Journal of Algebra 321 (2009) 1313–1325

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

The vanishing-off subgroup

Mark L. Lewis

Department of Mathematical Sciences, Kent State University, Kent, OH 44242, USA

ARTICLE INFO

Article history:
Received 1 July 2008
Available online 18 December 2008
Communicated by Luchezar L. Avramov

Keywords:
Character theory
Subgroups associated with characters
Central series

ABSTRACT

In this paper, we define the vanishing-off subgroup of a nonabelian group. We study the structure of the quotient of this subgroup and a central series obtained from this subgroup.

1. Introduction

Throughout this paper, G is a finite group and $\mathrm{Irr}(G)$ is the set of irreducible characters of G. Following Chapter 12 of [7], we define the *vanishing-off subgroup* $V(\chi)$ of a character χ to be the subgroup generated by the elements of G where χ is not 0. Mathematically, we write $V(\chi) = \langle g \in G \mid \chi(g) \neq 0 \rangle$. It is not difficult to see that $V(\chi)$ is a normal subgroup of G, and if H is any subgroup so that χ vanishes on $G \setminus H$, then $V(\chi) \leqslant H$. In other words, $V(\chi)$ is the smallest subgroup $V \leqslant G$ so that χ vanishes on $G \setminus V$.

When λ is a linear character of G, we know that λ never vanishes on G, so $V(\lambda) = G$. With this in mind, one should only look at $V(\chi)$ when χ is not linear. In fact, if G is a nonabelian group, we define the *vanishing-off subgroup* $V(G)$ *of* G to be the subgroup of G generated by the elements $g \in G$ where there is some nonlinear character $\chi \in \mathrm{Irr}(G)$ so that $\chi(g) \neq 0$. We write $\mathrm{nl}(G)$ for the nonlinear characters in $\mathrm{Irr}(G)$. With this in mind, we have

$$V(G) = \big\langle g \in G \;\big|\; \text{there exists } \chi \in \mathrm{nl}(G) \text{ such that } \chi(g) \neq 0 \big\rangle.$$

Notice that $V(G)$ will be a characteristic subgroup of G. Every character in $\mathrm{nl}(G)$ will vanish on $G \setminus V(G)$, and $V(G)$ is the smallest subgroup $V \leqslant G$ so that every character in $\mathrm{nl}(G)$ vanishes on $G \setminus V$.

E-mail address: lewis@math.kent.edu.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2008.11.024

It is often not the case that $V(G)$ is a proper subgroup of G. Using the remarks in Section 12 of [7], we deduce that $|G : V(G)|$ will divide $\chi(1)^2$ for all $\chi \in \mathrm{nl}(G)$. It follows that all of the primes that divide $|G : V(G)|$ will divide all the nonlinear degrees in $\mathrm{cd}(G)$. Hence, if G has characters in $\mathrm{nl}(G)$ of coprime degrees, then $G = V(G)$. In [1], we along with Bianchi, Chillag, and Pacifici showed that every nonsolvable group will have characters in $\mathrm{nl}(G)$ with coprime degrees. Therefore, $G = V(G)$ when G is nonsolvable. We will also provide examples where $G = V(G)$ when G is solvable and even nilpotent.

However, there are many examples where $V(G) < G$. The most prominent examples are the Camina groups that have been studied in many places including [3,8,12,13]. In fact, the condition for being a Camina group can be stated in terms of $V(G)$. In particular, a group G will be a Camina group if and only if $V(G) = G'$. We should note that this includes both the Frobenius groups with abelian Frobenius complements and extra-special groups. In [11], we generalized the definition of a Camina group. A group G will be a generalized Camina group if and only if $V(G) = \mathbf{Z}(G)G'$.

In this paper, we study some of the properties of $V(G)$. The first easy fact that we prove is the following. We should note that since $G/V(G)$ is trivial when G is not solvable, we do not actually need the hypothesis that G is solvable for the conclusion to hold.

Theorem 1. *Let G be a nonabelian solvable group. Then $G/V(G)$ is either cyclic or an elementary abelian p-group for some prime p.*

To obtain the deeper results regarding the vanishing-off subgroup of G, we look at a central series in terms of $V(G)$. We set $V_1(G) = V(G)$, and for $i \geqslant 2$, we set $V_i(G) = [V_{i-1}(G), G]$. We will compare this series with the lower central series for G. We set the following notation for the lower central series of G. We set $G_1 = G$, and for $i \geqslant 2$, we set $G_i = [G_{i-1}, G]$. Note that $G_2 = G'$. We show that $V_i(G) \leqslant G_i$, and we are interested in the case when $V_i(G) < G_i$ for some $i \geqslant 2$. In this next theorem, we study the case when $V_2(G) < G_2$. We obtain information regarding the terms of this series. In particular, we will show that all of the quotients $G_i/V_i(G)$ will be elementary abelian p-groups for some prime p. In addition, we will show that $G/V_2(G)$ will be what we called a VZ-group in [10] and [11] and those are the class of groups studied extensively in [5].

Theorem 2. *Let G be a group. Then $G_{i+1} \leqslant V_i(G) \leqslant G_i$ for all $i \geqslant 1$. If in addition $V_2(G) < G_2$, then the following hold:*

(1) *There is a prime p so that $G_i/V_i(G)$ is an elementary abelian p-group for all $i \geqslant 1$.*
(2) *There exist positive integers $m \leqslant n$ so that $|G : V(G)| = p^{2n}$ and $|G_2 : V_2(G)| = p^m$, and $\mathrm{cd}(G/V_2(G)) = \{1, p^n\}$.*

When $V_3(G) < G_3$, we can obtain further information regarding the structure of the group. The careful reader will recognize that many of the conclusions of this result are similar to the conclusions that we obtained for generalized Camina groups of nilpotence class 3 in [11] (and many of the results there are probably corollaries of the result here).

Theorem 3. *Suppose G is a group where $V_3(G) < G_3$. Let $Z/V_3(G) = \mathbf{Z}(G/V_3(G))$ and $C/V_3(G) = \mathbf{C}_{G/V_3(G)}(G'/V_3(G))$. Then the following are true:*

(1) $|G : V_1(G)| = |G' : V_2(G)|^2$.
(2) $V_2(G) = Z \cap G'$.
(3) *Either $|G : C| = |G' : V_2(G)|$ or $C = V_1(G)$.*
(4) $V(C) \leqslant V_1(G)$.
(5) *If $V_1(G) < C$, then $C' = V_2(G)$.*
(6) *If $V_1(G) < C$ and $[V_1(G), C] < V_2(G)$, then $|G : C|$ is a square.*

We should note that there are some important variances in this result from the results of [11]. In particular, we were able to prove that $|G : C| = |G' : V_2(G)| (= |G' : G_3|)$ is a square whenever G is a generalized Camina group of nilpotence class 3. When G is not a generalized Camina group but has $V_3(G) < G_3$, we see that either $|G : C| = |G' : V_2(G)|$ or $C = V_1(G)$. We will show that both cases can occur. When $V_1(G) < C$, we need an additional assumption to obtain the conclusion that $|G : C| = |G' : V_2(G)|$ is a square. This additional assumption is actually necessary, as we have examples where $|G : C|$ is not a square.

Following [11], it is tempting to conjecture that $|G_3 : V_3(G)| \leqslant |G' : V_2(G)|$, however, this need not be true since we have an example where $|G_3 : V_3(G)| = |G : V_1(G)| = 4$ and $|G' : V_2(G)| = 2$. In any case, we would like to obtain some bound for $|G_i : V_i(G)|$ in terms of $|G : V(G)|$ when $i \geqslant 3$.

One important fact regarding nilpotent, generalized Camina groups is that their nilpotence class is at most 3, and it is at most 2 if the associated prime is 2. With this in mind, we define the *vanishing height* of G to be the largest value of i so that $V_i(G) < G_i$. It would be tempting to conjecture that $i \leqslant 3$; however, we have examples where $i = 4$, and it seems likely that there is no bound on i.

If G is nilpotent with nilpotence class c, we will show that $Z_{c-1} \leqslant V(G)$ where Z_{c-1} is the $(c-1)$st term in the upper central series for G. We would like to determine if there is a largest nilpotence class c so that $V(G) = Z_{c-1}$. We have an example with $c = 4$. Notice that if G satisfies $V(G) = Z_{c-1}$, then G will have vanishing height c.

2. Generalized Camina pairs

Let N be a normal subgroup of a group G. Following the literature, we say that (G, N) is a Camina pair if every element $g \in G \setminus N$ is conjugate to every element of gN. These pairs were first studied in [3]. In this section, we will define a similar pair called a generalized Camina pair. We will show that generalized Camina pairs are useful in studying the vanishing-off subgroup. We begin with the following general lemma which is proved in [11]. (This result was suggested by work in [3] and [4].)

Lemma 2.1. *Let g be an element of a group G. Then the following are equivalent*:

(1) *The conjugacy class of g is gG'.*
(2) $|\mathbf{C}_G(g)| = |G : G'|$.
(3) *For every $z \in G'$, there is an element $y \in G$ so that* $[g, y] = z$.
(4) $\chi(g) = 0$ *for all nonlinear* $\chi \in \mathrm{Irr}(G)$.

We will say $g \in G$ is a *Camina element* of G if g satisfies one of the equivalent conditions of Lemma 2.1.

Lemma 2.2. *Let N be a subgroup of a group G. If N contains a Camina element for G, then $G' = [G, N]$.*

Proof. Let $g \in N$ be a Camina element for G. Then $G' = [g, G]$ by Lemma 2.1 and $[g, G] \leqslant [N, G] \leqslant G'$. The conclusion then follows. □

Let N be a normal subgroup of G. We say that (G, N) is a *generalized Camina pair* (abbreviated GCP) if every element in $G \setminus N$ is a Camina element for G. (We mention here that we allow $N = G$, although (G, G) is not a very interesting GCP.)

We note that GCPs are related to an idea defined in [9]. Following [9], we define (G, N, M) to be a *Camina triple* if for every element $g \in G \setminus N$, then g is conjugate to all of gM. Notice that (G, N, N) is a Camina triple if and only if (G, N) is a Camina pair, and (G, N, G') is a Camina triple if and only if (G, N) is a GCP.

At this time, we will gather facts about generalized Camina pairs. From Lemma 2.1, we have a number of equivalent conditions for a GCP.

Corollary 2.3. *Let N be a normal subgroup of a group G. The following are equivalent*:

(1) (G, N) *is a GCP.*
(2) $|\mathbf{C}_G(g)| = |G : G'|$ *for all* $g \in G \setminus N$.

(3) *Every character in* $\mathrm{nl}(G)$ *vanishes on* $G \setminus N$.
(4) *For every element* $g \in G \setminus N$ *and every element* $z \in G'$, *there is an element* $y \in G$ *so that* $[g, y] = z$.

Proof. We first suppose that (G, N) is a GCP. Thus, for every $g \in G \setminus N$, then $\mathrm{cl}(g) = gG'$. By Lemma 2.1, this is equivalent to $|\mathbf{C}_G(g)| = |G : G'|$ for every $g \in G \setminus N$. Also, by Lemma 2.1, this is equivalent to every nonlinear character in $\mathrm{Irr}(G)$ vanishing on $G \setminus N$. Finally, by Lemma 2.1, this is equivalent to having for element $z \in G'$ some element $y \in G$ so that $[g, y] = z$. $\square$

We now prove a number of basic facts regarding GCPs.

Lemma 2.4. *If* (G, N) *is a GCP, then the following are true*:

(1) $G' \leqslant N$.
(2) *If* $N \leqslant M < G$, *then* (G, M) *is a GCP.*
(3) *If* K *is a normal subgroup of* G *satisfying* $K \leqslant N$, *then* $(G/K, N/K)$ *is a GCP.*
(4) *If* G *is nonabelian, then* $\mathbf{Z}(G) \leqslant N$.

Proof. Suppose there exists $g \in G' \setminus N$. Then by definition of GCP, g is conjugate to all of $gG' = G'$. This implies that g is conjugate to 1, but this is not possible since $g \notin N$ implies $g \neq 1$. Thus, we have $G' \leqslant N$. Suppose $1 \leqslant G' \leqslant N \leqslant M < G$, and so, M is a normal subgroup of G. If $g \in G \setminus M$, then $g \in G \setminus N$, and so, $\mathrm{cl}(g) = gG'$. We conclude that (G, M) is a GCP. Now, suppose that K is a normal subgroup of G so that $K \leqslant N$. It follows that N/K is a normal subgroup of G/K. Since every nonlinear irreducible character of G vanishes on $G \setminus N$, it follows that every nonlinear character of G/K will vanish on $G/K \setminus N/K$. Thus, $(G/K, N/K)$ is a GCP. Finally, suppose that G is nonabelian. If $g \in \mathbf{Z}(G)$, then g is conjugate only to itself, and thus, g is not conjugate to all of the elements in gG' since $G' > 1$. This implies that $g \in N$. $\square$

We next consider abelian groups and GCPs.

Lemma 2.5. *Let* G *be a group. Then* G *is abelian if and only if* $(G, 1)$ *is GCP.*

Proof. If $(G, 1)$ is a GCP, then $G' = 1$ by Lemma 2.4, and so, G is abelian. On the other hand, if G is abelian, then $(G, 1)$ is a GCP, and the result is proved. $\square$

When G is nilpotent and (G, N) is a GCP with $N < G$, we see that G is essentially a p-group for some prime p.

Lemma 2.6. *Suppose* (G, N) *is a GCP where* G *is nonabelian and nilpotent and* $N < G$. *Then* G/N *is a* p*-group for some prime* p *and* $G = P \times Q$ *where* P *is a* p*-group and* Q *is an abelian* p'*-group.*

Proof. Let p be a prime divisor of $|G : N|$, and consider $\chi \in \mathrm{nl}(G)$. By Corollary 2.3, we know that χ vanishes on $G \setminus N$. This implies that $|G : N|$ divides $\chi(1)^2$ (see p. 200 in [7]). Hence, p divides every nonlinear degree in $\mathrm{cd}(G)$. If we take $G = P \times Q$ where P is the Sylow p-subgroup of G and Q is the Hall p-complement, then every character in $\mathrm{Irr}(Q)$ is linear. Therefore, Q is abelian. As P cannot be abelian, the same argument shows that no primes other than p can divide $|G/N|$. $\square$

3. The subgroup $V(G)$

In this section, we study some of the basic properties of the characteristic subgroup $V(G)$. We begin with a result comparing two GCPs.

Lemma 3.1. *If* (G, N_1) *and* (G, N_2) *are GCPs, then* $(G, N_1 \cap N_2)$ *is a GCP.*

Proof. If $g \in G \setminus (N_1 \cap N_2)$, then either $g \in G \setminus N_1$ or $g \in G \setminus N_2$. In either case, we have $\mathrm{cl}(g) = gG'$. We conclude that $(G, N_1 \cap N_2)$ is a GCP. $\square$

Let G be a group, and let $\mathcal{N}(G) = \{N \leqslant G \mid (G, N) \text{ is a GCP}\}$. We can now enumerate some of the properties of $V(G)$.

Lemma 3.2. *Let G be a nonabelian group. Then the following are true*:

(1) $(G, V(G))$ *is a GCP.*
(2) $G' \leqslant V(G)$.
(3) $\mathbf{Z}(G) \leqslant V(G)$.
(4) *Every element of $G \setminus V(G)$ is a Camina element for G.*
(5) *If (G, N) is a GCP, then $V(G) \leqslant N$.*
(6) $V(G) = \bigcap_{N \in \mathcal{N}(G)} N$.
(7) $V(G) = \prod_{\chi \in \mathrm{nl}(G)} V(\chi)$.
(8) *$V(G)$ is a characteristic subgroup of G.*
(9) *If $V(G) < M$, then $[G, M] = G'$.*

Proof. Let (G, N) be a GCP. By Corollary 2.3, we know that every character in $\mathrm{nl}(G)$ vanishes on $G \setminus N$ and so, $V(G) \leqslant N$. It follows that $V(G) \leqslant \bigcap_{N \in \mathcal{N}(G)} N$. The fact that $(G, V(G))$ is a GCP is an application of Corollary 2.3. Thus, $V(G) \in \mathcal{N}(G)$, and we conclude that $V(G) = \bigcap_{N \in \mathcal{N}(G)} N$. The facts that $G' \leqslant V(G)$, $\mathbf{Z}(G) \leqslant V(G)$, every element of $G \setminus V(G)$ is a Camina element, and every nonlinear character in $\mathrm{Irr}(G)$ vanishes on $G \setminus V(G)$ now follow. Notice that $V(\chi) \leqslant V(G)$ for all $\chi \in \mathrm{nl}(G)$. Thus, $\prod_{\chi \in \mathrm{nl}(G)} V(\chi) \leqslant V(G)$. On the other hand, every character in $\mathrm{nl}(G)$ will vanish on $G \setminus \prod_{\chi \in \mathrm{nl}(G)} V(\chi)$. This implies $V(G) \leqslant \prod_{\chi \in \mathrm{nl}(G)} V(\chi)$, and hence, we get $V(G) = \prod_{\chi \in \mathrm{nl}(G)} V(\chi)$. Any automorphism of G will just permute the elements of $\mathcal{N}(G)$, so $V(G)$ will be characteristic. Finally, if $V(G) < M$, then M must contain a Camina element of G, and so $[G, M] = G'$ by Lemma 2.2. $\square$

We now describe relationship between $V(G)$ and nonabelian quotients of G.

Lemma 3.3. *Let N be a normal subgroup of G so that G/N is nonabelian. Then $N \leqslant V(G)$ and $V(G/N) \leqslant V(G)/N$.*

Proof. We can find a character $\chi \in \mathrm{nl}(G/N)$. Now, observe that $N \leqslant \ker(\chi) \leqslant V(\chi) \leqslant V(G)$. By Lemma 3.2, we know that $(G, V(G))$ is a GCP, and thus, by Lemma 2.4, $(G/N, V(G)/N)$ is a GCP. The conclusion $V(G/N) \leqslant V(G)/N$ is obvious. $\square$

We describe the structure of $G/V(G)$ when G is nonabelian. When G has a quotient that is nonabelian and nilpotent, $G/V(G)$ will be an elementary abelian p-group for some prime p.

Lemma 3.4. *Let G be a group with a nonabelian nilpotent quotient. Then $G/V(G)$ is an elementary abelian p-group for some prime p.*

Proof. Let K be maximal among normal subgroups of G that have a nonabelian quotient. Since G has a nonabelian nilpotent quotient, we may assume that G/K is nilpotent. We know by Lemma 12.3 of [7] that G/K is a p-group for some prime p. Consider a character $\chi \in \mathrm{nl}(G/K)$. It is not difficult to see that $V(\chi)/K = \mathbf{Z}(G/K)$, and so, $G/V(\chi)$ is an elementary abelian p-group by Lemma 12.3(a) of [7]. By Lemma 3.2, we have $V(\chi) \leqslant V(G)$, and so, $G/V(G)$ is a quotient of $G/V(\chi)$. $\square$

When G has a quotient that is a Frobenius group with an abelian Frobenius complement, we show that $G/V(G)$ is cyclic.

Lemma 3.5. *Suppose G has a quotient that is a Frobenius group with an abelian Frobenius complement. Then $G/V(G)$ is cyclic.*

Proof. We now suppose that K is a normal subgroup where G/K is a Frobenius group with Frobenius kernel N/K. Without loss of generality, we may assume that N/K is a chief factor for G. Fix $\chi \in \mathrm{nl}(G/K)$. We know that χ is induced from an irreducible character of N/K, so $K \leqslant \ker(\chi) \leqslant V(\chi) \leqslant N$. Since an irreducible character cannot be a multiple of the regular character of some quotient, it follows that $\ker(\chi) < V(\chi)$. Since N/K is a chief factor, this implies that $K = \ker(\chi)$ and $V(\chi) = N$. Now, G/N is isomorphic to an abelian Frobenius complement, and so, $G/V(\chi)$ is cyclic. Since $V(\chi) \leqslant V(G)$ by Lemma 3.2, this gives the result. $\square$

Using Chapter 12 of [7], we know that if G is a nonabelian solvable group, then it must have a nonabelian quotient that is either nilpotent or a Frobenius group with an abelian Frobenius complement. Thus, we may combine the two previous lemmas to determine the structure of $G/V(G)$ when G is nonabelian and solvable. This yields Theorem 1.

Proof of Theorem 1. Take K to be maximal subject to K being normal and G/K being nonabelian. By Lemma 12.3 of [7], G/K is either a nonabelian p-group or G/K is a Frobenius group with an abelian Frobenius complement. Thus, the previous two lemmas apply and yield the result. $\square$

We now consider the terms of the upper central series of G. Let $Z_1 = \mathbf{Z}(G)$ and $Z_i/Z_{i-1} = \mathbf{Z}(G/Z_{i-1})$ for $i > 1$. We begin with an inductive step.

Lemma 3.6. *Suppose G is a group so that $G' \nleqslant Z_m$, then $Z_{m+1} \leqslant V(G)$.*

Proof. We will prove this lemma by induction on i where $1 \leqslant i \leqslant m+1$. The base case ($i = 1$) is Lemma 3.2. For the inductive case, we assume that $Z_i \leqslant V(G)$ for a given i with $1 \leqslant i \leqslant m$. By Lemma 2.4(3), $(G/Z_{i-1}, V(G)/Z_{i-1})$ is a GCP. Since $G' \nleqslant Z_i$, we can apply Lemma 2.4(4) to G/Z_{i-1}. We obtain $\mathbf{Z}(G/Z_{i-1}) \leqslant V(G)/Z_{i-1}$, and so, $Z_i \leqslant V(G)$. This proves the inductive step, and hence the lemma. $\square$

These next two lemmas show that $V(G)$ must be relatively large. First, when G is nilpotent with class c, we show that $V(G)$ must contain Z_{c-1}.

Corollary 3.7. *Suppose G is nilpotent of nilpotence class c. Then $Z_{c-1} \leqslant V(G)$.*

Proof. We know that $G' \nleqslant Z_{c-2}$, so by Lemma 3.6, we have $Z_{c-1} \leqslant V(G)$. $\square$

Also, when G is not nilpotent we get a similar result. We set $Z_\infty = Z_n$ when for some integer n we have $Z_n = Z_m$ for all $m \geqslant n$. Often Z_∞ is called the hyper-center of G. We show that if G is not nilpotent, then $V(G)$ must contain the hyper-center of G.

Corollary 3.8. *Suppose G is not nilpotent. Then $Z_\infty \leqslant V(G)$.*

Proof. Let n be the integer so that $Z_n = Z_\infty$. We know that $G' \nleqslant Z_n$, and so, $Z_{n+1} \leqslant V(G)$. Since $Z_{n+1} = Z_\infty$, we have the desired conclusion. $\square$

4. The central series associated with $V(G)$

In [11], we studied many of the properties of generalized Camina groups. We will show that many of the properties of generalized Camina groups can be found in general (nilpotent) groups using the vanishing-off subgroup. Much of the work in this section is based on the work in Section 5 of [12].

Our goal in this section is to prove Theorem 3. We prove Theorem 3 in a series of lemmas. When the parent group is clear, we will write V_i in place of $V_i(G)$. We begin with some immediate observations.

Lemma 4.1. *Let G be a group. Then the following facts are true*:

(1) *For all $i \geqslant 1$, then $G_{i+1} \leqslant V_i \leqslant G_i$.*
(2) *If $V_n < G_n$ for some n, then $V_i < G_i$ for all i with $1 \leqslant i \leqslant n$.*
(3) *If $V_n < G_n$, then G/V_n is nilpotent of nilpotence class n and $V_i(G/V_n) = V_i/V_n$ for $1 \leqslant i \leqslant n$.*

Proof. We work by induction on i. By Lemma 2.4, $G' = G_2 \leqslant V_1 \leqslant G_1$, and this gives the base case $i = 1$. We now suppose that $i > 1$, so that $G_i \leqslant V_{i-1} \leqslant G_{i-1}$. This yields $G_{i+1} = [G_i, G] \leqslant [V_{i-1}, G] = V_i$ and $V_i = [V_{i-1}, G] \leqslant [G_{i-1}, G] = G_i$, and (1) is proved. Suppose $V_i = G_i$ for some i. Then using induction one can show that $V_j = G_j$ for all $j \geqslant i$. Hence, if $i \leqslant n$ and $V_n < G_n$, then $V_i < G_i$.

Notice that $G_n \leqslant V_{n-1}$ and V_{n-1}/V_n is central in G/V_n. Hence, G_n/V_n is central in G/V_n and nontrivial. Thus, G/V_n has nilpotence class n.

To prove that $V_i(G/V_n) = V_i/V_n$, we work by induction on i. We start with $i = 1$. By Lemma 3.3, we know that $V_1(G/V_n) = V_1/V_n$. Suppose now that $i > 1$, and $V_{i-1}(G/V_n) = V_{i-1}/V_n$. We know that $V_i(G/V_n) = [V_{i-1}(G/V_n), G/V_n] = [V_{i-1}/V_n, G/V_n] = [V_{i-1}, G]V_n/V_n = V_i/V_n$ which is the desired result. □

We now consider the structure of G/V_2 when $V_2 < G_2 = G'$. First, we need some notation. As in [10] and [11], a group G is a VZ-group if all its nonlinear irreducible characters vanish off of the center. Hence, G is a VZ-group if and only if $V(G) = \mathbf{Z}(G)$. It is easy to see in a VZ-group that all nonlinear characters will be fully ramified with respect to the center. Many results regarding VZ-groups can be found in [5]. In Lemma 2.4 of [11], we noted the following fact:

Lemma 4.2. *(See [11].) Let G be a VZ-group, then $G/\mathbf{Z}(G)$ and G' are elementary abelian p-groups for some prime p. Furthermore, there exist positive integers $m \leqslant n$ so that $|G'| = p^m$ and $|G : \mathbf{Z}(G)| = p^{2n}$. In addition, $\operatorname{cd}(G) = \{1, p^n\}$.*

When $V_2 < G'$, we will see that G/V_2 has the structure of a VZ-group. Notice that this implies that if $|G : V_1|$ is not a square, then $|G' : V_2| = 1$.

Lemma 4.3. *Let G be a group so that $V_2 < G' = G_2$. Then G/V_2 is a VZ-group with $V(G/V_2) = V_1/V_2$. In particular, G/V_1 and G'/V_2 are elementary abelian p-groups for some prime p, and there exist positive integers $m \leqslant n$ so that $|G : V_1| = p^{2n}$ and $|G' : V_2| = p^m$. In addition, $\operatorname{cd}(G/V_2) = \{1, p^n\}$.*

Proof. Let $V = V_1$ and $W = V_2$. Since $W < G'$, it follows that G/W is a nonabelian group. By Lemma 4.1, $V/W = V(G/W)$. It is easy to see that V/W is contained in $\mathbf{Z}(G/W)$, and by Lemma 3.2, $V(G/W)$ contains $\mathbf{Z}(G/W)$. Therefore, $V(G/W) = \mathbf{Z}(G/W)$, and G/W is a VZ-group. The remaining results follow from Lemma 4.2. □

We still assume that $V_2 < G' = G_2$. Following Lemma 4.3, we write $|G : V_1| = p^{2n}$ and $|G' : V_2| = p^m$ (where $m \leqslant n$).

Lemma 4.4. *Assume $V_2 < G_2$. Then G_i/V_i is an elementary abelian p-group for all $i \geqslant 1$ for some prime p.*

Proof. We work by induction on i. If $i = 1, 2$, then this is Lemma 4.3. Thus, we may assume that $i \geqslant 3$. Also, by the inductive hypothesis, we have that G_{i-1}/V_{i-1} is an elementary abelian p-group. Without loss of generality, we may assume that $V_i = 1$. Notice that this implies that V_{i-1} is central in G. We have $G_i \leqslant V_{i-1}$, and so, G_i is central in G. As G_i is abelian, it suffices to show that it can be generated by elements of order p. Recall that G_i is generated by elements of the form $[x, w]$ where $x \in G_{i-1}$ and $w \in G$. By inductive hypothesis, we know that $x^p \in V_{i-1}$. Since G_i and V_{i-1} are central in G, we have $[x, w]^p = [x^p, w] = 1$. This implies that the generators of G_i all have order p, and the result follows. $\square$

We compute the index of an important centralizer in this next lemma.

Lemma 4.5. *Assume $V_2 < G_2$. Suppose $a \in G \setminus V_1$ and $A/V_2 = \mathbf{C}_{G/V_2}(aV_2)$. Then $|G : A| = |G' : V_2| = p^m$.*

Proof. Consider the map $x \mapsto [a, x]V_2$ from G to G'/V_2. We know that $G' \leqslant V_1$ and V_1/V_2 is central in G/V_2, so G'/V_2 is central in G/V_2. It follows that this map is a homomorphism, and it is easy to see that its kernel is A. Since $a \notin V_1$ and (G, V_1) is a GCP, this homomorphism will be onto, and the result is just the first isomorphism theorem. $\square$

For the rest of this section, we will consider the hypothesis that $V_3 < G_3$. Suppose that G has nilpotence class 3 and $V_1 = Z_2(G)$, then $V_2 \leqslant \mathbf{Z}(G)$, and hence, $V_3 = [V_2, G] = 1 < G_3$. Suppose now that we have a group G where all the nonlinear irreducible characters vanish off of the second center. This implies that all of the nonlinear irreducible characters of $G/Z(G)$ vanish off of the center of $G/Z(G)$, and so, $G/Z(G)$ is a VZ-group. In particular, $G/Z(G)$ has nilpotence class 2. This implies that G has nilpotence class 3. We now have $V_1 = Z_2(G)$ and thus G will satisfy $V_3 < G_3$.

For the remainder of this section, we set $C/V_3 = \mathbf{C}_{G/V_3}(G'/V_3)$. We now consider the elements in $G \setminus V_1$. In this next lemma, we use the Jacobi–Witt identity which says that if G has nilpotence class 3 and $w, x, y \in G$, then $[w, x, y][x, y, w][y, w, x] = 1$.

Lemma 4.6. *Assume $V_3 < G_3$. Suppose $b \in C \setminus V_1$ and let $B/V_2 = \mathbf{C}_{G/V_2}(bV_2)$. Then $B \leqslant C$.*

Proof. We may assume without loss of generality that $V_3 = 1$. Under this hypothesis, we have $C = \mathbf{C}_G(G')$. Also, we see that G has nilpotence class 3 and V_2 is central in G. Fix $x \in B$, and let $w \in G$ be arbitrary. Note that $B = \{g \in G \mid [b, g] \in V_2\}$. Thus, $[b, x] \in V_2 \leqslant Z_1$, so $[b, x, w] = 1$. Also, $[x, w] \in G'$ and $b \in C$, so $[x, w, b] = 1$. Applying the Jacobi–Witt identity, we have $[w, b, x] = 1$. Since $b \notin V_1$ and (G, V_1) is a GCP, we know that $[w, b]$ runs through all of G' as w runs through G. This implies that x centralizes G', and hence, $x \in C$. $\square$

The following lemma is important to understand the structure of G.

Lemma 4.7. *Assume $V_3 < G_3$. Suppose $a \in G \setminus C$ and $A/V_2 = \mathbf{C}_{G/V_2}(aV_2)$. Then $A \cap C = V_1$.*

Proof. Without loss of generality, we may assume that $V_3 = 1$, and hence, $C = \mathbf{C}_G(G')$. We first prove that $V_1 \leqslant C$. To see this consider $[G', V_1] = [G, G, V_1]$. Observe that $[G, V_1, G] \leqslant [V_2, G] = 1$ and similarly, $[V_1, G, G] = 1$. Hence, by the Three Subgroups Lemma, we have $[G', V_1] = 1$. This implies that $V_1 \leqslant C$.

Since $V_2 = [V_1, G]$, it follows that $V_1 \leqslant A$, so we have $V_1 \leqslant A \cap C$. If $V_1 < A \cap C$, then there exists an element $b \in (A \cap C) \setminus V_1$, and we set $B/V_2 = \mathbf{C}_{G/V_2}(bV_2)$. By Lemma 4.6, we know that $B \leqslant C$. Now, $[a, b] \in V_2$ since $b \in A$. This implies that $a \in B$. Thus, $a \in C$ which is a contradiction, and so, $A \cap C = V_1$. $\square$

In several of the next lemmas, we will assume that $|G_3| = p$. We will later see that for many of these results, this hypothesis can be removed. The first result determines some conjugacy classes. Notice that we do not need to assume anything regarding V_1 for this lemma. In fact, this next lemma does not impact V_1 at all.

Lemma 4.8. *Assume G is nilpotent and $|G_3| = p$. If $x \in G' \setminus \mathbf{Z}(G)$, then* $\mathrm{cl}(x) = xG_3$.

Proof. Because G is nilpotent, $G = P \times Q$ where P is a p-group and Q is a p'-group. Hence, $G' = P' \times Q'$, and as $|G_3| = p$, we have $G_3 = P_3$, so $Q_3 = 1$. In particular, $Q' \leqslant \mathbf{Z}(Q)$, so Q centralizes G'. Observe that G'/G_3 is central in G/G_3, and thus, it follows that $\mathrm{cl}(x) \subseteq xG_3$. We deduce that $|\mathrm{cl}(x)| \leqslant p$. Recall that $x \in G'$, which implies that $Q \leqslant \mathbf{C}_G(x)$. Now, $|\mathrm{cl}(x)| = |G : \mathbf{C}_G(x)|$ divides $|G : Q| = |P|$. Therefore, $|\mathrm{cl}(x)|$ is either 1 or p. Because x is not central, we must have $|\mathrm{cl}(x)| = p = |xG_3|$, and we conclude that $\mathrm{cl}(x) = xG_3$. □

We now can strengthen the relationship between m and n in $|G : V_1| = p^{2n}$ and $|G' : V_2| = p^m$ when $|G_3| = p$. In particular, we will show that $m = n$. Also, this shows that we have examples where $V_1 < C$ whenever $|G_3 : V_3| = p$.

Lemma 4.9. *Assume $V_3 = 1$ and $|G_3| = p$. Then $|G : V_1| = |G' : V_2|^2$ (i.e. $m = n$), $|G : C| = |G' : V_2| = p^n$, and $V_2 = G' \cap \mathbf{Z}(G)$.*

Proof. We know that $V_3 = 1$, so V_2 is central in G. As $G_3 \leqslant V_2$, we have that G_3 is central. Hence, G is nilpotent with nilpotence class 3. Also, $C = \mathbf{C}_G(G')$. Let $Z = \mathbf{Z}(G)$. It follows that $V_2 \leqslant G' \cap Z$, and since G'/V_2 is a p-group, we can set $|G' : G' \cap Z| = p^k$. Because $V_2 \leqslant G' \cap Z$, we have $p^k \leqslant |G' : V_2| = p^m$. Now, G'/V_2 is elementary abelian, so we can find $x_1, \dots, x_k \in G' \setminus Z$ so that $G' = \langle x_1, \dots, x_k, G' \cap Z\rangle$. It follows that $C = \bigcap_{i=1}^k \mathbf{C}_G(x_i)$. Thus, we have

$$|G : C| = \left| G : \bigcap_{i=1}^{k} \mathbf{C}_G(x_i) \right| \leqslant \prod_{i=1}^{k} \bigl|G : \mathbf{C}_G(x_i)\bigr| = p^k,$$

where the last equality follows from Lemma 4.8.

Since G has nilpotence class 3, G' is not central in G, so $C < G$. Thus, there is an element $a \in G \setminus C$, and let $A/V_2 = \mathbf{C}_{G/V_2}(aV_2)$. By Lemma 4.7, we have $V_1 = A \cap C$, so $|A : V_1| = |AC : C|$. Recall that $|G : A| = p^m$ via Lemma 4.5. Using Lemma 4.3 and the previous paragraph, we know that

$$p^{2m} \leqslant p^{2n} = |G : V_1| = |G : A||A : V_1| = p^m|AC : C| \leqslant p^m|G : C| \leqslant p^m p^k \leqslant p^{2m}.$$

We must have equality throughout. This implies that $m = n = k$ and $|G : C| = p^n$. We obtain $|G' : G' \cap Z| = |G' : V_2|$, and since $V_2 \leqslant G' \cap Z$, this implies that $V_2 = G' \cap Z$. □

To remove the hypothesis that $|G_3| = p$, we need to consider groups N with $V_3 \leqslant N < G_3$. Note that G_3/V_3 is central in G/V_3, so that N is normal in G. By Lemma 3.3, we know that $V_1(G/N) \leqslant V_1/N$ and $V_1(G/V_2) \leqslant \frac{V_1(G/N)}{V_2/N}$. On the other hand, using Lemma 4.1, we have $V_1(G/V_2) = V_1/V_2$. Combining these, we obtain the observation that $V_1(G/N) = V_1/N$. It follows that $V_2(G/N) = [V_1/N, G/N] = [V_1, G]N/N = V_2/N$. With this, we will be able to use an inductive hypothesis on G/N. We now obtain conclusion (1) in Theorem 3.

Lemma 4.10. *Assume $V_3 < G_3$. Then $|G : V_1| = |G' : V_2|^2$ (i.e. $m = n$).*

Proof. When $|G_3| = p$, this is Lemma 4.9. Thus, we may assume that $|G_3| > p$. We can find $V_3 \leqslant N < G_3$ so that $|G_3 : N| = p$. We apply Lemma 4.9 to G/N. Thus, we have $|G'/N : V_2/N| = |G' : V_2| = p^n$ and $|G : V_1| = |G/N : V_1/N| = p^{2n}$. □

We now determine the structure of C in the general case. This is conclusion (3) of Theorem 3.

Lemma 4.11. *Assume $V_3 < G_3$. Then either $|G : C| = |C : V_1| = |G' : V_2| = p^n$ or $C = V_1$.*

Proof. We may assume that $V_3 = 1$. We know that $G_3 > 1$, so G' is not central in G and $C < G$. Thus, we can apply Lemma 4.7 to see that $V_1 \leqslant C$. Also, since $G_3 > 1$, we can find $N \leqslant G_3$ so that $|G_3 : N| = p$. Let $D/N = \mathbf{C}_{G/N}(G'/N)$. By Lemma 4.9, we know that $|G : D| = |G' : V_2|$, and by Lemma 4.10, we have $|G : V_1| = |G' : V_2|^2 = |G : D|^2$. It is easy to see that $C \leqslant D$. If $C = D$, then we have $|G : C| = |C : V_1| = |G' : V_2|$ as desired.

We now assume that $C < D$. Thus, we can find $b \in D \setminus C$. Set $B/V_2 = \mathbf{C}_{G/V_2}(bV_2)$. Applying Lemma 4.7, we have that $B \cap C = V_1$. Since $V_1 \leqslant C$, we have $b \in D \setminus V_1$. Thus, we may apply Lemma 4.6 to see that $B \leqslant D$ and Lemma 4.5 to see that $|G : B| = |G' : V_2|$. We have already seen that $|G : D| = |G' : V_2|$. It follows that $B = D$. Hence, $C = D \cap C = B \cap C = V_1$, and this implies the result. $\square$

We now show that $V_2 = G' \cap Z$ where $Z/V_3 = \mathbf{Z}(G/V_3)$ when $V_3 < G_3$. This is conclusion (2) of Theorem 3.

Lemma 4.12. *Assume $V_3 < G_3$. If $Z/V_3 = \mathbf{Z}(G/V_3)$, then $V_2 = G' \cap Z$.*

Proof. Again, we can find a subgroup N so that $V_3 \leqslant N < G_3$ and $|G_3 : N| = p$. We know $V_2 \leqslant G'$ and $[V_2, G] = V_3$, so $V_2 \leqslant Z$. It follows that $V_2 \leqslant G' \cap Z$. By Lemma 4.9, we have $V_2/N = G'/N \cap \mathbf{Z}(G/N)$. Note that $Z/N \leqslant \mathbf{Z}(G/N)$. We now have

$$\frac{G' \cap Z}{N} = \frac{G'}{N} \cap \frac{Z}{N} \leqslant \frac{G'}{N} \cap \mathbf{Z}\left(\frac{G}{N}\right) = \frac{V_2}{N} \leqslant \frac{G' \cap Z}{N}.$$

Since we must have equality throughout, we obtain $V_2/N = (G' \cap Z)/N$, and therefore, $G' \cap Z = V_2$. $\square$

We obtain an equality of centralizers when $|G_3| = p$.

Lemma 4.13. *Assume $V_3 = 1$ and $|G_3| = p$. If $b \in C \setminus V_1$, then $C/V_2 = \mathbf{C}_{G/V_2}(bV_2)$.*

Proof. Let $B/V_2 = \mathbf{C}_{G/V_2}(bV_2)$. By Lemma 4.5, we have $|G : B| = p^n$. On the other hand, we proved in Lemma 4.9 that $|G : C| = p^n$. Finally, in Lemma 4.6, we showed that $B \leqslant C$. Hence, $B = C$, and the result is proved. $\square$

We find that (C, V_1) is a GCP when $V_3 = 1$ and $|G_3| = p$.

Lemma 4.14. *Assume $V_3 = 1$ and $|G_3| = p$. Then $C' = V_2$, C has nilpotence class 2, and (C, V_1) is a GCP.*

Proof. First, notice that $C' \leqslant G' \leqslant \mathbf{Z}(C)$, so C has nilpotence class at most 2. Consider $[C', G] = [C, C, G]$. Observe that $[C, G, C] \leqslant [G', C] = 1$, and similarly, $[G, C, C] = 1$. By the Three Subgroups Lemma, we deduce that $[C', G] = 1$, and thus, $C' \leqslant Z_1$. We also have $C' \leqslant G'$, and hence, $C' \leqslant Z_1 \cap G'$. By Lemma 4.9, this implies that $C' \leqslant V_2$.

By Lemma 4.9, we have $|G : V_1| = p^{2n}$ and $|G : C| = p^n$, so $|C : V_1| = p^n$. Hence, we can find an element $b \in C \setminus V_1$. Using Lemma 4.13, we have $C = \{x \in G \mid [b, x] \in V_2\}$. Since (G, V_1) is a GCP and $b \notin V_1$, for each $y \in V_2$, there is an element $g \in G$ so that $[b, g] = y$. We have just seen that this implies that $g \in C$, and $V_2 \leqslant [b, C] \leqslant C'$. We conclude that $C' = V_2$, C has nilpotence class 2, and (C, V_1) is a GCP. $\square$

We get the general version of Lemma 4.14. If $V_1 = C$, then obviously, $V(C) \leqslant V_1$. Hence, to prove conclusion (4) of Theorem 3, it suffices to prove that $V(C) \leqslant V_1$ when $V_1 < C$. In the next lemma, we prove that (C, V_1) is a GCP when $V_1 < C$, and therefore, Lemma 3.2 implies that $V(C) \leqslant V(G)$. We also obtain conclusion (5) of Theorem 3.

Lemma 4.15. *Assume $V_3 < G_3$. If $V_1 < C$, then $C' = V_2$, C/V_3 has nilpotence class 2, and (C, V_1) is a GCP.*

Proof. First, notice that $C' \leqslant G'$ and $G'/V_3 \leqslant \mathbf{Z}(C/V_3)$, so C/V_3 has nilpotence class at most 2. Consider a subgroup $V_3 \leqslant N < G_3$ so that $|G_3 : N| = p$. We know that $|G/N : \mathbf{C}_{G/N}(G'/N)| = p^n$ via Lemma 4.9. In light of Lemma 4.11, we have $|G : C| = p^n$. Since $C/N \leqslant \mathbf{C}_{G/N}(G'/N)$, this implies that $C/N = \mathbf{C}_{G/N}(G'/N)$. By Lemma 4.14, we have $C'N/N = V_2/N$, and hence, $C'N = V_2$. This implies that $C' \leqslant V_2$.

We know via Lemmas 4.10 and 4.11 that $|G : V_1| = p^{2n}$ and $|G : C| = p^n$, since we have $V_1 < C$. Hence, we can find $b \in C \setminus V_1$, and let $B/V_2 = \mathbf{C}_{G/V_2}(bV_2)$. By Lemma 4.6, we have $B \leqslant C$. Since $b \notin V_1$ and (G, V_1) is a GCP, there is for each $y \in V_2$ an element $g \in G$ so that $[b, g] = y$. Notice that this implies that $g \in B \leqslant C$. We now have $V_2 \leqslant [b, C] \leqslant C'$. We conclude that $V_2 = C'$, C/V_3 has nilpotence class 2, and (C, V_1) is a GCP. $\square$

Finally, we obtain conclusion (6) of Theorem 3.

Lemma 4.16. *Assume $V_3 < G_3$. If $V_1 < C$ and $[V_1, C] < V_2$, then $C/[V_1, C]$ is a VZ-group and n is even.*

Proof. By Lemma 4.15, we know that (C, V_1) is a GCP. It follows that $V(C) \leqslant V_1$. From Lemma 2.4, $\mathbf{Z}(C/[V_1, C]) \leqslant V(C)/[V_1, C] \leqslant V_1/[V_1, C] \leqslant \mathbf{Z}(C/[V_1, C])$. We deduce that $V(C) = V_1$. Now, $V_2(C) = [V_1, C] < V_2 = C'$ using Lemma 4.15. Applying Lemma 4.3, $C/[V_1, C]$ is a VZ-group. Since $|C : V_1| = p^n$ via Lemma 4.11, we deduce that n is even. $\square$

5. Examples

We now present some examples to demonstrate the phenomenon we have mentioned. Many of the examples we present were found using the small groups library in MAGMA (see [2]) or GAP (see [6]). For such groups, we will include their identification in the small groups library. We should note that the examples we present are in no way unique and that one can find many similar examples by exploring the small groups libraries. We would expect that it is possible to find constructions that work for any prime p for the groups in many of these examples. All the computations in this section were done using MAGMA or GAP, and so, we shall not explicitly state this.

We begin with an example that shows that $V(G)$ is not necessarily proper in G when G is solvable (nilpotent or not nilpotent). The easiest way to construct such an example is as follows. Let H and K be any pair of nonabelian groups (possibly isomorphic; they may or may not be nilpotent), and let $G = H \times K$. We can find nonlinear characters $\chi, \psi \in \mathrm{Irr}(G)$ so that $H \leqslant \ker(\chi)$ and $K \leqslant \ker(\psi)$. Notice that this implies that $H \leqslant V(\chi)$ and $K \leqslant V(\psi)$, and hence, $G = HK \leqslant V(G)$, and so, $G = V(G)$.

Looking at the small groups library, we see that $V(G) < G$ for every nonabelian 2-group whose order is 8, 16, or 32. Suppose $G = A \times B$ where each of A and B are either quaternion or dihedral of order 8, then $G = V(G)$, so there exist groups of order 64 for which the vanishing-off subgroup is not proper.

We now present an example of a group G of order 64 for which $G = V(G)$, but G is not a direct product of nonabelian groups. We set

$$G = \left\langle a, b, c, d, e, f \;\middle|\; a^2 = b^2 = c^2 = d^2 = e^2 = f^2 = 1,\ b^a = bd,\ c^a = ce,\ c^b = cf \right\rangle$$

where we follow the conventions of a polycyclic representation in GAP or MAGMA that commutators of generators that are not given must be trivial. (This is SmallGroup $(64, 73)$.) We computed that $V(G) = G$. We also computed the character degrees of G which were $\{1, 2\}$, so G cannot be the direct product of nonabelian groups.

We next include an example where $G = V(G)$ and G has Camina elements. Again, G has order 64. We set

$$G = \left\langle a, b, c, d, e, f \;\middle|\; a^2 = d,\ c^2 = f,\ b^2 = d^2 = e^2 = f^2 = 1,\ b^a = bc,\ c^a = ce,\ c^b = cf,\ d^b = def \right\rangle$$

and we have $G = V(G)$. (This is SmallGroup (64, 8).) Using MAGMA, one can see that b and bd are Camina elements of G.

We now present some examples of p-groups G where $|G : V(G)|$ is not a square. The first example is the dihedral (or semi-dihedral) group of order 2^n with $n \geqslant 4$. In this case, $V(G)$ is the cyclic subgroup of order 2^{n-1} which of course has index 2 in G.

Next, we present an example of a group of order 81 with a vanishing-off subgroup of index 3. In this example, we have

$$G = \langle a, b, c, d \mid a^3 = b^3 = c^3 = d^3 = 1,\ b^a = bc,\ c^a = cd \rangle$$

and $V(G) = \langle b, c, d\rangle$ which has index 3. (This is SmallGroup (81, 7).)

We also present an example of a group of order 256 where $|G : V(G)| = 8$. Also, this is an example where $V(G) < G$. This is a second example where $V(G)$ contains a Camina element for G. In this case, we have

$$G = \langle a, b, c, d, e, f, g, h \mid a^2 = g,\ c^2 = de,\ d^2 = e,\ f^2 = h,\ b^2 = e^2 = g^2 = h^2 = 1,\ b^a = be,$$
$$c^a = cf,\ c^b = cef,\ d^a = def,\ d^b = de,\ e^a = eh,\ f^b = fh,\ g^b = gh,\ g^c = gh \rangle.$$

(This is SmallGroup (256, 16349).) We compute that $V(G) = \langle d, e, f, g, h\rangle$, so $|G : V(G)| = 8$. We should also note that while every element in $G \setminus V(G)$ is a Camina element for G, this is an example where not all the Camina elements for G lie in $G \setminus V(G)$. In particular, $V(G)$ will itself contain a Camina element for G. We identify dg as a Camina element for G and $dg \in V(G)$.

We next present an example of a group G with vanishing height 3 where $|G' : V_2(G)|$ is not a square. In this case, we have $|G| = 32$. Take

$$G = \langle a, b, c, d, e \mid a^2 = d,\ b^2 = c^2 = d^2 = e^2 = 1,\ b^a = bc,\ c^a = ce,\ d^b = de \rangle.$$

(This is SmallGroup (32, 6).) In this case, $V(G) = \langle c, d, e\rangle$ so $V(G)$ has index 4. Also, $G' = \langle c, e\rangle$ and $V_2(G) = \langle e\rangle$, so $|G' : V_2(G)| = 2$. Finally, to see that G has vanishing height 3, we note that $G_3 = \langle e\rangle$ and $V_3(G) = 1$.

This next example we believe is instructive. It is an example where $|G : V(G)| = 4$ and $|G' : V_2(G)| = 2$ as shown in Lemma 4.10. Surprisingly, it has $|G_3 : V_3(G)| = 4$. In particular, this example shows that we do not have either $|G_3 : V_3(G)| \leqslant |G_2 : V_2(G)|$ or $|G_3 : V_3(G)|^2 \leqslant |G : V(G)|$. In fact, we do not even have $|G_3 : V_3(G)| < |G : V(G)|$. We also deduce that $C = V_1(G)$. In the examples we have looked at, it seems that $C = V_1(G)$ occurs exactly when $|G : V(G)| = |G_3 : V_3(G)|$, but we have not been able to prove this. In this case, we have $|G| = 128$, and we take

$$G = \langle a, b, c, d, e, f, g \mid a^2 = d,\ b^2 = e,\ c^2 = d^2 = e^2 = f^2 = g^2 = 1,\ b^a = bc,\ c^a = cf,$$
$$c^b = cg,\ d^b = df,\ e^a = eg \rangle.$$

(This is SmallGroup (128, 36).) We deduce that $V(G) = \langle c, d, e, f, g\rangle = \mathbf{C}_G(G') = C$. We also have that $G' = \langle c, f, g\rangle$ and $V_2(G) = \langle f, g\rangle$. Finally, $G_3 = \langle f, g\rangle$ and $V_3(G) = 1$.

We now present an example with vanishing height 4. This example will also have nilpotence class 4, and $Z_3(G) = V(G)$. We have a group G with $|G| = 128$. We take

$$G = \langle a, b, c, d, e, f, g \mid a^2 = e,\ c^2 = g,\ b^2 = d^2 = e^2 = f^2 = g^2 = 1,\ b^a = bd,\ c^b = cg,$$
$$d^a = df,\ e^b = ef,\ e^d = eg,\ f^a = fg\rangle.$$

(This is SmallGroup (128, 854).) In this case, $G_4 = \langle g \rangle$ and $V_4(G) = 1$. Observe that G has nilpotence class 4 and $V_4(G) < G_4$, so G has vanishing height 4. In fact, $V(G) = Z_3(G) = \langle c, d, e, f, g \rangle$. We note that there are no groups of order 256 with vanishing height 5.

For our final example, we present a group of order 3^7 with nilpotence class and vanishing height 4 with $Z_3(G) = V(G)$. In this case, we take

$$G = \big\langle a, b, c, d, e, f, g \;\big|\; a^3 = d,\ b^3 = c,\ c^3 = e,\ d^3 = f,\ e^3 = g,\ b^a = bc,\ c^a = ce,\ d^b = de^2,\\ d^c = dg^2,\ e^a = eg,\ f^b = fg^2 \big\rangle.$$

(This is SmallGroup $(3^7, 194)$.) We have $V(G) = Z_3(G) = \langle c, d, e, f, g \rangle$. This implies that G has nilpotence class and vanishing height 4 as desired.

References

[1] M. Bianchi, D. Chillag, M.L. Lewis, E. Pacifici, Character degree graphs that are complete graphs, Proc. Amer. Math. Soc. 135 (2007) 671–676.
[2] W. Bosma, J. Cannon, C. Playoust, The Magma algebra system I: The user language, J. Symbolic Comput. 24 (1997) 235–265.
[3] A.R. Camina, Some conditions which almost characterize Frobenius groups, Israel J. Math. 31 (1978) 153–160.
[4] D. Chillag, I.D. MacDonald, Generalized Frobenius groups, Israel J. Math. 47 (1984) 111–122.
[5] G.A. Fernández-Alcober, A. Moretó, Groups with two extreme character degrees and their normal subgroups, Trans. Amer. Math. Soc. 353 (2001) 2171–2192.
[6] The GAP Group, GAP – groups, algorithms, and programming, version 4.4; http://www.gap-system.org, 2006.
[7] I.M. Isaacs, Character Theory of Finite Groups, Academic Press, San Diego, CA, 1976.
[8] I.M. Isaacs, Coprime group actions fixing all nonlinear irreducible characters, Canad. J. Math. 41 (1989) 68–82.
[9] K.W. Johnson, S. Mattarei, S.K. Sehgal, Weak Cayley tables, J. London Math. Soc. (2) 61 (2000) 395–411.
[10] M.L. Lewis, Character tables of groups where all nonlinear irreducible characters vanish off the center, in: Proceeding of Ischia Group Theory, 2008, in press.
[11] M.L. Lewis, Generalizing Camina groups and their character tables, J. Group Theory, in press.
[12] I.D. MacDonald, Some p-groups of Frobenius and extra-special type, Israel J. Math. 40 (1981) 350–364.
[13] Y. Ren, Applications of the theory of Camina groups, Chinese Ann. Math. Ser. B 20 (1999) 39–50.