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# On the depth of invariant rings of infinite groups

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## ABSTRACT

Let  $K$  be an algebraically closed field. For a finitely generated graded commutative  $K$ -algebra  $R$ , let  $\text{cmdef } R := \dim R - \text{depth } R$  denote the Cohen–Macaulay defect of  $R$ . Let  $G$  be a linear algebraic group over  $K$  that is reductive but not linearly reductive. We show that there exists a faithful rational representation  $V$  of  $G$  (which we will give explicitly) such that  $\text{cmdef } K[V^{\oplus k}]^G \geq k - 2$  for all  $k \in \mathbb{N}$ .

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## 1. Introduction

Let us first fix some notation and conventions. The symbol  $G$  will always denote a linear algebraic group over the algebraically closed field  $K$  of characteristic  $p \geq 0$ , and  $V$  will always denote a finite dimensional rational  $G$ -module. Then  $G$  acts on the ring  $K[V]$  of polynomial functions  $V \rightarrow K$  by  $\sigma \cdot f := f \circ \sigma^{-1}$ , where  $f \in K[V]$ ,  $\sigma \in G$ , so we have  $K[V] \cong S(V^*)$ , the symmetric algebra of the dual  $V^* = \text{Hom}_K(V, K)$ . We will always assume that the algebra  $K[V]^G$  of invariants is finitely generated, which due to Nagata [18] is always true if  $G$  is reductive. We are interested in how far the invariant ring  $K[V]^G$  is from being Cohen–Macaulay. To this end we define the Cohen–Macaulay defect  $\text{cmdef } K[V]^G$  to be  $\dim K[V]^G - \text{depth } K[V]^G$ . Then  $\text{cmdef } K[V]^G \geq 0$  and  $K[V]^G$  is Cohen–Macaulay precisely when  $\text{cmdef } K[V]^G = 0$ . By Hochster and Roberts [12], the invariant ring  $K[V]^G$  is Cohen–Macaulay if  $G$  is linearly reductive.

Several papers have dealt with the depth of invariant rings of *finite* groups, see for example [4,6–10,14,19,21], but there are up to today no quantitative results for the depth of invariants of (infinite) algebraic groups.

The goal of this paper is to make a first step in this direction. To have a context for the results of this paper, we mention three results from the literature explicitly: Every overview has to start with

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the celebrated theorem of Ellingsrud and Skjelbred [6], who proved that for  $p > 0$  and  $G$  a cyclic  $p$ -group, we have  $\text{cmdef } K[V]^G = \max(\dim_K(V) - \dim_K(V^G) - 2, 0)$ . The main part of our paper is about bringing the following two results together:

**Theorem 1.1.** (See Gordeev, Kemper [10, Corollary 5.15].) *Let  $G$  be finite and  $p \nmid |G|$ . Then for every faithful  $G$ -module  $V$ , we have*

$$\lim_{k \rightarrow \infty} \text{cmdef } K[V^{\oplus k}]^G = \infty$$

(where  $V^{\oplus k}$  denotes the  $k$ -fold direct sum of  $V$ ).

**Theorem 1.2.** (See Kemper [13, Theorem 7].) *Let  $G$  be reductive, but not linearly reductive. Then there exists a  $G$ -module  $V$  such that  $K[V]^G$  is not Cohen–Macaulay.*

These two theorems should be compared with the following theorem, which is the main result of our paper:

**Theorem 1.3.** *Let  $G$  be reductive, but not linearly reductive. Then there exists a faithful  $G$ -module  $V$  (that will be given explicitly in the proof) such that*

$$\text{cmdef } K[V^{\oplus k}]^G \geq k - 2 \quad \text{for all } k \geq 1.$$

On the other hand, if  $G = \text{SL}_n$  acts on  $V = K^n$  by left multiplication, then  $V$  is a faithful  $\text{SL}_n$ -module, and if  $n \geq 2$  and  $p > 0$ , then  $\text{SL}_n$  is not linearly reductive. But by Hochster [11, Corollary 3.2], we have  $\text{cmdef } K[V^{\oplus k}]^{\text{SL}_n} = 0$  for all  $k \in \mathbb{N}$ , so Theorem 1.1 cannot be generalized to reductive groups that are not linearly reductive.

It would be interesting to know if  $\text{cmdef } K[V]^G > 0$  implies  $\lim_{k \rightarrow \infty} \text{cmdef } K[V^{\oplus k}]^G = \infty$ , but we have neither a proof nor a counterexample for this.

## 2. The depth of graded algebras

For the convenience of the reader, we have collected some standard facts about the depth of graded algebras that can be looked up in any better book on commutative algebra like [2,5]. The given references generally only treat the local case, but this case carries over to our graded situation. For this paper,  $R$  will denote a finitely generated graded commutative  $K$ -algebra  $R = \bigoplus_{d=0}^{\infty} R_d$  where  $R_0 = K$ . We call  $R_+ := \bigoplus_{d=1}^{\infty} R_d$  the *maximal homogeneous ideal* of  $R$ . A sequence of homogeneous elements  $a_1, \dots, a_k \in R_+$  is called a *partial homogeneous system of parameters* (phsop) if  $\text{height}(a_1, \dots, a_k)_R = k$ . If  $k = \dim R$ , then the sequence is called a *homogeneous system of parameters* (hsop), and then  $R$  is finitely generated as a module over  $A := K[a_1, \dots, a_k]$ . Due to the Noether normalization theorem, hsops always exist.

The following lemma makes it easier to find phsops in an invariant ring.

**Lemma 2.1.** (See Kemper [13, Lemma 4].) *Let  $G$  be reductive. If  $a_1, \dots, a_k \in K[V]^G$  form a phsop in  $K[V]$ , then they also form a phsop in  $K[V]^G$ .*

Let  $M$  always denote a non-zero, finitely generated  $\mathbb{Z}$ -graded  $R$ -module (the most important case is  $M = R$ ). Then a sequence of homogeneous elements  $a_1, \dots, a_k \in R_+$  is called a *homogeneous  $M$ -regular sequence of length  $k$*  if for each  $i = 1, \dots, k$  we have that  $a_i$  is not a zero divisor of  $M/(a_1, \dots, a_{i-1})_R M$ . If  $I \subseteq R_+$  is a proper homogeneous ideal, then a homogeneous  $M$ -regular sequence  $a_1, \dots, a_k \in I$  is called *maximal* (in  $I$ ), if it cannot be extended to a longer  $M$ -regular sequence lying in  $I$ . Due to the theorem of Rees, two such maximal  $M$ -regular sequences have the same length (which is finite in our setup), and we write  $\text{depth}(I, M)$  for that common length (the  *$I$ -depth*

of  $M$ ). We call  $\text{depth } M := \text{depth}(R_+, M)$  the *depth* of  $M$ . In the case of  $M = R$ , every regular sequence is a phsop, and  $R$  is Cohen–Macaulay if and only if every phsop is a regular sequence. We write  $\text{depth } I := \text{depth}(I, R)$  for the *depth* of  $I$ . We always have  $\text{depth } I \leq \text{height } I$ , so for the *Cohen–Macaulay defect* of  $I$ ,  $\text{cmdef } I := \text{height } I - \text{depth } I$ , we always have  $\text{cmdef } I \geq 0$ .

**Theorem 2.2.** (See [2, Exercise 1.2.23].) *Let  $I \subseteq R_+$  be a proper homogeneous ideal. Then*

$$\text{depth } R \leq \text{depth } I + \dim I.$$

*In particular we have  $\text{cmdef } R \geq \text{cmdef } I$ .*

To apply this theorem in order to get a good lower bound for  $\text{cmdef } R$ , one has to find ideals  $I \subseteq R_+$  of which one knows the depth. The following lemma, which is inspired by Shank and Wehlau [21, Theorem 2.1] is the proper tool.

**Lemma 2.3.** *Let  $I \subseteq R_+$  be a proper homogeneous ideal of  $R$ , and  $a_1, \dots, a_k \in I$  be a homogeneous  $M$ -regular sequence. Then  $\text{depth}(I, M) = k$  if and only if there exists an  $m \in M$  with  $m \notin (a_1, \dots, a_k)M$  and  $Im \subseteq (a_1, \dots, a_k)M$ .*

**Proof.** First assume that there exists a  $m \in M$  with  $m \notin (a_1, \dots, a_k)M$  and  $Im \subseteq (a_1, \dots, a_k)M$ . Then obviously  $I$  only consists of zero divisors of  $M/(a_1, \dots, a_k)M$ , hence  $a_1, \dots, a_k$  is a maximal  $M$ -regular sequence in  $I$ , hence  $\text{depth}(I, M) = k$ .

Conversely assume  $\text{depth}(I, M) = k$ . Then  $a_1, \dots, a_k \in I$  is a maximal homogeneous  $M$ -regular sequence. Then  $I$  only consists of zero divisors of  $N := M/(a_1, \dots, a_k)M$ , so by [5, Theorem 3.1.b] we have  $I \subseteq \bigcup_{\wp \in \text{Ass}_R(N)} \wp$ , and by prime avoidance [5, Lemma 3.3] there is a  $\wp \in \text{Ass}_R N$  with  $I \subseteq \wp$ . Since  $\wp$  is an associated prime ideal of  $N$ , there is a  $n \in N \setminus \{0\}$  with  $\wp = \text{Ann}_R n$ , and for a  $m \in M$  with  $n = m + (a_1, \dots, a_k)M$  we have  $m \notin (a_1, \dots, a_k)M$  and  $Im \subseteq (a_1, \dots, a_k)M$ .  $\square$

We will apply this lemma only in the case  $k = 2$ , since it is difficult to check if  $k \geq 3$  elements form a regular sequence. To check if two elements form a regular sequence, we have the following lemma.

**Lemma 2.4.** *Let  $a_1, a_2 \in K[V]_+$  be homogeneous. Then the following are equivalent:*

- (a)  $a_1, a_2$  form a phsop in  $K[V]$ .
- (b)  $a_1, a_2$  form a regular sequence in  $K[V]$ .
- (c)  $a_1, a_2$  are coprime in  $K[V]$ .

*If one (hence all) of the above conditions is satisfied and we additionally have  $a_1, a_2 \in K[V]^G$ , then  $a_1, a_2$  also form a regular sequence in  $K[V]^G$ .*

### 3. First cohomology of groups and the depth of invariants

The results of this section are a quantitative extension of the qualitative results of Kemper [13]. Let  $H$  be an arbitrary group and  $W$  be a  $KH$ -module (not necessarily finite dimensional). A map  $g: H \rightarrow W, \sigma \mapsto g_\sigma$  is called a (1)-cocycle, if we have  $g_{\sigma\tau} = \sigma g_\tau + g_\sigma$  for all  $\sigma, \tau \in H$ . The sum of two cocycles is again a cocycle, so the set of all cocycles  $Z^1(H, W)$  is an additive group. For any  $w \in W$ , the map  $H \rightarrow W$  given by  $\sigma \mapsto (\sigma - 1)w := \sigma w - w$  is also a cocycle, and we call a cocycle which is given by such a  $w$  a (1)-coboundary. The set of all coboundaries  $B^1(H, W)$  is obviously a subgroup of  $Z^1(H, W)$ , and we write  $H^1(H, W) := Z^1(H, W)/B^1(H, W)$  for the corresponding factor group. We call a cocycle  $g$  non-trivial, if it is not a coboundary, and we will often confuse an element  $g \in Z^1(H, W)$  with its image (also denoted  $g$ ) in  $H^1(H, W)$ . Thus  $g$  is non-trivial if and only if  $g \neq 0$  in  $H^1(H, W)$ . If  $H$  is a linear algebraic group and  $W$  a rational (not necessarily finite dimensional)

$H$ -module, then by  $Z^1(H, W)$  we will always mean the cocycles that are given by morphisms of  $H$  to  $W$  (this is automatic for  $B^1$ ). Now let  $a \in K[V]^G$  be an invariant. Then for any  $g \in Z^1(G, K[V])$ , we can define an element  $ag \in Z^1(G, K[V])$  by  $(ag)_\sigma := ag_\sigma$  for all  $\sigma \in G$ . Obviously, multiplication with  $a$  gives a group homomorphism  $Z^1(G, K[V]) \rightarrow Z^1(G, K[V])$ , and induces a group homomorphism  $H^1(G, K[V]) \rightarrow H^1(G, K[V])$ . Now for any  $g \in H^1(G, K[V])$ , we define its *annihilator* as

$$\text{Ann}_{K[V]^G}(g) := \{a \in K[V]^G : a \cdot g = 0 \in H^1(G, K[V])\} \trianglelefteq K[V]^G.$$

This ideal is proper if and only if  $g \neq 0 \in H^1(G, K[V])$ . We call a  $0 \neq g \in Z^1(G, K[V])$  *homogeneous of degree*  $d \geq 0$ , if  $g_\sigma \in K[V]_d$  for all  $\sigma \in G$ . An element of  $H^1(G, K[V]) \setminus \{0\}$  is called *homogeneous of degree*  $d \geq 0$ , if it can be represented by a homogeneous element of degree  $d$  of  $Z^1(G, K[V]) \setminus B^1(G, K[V])$  (this is well defined). If  $g \in H^1(G, K[V])$  is homogeneous, then its annihilator  $\text{Ann}_{K[V]^G}(g)$  is also homogeneous.

The proof of the following proposition has some overlap with the one of Kemper [13, Proposition 6], but we get a sharper result here.

**Proposition 3.1.** *Let  $0 \neq g \in H^1(G, K[V])$  be homogeneous, and assume there exist two homogeneous elements  $a_1, a_2 \in \text{Ann}_{K[V]^G}(g)$  of positive degree that are coprime in  $K[V]$ . Then*

$$\text{depth}(\text{Ann}_{K[V]^G}(g)) = 2.$$

**Proof.** Because of Lemma 2.4,  $a_1, a_2$  form a  $K[V]^G$ -regular sequence in  $\text{Ann}_{K[V]^G}(g)$ . Since  $a_i g = 0 \in H^1(G, K[V])$  ( $i = 1, 2$ ), there are elements  $b_1, b_2 \in K[V]$  such that

$$a_i g_\sigma = (\sigma - 1)b_i \quad \text{for all } \sigma \in G, \quad i = 1, 2.$$

Now set  $m := a_1 b_2 - a_2 b_1 \in K[V]^G$ . We will show that  $m$  fulfills the hypotheses of Lemma 2.3 with  $R = M = K[V]^G$ ,  $I = \text{Ann}_{K[V]^G}(g)$  and  $k = 2$ , i.e.  $m \notin (a_1, a_2)_{K[V]^G}$  and  $m \text{Ann}_{K[V]^G}(g) \subseteq (a_1, a_2)_{K[V]^G}$ . Then Lemma 2.3 yields  $\text{depth}(\text{Ann}_{K[V]^G}(g)) = 2$ .

Assume by way of contradiction  $m \in (a_1, a_2)_{K[V]^G}$ . Then there are  $f_1, f_2 \in K[V]^G$  with

$$m = a_1 b_2 - a_2 b_1 = f_1 a_1 + f_2 a_2.$$

Then  $a_1(b_2 - f_1) = a_2(f_2 + b_1)$ , and  $a_1, a_2$  being coprime in  $K[V]$  yields that  $a_1$  is a divisor of  $f_2 + b_1$ , hence  $f_2 + b_1 = a_1 \cdot h$  with  $h \in K[V]$ . Now

$$a_1 \cdot (\sigma - 1)h = (\sigma - 1)(a_1 h) = (\sigma - 1)(f_2 + b_1) = (\sigma - 1)b_1 = a_1 g_\sigma \quad \text{for all } \sigma \in G,$$

hence  $g_\sigma = (\sigma - 1)h$  for all  $\sigma \in G$ . But then  $g = 0 \in H^1(G, K[V])$  contradicting the hypotheses of the proposition. Hence we really have  $m \notin (a_1, a_2)_{K[V]^G}$ . Now we show  $m \text{Ann}_{K[V]^G}(g) \subseteq (a_1, a_2)_{K[V]^G}$ . Let  $a_3 \in \text{Ann}_{K[V]^G}(g)$ . Then there is a  $b_3 \in K[V]$  with  $a_3 g_\sigma = (\sigma - 1)b_3$  for all  $\sigma \in G$ . Let

$$u_{ij} := a_i b_j - a_j b_i \in K[V]^G \quad \text{for } 1 \leq i < j \leq 3.$$

Obviously,  $m = u_{12}$  and we have

$$u_{23}a_1 - u_{13}a_2 + ma_3 = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0,$$

hence  $ma_3 \in (a_1, a_2)_{K[V]^G}$ . Since  $a_3 \in \text{Ann}_{K[V]^G}(g)$  was arbitrary, we have  $m \text{Ann}_{K[V]^G}(g) \subseteq (a_1, a_2)_{K[V]^G}$ .  $\square$

Technically, the following theorem is our main result.

**Theorem 3.2.** Assume there is a  $0 \neq g \in H^1(G, K[V])$ . Let  $a_1, \dots, a_k \in K[V]^G$  with  $k \geq 2$  and  $a_i g = 0 \in H^1(G, K[V])$  for  $i = 1, \dots, k$ . If one of the following two conditions

- (a)  $G$  is reductive and  $a_1, \dots, a_k$  form a phsop in  $K[V]$ ,
- (b)  $a_1, a_2$  are coprime in  $K[V]$ , and  $a_1, \dots, a_k$  form a phsop in  $K[V]^G$ ,

is true, then

$$\text{cmdef } K[V]^G \geq k - 2.$$

In the case of (a) and  $k = 3$ , this is Kemper [13, Proposition 6].

**Proof.** Condition (a) implies condition (b) by Lemma 2.4 and Lemma 2.1, so let us assume condition (b). By hypothesis,  $\text{height Ann}_{K[V]^G}(g) \geq k$ , and by Proposition 3.1, we have  $\text{depth Ann}_{K[V]^G}(g) = 2$ . Thus  $\text{cmdef } K[V]^G \geq \text{cmdef Ann}_{K[V]^G}(g) \geq k - 2$ .  $\square$

#### 4. Invariant rings with big Cohen–Macaulay defect

In this section, given a reductive, but not linearly reductive group  $G$  and a  $k \in \mathbb{N}$ , we will explicitly construct a  $G$ -module  $V$  that fulfills the hypotheses of Theorem 3.2, hence  $\text{cmdef } K[V]^G \geq k - 2$ . The main step is the construction of a  $G$ -module  $U$  with a  $0 \neq g \in H^1(G, U)$ . For some classical groups, this has been done in Kohls [15] by explicit calculation. With the help of a result of Nagata, we can give a general construction.

**Definition 4.1.** Let  $W, V$  be  $G$ -modules with  $W \subseteq V$ . Then

$$\text{Hom}_K(V, W)_0 := \{f \in \text{Hom}_K(V, W) : f|_W = 0\}$$

is a submodule of  $\text{Hom}_K(V, W)$ , and we have

$$\text{Hom}_K(V, W)_0 \cong W \otimes (V/W)^*.$$

**Proposition 4.2.** (See Kohls [15, Proposition 6].) Let  $W$  be a submodule of a  $G$ -module  $V$  and  $\iota \in \text{Hom}_K(V, W)$  with  $\iota|_W = \text{id}_W$ . Then  $\sigma \mapsto g_\sigma := (\sigma - 1)\iota$  is a cocycle in  $Z^1(G, \text{Hom}_K(V, W)_0)$ , which is a coboundary if and only if there exists a  $G$ -invariant complement for  $W$ .

Regarding this proposition, we see that in order to construct a non-trivial cocycle, one has to find a  $G$ -module  $V$  that contains a submodule which has no complement. By definition, such a  $G$ -module  $V$  exists if and only if the group  $G$  is not linearly reductive. Next, we will show how to find such a  $V$ .

**Definition 4.3.** Assume  $p > 0$ . We call the  $G$ -submodule

$$F^p(V) := \{f \in S^p(V) : \text{there exists a } v \in V \text{ with } f = v^p\}$$

of  $S^p(V)$  the  $p$ th Frobenius power of  $V$ .

Recall that every linear algebraic group  $G$  is isomorphic to a closed subgroup of a suitable  $\text{GL}_n(K)$  [22, Theorem 2.3.6]. Then  $K^n$  is a faithful  $G$ -module.

**Theorem 4.4.** (See Nagata [17, Proof of Theorem 1].) Let  $p > 0$ ,  $G$  be a connected linear algebraic group, and  $V$  be a faithful  $G$ -module. The following are equivalent:

- (a)  $G$  is linearly reductive.
- (b)  $G$  is a torus.
- (c) The submodule  $F^p(V)$  of  $S^p(V)$  has a complement in  $S^p(V)$ .

**Corollary 4.5.** Let  $p > 0$ ,  $G$  be a linear algebraic group such that the connected component of the unit element  $G^0$  is not a torus, and  $V$  be a faithful  $G$ -module. Then the submodule  $F^p(V)$  of  $S^p(V)$  has no complement in  $S^p(V)$ . In particular,  $G$  is not linearly reductive.

Corollary 4.5 together with Proposition 4.2 explicitly leads to the construction of a  $G$ -module  $U$  and a non-trivial cocycle  $g \in Z^1(G, U)$ . All we need to start is a faithful  $G$ -module (which always exists). So the next step is to find annihilators of  $g$ . If  $W$  is another  $G$ -module and  $w \in W^G$ , then for a  $g \in Z^1(G, U)$  we can define in an obvious manner  $w \otimes g \in Z^1(G, W \otimes U)$ , and we also get a map  $w \otimes : H^1(G, U) \rightarrow H^1(G, W \otimes U)$ .

Let  $g \in Z^1(G, V)$ . Then the  $K$ -vector space  $\tilde{V} := V \oplus K$  can be turned into a  $G$ -module with the  $G$ -action given by  $\sigma \cdot (v, \lambda) := (\sigma v + \lambda g_\sigma, \lambda)$  for all  $(v, \lambda) \in \tilde{V}$ ,  $\sigma \in G$ . Up to  $G$ -module isomorphism,  $\tilde{V}$  only depends on  $g + B^1(G, V)$ . We call  $\tilde{V}$  the (corresponding) extended  $G$ -module of  $V$  (by  $g$ ).

**Proposition 4.6.** (See Kemper [13, Proposition 2].) Let  $U$  be a  $G$ -module,  $g \in Z^1(G, U)$  be a cocycle, and let  $\tilde{U} = U \oplus K$  be the extended  $G$ -module corresponding to  $g$ . Let further be  $\pi : \tilde{U} \rightarrow K$ ,  $(u, \lambda) \mapsto \lambda$  (with  $u \in U$ ,  $\lambda \in K$ ). Then  $\pi$  is invariant,  $\pi \in \tilde{U}^{*G}$ , and  $\pi \otimes g = 0 \in H^1(G, \tilde{U}^* \otimes U)$ .

**Theorem 4.7.** Let  $G$  be a reductive group,  $U$  be a  $G$ -module such that there is a  $0 \neq g \in H^1(G, U)$  and  $\tilde{U}$  be the corresponding extended  $G$ -module. If  $V := U^* \oplus \bigoplus_{i=1}^k \tilde{U}$ , then we have  $\text{cmdef } K[V]^G \geq k - 2$ .

**Proof.** Since  $U$  is a direct summand of  $K[V] = S(V^*)$ , we have (after an embedding)  $0 \neq g \in H^1(G, K[V])$ . Let  $a_1, \dots, a_k \in K[V]^G$  be the  $k$  copies of the element  $\pi \in \tilde{U}^{*G}$  of Proposition 4.6 in the  $k$  summands  $\tilde{U}^*$  of  $K[V]$ . Then  $a_1, \dots, a_k$  form a phsop in  $K[V]$  and we have  $a_i g = 0 \in H^1(G, K[V])$  for  $i = 1, \dots, k$ . Now the result follows from Theorem 3.2, case (a).  $\square$

**Theorem 4.8.** Under the hypotheses of Theorem 4.7, with  $V := U^* \oplus \tilde{U}$ , we have

$$\text{cmdef } K[V^{\oplus k}]^G \geq k - 2 \quad \text{for all } k \in \mathbb{N}.$$

If  $W$  is any faithful  $G$ -module, then  $V := W \oplus U^* \oplus \tilde{U}$  is faithful, and the theorem above remains valid with this  $V$ . Now let  $G$  be a reductive group that is not linearly reductive. Then by definition, there exists a  $G$ -module  $M$  with a submodule  $N \subseteq M$  without complement. By Proposition 4.2, the module  $U := \text{Hom}_K(M, N)_0$  satisfies the hypotheses of Theorem 4.7 or 4.8, so we have proved Theorem 1.3. Together with the theorem of Hochster and Roberts, this immediately leads to the following characterization of linearly reductive groups:

**Corollary 4.9.** A reductive group  $G$  is linearly reductive if and only if there is a global Cohen–Macaulay defect bound, i.e. a number  $b \in \mathbb{N}$  with  $\text{cmdef } K[V]^G \leq b$  for all  $G$ -modules  $V$ .

Bringing all construction steps together, we get the following explicit result. We will restrict ourselves to the case that  $G^0$  is not a torus. See [16, Satz 4.2] for the other case.

**Theorem 4.10.** Let  $p > 0$  and  $G$  be a reductive group such that  $G^0$  is not a torus, and  $V$  be a faithful  $G$ -module. Let

$$U := \text{Hom}_K(S^p(V), F^p(V))_0 \cong F^p(V) \otimes (S^p(V)/F^p(V))^*,$$

(see Definition 4.1) and  $\iota \in \text{Hom}_K(S^p(V), F^p(V))$  with  $\iota|_{F^p(V)} = \text{id}_{F^p(V)}$ . Then with  $g : G \rightarrow U$ ,  $\sigma \mapsto (\sigma - 1)\iota$  we have a  $0 \neq g \in H^1(G, U)$ . Let  $\tilde{U}$  be the corresponding extended  $G$ -module. Then the  $G$ -module

$$M_k := F^p(V)^* \oplus (S^p(V)/F^p(V)) \oplus \bigoplus_{i=1}^k \tilde{U}$$

is faithful, and we have

$$\text{cmdef } K[M_k]^G \geq k - 2 \quad \text{for all } k \geq 0.$$

**Proof.** By Proposition 4.2 and Corollary 4.5, we have  $0 \neq g \in H^1(G, U)$ . The direct summand  $F^p(V)^*$  of  $M_k$  is faithful since  $V$  is, hence  $M_k$  is faithful. Since the module  $U = F^p(V) \otimes (S^p(V)/F^p(V))^*$  is a direct summand of the second symmetric power  $S^2(F^p(V) \oplus (S^p(V)/F^p(V))^*)$ , it is also a direct summand of  $K[M_k] = S(M_k^*)$ , hence after an embedding we have  $0 \neq g \in H^1(G, K[M_k])$ . Now the proof proceeds like the one of Theorem 4.7.  $\square$

**Remark.** Comparing with Theorem 4.7, we see that in the definition of  $M_k$  we replaced the summand  $U^* = F^p(V)^* \otimes (S^p(V)/F^p(V))$  by  $F^p(V)^* \oplus (S^p(V)/F^p(V))$ . This makes  $M_k$  faithful and leads in most cases to a lower dimension of  $M_k$ .

## 5. Examples for $\text{SL}_2$ and $\mathbb{G}_a$ invariants in positive characteristic

The group  $\text{SL}_2$  acts faithfully by left multiplication on  $U := K^2$ . Let  $\{X, Y\}$  be the standard basis of  $U$ , so we have  $U = \langle X, Y \rangle_K$ . We use the notation like  $S^2(U) =: \langle X^2, Y^2, XY \rangle$  and  $F^p(U) =: \langle X^p, Y^p \rangle$ . Explicit calculations of the modules  $M_k$  of Theorem 4.10 lead to the following examples. These calculations can be found in Kohls [15, Section 3], where we had to restrict ourselves to the case  $k = 3$  since we did not have Theorem 3.2.

**Example 5.1.** Let  $p = 2$ . Then

$$\text{cmdef } K \left[ \langle X^2, Y^2 \rangle \oplus \bigoplus_{i=1}^k \langle X^2, Y^2, XY \rangle \right]^{\text{SL}_2} \geq k - 2 \quad \text{for all } k \in \mathbb{N}.$$

**Example 5.2.** Let  $p = 3$ . Then with

$$M_k := \langle X^3, Y^3 \rangle \oplus \langle X, Y \rangle \oplus \bigoplus_{i=1}^k S^4(\langle X, Y \rangle) \quad \text{or}$$

$$M_k := \langle X^3, Y^3 \rangle \oplus \langle X, Y \rangle \oplus \bigoplus_{i=1}^k \langle X^2, Y^2, XY \rangle$$

we have  $\text{cmdef } K[M_k]^{\text{SL}_2} \geq k - 2$  for all  $k \in \mathbb{N}$ . In the second case,  $M_k$  is self-dual and completely reducible, since its summands are.

One can use Roberts' isomorphism [20] to turn an example for the group  $\text{SL}_2$  into an example for the additive group  $\mathbb{G}_a = (K, +)$ : Every  $\text{SL}_2$ -module  $V$  can be regarded as a module of the additive group  $\mathbb{G}_a$  by the embedding  $\mathbb{G}_a \hookrightarrow \text{SL}_2$ ,  $t \mapsto \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ . Roberts' isomorphism (see [3, Example 3.6]) says we then have

$$K[\langle X, Y \rangle \oplus V]^{\text{SL}_2} \cong K[V]^{\mathbb{G}_a}.$$

It is probably worth remarking that in positive characteristic, it is not known if for every  $\mathbb{G}_a$ -module  $V$  the invariant ring  $K[V]^{\mathbb{G}_a}$  is finitely generated, while in characteristic zero this is Weitzenböck's Theorem [23]. If  $V$  is a  $\mathrm{SL}_2$ -module and we have used Theorem 3.2 in case (a) to show  $\mathrm{cmdef} K[V]^{\mathrm{SL}_2} \geq k - 2$ , then we also have  $\mathrm{cmdef} K[\langle X, Y \rangle \oplus V]^{\mathrm{SL}_2} \geq k - 2$ , because all the hypotheses of the theorem made for  $K[V]$  will still hold for  $K[\langle X, Y \rangle \oplus V]$ . Then by Roberts' isomorphism, we also have  $\mathrm{cmdef} K[V]^{\mathbb{G}_a} \geq k - 2$ . In the case that  $\langle X, Y \rangle$  already is a direct summand of  $V$ , so  $V \cong \langle X, Y \rangle \oplus V'$  with a  $\mathrm{SL}_2$ -module  $V'$ , then Roberts' isomorphism directly tells us  $\mathrm{cmdef} K[V']^{\mathbb{G}_a} \geq k - 2$ . In particular, Examples 5.1 and 5.2 for the group  $\mathrm{SL}_2$  can easily be turned into examples for the group  $\mathbb{G}_a$ —e.g. for  $p = 3$  we have  $\mathrm{cmdef} K[X^3, Y^3] \oplus \bigoplus_{i=1}^k \langle X^2, Y^2, XY \rangle^{\mathbb{G}_a} \geq k - 2$  for all  $k \geq 1$ .

If  $G$  is a non-trivial, connected unipotent group, there is a surjective algebraic homomorphism  $G \rightarrow \mathbb{G}_a$  (see [3, last paragraph before Section 3] for a proof of this well-known result). So if  $V$  is any  $\mathbb{G}_a$ -module, it can be regarded as a  $G$ -module with the same invariant ring. In particular, we have

**Theorem 5.3.** *For every non-trivial, connected unipotent group  $G$  over an algebraically closed field  $K$  of characteristic  $p > 0$ , there exists a  $G$ -module  $V$  such that  $K[V^{\oplus k}]^G$  is finitely generated and  $\mathrm{cmdef} K[V^{\oplus k}]^G \geq k - 2$  for all  $k \geq 1$ .*

The modules  $M_k$  of Theorem 4.10 often are not very “nice”, in particular they have big dimensions. With some more effort, we succeeded to construct “nicer” modules for the groups  $\mathrm{SL}_2$  and  $\mathbb{G}_a$  such that the invariant rings have big Cohen–Macaulay defect. We just state the result here, and refer to my thesis [16, pp. 113–126] for the proof.

**Theorem 5.4.** *Let  $\langle X, Y \rangle$  be the natural representation of  $\mathrm{SL}_2$  and  $p > 0$ . Let*

$$V := \langle X^p, Y^p \rangle \oplus \bigoplus_{i=1}^k \langle X, Y \rangle.$$

*Then we have*

$$\mathrm{cmdef} K[V]^{\mathbb{G}_a} \geq k - 2 \quad \text{and} \quad \mathrm{cmdef} K[V]^{\mathrm{SL}_2} \geq k - 3.$$

*As a direct sum of self dual  $\mathbb{G}_a$ - or  $\mathrm{SL}_2$ -modules,  $V$  is self-dual, too. Regarded as  $\mathrm{SL}_2$ -module,  $V$  is completely reducible as a direct sum of irreducible  $\mathrm{SL}_2$ -modules. Furthermore,*

$$\dim K[V]^{\mathbb{G}_a} = 2k + 1 \quad \text{and} \quad \dim K[V]^{\mathrm{SL}_2} = 2k - 1,$$

*so we have*

$$\mathrm{depth} K[V]^{\mathbb{G}_a} \leq k + 3 \quad \text{and} \quad \mathrm{depth} K[V]^{\mathrm{SL}_2} \leq k + 2.$$

We have the conjecture that all inequations in this theorem in fact are equations if  $k \geq 2$  and  $k \geq 3$  for  $\mathbb{G}_a$  and  $\mathrm{SL}_2$  respectively. We were able to verify this conjecture with computational methods using MAGMA [1] for the group  $\mathbb{G}_a$  in the cases  $(p, k) \in \{(2, 2), (2, 3), (2, 4), (3, 2), (3, 3)\}$ —then by Roberts' isomorphism, the conjecture is also true for  $\mathrm{SL}_2$  and the corresponding pair  $(p, k + 1)$ . See [16, pp. 129–140] for the computational details. It is interesting to compare this theorem with the result of Hochster that we mentioned in the text below Theorem 1.3.

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