' = q'(f)$ stand for the analogous numbers in the level $i-1$ (as before, we use the sign $-\infty$ or ∞ if such a multinode does not exist).

Verification of (A2). When ϕ_i applies to f (at $V_i(p)$), the value $\delta_{i-1}(p-1)$ decreases by 1. (Recall that for $v \in V_i(j)$ and $(u, v) = e^{\mathrm{NW}}(v)$, u belongs to $V_{i-1}(j-1)$.) In the above interpretation, this means that one black ball is removed from the arrangement for the level $i-1$. Then (5.2) implies that in case $p - 1 < q'$, the sum of values $\tilde{\epsilon}_{i-1}(j)$ over j (equal to $h_{i-1}(f)$) increases by 1, while all $\tilde{\delta}_{i-1}(j)$ preserve. And if $p - 1 \geqslant q'$, then the sum of values $\tilde{\delta}_{i-1}(j)$ (equal to $t_{i-1}(f)$) decreases by 1, while all $\tilde{\epsilon}_{i-1}(j)$ preserve. Also in the former case, we obtain $p(f') \leqslant p(f)$ and $q'(f') = q'(f)$, where

$f' := \phi_i(f)$, and therefore, the next application of ϕ_i will fall in the former case as well (further increasing h_{i-1}). Next, when ϕ_{i-1} applies to f, we observe from (5.3) that: in case $p' \leqslant q-1$, the sum of $\tilde{\epsilon}_i(j)$ increases by one, while all $\tilde{\delta}_i(j)$ preserve, and in case $p' \geqslant q$, the sum of $\tilde{\delta}_i(j)$ decreases by one, while all $\tilde{\epsilon}_i(j)$ preserve. Also in the former case, $p'(f') \leqslant p'(f)$ and $q(f') = q(f)$, where $f' := \phi_{i-1}(f)$, so the next application of ϕ_{i-1} increases h_i as well.

Verification of (A3). This is also easy. Let $f' := \phi_i(f)$ and $f'' := \phi_{i-1}(f)$. Suppose (f, f') has label 0. Then $p-1 \geqslant q'$ and $p'(f') = p'(f)$ (see the previous verification). Moreover, the switch-node u in $V_{i-1}(p')$ for f remains the switch-node for f'. (Indeed, since $p' \leqslant p-1$, the slacks of the SW-edges of all nodes in $V_{i-1}(p')$ preserve, and the slacks of their SE-edges do not increase.) In its turn, $p' \leqslant p-1 \leqslant q-1$ implies that (f, f'') has label 1, as required in the axiom. Also neither the active multinode in the level i nor the switch-node v in it can change when ϕ_{i-1} applies to f. Thus, both $\phi_{i-1}\phi_i$ and $\phi_i\phi_{i-1}$ increase the original function f by 1 on the same elements u, v. A verification of the relation $\phi_{i-1}\phi_i(f) = \phi_i\phi_{i-1}(f)$ in the case when (f, f'') has label 0 is similar.

Verification of (A4). This is somewhat more involved. Assuming that both ϕ_i and ϕ_{i-1} are applicable to a feasible function f, define $f_1 := \phi_i(f)$ and $g_1 := \phi_{i-1}(f)$, and let both (f, f_1) and (f, g_1) have label 1. Then $p-1 < q'$ and $p'+1 \leqslant q$ (where $p = p(f)$, and similarly for q, p', q').

Since $\ell(f, f_1) = 1$, we have $h_{i-1}(f_1) = h_{i-1}(f) + 1 \geqslant 2$. Therefore, we can define $f_2 := \phi_{i-1}(f_1)$ and $f_3 := \phi_{i-1}(f_2)$. Similarly, we can define $g_2 := \phi_i(g_1)$ and $g_3 := \phi_i(g_2)$. Our aim is to show that ϕ_i is applicable to f_3, that ϕ_{i-1} is applicable to g_3, and that $\phi_i(f_3) = \phi_{i-1}(g_3)$. Two cases are possible: $p' \leqslant p-1$ and $p' \geqslant p$.

Case $p' \leqslant p-1$. For $k = 1, 2, 3$, we denote $p(f_k)$, $q(f_k)$, $p'(f_k)$, $q'(f_k)$ by p_k, q_k, p'_k, q'_k, respectively; similar numbers for g_k are denoted by $\bar{p}_k, \bar{q}_k, \bar{p}'_k, \bar{q}'_k$. We use the above interpretation and associate to each current function f' the corresponding arrangement $(C = C(f'), W = W(f'), B = B(f'))$ in the level i and the corresponding arrangement $(C' = C'(f'), W' = W'(f'), B' = B'(f'))$ in the level $i-1$.

Since $\tilde{\epsilon}_i(p) > 0$, there is a white ball $w \in W(f)$ with $j(w) = p$. In view of $p-1 < q'$, w corresponds to a coupled black ball b' with $j(b') = p-1$ in the level $i-1$; let $[\alpha', \beta']$ be the maximal interval for $C'(f)$ that contains b'. Then $p-1 < \beta' \leqslant q'$. We also define the number β as follows: if the point $p'+1$ lies in the interior of some maximal interval $[\tilde{\alpha}, \tilde{\beta}]$ for $C(f)$, put $\beta := \tilde{\beta}$; otherwise put $\beta := p'+1$. (The meaning of β is: in view of $p'+1 \leqslant q$, if a new white ball $\hat{w}$ with $j(\hat{w}) = p'+1$ is added in the level i, then the arrangement in this level changes so that there appears a free ball w' with $j(w') = \beta$; see (5.3).) Appealing to the interpretation, we can precisely characterize the changes of $\tilde{\epsilon}_i$, $\tilde{\delta}_i$, $\tilde{\epsilon}_{i-1}$, $\tilde{\delta}_{i-1}$ when the above-mentioned transformations of our functions are carried out.

(i) The transformation $f \to f_1$ decreases $\tilde{\epsilon}_i(p)$ by 1 and increases $\tilde{\delta}_i(p)$ by 1. Also $\tilde{\epsilon}_{i-1}(\beta')$ becomes equal to 1; cf. (5.2)(a).

In particular, $p'_1 = \beta'$, i.e., $V_{i-1}(\beta')$ becomes the active multinode in the level $i-1$.

(ii) The transformation $f_1 \to f_2$ reduces $\tilde{\epsilon}_{i-1}(\beta')$ to 0 and increases $\tilde{\delta}_{i-1}(\beta')$ by 1. Also $\tilde{\delta}_i(r)$ decreases by 1 for some $r \geqslant p = q_1$; cf. (5.3)(c).

This gives $p'_2 = p'$ and $q_2 \geqslant p$ and preserves all intervals for C that lie before p.

(iii) The transformation $f_2 \to f_3$ decreases $\tilde{\epsilon}_{i-1}(p')$ by 1 and changes $\tilde{\delta}_{i-1}(p')$ from 0 to 1. Also $\tilde{\epsilon}_i(\beta)$ increases by 1; cf. (5.3)(a), (b).

The latter property implies $p'+1 \leqslant \beta \leqslant p_3 \leqslant p$. Then ϕ_i is applicable to f_3; define $f_4 := \phi_i(f_3)$. (Furthermore, one can see that $V_i(p_3)$ is the active multinode in the level i for the function $\phi_i\phi_{i-1}(f)$ as well.)

Thus, the combined transformation $\phi_i\phi_{i-1}\phi_{i-1}\phi_i$ consecutively increases f by 1 in the switch-nodes v_0, v_1, v_2, v_3 of $V_i(p)$, $V_{i-1}(\beta')$, $V_{i-1}(p')$, $V_i(p_3)$, respectively, where each switch-node is defined for the current function at the moment of the corresponding transformation. (Note that p' and β' are different, while p and p_3 may coincide.)

Next we examine the other chain of transformations.

(iv) The transformation $f \to g_1$ decreases $\tilde{\epsilon}_{i-1}(p')$ by 1 and increases $\tilde{\delta}_{i-1}(p')$ by 1. Also $\tilde{\epsilon}_i(\beta)$ increases by 1.

From (5.3)(a), (b) it follows that $\beta \leqslant p$, implying $\bar{p}_1 = p$.

(v) The transformation $g_1 \to g_2$ decreases $\tilde{\epsilon}_i(p)$ by 1 and increases $\tilde{\delta}_i(p)$ by 1. Also $\tilde{\delta}_{i-1}(p')$ reduces to 0.

Moreover, (5.2)(b) implies the following important property (∗): $[p', \beta']$ becomes a maximal interval in the new arrangement in the level $i-1$. Also (as mentioned after (iii)) $\bar{p}_2$ coincides with p_3.

(vi) The transformation $g_2 \to g_3$ decreases $\tilde{\epsilon}_i(\bar{p}_2)$ by 1 and increases $\tilde{\delta}_i(\bar{p}_2)$ by 1. Also, in view of $p' + 1 \leqslant \bar{p}_2 \leqslant p$, the interval $[p', \beta']$ in the level $i-1$ (see (∗) above) is destroyed and $\tilde{\epsilon}_{i-1}(\beta')$ becomes equal to 1; cf. (5.2)(a).

So $\bar{p}'_3 = \beta'$ and we can apply ϕ_{i-1} to g_3; let $g_4 := \phi_{i-1}(g_3)$. We assert that $g_4 = f_4$.

To see this, notice that the combined transformation $\phi_{i-1}\phi_i\phi_i\phi_{i-1}$ increases the initial f within the same multinodes as those in the transformation $\phi_i\phi_{i-1}\phi_{i-1}\phi_i$, namely, $V_{i-1}(p')$, $V_i(p)$, $V_i(\bar{p}_2 = p_3)$, $V_{i-1}(\beta')$ (but now the order is different). Let $\bar{v}_0$, $\bar{v}_1$, $\bar{v}_2$, $\bar{v}_3$ be the switch-nodes in these multinodes, respectively (each being taken at the moment of the corresponding transformation). Since no change in the level $i-1$ affects the slacks of SW- and SE-edges in the level i, we have $\bar{v}_1 = v_0$ and $\bar{v}_2 = v_3$. Also $p' + 1 \leqslant \beta$, p implies that the transformations in the level i do not decrease the slacks of the SE-edges of nodes in $V_{i-1}(p')$ and do not change the slacks of their SW-edges, whence $\bar{v}_0 = v_2$.

It remains to check that $\bar{v}_3 = v_1$. Let u be the switch-node in $V_{i-1}(\beta') =: X$ for the initial function f. We have $p \leqslant \beta'$. Therefore, the increase at $v_0 = \bar{v}_1$ can change the switch-node in X only if $p = \beta'$ and if the end u' of the edge $e^{\mathrm{NE}}(v_0)$ is situated after u in the ordering on X. If this is the case, then under each of the transformations $f \to f_1$ and $g_1 \to g_2$ (concerning $V_i(p)$) the switch-node u in X is replaced by u'. Besides these, there is only one transformation in the level i that preceedes the transformation within X, namely, $g_2 \to g_3$. We know that $\bar{p}_2 \leqslant p$ and that if $\bar{p}_2 = p$ then $\bar{v}_2$ coincides with or preceedes v_0 (taking into account that the transformation $g_1 \to g_2$ concerning $V_i(p)$ was applied earlier). This easily implies that $g_2 \to g_3$ can never change the switch-node in X. Thus, $\bar{v}_3 = v_1$.

The case $p' \geqslant p$ is examined in a similar fashion, and we leave it to the reader.

Finally, due to Proposition 2.3, verifying the second part of axiom (A4) (concerning the operators ϕ_i^{-1} and ϕ_{i-1}^{-1}) is not necessary.

This completes the proof of Theorem 5.1. □

Remark 3. In light of the second claim in Proposition 2.3, instead of the tiresome verification of axiom (A4) in the above proof, one may attempt to show that a maximal connected subgraph with colors i and $i-1$ in $\mathcal{K}$ has only one zero-indegree vertex. However, no direct method to show this is known to us.

Clearly the source of the crystal $\mathcal{K}(\mathbf{c})$ is the identically zero function f_0 on $V(G)$, and the sink is the function $f_{\mathbf{c}}$ taking the constant value c_k within each subgraph G^k, $k = 1, \ldots, n$. In particular, this implies that

(5.4) the distance (viz. the number of edges of a path) from the source to the sink, or the *length* of $\mathcal{K}(\mathbf{c})$, is equal to $\sum_{k=1}^{n} c_k |V(G^k)|$, or $\sum_{k=1}^{n} c_k k(n-k+1)$.

Also one can see that for the source function f_0 and a level i, one has $\tilde{\epsilon}_i(1) = c_i$ and $\tilde{\epsilon}_i(j) = 0$ for $j = 2, \dots, i$ (moreover: starting from f_0, each application of ϕ_i increases the weight of $v_i^i(1)$ by 1 until the weight becomes c_i). So $h_i(f_0) = c_i$ for each color i. This means that $\mathcal{K}(\mathbf{c})$ is the crystal $K(\mathbf{c})$, and now the result of Stembridge [16] that there exists exactly one RAN-crystal with source having a prescribed n-tuple $\mathbf{c}$ of parameters (see (2.3)) and Corollary 2.5 enable us to conclude with the following

Theorem 5.2. *The crossing model $\mathcal{M}_n$ generates precisely the set of regular A_n-crystals.*

6. Principal lattice, principal subcrystals, and skeleton

In this section we apply the crossing model to establish certain structural properties of RAN-crystals. We consider the initial setting for the crossing model, i.e., when the upper bounds are non-negative integers and the lower bounds are zeros. So we deal with a parameter tuple $\mathbf{c} = (c_1, \dots, c_n) \in \mathbb{Z}_+^n$ and the set $\mathcal{F}(\mathbf{c})$ of feasible functions in the model. As before, $G = (V(G), E(G))$ is the supporting graph, and $G^k = (V(G^k), E(G^k))$ is kth base subgraph (component) in G. The pair $(\mathcal{F}(\mathbf{c}), \mathcal{E}(\mathbf{c}))$ is isomorphic to the crystal $K = K(\mathbf{c}) = (V, E)$. Recall that F_i denotes ith partial operator on V (corresponding to the partial operator ϕ_i on $\mathcal{F}(\mathbf{c})$), and $K^k = K_n^k(c_k)$ denotes kth base crystal. We will also use the following additional notation:

- $v(f)$ denotes the vertex of K corresponding to a feasible function f;
- $f^1 \sqcup f^2 \sqcup \cdots \sqcup f^n$, where $f^k : V(G^k) \to \mathbb{Z}$ $(i = 1, \dots, n)$, denotes the function on $V(G)$ coinciding with f^k within each G^k;
- $v^k(f^k)$ denotes the vertex of K^k corresponding to a feasible function f^k on $V(G^k)$;
- $C^k a$ denotes the function on $V(G^k)$ taking a constant value $a \in \mathbb{Z}$.

6.1. Principal lattice and principal subcrystals

Among the variety of feasible functions, certain functions are of most interest to us. These are functions f of the form $C^1 a_1 \sqcup \cdots \sqcup C^n a_n$, where each a_k is an integer satisfying $0 \leqslant a_k \leqslant c_k$. Such an f is feasible (since all edges of G are f-tight); we call it a *principal function* and denote by $f[\mathbf{a}]$, where $\mathbf{a} = (a_1, \dots, a_n)$. The corresponding vertex $v(f)$ is called a *principal vertex* of the crystal and denoted by $v[\mathbf{a}]$. In particular, the source and sink of K are the principal vertices $v[\mathbf{0}]$ and $v[\mathbf{c}]$, respectively. So there are $(c_1 + 1) \times \cdots \times (c_n + 1)$ principal vertices; their set is denoted by $\Pi = \Pi(\mathbf{c})$ and called the *principal lattice* in K.

The principal lattice possesses a number of nice properties, described throughout this and next sections. One of them is that the intervals between pairs of principal vertices are RAN-crystals as well, where for vertices u, v in an (acyclic) digraph, the *interval* from u to v is the subgraph $\mathrm{Int}(u, v)$ formed by the vertices and edges lying on paths from u to v.

To show this (and also for purposes of Subsection 6.3), we first consider the crossing model with tuples $\mathbf{c}', \mathbf{d}' \in \mathbb{Z}^n$ of upper and lower bounds, $\mathbf{c}' \geqslant \mathbf{d}'$. This gives the crystal $K(\mathbf{c}' - \mathbf{d}')$, also denoted as $K(\mathbf{c}', \mathbf{d}')$. Let $\mathbf{c}'', \mathbf{d}'' \in \mathbb{Z}^n$ be such that $\mathbf{c}'' \geqslant \mathbf{c}'$ and $\mathbf{d}'' \leqslant \mathbf{d}'$. Clearly

(6.1) any feasible function f for $(\mathbf{c}', \mathbf{d}')$ is feasible for $(\mathbf{c}'', \mathbf{d}'')$ as well.

This gives an injective map γ from the vertex set of $K' = K(\mathbf{c}', \mathbf{d}')$ to the vertex set of the crystal $K'' = K(\mathbf{c}'', \mathbf{d}'')$. Comparing the residual slacks $\tilde{\epsilon}_i'(j)$ and $\tilde{\delta}_i'(j)$ for the function f in the model with the bounds $\mathbf{c}'$, $\mathbf{d}'$ and the residual slacks $\tilde{\epsilon}_i''(j)$ and $\tilde{\delta}_i''(j)$ for f in the model with the bounds $\mathbf{c}''$, $\mathbf{d}''$, one can see that

$$\begin{aligned} &\tilde{\epsilon}_i''(1) = \tilde{\epsilon}_i'(1) + c_i'' - c_i' \quad \text{and} \quad \tilde{\epsilon}_i''(j) = \tilde{\epsilon}_i'(j) \quad \text{for } j = 2, \dots, i; \\ &\tilde{\delta}_i''(i) = \tilde{\delta}_i'(i) + d_i' - d_i'' \quad \text{and} \quad \tilde{\delta}_i''(j') = \tilde{\delta}_i'(j') \quad \text{for } j' = 1, \dots, i-1. \end{aligned} \tag{6.2}$$

Moreover, for each multinode, the switch-nodes concerning f in both models are the same, and similarly for the switch-nodes in backward direction. Also the situation when an active multinode $V_i(j)$ for $f, \mathbf{c}'', \mathbf{d}''$ is not active for $f, \mathbf{c}', \mathbf{d}'$ can arise only if: $j = 1$, the switch-node in $V_i(j)$ is $v_i^i(1)$, and $f(v_i^i(1)) = c_i$; and symmetrically for the active multinodes in backward direction. These observations show that γ is extendable to the edges of K', and moreover,

(6.3) the image of K' by γ is a subcrystal of K'' isomorphic to K', and any path in K'' connecting vertices of $\gamma(K')$ is entirely contained in $\gamma(K')$. Therefore, $\gamma(K')$ is the interval $\mathrm{Int}(\gamma(s_{K'}), \gamma(\bar{s}_{K'}))$ of K'', where $s_{K'}$ and $\bar{s}_{K'}$ are the source and sink of K', respectively.

Note that $\gamma(s_{K'})$ and $\gamma(\bar{s}_{K'})$ are the principal vertices $v[\mathbf{d}' - \mathbf{d}'']$ and $v[\mathbf{c}' - \mathbf{d}'']$ in $K(\mathbf{c}'' - \mathbf{d}'')$, respectively. So we obtain the following

Proposition 6.1. *For $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{Z}_+^n$ with $\mathbf{a} \leqslant \mathbf{b} \leqslant \mathbf{c}$, the interval of $K(\mathbf{c})$ between the principal vertices $v[\mathbf{a}]$ and $v[\mathbf{b}]$ is isomorphic to the RAN-crystal $K(\mathbf{b} - \mathbf{a})$.*

6.2. Skeleton

This is a certain part of a RAN-crystal $K = K(\mathbf{c})$ related to so-called 1-*relaxations* of principal functions. We use notation $\mathbf{a}^{(-k)}$ for an $(n-1)$-tuple of integers a_i where the index i ranges $1, \dots, k-1, k+1, \dots, n$. For $\mathbf{a}^{(-k)}$ satisfying $\mathbf{0}^{(-k)} \leqslant \mathbf{a}^{(-k)} \leqslant \mathbf{c}^{(-k)}$, define $\mathcal{F}[\mathbf{a}^{(-k)}]$ to be the set of all feasible functions $f = f^1 \sqcup \cdots \sqcup f^n$ on $V(G)$ such that $f^i = C^i a_i$ for each $i \neq k$. In other words, the nonfixed part f^k of f is any feasible function for G^k. (The latter is an arbitrary nonnegative integer function g on $V(G^k)$ bounded by c_k and satisfying the monotonicity condition $g(u) \geqslant g(v)$ for each edge $(u, v) \in E(G^k)$. Since the switch condition becomes redundant for G^k taken separately, just all these functions g generate the vertices v of K^k: $v = v^k(g)$.)

Let $K[\mathbf{a}^{(-k)}]$ denote the subgraph of K induced by the set of vertices $v(f)$ for all $f \in \mathcal{F}[\mathbf{a}^{(-k)}]$. For any $f \in \mathcal{F}[\mathbf{a}^{(-k)}]$, all edges in the subgraphs G^i with $i \neq k$ are f-tight. Also each multinode X of G contains at most one node of G^k. These facts imply that the moves from f do not depend on the entries of $\mathbf{a}^{(-k)}$, unless f is transformed within a leftmost multinode $V_i(1)$ not intersecting G^k (i.e., with $i \neq k$). This leads to the following property.

Proposition 6.2. *For any $\mathbf{a}^{(-k)} \leqslant \mathbf{c}^{(-k)}$, the subgraph $K[\mathbf{a}^{(-k)}]$ of $K(\mathbf{c})$ is isomorphic to the base crystal $K^k(c_k)$.*

The union $\mathcal{C}$ of these subgraphs $K[\mathbf{a}^{(-k)}]$ over all k and all $\mathbf{a}^{(-k)} \leqslant \mathbf{c}^{(-k)}$ constitutes the object that we call the *skeleton* of K. Each $K[\mathbf{a}^{(-k)}]$ contains $c_k + 1$ principal vertices $v[\mathbf{a}']$ of $K(\mathbf{c})$; here $a'_i = a_i$ for $i \neq k$ and a'_k runs $0, \dots, c_k$. The corresponding set of $c_k + 1$ vertices in the base crystal $K_n^k(c_k)$ is referred as its *axis* and denoted by $S^k = S^k(c_k)$. (In case $n = 2$, [3] uses the name "diagonal" rather than "axis.")

The proposition below asserts that the skeleton of $K(\mathbf{c})$ is obtained from the base crystals $K^k(c_k)$ by use of a construction which is a natural generalization of the diagonal-product construction for RA2-crystals (see Theorem 2.2) to the case of n colors.

Again (like for $n = 2$) we can describe such a construction for arbitrary graphs $H^1, \dots, H^n$ with distinguished vertex subsets $S^1, \dots, S^n$ (respectively). Let $\mathcal{V}$ be the collection of all n-element sets containing exactly one vertex from each S^i. For $k = 1, \dots, n$, let $\mathcal{V}^{(-k)}$ be the collection of all $(n-1)$-element sets containing exactly one vertex from each S^i with $i \neq k$. For each k, take $|\mathcal{V}^{(-k)}|$ copies of H^k, each being indexed as H_q^k for $q \in \mathcal{V}^{(-k)}$. We glue these copies together by identifying, for each $q = \{v_1, \dots, v_n\} \in \mathcal{V}$ (where $v_k \in S^k$), the copies of vertices v_k in $H_{q\setminus\{v_k\}}^k$, $k = 1, \dots, n$, into one vertex. The resulting graph is denoted as $(H^1, S^1) \bowtie \cdots \bowtie (H^n, S^n)$.

In our case we take as H^k the base crystal $K_n^k(c_k)$, and as the distinguished subset S^k the axis in it. The graph $(H^1, S^1) \bowtie \cdots \bowtie (H^n, S^n)$ is called the *axis-product* and denoted by $\mathcal{A}(\mathbf{c})$ (this is an n-colored digraph where the edge colors are inherited from the base crystals). The principal vertices

in $\mathcal{A}(\mathbf{c})$ are defined to be those obtained by gluing together vertices from the axes of graphs K^k. So the principal vertices of $\mathcal{A}(\mathbf{c})$ one-to-one correspond to the principal functions in the model, or to the n-tuples $\mathbf{a} \in \mathbb{Z}_+^n$ with $\mathbf{a} \leqslant \mathbf{c}$.

Summing up the above explanations, we have the following

Proposition 6.3. *$K(\mathbf{c})$ contains an induced subgraph K' isomorphic to $\mathcal{A}(\mathbf{c}) = (H^1, S^1) \bowtie \cdots \bowtie (H^n, S^n)$ (respecting edge colors). Moreover, K' is determined uniquely and its vertices correspond to the feasible functions $f^1 \sqcup \cdots \sqcup f^k$ for $(G, \mathbf{c})$ such that each f^i is a constant function on $V(G^i)$, except possibly for one function f^k, which is an arbitrary feasible function for (G^k, c_k).*

Here the uniqueness can be shown as follows. The length of a path in K from the source $v[\mathbf{0}]$ to the sink $v[\mathbf{c}]$ is equal to $\sum_{k=1}^n c_k |V(G^k)|$ (see (5.4)). The length of a path from the source to the sink in $\mathcal{A}(\mathbf{c})$ is the same. Therefore (since K is graded), the source of K' must be at $v[\mathbf{0}]$ and the sink of K' must be at $v[\mathbf{c}]$. Now it is easy to realize that K' is reconstructed in K in a unique way.

Next, for two principal vertices $v[\mathbf{a}]$ and $v[\mathbf{b}]$, let us say that the latter is the kth *immediate successor* of the former if $b_k = a_k + 1$ and $a_i = b_i$ for all $i \neq k$. One can see that any possible transformation of the function $f[\mathbf{a}]$ into $f[\mathbf{b}]$ (by use of forward moves in the model) consists of a sequence of $|V(G^k)|$ moves, and the corresponding sequence of nodes where the current function changes forms a *linear order* on $V(G^k)$ agreeable with the poset structure of G^k. In other words, this is an ordering $(u_1, \ldots, u_d)$ of the nodes of G^k such that for each $p = 1, \ldots, d$, the set $U_p = \{u_1, \ldots, u_p\}$ is an ideal in G^k (i.e., no edge goes to U_p from the complement). Each U_p determines the function g_p on $V(G^k)$ taking the value $a_k + 1$ within U_p, and a_k on the rest. Let $\ell(p)$ denote the level number of u_p in G, and let f_p denote the function on $V(G)$ formed from $f[\mathbf{a}]$ by replacing $C^k a_k$ on $V(G^k)$ by g_p. One can check that f_p coincides with the function obtained from f_{p-1} by the move in the level $\ell(p)$ (which just increases the weight of u_p by one). Thus, we have the following

Proposition 6.4. *For $k = 1, \ldots, n$ and a principal vertex v of K, if the kth immediate successor w of v exists, then each paths from v to w in K one-to-one corresponds to a linear order $(u_1, \ldots, u_d)$ for G^k (where $d = |V(G^k)|$). Under this correspondence, the node w can be expressed as $F_{\ell(d)} \cdots F_{\ell(1)} v$, where $\ell(p)$ is the level number of u_p.*

For $k = 1, \ldots, n$, the set of strings $\ell(d) \cdots \ell(1)$ as in this proposition is denoted by $FS_n(k)$; this is invariant for all principal vertices having the kth immediate successor. We refer to any of such strings as a *fundamental* one. As a special case, $FS_n(k)$ contains the fundamental string

(6.4) $S_{n,k} = w_{n,k,n-k+1} \cdots w_{n,k,2} w_{n,k,1}$, where the substring $w_{n,k,i}$ is of the form $(i)(i+1)\cdots(i+k-1)$.

(This corresponds to a route in G^k (according to which the weights of nodes are consecutively increased by 1) consisting of $n - k + 1$ paths, as follows. We starts from the source left^k and go as long as possible in the NE direction, up to the topmost node top^k (obtaining the string $w_{n,k,1}$ of levels). Then we begin at the next node on the SW-side of G^k, namely, $v^k_{k+1}(2)$, and again go in the NE direction (yielding $w_{n,k,2}$), and so on. At the final stage, we begin at the last node on the SW-side, namely, bottom^k, and go up to the sink right^k (yielding $w_{n,k,n-k+1}$).)

Example. Let $n = 3$. Since the graph G^1 forms a path, there is only one fundamental string for $k = 1$, namely, 321. Similarly, $FS_3(3)$ consists of a unique string, namely, 123. The set $FS_3(2)$ for the graph (rhombus) G^2 consists of two strings: 2312 and 2132.

6.3. Infinite crystals

So far, we have dealt with n-colored crystals having a finite set of vertices, or finite crystals. However, by use of the crossing model one can generate infinite analogs of RAN-crystals (arising when

we admit infinite monochromatic paths). Some applications of "crystals" of this sort are indicated in [10] in connection with modified quantized enveloping algebras. Infinite analogs of RA2-crystals are discussed in [3, Sec. 6].

To obtain infinite RAN-crystals, we use the crossing model with double-sided bounds and consider an upper bound $\mathbf{c} \in (\mathbb{Z} \cup \{\infty\})^n$ and a lower bound $\mathbf{d} \in (\mathbb{Z} \cup \{-\infty\})^n$ with $\mathbf{c} \geqslant \mathbf{d}$. More strictly: for a variable $M \in \mathbb{Z}_+$ and each color i, define c_i^M to be c_i if $c_i < \infty$, and $\max\{M, d_i\}$ otherwise, and define d_i^M to be d_i if $d_i > -\infty$, and $\min\{-M, c_i\}$ otherwise. When M grows, there appears a sequence of finite crystals $K(\mathbf{c}^M, \mathbf{d}^M)$, each containing the previous crystal $K(\mathbf{c}^{M-1}, \mathbf{d}^{M-1})$ as a principal interval, by (6.3). At infinity we obtain the desired (well-defined) "infinite crystal" $K(\mathbf{c}, \mathbf{d})$ (when $\mathbf{c}$ or/and $\mathbf{d}$ is not finite).

Some trivial consequences of this construction are as follows. The largest "infinite crystal," denoted by $K_{-\infty}^{\infty}$, arises when $c_i = \infty$ and $d_i = -\infty$ for all i. Among the variety of "crystals" produced by the construction, $K_{-\infty}^{\infty}$ is distinguished by the property that any monochromatic path in it is fully infinite, i.e., infinite in both forward and backward directions (this object was introduced as the free combinatorial A-type crystal by Berenstein and Kazhdan [1]).

Equivalently: the principal lattice of $K_{-\infty}^{\infty}$ is formed by the vertices $v[\mathbf{a}]$ for all $\mathbf{a} \in \mathbb{Z}^n$. Also $K_{-\infty}^{\infty}$ can be regarded as the "universal" RAN-crystal (with n colors), due to the following property:

(6.5) any finite or "infinite" RAN-crystal is a (finite or infinite) principal interval of $K_{-\infty}^{\infty}$, and vice versa.

(An infinite principal interval of the form $\mathrm{Int}(v[\mathbf{a}], \infty)$ (resp. $\mathrm{Int}(-\infty, v[\mathbf{a}])$) is the union of all paths beginning at $v[\mathbf{a}]$ (resp. ending at $v[\mathbf{a}]$).)

7. Subcrystals with $n-1$ colors

In this section we apply the crossing model to study $(n-1)$-colored subcrystals of an RAN-crystal $K = K(\mathbf{c}) = (V, E_1 \cup \cdots \cup E_n)$.

For a subset $J \subset \{1, \dots, n\}$ of colors, let $\mathcal{K}(J)$ denote the set of maximal connected subgraphs of K whose edges have colors from J, i.e., the components of the graph $(V, \bigcup(E_i\colon\ i \in J))$. When the colors in J go in succession, i.e., J is an interval of $(1, \dots, n)$, each member K' of $\mathcal{K}(J)$ is a regular $A_{|J|}$-crystal. (When J has a gap, K' becomes the Cartesian product of several regular crystals. For example, for $J = \{1, 3\}$, K' is the Cartesian product of two paths, with the color 1 and the color 3, or a regular $A_1 \times A_1$-crystal.)

We are interested in the case when J is either $\{1, \dots, n-1\}$ or $\{2, \dots, n\}$, denoting $\mathcal{K}(J)$ by $\mathcal{K}^{(-n)}$ in the former case, and by $\mathcal{K}^{(-1)}$ in the latter case. In other words, $\mathcal{K}^{(-n)}$ (resp. $\mathcal{K}^{(-1)}$) is the set of $(n-1)$-colored crystals arising when the edges with the color n (resp. 1) are removed from K.

Consider $K' \in \mathcal{K}^{(-n)}$ and let $\mathcal{F}(K')$ denote the set of feasible functions corresponding to the vertices of K'. Since K' is connected, any $f \in \mathcal{F}(K')$ can be obtained from any other $f' \in \mathcal{F}(K')$ by a series of forward and backward moves in levels $1, \dots, n-1$. So all functions in $\mathcal{F}(K')$ have one and the same tuple of values within the level n of G. This level consists of nodes $v_n^1(n), v_n^2(n-1), \dots, v_n^n(1)$ (from right to left), and we denote the n-tuple $(f(v_n^1(n)), \dots, f(v_n^n(1)))$ by $\mathbf{a}(K')$, where $f \in \mathcal{F}(K')$. Thus, we have the following property: each subcrystal $K' \in \mathcal{K}^{(-n)}$ contains at most one principal vertex v of K, in which case $v = v[\mathbf{a}]$ for $\mathbf{a} = \mathbf{a}(K')$. Also the members of $\mathcal{K}^{(-n)}$ cover all principal vertices of K.

Similarly, for $K'' \in \mathcal{K}^{(-1)}$ and for the set $\mathcal{F}(K'')$ of feasible functions corresponding to the vertices of K'', the tuple $\mathbf{a}(K'') := (f(v_1^1(1)), \dots, f(v_1^n(1)))$ (where the nodes follow from left to right in the level 1) is the same for all $f \in \mathcal{F}(K'')$. So each subcrystal $K'' \in \mathcal{K}^{(-1)}$ contains at most one principal vertex of K as well, and the members of $\mathcal{K}^{(-1)}$ cover all principal vertices of K.

We show a sharper property.

Proposition 7.1. *Each subcrystal in $\mathcal{K}^{(-n)}$ contains precisely one principal vertex of $K(\mathbf{c})$, and similarly for the subcrystals in $\mathcal{K}^{(-1)}$. In particular, $|\mathcal{K}^{(-n)}| = |\mathcal{K}^{(-1)}| = (c_1 + 1) \times \cdots \times (c_n + 1)$.*

(This property need not hold when an $(n-1)$-element subset of colors is different from $\{1,\dots,n-1\}$ and $\{2,\dots,n\}$.)

Proof. For a node v of the supporting graph G, the maximal path beginning at v and going in the NE direction is called the *NE-path* from v and denoted by $P^{\mathrm{NE}}(v)$. Similarly, the maximal path beginning at v and going in the SE direction is called the *SE-path* from v and denoted by $P^{\mathrm{SE}}(v)$.

Let $K' \in \mathcal{K}^{(-n)}$ and let $\mathbf{a} = (a_1,\dots,a_n) := \mathbf{a}(K')$. Consider an arbitrary function $f \in \mathcal{F}(K')$. We show that the principal function $f[\mathbf{a}]$ can be reached from f by a series of forward moves, followed by a series of backward moves, all in levels $\neq n$, whence the desired inclusion $f[\mathbf{a}] \in \mathcal{F}(K')$ will follow.

To show this, let $\mathcal{F}_0$ be the set of functions $f' \in \mathcal{F}(K')$ that can be obtained from f by (a series of) forward moves in levels $\neq n$ and such that $f'(v_k^k(1)) = f(v_k^k(1)) =: b_k$ for $k = 1,\dots,n$. Take a maximal function f_0 in $\mathcal{F}_0$. We assert that

(7.1) the SW-edges of all nodes in G (where such edges exist) are tight for f_0.

Suppose this is not so for some node, and among such nodes choose a node $v = v_i^k(j)$ with i minimum. Acting as in Section 4, extend G to the graph $\bar{G}$ and extend f_0 to the corresponding function $\bar{f}_0$ on $V(\bar{G})$ by setting the upper bound $\mathbf{b}$ and the lower bound $\mathbf{0}$ (then $\bar{f}_0$ satisfies both the monotonicity condition and the switch condition at each multinode and its values within each subgraph G^k lie between 0 and b_k).

Consider an arbitrary node $v' = v_i^{k'}(j')$ with $1 \leqslant j' \leqslant i$ in the level i and take the rhombus ρ containing v' as the right node; let z', u', w' be the left, upper and lower nodes of ρ, respectively. Then $\partial \bar{f}_0(z',u') = 0$ (this follows from $\bar{f}_0(z') = \bar{f}_0(u') = c_{k'}$ when $j' = 1$, and follows from the minimality of i when $j' > 1$, in view of $(z',u') = e^{\mathrm{SW}}(u')$). This implies $\epsilon(v') \geqslant \delta(z')$ (where these numbers concern the bound $\mathbf{b}$); cf. (4.7). Moreover, this inequality is strict when $v' = v$ (since $(w',v') = e^{\mathrm{SW}}(v')$ and $e^{\mathrm{SW}}(v)$ is not tight).

These observations imply $\tilde{\epsilon}_i(j) > 0$, where $\tilde{\epsilon}$ concerns the bound $\mathbf{b}$. So the level i contains an active multinode, and therefore, $\bar{f}_0$ can be increased by a forward move in this level. This move remains applicable when the bound changes to $\mathbf{c}$; cf. (6.2). Thus, f_0 is not maximal, and this contradiction proves (7.1).

From (7.1) it follows that for each k, all edges of the NE-path from the bottommost node bottom^k in G^k (going to the sink right^k) are f_0-tight. Hence $f(\mathit{right}^k) = a_k$.

Now we apply (a series of) backward moves from f_0 in levels $\neq n$. Let $\mathcal{F}_1$ be the set of functions $f' \in \mathcal{F}(K')$ that can be obtained by such moves and satisfy $f'(\mathit{right}^k) = a_k$ for $k = 1,\dots,n$. Let f_1 be a minimal function in $\mathcal{F}$. Arguing in a similar fashion, one shows that

(7.2) the NW-edges of all nodes in G (where such edges exist) are tight for f_1.

Now (7.2) implies that f_1 is constant within each G^k, i.e., $f_1 = f[\mathbf{a}]$, as required.

To show the assertion concerning $\mathcal{K}^{(-1)}$, we can simply renumber the colors, by regarding each color i as $n-i+1$, and apply the model for this numeration. Clearly the set of principal vertices preserves under this renumbering, and now the result for $\mathcal{K}^{(-1)}$ follows from that for $\mathcal{K}^{(-n)}$. □

Remark 4. Renumbering the colors as above causes a "turn-over" of the original model, so that level i turns into level $n-i+1$. (Note that the model does not maintain this transformation since the switch condition (3.1)(iii) is imposed on SW- and SE-edges of nodes, but not on NW- and NW-ones.) A feasible function f in the original model corresponds to a feasible function f' in the new model, so that f and f' determine the same vertex of the crystal. (In fact, the transformation $f \mapsto f'$ is related to the Schützenberger involution in a crystal.) It seems to be a nontrivial task to explicitly express f' via f (for $n = 2$ an explicit piece-wise linear relation is pointed out in [3]).

We denote the member of $\mathcal{K}^{(-n)}$ (of $\mathcal{K}^{(-1)}$) containing a given principal vertex $v[\mathbf{a}]$ by $K^{(-n)}[\mathbf{a}]$ (resp. $K^{(-1)}[\mathbf{a}]$) and call it the *upper* (resp. *lower*) *subcrystal of* $K(\mathbf{c})$ *determined by* $\mathbf{a}$.

It turns out that one can explicitly express the parameter of $K' = K^{(-n)}[\mathbf{a}]$.

To do this, note that the source and sink of K' correspond to the minimal function $f_{\min}(K')$ and the maximum function $f_{\max}(K')$ in $\mathcal{F}(K')$, respectively. One can see that in each G^k, $f_{\min}(K')$ takes value 0 on all nodes, except for those on the path $P^{\mathrm{SE}}(\mathit{left}^k)$ (from left^k to bottom^k), where the values are identically a_k. (The paths $P^{\mathrm{NE}}(\cdot)$ and $P^{\mathrm{SE}}(\cdot)$ are defined in the proof of Proposition 7.1.) In its turn, $f_{\max}(K')$ takes value c_k on all nodes, except for those on the path $P^{\mathrm{NE}}(\mathit{bottom}^k)$ (from bottom^k to right^k), where the value is a_k. Symmetrically: the source and sink of a subcrystal $K'' = K^{(-1)}[\mathbf{a}]$ correspond to the minimal and maximum functions in $\mathcal{F}(K'')$, respectively, and in each G^k, the former takes value 0 on all nodes, except for those on $P^{\mathrm{NE}}(\mathit{left}^k)$, where the value is a_k, while the latter takes value c_k on all nodes, except for those on $P^{\mathrm{SE}}(\mathit{top}^k)$, where the value is a_k.

Proposition 7.2. *The subcrystal $K^{(-n)}[\mathbf{a}]$ is isomorphic to the crystal $K_{n-1}(\mathbf{q})$ with colors $1,\dots,n-1$, where $q_i = c_i - a_i + a_{i+1}$ for each i. In its turn, $K^{(-1)}[\mathbf{a}]$ is isomorphic to the crystal $K_{n-1}(\mathbf{q}')$ with colors $2,\dots,n$, where $q'_i = c_i - a_i + a_{i-1}$.*

Proof. Consider $f = f_{\min}(K^{(-n)}[\mathbf{a}])$ and $i \in \{1,\dots,n-1\}$. From the above description of f it follows that for each node $v_i^k(j)$ with $j \geqslant 1$ in the extended supporting graph $\bar{G}$, at least one of its NW- and SW-edges is tight for f (extended to $\bar{G}$), except possibly for two nodes in the multinode $V_i(1)$: the first node $v = v_i^i(1)$, in which $\partial f(e^{\mathrm{SW}}(v)) = \partial f(e^{\mathrm{NW}}(v)) = c_i - a_i$, and the second node $v' = v_i^{i+1}(1)$, in which $\partial f(e^{\mathrm{SW}}(v')) = a_{i+1}$ and $\partial f(e^{\mathrm{NW}}(v')) = c_{i+1}$. So, maintaining the monotonicity condition (3.1)(i), one can increase the function (by the operator ϕ_i) only at v or v'. More precisely, the active multinode in the level i is $V_i(1)$ (unless $q_i = 0$) and the switch-node in it is either v or v'. If $a_{i+1} > 0$, then v cannot be the switch-node (since $e^{\mathrm{SW}}(v')$ is not tight). So the switch-node is v', and the operator ϕ_i acts a_{i+1} times at v', making the edge $e^{\mathrm{SW}}(v')$ tight. After that the switch-node becomes v and ϕ_i acts $c_i - a_i$ times at this node. This gives the desired parameter of $K^{(-n)}[\mathbf{a}]$.

(One can argue more formally. For each rhombus ρ of $\bar{G}$ with the left and right nodes in the level i, the value $\partial f(\rho)$ (defined in (4.7)) is zero, except possibly for two rhombi: the rhombus ρ whose right node is v, where $\partial f(\rho) = c_i - a_i$, and the rhombus ρ whose right node is v', where $\partial f(\rho) = a_{i+1}$. This implies that the total residual upper slack $\tilde{\epsilon}_i(1) + \cdots + \tilde{\epsilon}_i(i)$ for f in the level i is just $q_i = c_i - a_i + a_{i+1}$.)

The assertion concerning the lower subcrystal $K^{(-1)}[\mathbf{a}]$ follows by symmetry (when each color i is renumbered as $n - i + 1$). □

Remark 5. This proposition implies that all possible parameters $\mathbf{q}$ of the upper subcrystals of $K(\mathbf{c})$ give the set of integer points of some polytope in $\mathbb{R}^{n-1}$. Note also that for corresponding tuples $\mathbf{q}$ and $\mathbf{a}$, the numbers $a_1,\dots,a_{n-1}$ are determined by $\mathbf{q}$ and a_n, namely: $a_i = c[i:n-1] - q[i:n-1] + a_n$ for $i < n$. This enables us to compute the quantity $\eta(q)$ of crystals in $\mathcal{K}^{(-n)}$ having a prescribed parameter $\mathbf{q}$: this is as large as the set of numbers $a_n \in \mathbb{Z}$ that together with $\mathbf{q}$ determine $\mathbf{a}$ satisfying $0 \leqslant a_i \leqslant c_i$ for all $i = 1,\dots,n$. (One can express $\eta(q)$ as the difference between $\min\{c_n, q[i:n-1] - c[i+1:n-1]\colon\ i = 1,\dots,n-1\}$ and $\max\{0, q[i:n-1] - c[i:n-1]\colon\ i = 1,\dots,n-1\}$. In particular, if $c_i = 0$ takes place for some i, then all upper subcrystals of $K(\mathbf{c})$ are different.) This gives a *branching rule* for decomposing an irreducible sl_{n+1}-module into the sum of irreducible sl_n-modules. In the above expression, the branching rule looks simpler than the rule indicated in [2, Corollary 2.11].

Next, we are able to indicate where a principal vertex $v[\mathbf{a}]$ of $K(\mathbf{c})$ is located in the subcrystals $K^{(-n)}[\mathbf{a}]$ and $K^{(-1)}[\mathbf{a}]$.

Proposition 7.3. *Let Π' be the principal lattice of the upper subcrystal $K' = K^{(-n)}[\mathbf{a}]$. Then the principal vertex $v = v[\mathbf{a}]$ of $K = K(\mathbf{c})$ is contained in Π' and the $(n-1)$-tuple $\mathbf{a}'$ of its coordinates in Π' satisfies $a'_i = a_{i+1}$ for $i = 1,\dots,n-1$. Symmetrically, v is contained in the principal lattice Π'' of the lower subcrystal $K^{(-1)}[\mathbf{a}]$ and the $(n-1)$-tuple $\mathbf{a}''$ of its coordinates in Π'' satisfies $a''_i = a_{i-1}$, $i = 2,\dots,n$.*

Proof. We have to show that the principal vertex v' of K' with the coordinates $\mathbf{a}'$ in Π' coincides with v. By explanations in Subsection 6.2 (applied to K' in place of K), the vertex v' can be obtained from the source of K' by applying the sequence of forward moves (in $\mathcal{M}_{n-1}$) corresponding to the combined string

$$(S_{n-1,n-1})^{a_n}(S_{n-1,n-2})^{a_{n-1}}\cdots(S_{n-1,2})^{a_3}(S_{n-1,1})^{a_2}.$$

For $k=1,\dots,n$, partition $V(G_n^k)$ into two subsets L_k, R_k, where L_k is the set of nodes of the path $P^{\mathrm{SE}}(\mathit{left}^k)$ and R_k is the rest. Note that $R_1=\emptyset$ and that for $k>1$, the subgraph of G_n^k induced by R_k is isomorphic to G_{n-1}^{k-1}. Also: (a) the minimal feasible function $f_{\min}(K')$ for the subcrystal K' (in $\mathcal{M}_n$) takes the constant value a_k on L_k and 0 on R_k, for each k; and (b) the principal function $f[\mathbf{a}]$ takes the value a_k on each $L_k\cup R_k$.

Suppose that $\mathbf{b}\in\mathbb{Z}_+^n$ is such that $\mathbf{b}\leqslant\mathbf{a}$, and that f is the feasible function taking the constant values a_k and b_k within L_k and R_k, respectively, for each k. To obtain the desired result, it suffices to show the following:

(7.3) let $b_k<a_k$ for some $k>1$, and let f' be the feasible function (in $\mathcal{M}_n$) taking the constant value b_k+1 within R_k and coinciding with f on the rest; then f' is obtained from f by applying the sequence of moves (in $\mathcal{M}_n$) corresponding to the fundamental string $S_{n-1,k-1}$.

According to (6.4), $S_{n-1,k-1}=w_{n-1,k-1,n-k+1}\cdots w_{n-1,k-1,2}w_{n-1,k-1,1}$, and for each i, the substring $w_{n-1,k-1,i}=:w'_i$ is $(i)(i+1)\cdots(i+k-2)$. Observe that each w'_i corresponds to the (maximal) NE-path P_i in G_n^k beginning at the node $v^k_{i+k-2}(i)$ (which is the ith node on the SW-side of the rectangular indiced by R_k, viz. on the path $P^{\mathrm{SE}}(v^k_{k-1}(1))$). One can check that the action corresponding to $S_{n-1,k-1}$ changes f only within the base subgraph G_n^k and the action corresponding to w'_i consecutively increases the current function along the path P_i. This results in the function f' as required in (7.3). A verification in details is left to the reader.

The assertion concerning $K^{(-1)}[\mathbf{a}]$ follows by symmetry. □

Note that for $K'=K^{(-n)}[\mathbf{a}]$ and $\mathbf{a}'$ as in Proposition 7.3, if we apply the first part of Proposition 7.2 to K' and the second part to the lower subcrystal $\tilde{K}$ of K' determined by $\mathbf{a}'$, then we obtain that the $(n-2)$-colored crystal $\tilde{K}$ with colors $2,\dots,n-1$ that contains the principal vertex $v[\mathbf{a}]$ of $K(\mathbf{c})$ has the parameter $\mathbf{c}''$ such that $c''_i=(c_i-a_i+a_{i+1})-a'_i+a'_{i-1}=c_i-a_i+a_{i+1}-a_{i+1}+a_i=c_i$ for $i=2,\dots,n-1$. (This $\tilde{K}$ is the component of $K^{(-n)}[\mathbf{a}]\cap K^{(-1)}[\mathbf{a}]$ that contains $v[\mathbf{a}]$.) This leads to a rather surprising property:

(7.4) for any $1\leqslant r\leqslant\lceil n/2\rceil$, all $(n-2r+2)$-colored subcrystals of $K(\mathbf{c})$ with the colors $r,r+1,\dots,n-r+1$ that meet the principal lattice $\Pi(\mathbf{c})$ have the same parameter, namely, $(c_r,\dots,c_{n-r+1})$, and therefore, they are isomorphic.

Finally, using the crossing model, one can compute the lengths of maximal monochromatic paths in $K^{(-n)}[\mathbf{a}]$ (or in $K^{(-1)}[\mathbf{a}]$) that go through the principal vertex $v[\mathbf{a}]$ of K (one can say that the length concerning a color i expresses the "i-width" of the subcrystal at this vertex).

In conclusion of this paper we can add that the crossing model can be used to reveal more structural properties of RAN-crystals. A nontrivial problem on this way is to characterize the intersection of the upper subcrystal $K^{(-n)}[\mathbf{a}]$ and the lower subcrystal $K^{(-1)}[\mathbf{b}]$ of $K(\mathbf{c})$ for any $\mathbf{a},\mathbf{b}\in\mathbb{Z}_+^n$ (this intersection may be empty or consist of one or more subcrystals with colors $2,\dots,n-1$). This problem is solved in the forthcoming paper [5], giving rise to an efficient recursive algorithm of assembling the RAN-crystal $K(\mathbf{c})$ for a given parameter $\mathbf{c}$. Also using the model, we explain there that a regular B_n-crystal (C_n-crystal) with parameter $\mathbf{c}=(c_1,\dots,c_n)$ can be extracted from the "symmetric part" of the regular A_{2n-1}-crystal with parameter $(c_1,\dots,c_n,\dots,c_1)$ (resp. the regular A_{2n}-crystal with parameter $(c_1,\dots,c_n,c_n,\dots,c_1)$).

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