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Restricted representations of the Witt superalgebras ☆

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ABSTRACT

Let k be an algebraically closed field of characteristic $p > 3$, and $W(n)$ the superversion of the Witt algebra over k, i.e. the Lie superalgebra of superderivations of the Grassmann algebra of rank n over k. The simple modules and projective indecomposable modules in the restricted supermodule category for $W(n)$ are studied. The character formulas for such simple modules are given, and the Cartan invariants for this category are presented.

1. Preliminaries

1.1. Introduction

In [BW], Block and Wilson proved the conjecture raised by Kostrikin and Shafarevich in [KS] which asserts all simple finite-dimensional Lie algebras over an algebraically closed field of characteristic $\geqslant 7$ fall into two types of infinite series: classical type and Cartan type. The former are analogue of complex simple Lie algebras, and the latter are Lie algebras of derivations on the truncated polynomial algebras. Kac's classification result on finite-dimensional simple Lie superalgebras over the complex numbers in [K] shows that a complex superversion of Kostrikin–Shafarevich conjecture is true. Cartan type Lie superalgebras in prime characteristic can be given naturally (cf. [L] and [ZL]). One naturally expects a supercounterpart of Kostrikin–Shafarevich conjecture on the simple finite-dimensional Lie superalgebras over k.

Let $\mathfrak{g} = W(n)$ be the Lie superalgebra of superderivations on the Grassmann algebra of rank n over an algebraically closed filed k of characteristic $p > 3$, which is of simplest series in Cartan type

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Lie superalgebras over k. The aim of this paper is to study restricted representations of $\mathfrak{g}$, i.e. representations of the restricted enveloping algebra $u(\mathfrak{g})$. Our work is motivated by Serganova's work on the complex Witt superalgebras. Recall that Bernstein and Leites studied the representations of this complex Lie superalgebra $W(n)$ early in 1983, giving a construction for all simple modules and giving their dimensions (cf. [BL]). Serganova reworked with this subject, classifying all simple modules and giving their character formulas for the $\mathbb{Z}$-graded module category of Cartan type Lie supalgebras over the complex numbers (cf. [S]). We apply Serganova's arguments to the category $(\mathfrak{T}, u(\mathfrak{g}))$-**mod** (to see Definition 2.8) in the modular case, which is a refined version of the restricted module category for $\mathfrak{g}$, subject to the admissibility with $\mathfrak{T}$-action for the canonical maximal torus $\mathfrak{T}$ of $\mathrm{GL}(n,k)$. We obtain the character formulas for simple modules in this category. Then we study projective modules in $u(\mathfrak{g})$-module category, finally we obtain the Cartan invariants for this category, extending the result on the restricted module category for the ordinary Witt algebra obtained by Nakano (cf. [N]).

1.2. *The Witt superalgebra*

In this subsection and next subsection, we record some basic properties of the Witt superalgebras and fix some notations. Let k be an algebraically closed field of characteristic $p>0$. Let $\bigwedge(n)$ be the free commutative superalgebra with n odd generators $\xi_1,\dots,\xi_n$ (isomorphic to the Grassmann algebra), and let $W(n)$ be the Lie superalgebra of superderivations of $\bigwedge(n)$. Then $W(n) := \{\sum_{i=1}^n f_i\partial_i \mid f_i \in \bigwedge(n)\}$, where ∂_i is a superderivation defined via $\partial_i(\xi_j)=\delta_{ij}$. The superstructure on $W(n)$ arises from the $\mathbb{Z}$-grading over $W(n)=\sum_{j=-1}^{n-1} W(n)_j$ with $W(n)_j = \text{span-}\{\xi_{t_1}\cdots\xi_{t_{j+1}}\partial_s \mid t_1<t_2<\cdots<t_{j+1},\ s=1,\dots,n\}$, more precisely with convention $\deg(\xi_i)=1$ and $\deg(\partial_i)=-1$. Set $W(n)^i=\sum_{j=i}^{n-1} W(n)_j$, $i=-1,0,1,\dots,n-1$. There is a natural filtered structure on W_n: $W(n) = W(n)^{-1} \supset W(n)^0 \supset W(n)^1 \supset \cdots \supset W(n)^i \supset \cdots$.

If $n=1$, then $W(n)$ becomes a 2-dimensional Lie algebra. *So we will always assume $n \geqslant 2$ throughout the paper*.

Recall that a Lie superalgebra $\mathfrak{g}=\mathfrak{g}_{\bar 0}\oplus\mathfrak{g}_{\bar 1}$ is called *restricted* if $\mathfrak{g}_{\bar 0}$ is a restricted Lie algebra in the usual sense (cf. [J] and [SF]), and $\mathfrak{g}_{\bar 1}$ is a restricted $\mathfrak{g}_{\bar 0}$-module under the adjoint action. The restricted enveloping algebra $u(\mathfrak{g})$ of $\mathfrak{g}$ is defined to be a quotient of $U(\mathfrak{g})$ by the ideal generated by $\{X^p - X^{[p]} \mid X\in\mathfrak{g}_{\bar 0}\}$. A supermodule (ρ, V) of $\mathfrak{g}$ is said to be *restricted* if ρ satisfies for all $X\in\mathfrak{g}_{\bar 0}$

$$\rho(X)^p - \rho\big(X^{[p]}\big) = 0.$$

All restricted modules of $\mathfrak{g}$ constitute a full subcategory of the $\mathfrak{g}$-module category, which coincides with the $u(\mathfrak{g})$-module category, denoted by $u(\mathfrak{g})$-**mod**.

By a direct computation, we know that $W(n)$ is a restricted Lie superalgebra. Especially, the p-mapping $[p]$ on $W(n)_{\bar 0}$ is just given as the usual pth power of derivations.

1.3. *More structural information*

Let $\mathfrak{g}=W(n)$, and $\mathfrak{g}_{-1}=W_{-1}$ and $\mathfrak{g}_1=\sum_{i\geqslant 0} W_{2i+1}$, $\mathfrak{g}_0=W_0$. Then $\mathfrak{g}_{\bar 0}=\mathfrak{g}_0+\sum_{i>0} W_{2i}$, and $\mathfrak{g}_{\bar 1}=\mathfrak{g}_{-1}+\mathfrak{g}_1$. Recall that $\mathfrak{g}_0\cong\mathfrak{gl}(n)$ under the map $\mathfrak{g}_0\to\mathfrak{gl}(n)$, $\xi_iD_j\mapsto E_{ij}$. Furthermore, $\mathfrak{g}_0=\mathfrak{n}^-+\mathfrak{h}+\mathfrak{n}^+$ for $\mathfrak{n}^-=\sum_{i>j}k\xi_iD_j$, $\mathfrak{h}=\sum_i kh_i$ for $h_i=\xi_iD_i$ and $\mathfrak{n}^+=\sum_{i<j}\xi_iD_j$, $\mathfrak{b}^\pm=\mathfrak{n}^\pm+\mathfrak{h}$. (By convention, $\mathfrak{b}^+$ can be simply denoted by $\mathfrak{b}$.) Set $N^- := \mathfrak{g}_{-1}+\mathfrak{n}^-$, $N^+ := \mathfrak{n}^+ + W^1$; $B^- := N^- + \mathfrak{h}$ and $B^+ = N^+ + \mathfrak{h}$; $\mathfrak{g}^+ := \mathfrak{g}_0 + W^1$ $(=W^0)$, $\mathfrak{g}^- := \mathfrak{g}_0+\mathfrak{g}_{-1}$. Then $N^\pm$, $\mathfrak{g}^+=\mathfrak{g}_{\bar 0}+\mathfrak{g}_1$ and $\mathfrak{g}^-$ are all restricted supersubalgebras of $\mathfrak{g}$.

The Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}_0$ is also a Cartan subalgebra of $\mathfrak{g}$. We have a root decomposition $\mathfrak{g}=\mathfrak{h}\oplus\sum_{\alpha\in\Delta}\mathfrak{g}_\alpha$. Taking the standard basis $\{\epsilon_1,\dots,\epsilon_n\}\in\mathfrak{h}^*$ with $\epsilon_i(h_j)=\delta_{ij}$, we have

$$\Delta=\{\epsilon_{i_1}+\cdots+\epsilon_{i_t}-\epsilon_j \mid 1\leqslant i_1<\cdots<i_t\leqslant n;\ 1\leqslant t, j\leqslant n\}\cup\{-\epsilon_i \mid i=1,\dots,n\}.$$

We recall some facts on the roots. A root α is either satisfying $-\alpha\notin\Delta$, or satisfying $-\alpha\in\Delta$, which is by definition called *nonessential*, or *essential*, dependent on the corresponding situation. So all roots are of three basic types: nonessential, even essential and odd essential. If α is even essential, then both $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ are in $\mathfrak{g}_0$, the multiplicity of α and $-\alpha$ is 1 with $\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}$ generating a subalgebra isomorphic to $\mathfrak{sl}(2)$. If α is odd essential, then one of the two spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ has dimension 1 and the other one has dimension $n-1$. The corresponding root space belongs to $\mathfrak{g}_{-1}$ in the former case, or belongs to W_1 in the later case. For an essential root α, $\dim \mathfrak{h}_\alpha = n-1$ if α is old, and 1 if α is even, where $\mathfrak{h}_\alpha = [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$. Clearly, an essential root is some one from the set $\{\pm\epsilon_i, \pm(\epsilon_i - \epsilon_j)\}$.

Another important fact on $\mathfrak{g}$ is that the restricted enveloping algebra $u(\mathfrak{g})$ can be formulated as

$$\bigwedge(\mathfrak{g}_{-1}) \otimes_k u\big(\mathfrak{g}^+\big).$$

Convention. The terminology of ideals, subalgebras, modules etc. of a Lie superalgebra instead of superideals, subsuperalgebras, supermodules, etc. is adopted in this paper. The notation A-**mod** will stand for the module category for an algebraic object A.

2. Restricted representations and character formulas of irreducible modules for $W(n)$

Keep the notations as in the previous section. In particular, we always assume that k is an algebraically closed field of characteristic $p > 3$, and all vector spaces are defined over k. Assume $\mathfrak{g} = W(n)$ in the sequel.

2.1. Restricted irreducible representations of $W(n)$

Recall that the iso-classes of all restricted simple modules of $\mathfrak{g}_0$ are parameterized by $\Lambda := \{\lambda = (\lambda_1, \ldots, \lambda_n) \mid \lambda_i \in \mathbb{F}_p,\ i = 1, \ldots, n\}$ which coincides with the set $\mathbb{F}_p^n$ (cf. [J2]). Take a set of representatives of restricted simple modules: $\{L^0(\lambda) \mid \lambda \in \Lambda\}$. Precisely, $L^0(\lambda)$ can be regarded as the simple head of the baby Verma module $Z(\lambda) = u(\mathfrak{g}_0) \otimes_{u(\mathfrak{b})} k_\lambda$. Similarly, we can define $L^0_{\min}(\lambda)$ to be the simple head of $Z^-(\lambda) = u(\mathfrak{g}_0) \otimes_{u(\mathfrak{b}^-)} k_\lambda$. And $\{L^0_{\min}(\lambda) \mid \lambda \in \Lambda\}$ becomes another set of representatives of the iso-classes of restricted simple modules of $\mathfrak{g}_0$.

Since every element in W^1 nilpotently acts on $\mathfrak{g}$. So each simple restricted module of $\mathfrak{g}_0$ can be extended to the one of $\mathfrak{g}^+ = \mathfrak{g}_0 + W^1$ with trivial action of W^1. Define the Kac modules

$$K^+(\lambda) = u(\mathfrak{g}) \otimes_{u(\mathfrak{g}^+)} L^0(\lambda) \overset{\text{as } k\text{-superspace}}{=} \bigwedge(\mathfrak{g}_{-1}) \otimes_k L^0(\lambda), \tag{2.1}$$

$\lambda \in \Lambda$. In some cases, $K^+(\lambda)$ is written as $K(\lambda)$ for simplicity (note: both notations are used in the text, dependent on different situations). The weight λ can be expressed as $\lambda = \sum_{i=1}^n \lambda_i \epsilon_i$ for $\epsilon_i = (\delta_{i1}, \ldots, \delta_{in})$.

Lemma 2.1. *Let v_λ be a fixed maximal vector of $L^0(\lambda)$.*

(1) *For any $f \in u(N^-)N^-$, there exists a positive integer r such that $f^r = 0$.*
(2) *Set $J(\lambda)$ to be the sum of all proper submodules of $K(\lambda)$. Then $J(\lambda)$ is a proper submodule of $K(\lambda)$.*

Proof. (1) We observe that N^- has a decomposition of root spaces under the (adjoint) action of $\mathfrak{h}$: $N^- = \sum_{\alpha \in \Delta^-} N_\alpha$ where $\Delta^- = \{-\epsilon_i,\ 1 \leqslant i \leqslant n;\ \epsilon_i - \epsilon_j,\ 1 \leqslant i < j \leqslant n\}$, and $[N_\alpha, N_\beta] = \delta_{\alpha+\beta \in \Delta^-, \mathrm{true}} N_{\alpha+\beta}$, where $\delta_{\alpha+\beta \in \Delta^-, \mathrm{true}}$ equals 1 or 0 dependent on whether $\alpha + \beta$ lies in Δ^-, or not. Hence N^- is nilpotent. On the other hand, $D_i^2 = 0$ and $(\xi_i D_j)^p = 0$ in $u(N^-)$. Hence, $u(N^-)$ is a local algebra with maximal ideal $u(N^-)N^-$ which is nilpotent.

(2) We first observe that any nonzero proper submodule M in $K(\lambda)$ is included in $u(N^-)N^- v_\lambda$, where v_λ is the maximal vector of $L^0(\lambda)$. Actually, suppose $M \not\subset u(N^-)N^- v_\lambda$, then there is a nonzero vector $w = (1-f)v_\lambda \in M$, $f \in u(N^-)N^-$. Accordingly, by (1), $f \in u(N^-)N^-$ satisfies $f^r = 0$ for some positive integer r. So we have for $f' = 1 + f + \cdots + f^{r-1} \in u(N^-)$, $f'w = v_\lambda$, and then $M = K(\lambda)$. Obviously, $u(N^-)N^- v_\lambda$ is a proper subspace of $K(\lambda)$. So $J(\lambda)$ is really a proper submodule. $\square$

Thus, by the above lemma we know that $K(\lambda)$ has a unique simple quotient, denoted by $L(\lambda)$, $\lambda \in \Lambda$. In the same way, we can define

$$K^{\vee}(\lambda) = u(\mathfrak{g}) \otimes_{u(\mathfrak{g}^{+})} L^{0}_{\min}(\lambda), \tag{2.2}$$

which admits a unique simple quotient denoted by $L^{\vee}(\lambda)$. Note that there is a bijection σ on Λ: $\lambda \mapsto \sigma(\lambda)$ which is defined via

$$L^{0}(\lambda) \cong L^{0}_{\min}\big(\sigma(\lambda)\big).$$

Obviously, $L^{0}(\lambda)$ admits a unique lowest weight vector $f_{\lambda} v_{\lambda}$ of weight $\sigma(\lambda)$, up to scalar, where $f_{\lambda} \in u(\mathfrak{n}^{-})$. Additionally, we have

$$K^{-}(\lambda) := U(\mathfrak{g}) \otimes_{u(\mathfrak{g}^{-})} L^{0}(\lambda). \tag{2.3}$$

By the same reason, $K^{-}(\lambda)$ has a unique simple quotient denoted by $L^{\triangledown}(\lambda)$.

Proposition 2.2. *The family $L(\lambda)$ constitute the set of iso-classes of restricted irreducible modules of $\mathfrak{g}$. Similarly, the family $L^{\vee}(\lambda)$ (resp. $L^{\triangledown}(\lambda)$) constitute the set of iso-classes of restricted irreducible modules of $\mathfrak{g}$.*

Proof. It suffices to prove that $\lambda = \mu$ if $L(\lambda) \cong L(\mu)$ for $\lambda, \mu \in \Lambda$.

By the hypothesis $L(\lambda) \cong L(\mu)$, there is an isomorphism $K(\lambda)/J(\lambda) \cong L(\mu)$. Thus there must exist a vector $w_{\mu} \in K(\lambda)\backslash J(\lambda)$ with $(W^{1} + \mathfrak{n}^{+})w_{\mu} \in J(\lambda)$ and $hw_{\mu} \in \mu(h)w_{\mu} + J(\lambda)$, for $h \in \mathfrak{h}$. We can write

$$w_{\mu} = av_{\lambda} + \sum_{i} \omega_{i} \otimes f_{i} v_{\lambda},$$

where $a \in k^{\times}$ (here and further, $k^{\times}$ *means the multiplicative group* $k\backslash\{0\}$) and $\omega_{i} \otimes f_{i} \in \bigwedge(\mathfrak{g}_{-1})\mathfrak{g}_{-1} \otimes u(\mathfrak{n}^{-}) + \bigwedge(\mathfrak{g}_{-1}) \otimes u(\mathfrak{n}^{-})\mathfrak{n}^{-}$. From $hw_{\mu} - \mu(h)w_{\mu} \in J(\lambda)$, it follows that $\lambda(h) = \mu(h)$ for all $h \in \mathfrak{h}$, thereby $\lambda = \mu$. $\square$

Thus, we can define two bijections $\vee$ and $\triangledown$ on Λ via: $L(\lambda) \cong L^{\vee}(\lambda^{\vee})$ and $L(\lambda) \cong L^{\triangledown}(\lambda^{\triangledown})$.

Next, let us analyze the structure of $L(\lambda)$. Consider $u(\mathfrak{g}_0)$-submodules $L^{i}_{\min}$, $i = 0, 1, \ldots, n$ which is generated by the homomorphic image $\overline{v^{i}_{\lambda}}$ of the vector

$$v^{i}_{\lambda} = (D_1 \wedge D_2 \wedge \cdots \wedge D_i) \otimes f_{\lambda} v_{\lambda}$$

under the canonical projection of $K(\lambda)$ onto $L(\lambda)$.

Call λ regular, if $\lambda(h_i) \neq 0$, $i = 2, \ldots, n$, and $\lambda(h_1, \ldots, h_{n-1}) \neq (1, 1, \ldots, 1)$. We first have the following lemma.

Lemma 2.3. *Assume $\sigma(\lambda)$ is regular. Then all $L^{i}_{\min}$ are nonzero $\mathfrak{g}_0$-submodules in $L(\lambda)$, $i = 0, 1, \ldots, n$.*

Proof. We need to prove that v^{i}_{λ} does not lie in $J(\lambda)$. This is to say, the $\mathfrak{g}$-submodule generated by v^{i}_{λ} is the whole $K(\lambda)$. That is obvious for $i = 0$. Now fix $i \in \{1, \ldots, n-1\}$. Denote $K(\lambda)^{i}$ the $\mathfrak{g}$-submodule generated by v^{i}_{λ}. Recall that $\mathfrak{n}^{-} f_{\lambda} v_{\lambda} = 0$, and $W^{1} L^{0}(\lambda) = 0$. We have the following calculation:

$$\begin{aligned}(\xi_i \xi_{i+1} D_{i+1}) v^{i}_{\lambda} &= (D_1 \wedge \cdots \wedge D_i) \otimes \xi_{i+1} D_{i+1} f_{\lambda} v_{\lambda} \\ &= \sigma(\lambda)(h_{i+1}) v^{i-1}_{\lambda}.\end{aligned}$$

By induction, we know that $K(\lambda)^i = K(\lambda)$ for $i = 0, 1, \dots, n-1$.

Now consider the case when $i = n$. We have for $j = 1, \dots, n-1$

$$(\xi_n \xi_j D_j) v_\lambda^n = (-1)^{n-j-1}\big(1 - \sigma(\lambda)(h_j)\big) v_\lambda^{n-1},$$

which means that $K(\lambda)^n = K(\lambda)$. Thus, all $\mathfrak{g}_0$-submodules $L^i_{\min}$ are nonzero because the generators $\overline{v^i_\lambda}$ are all nonzero. $\square$

Furthermore, it's not hard to see that $L^i_{\min}$ is a lowest weight module with the lowest weight vector $\overline{v^i_\lambda}$ of weight λ^i where $\lambda^i = \sigma(\lambda) - \sum_{j=1}^i \epsilon_j$, i.e. $\mathfrak{n}^- \overline{v^i_\lambda} = 0$ and $h\overline{v^i_\lambda} = \lambda^i(h)\overline{v^i_\lambda}$ for $h \in \mathfrak{h}$. So there is a canonical surjective homomorphism of $\mathfrak{g}_0$-module from $L^i_{\min}$ to the irreducible lowest weight module $L^0_{\min}(\lambda^i)$. Set $K^i_{\min}$ the $\mathfrak{g}_0$-submodule in $K(\lambda)$ generated by v^i_λ. Then the above argument for $L^i_{\min}$ is true for $K^i_{\min}$. Furthermore, $L^i_{\min}$ is the homomorphic image of $K^i_{\min}$ under the canonical surjective $\mathfrak{g}_0$-homomorphism from $K(\lambda)$ onto $L(\lambda)$.

Proposition 2.4. *Maintain the above notations.*

(1) $K^i_{\min} \cong L^i_{\min} \cong L^0_{\min}(\lambda^i)$.

(2) *Assume $\sigma(\lambda)$ is regular, then as a $\mathfrak{g}_0$-module,* $\mathrm{Soc}_{u(\mathfrak{g}_0)}(K(\lambda)) = \bigoplus_{i=0}^n K^i_{\min}$ *and* $\mathrm{Soc}_{u(\mathfrak{g}_0)}(L(\lambda)) = \bigoplus_{i=0}^n L^i_{\min}$.

Proof. (1) Observe the fact that $K(\lambda)$ is a direct sum of all $\mathfrak{g}_0$-submodules $\Omega^i(\lambda) := \bigwedge^i(\mathfrak{g}_{-1}) \otimes L^0(\lambda)$, $i = 0, 1, \dots, n$. Obviously, $K^i_{\min} \subset \Omega^i(\lambda)$. Next, we assert that $K^i_{\min}$ is the simple socle of $\Omega^i(\lambda)$ as $\mathfrak{g}_0$-module. For this, we consider $\mathcal{A}_{\mathfrak{n}^-}(\Omega^i(\lambda)) := \{v \in \Omega^i(\lambda) \mid \mathfrak{n}^- v = 0\}$. Clearly, $kv^i_\lambda \subset \mathcal{A}_{\mathfrak{n}^-}(\Omega^i(\lambda))$.

Claim 1. $\mathcal{A}_{\mathfrak{n}^-}(\Omega^i(\lambda)) = kv^i_\lambda$.

Assume w is a nonzero vector in $\mathcal{A}_{\mathfrak{n}^-}(\Omega^i(\lambda))$. Set $\omega_{\underline{j}} := D_{j_1} \wedge \cdots \wedge D_{j_i}$ for $\underline{j} = \{1 \leqslant j_1 < j_2 < \cdots < j_i \leqslant n\}$. We may write

$$w = \sum_{1 \leqslant j_1 < j_2 < \cdots < j_i \leqslant n} c_{\underline{j}} \omega_{\underline{j}} \otimes u_{\underline{j}} v_\lambda,$$

where $c_{\underline{j}} \in k$, and $u_{\underline{j}} \in u(\mathfrak{n}^-)$.

We first assert that $\underline{j} = \{1 < 2 < \cdots < i\}$ if $c_{\underline{j}} \neq 0$. Set $d(w) = \max\{q \mid j_q = q \text{ for all } \underline{j}, \text{ with } c_{\underline{j}} \neq 0\}$ (appoint $d(w) = 0$ if no such a q exists). Then the assertion is equivalent to say that $d(w) = i$. Suppose $d < i$, then there is $\underline{j}$ with $c_{\underline{j}} \neq 0$ and $j_{d+1} > d+1$. Fix $\underline{j}^0$ with $j^0_{d+1} = \min\{j_{d+1}(> d+1) \mid c_{\underline{j}} \neq 0\}$, and denote $d(w)$ by d for simplicity. Then we have

$$\begin{aligned}(\xi_{j^0_{d+1}} D_{d+1})w &= \sum_{\underline{j}} -c_{\underline{j}}(D_1 \wedge \cdots \wedge D_d \wedge \delta_{j_{d+1}, j^0_{d+1}} D_{d+1} \wedge D_{j_{d+2}} \wedge \cdots \wedge D_{j_i}) \otimes u_{\underline{j}} v_\lambda \\ &\quad + \sum_{\underline{j}} c_{\underline{j}} \omega_j \otimes \xi_{j^0_{d+1}} D_{d+1} u_{\underline{j}} v_\lambda \\ &= w' + w''\end{aligned}$$

where w' and w'' respectively denote the first and the second summation appearing on the RHS of the first equation. By the choice of $\underline{j}^0$, we know $w' \neq 0$ and $d(w') = d(w) + 1$. And $d(w'') = d(w)$ if $w'' \neq 0$, which implies w' and w'' are linearly independent. Thus we prove that $(\xi_{j^0_{d+1}} D_{d+1})w \neq 0$

which contradicts with the assumption of w. Hence our assertion is true. This is to say, w is of the form

$$w = (D_1 \wedge \cdots \wedge D_i) \otimes f v_\lambda$$

where $f \in u(\mathfrak{n}^-)$. Then $w \in \mathcal{A}_{\mathfrak{n}^-}(\Omega^i(\lambda))$ means that $f v_\lambda$ is the minimal vector of $L^0(\lambda^i)$, thereby $f v_\lambda \in k f_\lambda v_\lambda$. Hence $w \in k v_\lambda^i$.

Thus, we complete the proof of Claim 1.

Claim 2. $K^i_{\min}$ is a simple $\mathfrak{g}_0$-module, thereby isomorphic to $L^i_{\min}$, both of which are isomorphic to $L^0_{\min}(\lambda^i)$. Furthermore, $\mathrm{Soc}_{u(\mathfrak{g}_0)}(\Omega^i(\lambda)) = K^i_{\min}$.

For any irreducible $\mathfrak{g}_0$-submodule L in $\Omega^i(\lambda)$ must admit a nonzero vector annihilated by $\mathfrak{n}^-$. Thanks to the argument above, L contains $k v^i_\lambda$. Hence L includes $K^i_{\min}$, which means that L coincides with $K^i_{\min}$, and $K^i_{\min}$ is an irreducible $\mathfrak{g}_0$-module. Thus, $K^i_{\min}$ is a unique simple $\mathfrak{g}_0$-submodule in $\Omega^i(\lambda)$, i.e. naturally becoming the simple socle. Furthermore, the analysis before the lemma implies that both $K^i_{\min}$ and $L^i_{\min}$ are isomorphic to $L^0_{\min}(\lambda^i)$.

(2) Recall that as a $\mathfrak{g}_0$-module, $K(\lambda) = \bigoplus_{i=0}^n \Omega^i(\lambda)$. Hence

$$\mathrm{Soc}_{u(\mathfrak{g}_0)}\bigl(K(\lambda)\bigr) = \bigoplus_{i=0}^{n} \mathrm{Soc}_{u(\mathfrak{g}_0)}\bigl(\Omega^i(\lambda)\bigr) = \bigoplus_{i=0}^{n} K^i_{\min}.$$

The proof is completed. □

Proposition 2.5.

(1) $\mathrm{Soc}(K(\lambda)) \cong L^\nabla(\lambda - \sum_{i=1}^n \epsilon_i)$.
(2) $L^\vee(\sigma(\lambda)) \cong L(\lambda)$.

Proof. (1) Observe that for any nonzero vector $w \in K(\lambda)$, $u(\mathfrak{g})w$ contains v^s_λ. We can write $w = \sum_{i=1}^s w_i$ for $w_i (\neq 0) \in K^{j_i}$, $j_1 < j_2 < \cdots < j_s$. By the arguments in the proof of Proposition 2.4, $v^s_\lambda \in u(\mathfrak{g}_0)w$. Hence, $v^n_\lambda \in u(\mathfrak{g})w$. This means that $u(\mathfrak{g})v^n_\lambda = \mathrm{Soc}(K(\lambda))$. Furthermore, v^n_λ is annihilated by N^-, with weight $\lambda - \sum_{i=1}^n \epsilon_i$. Hence $u(\mathfrak{g})v^n_\lambda$ is a homomorphic image of $K^-(\lambda - \sum_{i=1}^n \epsilon_i)$. Thus, $\mathrm{Soc}(K(\lambda)) \cong L^\nabla(\lambda - \sum_{i=1}^n \epsilon_i)$.

(2) This is because $L^0_{\min}(\sigma(\lambda)) \cong L^0(\lambda)$. □

Call $\lambda \in \Lambda$ *typical* if $K(\lambda) = L(\lambda)$. Otherwise, it is called *atypical*. By Proposition 2.5, we know that $\lambda^\nabla = \lambda - \sum_i \epsilon_i$ if λ is typical (*here and further, simply write* $\sum_i$ *for* $\sum_{i=1}^n$). Conversely, suppose $\lambda^\nabla = \lambda - \sum_i \epsilon_i$. Then $L(\lambda) \cong L^\nabla(\lambda - \sum_i \epsilon_i)$. According to the structure of $L(\lambda)$, the weight vector of weight $\lambda - \sum_i \epsilon_i$ must have a nonzero inverse image $w_{\lambda - \sum_i \epsilon_i}$ in $K(\lambda)$ of weight $\lambda - \sum_i \epsilon_i$. By the previous analysis, such a weight vector in $K(\lambda)$ is contained in $\mathrm{Soc}\, K(\lambda)$, which implies that $J(\lambda)$ must be zero. Hence $K(\lambda) = L(\lambda)$. Furthermore, we have by the same arguments as in [S] without any change:

Proposition 2.6. *Let $\lambda \in \Lambda$. Then λ is atypical if and only if λ is of the form $\lambda = a\epsilon_i + \sum_{j=i+1}^n \epsilon_j$ for $a \in \{0, 1, 2, \ldots, p-1\}$ and $i \in \{1, \ldots, n\}$. In this case, $\lambda^\nabla = -\epsilon_1 - \cdots - \epsilon_{i-1} + a\epsilon_i$.*

Remark 2.7. The above result on simple modules and weights can be generalized to more general case $W(m; n)$ (cf. [SZ]).

2.2. Rational $\mathfrak{T}$-module category

Let $\mathfrak{T}$ be the canonical maximal torus of $\mathrm{GL}(n,k)$, which consists of diagonal matrices $\mathrm{Diag}(t_1,\dots,t_n)$, $t_i \in k^\times$. Recall that the character group $X(\mathfrak{T})$ of $\mathfrak{T}$ is a free abelian group of rank n, identified with $\mathbb{Z}^n$. By definition, a rational $\mathfrak{T}$-module V means that $V = \bigoplus_{\lambda\in X(\mathfrak{T})} V_\lambda$, where $V_\lambda = \{v \in V \mid T(v) = t_1^{\lambda_1}\cdots t_n^{\lambda_n} v\}$ for $\lambda = (\lambda_1,\dots,\lambda_n) \in \mathbb{Z}^n$ and $T = \mathrm{Diag}(t_1,\dots,t_n)$. Set $\mathcal{W}_{\mathfrak{T}}(V) = \{\lambda \in X(\mathfrak{T}) \mid V_\lambda \neq 0\}$ (sometimes simply denoted by $\mathcal{W}(V)$). Set $\mathcal{T} := \{\mathbf{t} := \mathrm{Diag}(t,t,\dots,t) \in \mathfrak{T} \mid t \in k^\times\} \cong k^\times$. Then we have a $\mathbb{Z}$-graded decomposition for a rational $\mathfrak{T}$-module $V = \sum_s V_s$ with $V_s = \{v \in V \mid \mathbf{t}v = t^s v\}$ and $\mathcal{W}_{\mathcal{T}}(V) := \{s \in \mathbb{Z} \mid V_s \neq 0\}$.

Recall that the automorphism group of $\mathfrak{g}$ contains a closed subgroup $\mathrm{GL}(n,k)$ which admits a natural representation on the space k-span$\{\xi_1,\dots,\xi_n\}$. The action of $\mathfrak{T}$ on $\mathfrak{g}$ is given via $T\xi_i = t_i\xi_i$, and then

$$T(\xi_{i_1}\cdots\xi_{i_s}\partial_j) = t_{i_1}\cdots t_{i_s} t_j^{-1}\xi_{i_1}\cdots\xi_{i_s}\partial_j,$$

where $T = \mathrm{Diag}(t_1,\dots,t_n)$. Obviously, $\mathrm{Lie}(\mathfrak{T}) = \langle E_{ii} = \mathrm{Diag}(\delta_{i1},\dots,\delta_{in}),\ i = 1,\dots,n\rangle \cong \mathfrak{h}$, we will identify both with each other, under the map $E_{ii} \mapsto \xi_i\partial_i$. So for $\tau \in X(\mathfrak{T}) = \mathrm{Hom}_{\text{algebraic groups}}(\mathfrak{T}, k^\times)$, its differential $\mathrm{d}\tau : \mathfrak{h} \to k$, is a homomorphism of restricted Lie algebras and satisfies $\mathrm{d}\tau(h^{[p]}) = (\mathrm{d}\tau(h))^{[p]}$. This means that $\mathrm{d}\tau \in \Lambda$. The map $\varphi : \tau \mapsto \mathrm{d}\tau$ has kernel $pX(\mathfrak{T})$. And this induces a bijection $X(\mathfrak{T})/pX(\mathfrak{T}) \cong \Lambda$. With identification $\mathfrak{h} = \mathrm{Lie}(\mathfrak{T})$, we may identify $X(\mathfrak{T})/pX(\mathfrak{T})$ with Λ. Sending $\tau \in X(\mathfrak{T})$ to $\overline{\tau} \in \Lambda = X(\mathfrak{T})/pX(\mathfrak{T})$, we write $\mathrm{d}\tau(h)$ directly as $\overline{\tau}(h)$ without any confusion, and call them *restricted weights*. (Sometimes, $\mathrm{d}\lambda$ and λ are not discriminated in use if no confusion happens in context.)

Naturally, $u(\mathfrak{g})$ and its canonical subalgebras which will be used later become rational $\mathfrak{T}$-modules with the action denoted by $\mathrm{Ad}(T)a$ for $T \in \mathfrak{T}$ and $a \in u(\mathfrak{g})$.

Let us introduce the full subcategory $(u(\mathfrak{g}),\mathfrak{T})$-**mod** of the $u(\mathfrak{g})$-module category $u(\mathfrak{g})$-**mod**:

Definition 2.8.[1] The category $(u(\mathfrak{g}),\mathfrak{T})$-**mod** is defined as such a category whose objects are finite-dimensional k-superspaces endowed with both $u(\mathfrak{g})$-module and rational $\mathfrak{T}$-module structure satisfying the following compatibility conditions for $V \in (u(\mathfrak{g}),\mathfrak{T})$-**mod**:

(i) The action of $u(\mathfrak{h})$ coincides with the action of $\mathrm{Lie}(\mathfrak{T})$ induced from $\mathfrak{T}$.
(ii) For $a \in u(\mathfrak{g})$, $T \in \mathfrak{T}$, and $v \in V$: $T(av) = (\mathrm{Ad}\,T(a))Tv$.

The morphisms of $(u(\mathfrak{g}),\mathfrak{T})$-**mod** are defined to be linear maps of k-superspaces acting as both $u(\mathfrak{g})$-module homomorphisms, and rational $\mathfrak{T}$-module homomorphisms.

Remark 2.9. (1) Let u be a Hopf subalgebra of $u(\mathfrak{g})$ containing $u(\mathfrak{h})$. We can define a category of $(u,\mathfrak{T})$-**mod** in the same way as above, of which each object is simply called a $\hat{u}$-module.

(2) The $\hat{u}$-module category can be realized a module category of the precise Hopf algebra $\hat{u} = u\,\#\,\mathrm{Dist}(\mathfrak{T})$, where $\mathrm{Dist}(\mathfrak{T})$ denotes the distribution algebra of $\mathfrak{T}$ as defined in [LN].

Example 2.10. According to the arguments in Section A.2 of Appendix A, we have rational $\mathfrak{T}$-modules $\hat{K}^{\mathfrak{B}}(\lambda)$, $\hat{K}^\bullet(\lambda)$, and $\hat{L}(\lambda)$ and $\hat{L}^\bullet(\lambda)$, where $\bullet \in \{\vee, \triangledown\}$, for $\lambda \in X(\mathfrak{T})$.

For illustration, we explain the structure on one of them: $\hat{K}(\tau) = \hat{u}(\mathfrak{g}) \otimes_{\hat{u}(\mathfrak{g}^+)} \hat{L}^0(\tau)$ for $\tau \in X(\mathfrak{T})$ where $\hat{L}^0(\tau)$ is a simple $\hat{u}(\mathfrak{g}_0)$-module (i.e. a simple $G_1\mathfrak{T}$-module in [J1], for $G = \mathrm{GL}(n,k)$). Naturally, $\hat{K}(\tau)$ is isomorphic to, as a $u(\mathfrak{g})$-module, $K(\overline{\tau})$, and is endowed with $\mathfrak{T}$-structure as long as defining T-action as diagonal action for $T \in \mathfrak{T}$, i.e. $T \cdot (u \otimes v) = \mathrm{Ad}(T)u \otimes Tv$ for $u \in u(\mathfrak{g})$ and $v \in L^0(\tau)$.

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[1] With the same spirit of Jantzen's idea, there are other formulations different from the category $(u(\mathfrak{g}),\mathfrak{T})$-**mod**, such as the $G_1\mathfrak{T}$-module category (cf. [J1]), the $\mathbb{Z}$-graded $U_0(\mathfrak{g})$-module category (cf. [J2, §11]), and the $u(\mathfrak{g})\,\#\,\mathrm{Dist}(\mathfrak{T})$-module category (cf. [LN], see Remark 2.9(2)).

For more information about the category $(u(\mathfrak{g}), \mathfrak{T})$-**mod**, see Appendix A. The following basic facts are clear.

Proposition 2.11.

(1) *$\hat{K}(\tau)$ is irreducible if and only if $K(\overline{\tau})$ is irreducible.*
(2) *The iso-classes of irreducible modules in $(u(\mathfrak{g}), \mathfrak{T})$-**mod** are in one-to-one correspondence with $X(\mathfrak{T})$. Precisely, each simple objects in $(u(\mathfrak{g}), \mathfrak{T})$-**mod** is isomorphic to $\hat{L}(\tau)$ for $\tau \in X(\mathfrak{T})$.*
(3) *$\hat{K}(\tau)|_{u(\mathfrak{g})} \cong K(\overline{\tau})$, and $\hat{L}(\tau)|_{u(\mathfrak{g})} \cong L(\overline{\tau})$. Furthermore, sending $\tau \in X(\mathfrak{T})$ to $\overline{\tau} \in \Lambda = X(\mathfrak{T})/pX(\mathfrak{T})$ gives rise to the map $\hat{L}(\tau) \mapsto L(\overline{\tau})$ from the set of iso-classes of simple objects of $(u(\mathfrak{g}), \mathfrak{T})$-**mod** and to those of $u(\mathfrak{g})$-**mod**.*

By the above proposition, good understanding of $(u(\mathfrak{g}), \mathfrak{T})$-**mod** can provide us the complete information on restricted simple modules of $\mathfrak{g}$. Naturally, we may call $\tau \in X(\mathfrak{T})$ typical (*resp.* atypical) if $\overline{\tau}$ is typical (*resp.* atypical). Set $\Omega_a := \{\tau \in X(\mathfrak{T}) \mid \tau \text{ is atypical}\}$. By the same argument as in [S] with aid of odd reflections (cf. [S, §5–7], or see Appendix A, Section A.1), we have:

Lemma 2.12.

(1) *$\Omega_a = \{t\epsilon_i + \epsilon_{i+1} + \cdots + \epsilon_n \mid t \in \mathbb{Z},\ i = 1, \ldots, n\}$.*
(2) *For any nonzero $\tau = t\epsilon_i + \epsilon_{i+1} + \cdots + \epsilon_n \in \Omega_a$, $\tau^\triangledown = -\epsilon_1 - \cdots - \epsilon_{i-1} + t\epsilon_i$.*

Definition 2.13. For $V \in (u(\mathfrak{g}), \mathfrak{T})$-**mod**, define the weight length $\ell(V) = \#\mathcal{W}_{\mathcal{T}}(V) - 1$.

Lemma 2.14. *Let $\tau \in X(\mathfrak{T})$.*

(1) *$\ell(\hat{L}(\tau)) \leqslant n$. Furthermore, τ is typical if and only if $\ell(\hat{L}(\tau)) = n$.*
(2) *If the nontrivial character τ is atypical, then $\ell(\hat{L}(\tau)) = n - 1$.*

Proof. (1) Note $\hat{L}(\tau)$ is the unique simple quotient of $\hat{K}(\tau)$, the latter of which has weight length n. Hence $\ell(\hat{L}(\tau)) \leqslant n$. Furthermore, $\mathcal{W}_{\mathcal{T}}(\hat{L}(\tau)) = \{|\tau| := \sum_i \tau_i, |\tau| - 1, \ldots, |\tau^\vee| = \sum_i \tau_i^\vee\}$. So

$$\ell\big(\hat{L}(\tau)\big) = |\tau| - \big|\tau^\vee\big|. \tag{2.4}$$

On the other hand, the number $|\tau| - |\tau^\vee|$ equals the number of typical odd reflections on the way from τ to $\tau^\vee$. Since n is the number of all odd essential reflections (see Section A.2 of Appendix A), it follows that $|\tau| - |\tau^\vee| = n$ if and only if τ is typical.

(2) By Lemma 2.12(2), $\tau^\triangledown = -\epsilon_1 - \cdots - \epsilon_{i-1} + t\epsilon_i$ while $\tau = t\epsilon_i + \epsilon_{i+1} + \cdots + \epsilon_n$. By (2.4), we have $\ell(\hat{L}(\tau)) = n - 1$. □

Set $m(\tau, \nu) := [\hat{K}(\tau) : \hat{L}(\nu)]$.

Lemma 2.15. *Let $\tau = t\epsilon_i + \epsilon_{i+1} + \cdots + \epsilon_n \in \Omega_a$. The following statements on $m(\tau, \nu)$ hold.*

(1) *For $\nu \neq 0$, $m(\tau, \nu) \neq 0$ if and only if $\tau = \nu$ or $\nu^\triangledown = \tau - \sum_{i=1}^n \epsilon_i$. In such cases, $m(\tau, \nu) = 1$.*
(2) *For $\nu = 0$, $m(\tau, 0) = 0$ if $t \neq 1$; $m(\tau, 0) = 1$ if $t = 1$.*

Proof. (1) This follows directly from Lemmas 2.12 and 2.14.

(2) Suppose $m(\tau, 0) \neq 0$. Then $\hat{K}(\tau)$ must contain a nonzero $\hat{u}(\mathfrak{b})$-module primitive vector v of weight 0, i.e. the submodule $\hat{u}(\mathfrak{b})v$ contains a maximal submodule $M(v)$ such that $u(\mathfrak{b})v/M(v)$ is a one-dimensional trivial $\mathfrak{b}$-module. Note that as $\hat{u}(\mathfrak{g}_0)$-module, $\hat{K}(\tau) = \bigwedge(\mathfrak{g}_{-1}) \otimes \hat{L}^0(\tau)$, and $\bigwedge(\mathfrak{g}_{-1})$

is a $\hat{u}(\mathfrak{b})$-module. Especially, the weights of all possible $\hat{u}(\mathfrak{b})$-module primitive vectors in $\bigwedge(\mathfrak{g}_{-1})$ are $-(\epsilon_{j_1}+\cdots+\epsilon_{j_s})$ with $j_1<\cdots<j_s$, the set of which is denoted by $P(\bigwedge(\mathfrak{g}_{-1}))$.

Recall that the weights of all possible $\hat{u}(\mathfrak{b})$-module primitive vectors in $\hat{K}(\tau)$ are of the form $\delta+\tau$ for $\delta\in P(\bigwedge(\mathfrak{g}_{-1}))$ by the modular version of Littlewood–Richardson rule (cf. [FH, p. 456] and [G, D10]). Thus, $m(\tau,0)\neq 0$ implies that τ coincides with the negative one of a certain element from $P(\bigwedge(\mathfrak{g}_{-1}))$. Combining with $\tau=t\epsilon_i+\sum_{j=i+1}^n\epsilon_j$, we know that $t=1$, i.e. $\tau=\sum_{j=i}^n\epsilon_j$.

Next for $\tau=\sum_{j=i}^n\epsilon_j$, we assert that $m(\tau,0)=1$. This is because the following items are true:

(1°) In $\hat{K}(\tau)$, $w=\partial_i\wedge\cdots\wedge\partial_n\otimes v_\tau$ is a $\hat{u}(\mathfrak{b})$-module primitive vector. Hence $m(\tau,0)\geqslant 1$.

(2°) On the other hand, Littlewood–Richardson rules tell us that $m(\tau,0)\leqslant 1$. □

Corollary 2.16.

(1) *For $\tau=t\epsilon_i+\epsilon_{i+1}+\cdots+\epsilon_n\in\Omega_a$, with $t\neq 1,0$, there is the following exact sequence*:

$$0\longrightarrow\hat{L}(\tau-\epsilon_i)\longrightarrow\hat{K}(\tau)\longrightarrow\hat{L}(\tau)\longrightarrow 0.\tag{2.5}$$

(2) *For $\tau=\sum_{j=i}^n\epsilon_j$, with $i>1$, there are two exact sequences*:

$$0\longrightarrow\hat{J}(\tau)\longrightarrow\hat{K}(\tau)\longrightarrow\hat{L}(\tau)\longrightarrow 0;\tag{2.6}$$

$$0\longrightarrow\hat{L}(\tau-\epsilon_{i-1})\longrightarrow\hat{J}(\tau)\longrightarrow\hat{L}(0)\longrightarrow 0.\tag{2.7}$$

(3) *For $\tau=\sum_{j=1}^n\epsilon_j$, there are the following two exact sequences*:

$$0\longrightarrow\hat{L}(-\epsilon_n)\longrightarrow\hat{K}(0)\longrightarrow\hat{L}(0)\longrightarrow 0;\tag{2.8}$$

$$0\longrightarrow\hat{L}(0)\longrightarrow\hat{K}(\tau)\longrightarrow\hat{L}(\tau)\longrightarrow 0.\tag{2.9}$$

Proof. (1) It's clear, by Proposition 2.5 and Lemma 2.15.

(2) By Proposition 2.5 and Lemma 2.12, $\operatorname{Soc}\hat{K}(\tau)=\hat{L}^{\nabla}(\tau-\sum_{i=1}^n\epsilon_i)=\hat{L}(\tau-\epsilon_{i-1})$. Then Lemma 2.15 gives rise to the statement.

(3) The exact sequence (2.9) mainly follows from Proposition 2.5 because $\operatorname{Soc}\hat{K}(\tau)=\hat{L}^{\nabla}(0)=\hat{L}(0)$. Next Lemma 2.15 tells us that $\hat{K}(\tau)$ has two factors both of which have multiplicity one.

There is the same reason for $\operatorname{Soc}\hat{K}(0)=\hat{L}^{\nabla}(-\sum_{j=1}^n\epsilon_i)\cong\hat{L}(-\epsilon_n)$, which gives rise to the exact sequence (2.8). □

2.3. Character formula

Let V be an object in $(u(\mathfrak{g}),\mathfrak{T})$-**mod** with $V=\sum_{\tau\in\mathcal{W}(V)}V_\tau$. We denote the character of V by $\operatorname{ch}(V)$ which is by definition equal to $\sum_\tau(\dim V_\tau)e^\tau$.

Set $\Pi:=\prod_{i=1}^n(1+e^{-\epsilon_i})$.

Theorem 2.17. *Let τ be typical. Then* $\operatorname{ch}(\hat{L}(\tau))=\operatorname{ch}(\hat{K}(\tau))=\Pi\operatorname{ch}(\hat{L}^0(\tau))$.

Proof. It follows directly from the definition of $\hat{K}(\tau)$. □

Theorem 2.18. *Let $\tau=t\epsilon_i+\epsilon_{i+1}+\cdots+\epsilon_n$, with $t\neq 0$.*

(1) *If $t\in\mathbb{Z}_-$, then*

$$\operatorname{ch}\hat{L}(\tau)=\Pi\sum_{j=1}^{\infty}(-1)^j\operatorname{ch}\big(\hat{L}^0(\tau-j\epsilon_i)\big).$$

(2) *If $t \in \mathbb{Z}_+$ and $i > 1$, then*

$$\operatorname{ch}\hat{L}(\tau) = \Pi\left(\sum_{j=0}^{t-1}(-1)^j \operatorname{ch}\hat{L}^0(\tau - j\epsilon_i) + \sum_{s=1}^{\infty}(-1)^{s+t}\operatorname{ch}\hat{L}^0\big(\tau - s\epsilon_{i-1} - (t-1)\epsilon_i\big)\right) + (-1)^t \operatorname{ch}\hat{L}(0).$$

(3) *If $t \in \mathbb{Z}_+$ and $i = 1$, then*

$$\operatorname{ch}\hat{L}(\tau) = \Pi\left(\sum_{j=0}^{t-1}(-1)^j \operatorname{ch}L^0(\tau - j\epsilon_1) + \sum_{s=0}^{\infty}(-1)^{s+t}\operatorname{ch}\hat{L}^0(-s\epsilon_n)\right).$$

Proof. (1) In this case, we have the following complex:

$$\cdots \longrightarrow \hat{K}(\tau - s\epsilon_i) \longrightarrow \cdots \longrightarrow \hat{K}(\tau - \epsilon_i) \longrightarrow \hat{K}(\tau) \longrightarrow 0$$

which gives a resolution of $\hat{L}(\tau)$. Hence we obtain the desired character formula.

(2) By Corollary 2.16(1) and (2), we have for $\tau = t\epsilon_i + \epsilon_{i+1} + \cdots + \epsilon_n$, with $i > 1$, the following complexes:

$$\cdots \longrightarrow \hat{K}\big(\tau - (t-1)\epsilon_i - s\epsilon_{i-1}\big) \longrightarrow \cdots \longrightarrow \hat{K}\big(\tau - (t-1)\epsilon_i - \epsilon_{i-1}\big) \longrightarrow 0$$
$$0 \longrightarrow \hat{L}\big(\tau - (t-1)\epsilon - \epsilon_{i-1}\big) \longrightarrow \hat{J}\big(\tau - (t-1)\epsilon_i\big) \longrightarrow \hat{L}(0) \longrightarrow 0$$
$$0 \longrightarrow \hat{J}\big(\tau - (t-1)\epsilon_i\big) \longrightarrow \hat{K}\big(\tau - (t-1)\epsilon_i\big) \longrightarrow \cdots \longrightarrow \hat{K}(\tau - \epsilon_i) \longrightarrow \hat{K}(\tau) \longrightarrow 0.$$

Those complexes give rise to the desired character formula.

(3) When $i = 1$, we have the following complex by Corollary 2.16(1) and (3):

$$\cdots \longrightarrow \hat{K}(-s\epsilon_n) \longrightarrow \cdots \longrightarrow \hat{K}(0) \longrightarrow \hat{K}(\epsilon_1 + \cdots + \epsilon_n) \longrightarrow \cdots \longrightarrow \hat{K}(t\epsilon_1 + \cdots + \epsilon_n) \longrightarrow 0.$$

Such a complex gives a resolution of $\hat{L}(\tau)$. Hence we have the character formula. □

Corollary 2.19. *The following formulas give rise to the complete calculation of characters, combining with Theorem* 2.18(1):

$$\operatorname{ch}\hat{L}(0) = \operatorname{ch}\hat{K}(0) - \operatorname{ch}\hat{L}(-\epsilon_n) = \operatorname{ch}\hat{K}(\mathcal{E}) - \operatorname{ch}\hat{L}(\mathcal{E})$$

for $\mathcal{E} = \sum_{j=1}^{n}\epsilon_j$.

3. Projective representations and Cartan invariants for restricted representations of *W*(*n*)

3.1. General facts on projective supermodules

Let $\mathcal{A}$ be a finite-dimensional superalgebra over k, and $\mathcal{A}$-**mod** the supermodule category of $\mathcal{A}$. We denote by $|\mathcal{A}|$ the underlying k-algebra of the superalgebra $\mathcal{A}$ and by $|M|$ the underlying $|\mathcal{A}|$-module of the $\mathcal{A}$-supermodule M.

Recall that there is a parity change functor $\pi : \mathcal{A}\text{-}\mathbf{mod} \to \mathcal{A}\text{-}\mathbf{mod}$ [Ma, 3.1.5], with $\pi(M)$ being the same underlying vector space but the new left $\mathcal{A}$-action via $a \cdot m = (-1)^{|a|}am$ for homogeneous

$a \in \mathcal{A}$ (of parity $|a| \in \mathbb{Z}_2$) and $m \in M$. There is a linear map $\delta_M : M \to M$ on homogeneous vectors by $\delta_M(v) = (-1)^{|v|}v$. Let $N \subset M$ be a subspace. There is obviously a judgement that N is a subsuperspace of M if and only if N is δ_M-stable.

Recall the Jacobson radical $\mathcal{R}$ of $|\mathcal{A}|$ can be characterized as the unique smallest superideal of $\mathcal{A}$ such that $A := \mathcal{A}/\mathcal{R}$ is a semisimple superalgebra (cf. [BK, 2.6]).

Recall that the semisimple superalgebra A can be decomposed into a direct product of some simple superalgebras of type M and of some simple algebras of type Q (cf. [Jo, 2.11]),

$$A \cong \prod_{i=1}^{m} M(r_i|s_i) \times \prod_{j=1}^{n} Q(t_j). \tag{3.1}$$

And up to isomorphisms, simple supermodules over A are parameterized by $I_l^{(r_i|s_i)} := M(r_i|s_i)E_{l,l}^{(i)}$, $l = 1, \dots, r_i + s_i$, where $E_{l,l}^{(i)}$ is an $(r_i|s_i)$-matrix with 1 at the (l,l) position and zero elsewhere, and $J_l^{(n_j)} := I_l^{(n_j|0)} + I_l^{(n_j|0)}\mathbf{t}$, $l = 1, \dots, n_j$, where $\mathbf{t}^2 = -1$. Here we use the isomorphism $Q(n_j) \cong M(n_j|0) \otimes Q(1)$, $Q(1) = k + k\mathbf{t}$. Note that $U_{j,l}^{+} := I_l^{(n_j|0)}$ and $U_{j,l}^{-} := I_l^{(n_j|0)}\mathbf{t}$ are simple $|\mathcal{A}|$-modules, but not supermodule.

Furthermore, $\{V_{i,l} := I_l^{(r_i|s_i)},\ l = 1, \dots, r_i + s_i\}$, $i = 1, \dots, m$, are irreducible supermodules of $\mathcal{A}$, as $|\mathcal{A}|$-modules, and $\{U_{j,l} := J_l^{(n_j)},\ l = 1, \dots, n_j\}$, $j = 1, \dots, m$, are self-associative irreducible supermodules. All $V_{i,l}, U_{j,l}$ above constitute a complete set of pairwise non-isomorphic simple supermodules in $\mathcal{A}$-**mod**.

Let $P(S)$ denote the projective cover of simple module S in the $|\mathcal{A}|$-module category. We assert that $P(V_{i,l})$ is just a projective cover in the $\mathcal{A}$-supermodule category, and that $P(U_{j,l}) := P(U_{j,l}^{+}) + P(U_{j,l}^{+})\mathbf{t}$ is just a projective cover of $U_{j,l}$ in the $\mathcal{A}$-supermodule category. For this, we only need to show that $P(V_{i,l})$ and $P(U_{j,l})$ are all supermodules of $\mathcal{A}$. It can be seen from that $|\mathcal{A}|$ can be decomposed into a direct sum of left ideas of $|\mathcal{A}|$, $\sum_i \mathcal{A}e_{i,l} + \sum_j \mathcal{A}e_{j,l}^{+} + \mathcal{A}e_{j,l}^{-}$, where $e_{i,l}$ and $e_{j,l}^{\pm}$ are lifted primitive idempotents with respect to the primitive idempotent decomposition of identity, corresponding to the primitive idempotents of the corresponding identity decomposition for A in (3.1) (cf. [CR, §6A]). The assertion follows from the structures of $V_{i,l}$ and $U_{j,l}$, and from the judgement of supermodules. So we have the following facts:

Lemma 3.1. *Keep the notations as above. Each simple module S in $\mathcal{A}$-**mod** has a projective cover, which is unique, up to isomorphism, denoted by $Q(S)$.*

Now, let $\mathcal{A} = u(\mathfrak{g})$. Then the $\mathcal{A}$-**mod** is just the restricted supermodule category of $\mathfrak{g}$. The above lemma holds in the $u(\mathfrak{g})$-**mod**. It's easily seen that the above lemma also holds for $(u(\mathfrak{g}), \mathfrak{T})$-**mod**.

3.2. $\hat{K}$-filtrations

We first need the following notations and facts:

Notations and Facts 3.2.

(1) Denote by $\hat{Q}(\tau)$, $\hat{Q}^0(\tau)$ the projective cover of $\hat{L}(\tau)$ in $(u(\mathfrak{g}), \mathfrak{T})$-**mod**, and the projective cover of $\hat{L}^0(\tau)$ in $(u(\mathfrak{g}_0), \mathfrak{T})$-**mod** respectively.
(2) Set $\hat{I}(\tau) := u(\mathfrak{g}) \otimes_{u(\mathfrak{g}_0)} \hat{Q}^0(\tau)$, which is equal to

$$u(\mathfrak{g}) \otimes_{u(\mathfrak{g}^{+})} \big(u\big(\mathfrak{g}^{+}\big) \otimes_{u(\mathfrak{g}_0)} \hat{Q}^0(\tau)\big).$$

(3) Set $\hat{K}_Q^{\pm}(\tau) := u(\mathfrak{g}) \otimes_{u(\mathfrak{g}^{\pm})} \hat{Q}^0(\tau)$, with trivial W^1-action on $\hat{Q}^0(\tau)$ for $\hat{K}_Q^+(\tau)$, and with trivial $\mathfrak{g}_{-1}$-action on $\hat{Q}^0(\tau)$ for $\hat{K}_Q^-(\tau)$. It's obvious that $\hat{K}_Q^{\pm}(\tau)$ is the projective cover of the simple object $\hat{L}(\lambda)$ in the $\hat{u}(\mathfrak{g}^{\mp})$-module category.

(4) Set $\mathcal{G} := \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_{(1)}$, where $\mathfrak{g}_{(1)} := k$-span of $\{\xi_i \hbar \mid i = 1, \ldots, n\}$ for $\hbar = \sum_{j=1}^n \xi_j \partial_j$. One can show that $\mathcal{G} \cong \mathfrak{sl}(n|1)$ is a classical Lie-subsuperalgebra of $\mathfrak{g}$ with $\mathcal{G}_{\bar{0}} = \mathfrak{g}_0$, and $\mathcal{G}_{\bar{1}} = \mathfrak{g}_{-1} + \mathfrak{g}_{(1)}$. Further set $B_{\mathcal{G}}^+ = \mathfrak{b}^+ + \mathfrak{g}_{(1)}$, and $B_{\mathcal{G}}^- = \mathfrak{g}_{-1} + \mathfrak{b}^-$. Denote in $\hat{u}(\mathcal{G})$-**mod**

$$\hat{V}_{\mathcal{G}}^{\pm}(\tau) = u(\mathcal{G}) \otimes_{u(B_{\mathcal{G}}^{\pm})} k_\tau.$$

The iso-classes of simple $\hat{u}(\mathcal{G})$-modules can be parameterized by

$$\{\hat{L}_{\mathcal{G}}(\tau) = \text{the simple head of } \hat{V}_{\mathcal{G}}^+(\tau) \mid \tau \in X(\mathfrak{T})\}.$$

(5) Recall that $B^- = N^- + \mathfrak{h} = \mathfrak{g}_{-1} + \mathfrak{n}^- + \mathfrak{h}$ and $B^+ = N^+ + \mathfrak{h} = W^1 + \mathfrak{n}^+ + \mathfrak{h}$. Both $B^{\pm}$ are restricted Lie superalgebras. The iso-classes of restricted irreducible $B^{\pm}$-modules are represented by $\{k_{\bar{\tau}} \mid \tau \in X(\mathfrak{T})\}$. The iso-classes of restricted irreducible $\mathfrak{g}^{\pm}$-modules coincide with those of restricted irreducible $\mathfrak{g}_0$-modules represented by $\{L^0(\bar{\tau}) \mid \tau \in X(\mathfrak{T})\}$, endowed with trivial $\mathfrak{g}_{-1}$-action (*resp.* W^1-action). In comparison with baby Verma modules for restricted Lie algebra $\mathfrak{g}_0$: $V_{\pm}^0(\tau) = u(\mathfrak{g}_0) \otimes_{u(\mathfrak{b}^{\pm})} k_\tau$, $\tau \in X(\mathfrak{T})$, we can define generalized baby Verma modules $V_{\mathfrak{g}^{\pm}}(\bar{\tau}) := u(\mathfrak{g}^{\pm}) \otimes_{u(B^{\pm})} k_{\bar{\tau}}$, and $V^{\pm}(\bar{\tau}) := u(\mathfrak{g}) \otimes_{u(B^{\pm})} k_{\bar{\tau}}$. Similarly, we have $\hat{V}_{\mathfrak{g}^{\pm}}(\tau)$, $\hat{V}^{\pm}(\tau)$.

Then we turn to a definition which will be necessary in the sequel.

Definition 3.3. Let $M \in (u(\mathfrak{g}), \mathfrak{T})$-**mod**. We say that M has a $\hat{\mathcal{K}}^+$-filtration (*resp.* $\hat{\mathcal{K}}_Q^+$-filtration) if there is a submodule filtration of M:

$$0 = M_l \subset M_{l-1} \subset \cdots \subset M_1 \subset M_0 = M \tag{3.2}$$

such that each sub-quotient $M_i/M_{i+1} \cong \hat{K}^+(\tau_i)$ (*resp.* $\hat{K}_Q^+(\tau_i)$) for some $\tau_i \in X(\mathfrak{T})$, $i = 0, 1, \ldots, l-1$. Denote by $[M : \hat{K}^+(\tau_i)]$ (*resp.* $[M : \hat{K}_Q^+(\tau_i)]$) the times of occurrence of $\hat{K}^+(\tau_i)$ (*resp.* $\hat{K}_Q^+(\tau_i)$) in the sub-quotients of the above filtration, called the multiplicity of $\hat{K}^+(\tau_i)$. Those numbers are independent of the choice of $\hat{\mathcal{K}}^+$-filtration (*resp.* $\hat{\mathcal{K}}_Q^+$-filtration).[2] The number l in (3.2) is called the filtration length, denoted by $\mathbf{l}(M)$ (*resp.* $\mathbf{l}_Q(M)$). In the same way, one can define $\hat{\mathcal{K}}^-$-filtration (*resp.* $\hat{\mathcal{K}}_Q^-$-filtration); and $\mathbf{l}^-(M)$ (*resp.* $\mathbf{l}_Q^-(M)$).

In the same sense, we can say a module in $\hat{u}(\mathfrak{g}^{\pm})$-**mod** (*resp.* in $\hat{u}(\mathfrak{g})$-**mod**) to admit a filtration or to say it to be filtrable by $\hat{V}_{\mathfrak{g}^{\pm}}$ (*resp.* by $\hat{V}^{\pm}$).

Next, we continue to list the last items of Notations and Facts 3.2:

(6) Regarded as a $\hat{u}(\mathfrak{g}^{\pm})$-module with trivial $\mathfrak{g}_{\pm 1}$-action, $\hat{Q}^0(\tau)$ is filtrable by $\hat{V}_{\mathfrak{g}^{\pm}}(\upsilon)$, $\upsilon \in X(\mathfrak{T})$. Hence $\hat{K}_Q^{\pm}(\tau) = u(\mathfrak{g}) \otimes_{u(\mathfrak{g}^{\pm})} \hat{Q}^0(\tau)$ can be filtrable by $\hat{V}^{\pm}(\upsilon)$, which equals $u(\mathfrak{g}) \otimes_{u(\mathfrak{g}^{\pm})} \hat{V}_{\mathfrak{g}^{\pm}}(\upsilon)$, $\upsilon \in X(\mathfrak{T})$.

(7) For $M \in \mathrm{Ob}(u(\mathfrak{g})$-**mod**$)$ (*resp.* $\hat{M} \in \mathrm{Ob}(\hat{u}(\mathfrak{g})$-**mod**$)$), we have the corresponding element $[M]$ in the Grothendieck group $G[u(\mathfrak{g})$-**mod**$]$ (*resp.* the corresponding element $[\hat{M}]$ in the Grothendieck group $G[\hat{u}(\mathfrak{g})$-**mod**$]$).

[2] This statement can be known from the fact that in the Grothendieck group $G[(u(\mathfrak{g}), \mathfrak{T})$-**mod**$]$, $\{\hat{K}(\tau) \mid \tau \in X(\mathfrak{T})\}$ constitute a basis because of Theorems 2.17 and 2.18, along with Corollary 2.19.

(8) In the category $\hat{u}_0(\mathfrak{g}_0)$-**mod**, $[\hat{V}^0_-(\lambda)] = [\hat{V}^0(-w_0\lambda + 2(p-1)\rho)]$, where w_0 is the longest element of the Weyl group of $\mathfrak{g}_0$, and ρ is half-sum of all positive roots of $\mathfrak{g}_0$, and $[M]$ shares the same meaning in the above item (7) for $M \in \mathrm{Ob}(\hat{u}_0(\mathfrak{g}_0)$-**mod**).

Lemma 3.4.

(1) *The one-to-one correspondence $\tilde{\sigma}$ on $X(\mathfrak{T})$*

$$\tilde{\sigma}: \tau \mapsto -w_0(\tau) + \sum_{i=1}^{n} \epsilon_i + 2(p-1)\rho$$

gives rise to

$$\big[\hat{V}^-_{\mathcal{G}}(\tau)\big] = \big[\hat{V}^+_{\mathcal{G}}\big(\tilde{\sigma}(\tau)\big)\big], \tag{3.3}$$

where w_0 is the longest element of the Weyl group W of $\mathfrak{g}_0$, and ρ is half-sum of all positive roots of $\mathfrak{g}_0$.

(2) $[\hat{V}_{\mathfrak{g}^\pm}(\tau) : \hat{L}^0(\eta)] = [\hat{V}^0_\pm(\tau) : \hat{L}^0(\eta)]$ *and* $[\hat{K}^\pm_Q(\tau) : \hat{V}^\pm(\upsilon)] = [\hat{Q}^0(\tau) : \hat{V}_{\mathfrak{g}^\pm}(\upsilon)]$.

Proof. (1) Note that $\mathcal{G} \cong \mathfrak{sl}(n|1)$. The proof is similar to the one for the ordinary case as in Notations and Facts 3.2(8). We give a sketchy argument here. Let ω be an automorphism of $\mathrm{GL}(n,k)$ such that $\omega(\mathfrak{T}) = \mathfrak{T}$ with (extended) derivative that acts on the $\mathcal{G}$ in the way $\omega(x_\alpha) = x_{-w_0\alpha}$ and $\omega(h_\alpha) = h_{-w_0\alpha}$. Then we can define for $M \in \hat{u}(\mathcal{G})$-**mod** an ω-dual ${}^\omega M$ of M, which has the ground space M^* and has ω-twisted action defined via $g\varphi = (-1)^{|g||\varphi|}\varphi \circ (-\omega(g))$ (we denote $|g|, |\varphi|$ by the parity of $g \in \mathcal{G}$ and $\varphi \in M^*$). Similar to the arguments of [AJS, 1.13], we have $\hat{V}_{\mathcal{G}}(\tau) \cong {}^\omega(\hat{V}^-_{\mathcal{G}}(\tau - \sum_{i=1}^n \epsilon_i - 2(p-1)\rho)^*)$. On the other hand, by analogy of the arguments in the ordinary cases (cf. [J, II.2.13]), we can prove that there is an isomorphism in $\hat{u}(\mathcal{G})$-**mod**: $\hat{V}_{\mathcal{G}}(\tau)^* \cong {}^\omega\hat{V}_{\mathcal{G}}(-w_0\tau)$. Combining the above arguments, we can get the desired formula.

(2) Recall as k-vector spaces, $\hat{V}_{\mathfrak{g}^\pm}(\tau) = u(\mathfrak{g}^\pm) \otimes_{u(B^\pm)} k_\tau = u(\mathfrak{n}^\mp) \otimes_k k_\tau = \hat{V}^0_\pm(\tau)$. On the other hand, $\hat{L}^0(\tau)$ can be extended a $\hat{u}(\mathfrak{g}^\pm)$-module by trivial W^1-action or trivial $\mathfrak{g}_{-1}$-action. So, the multiplicity of $\hat{V}_{\mathfrak{g}^\pm}(\tau)$ by $\hat{L}^0(\tau)$ in $\hat{u}(\mathfrak{g}^\pm)$-**mod** is clearly equal to the multiplicity of $\hat{V}^0_\pm(\tau)$ by $\hat{L}^0(\tau)$ in $\hat{u}(\mathfrak{g}_0)$-**mod**. So the first equation holds. The argument for the second one is the same. □

We have some further results on filtrable modules.

Lemma 3.5.

(1) *The following formulas hold:*[3]
  (i) $[\hat{V}^-(\tau)] = [\hat{V}^+(\tilde{\sigma}(\tau))]$ *if $n = 2$, and*
  (ii) $[V^-(\bar{\tau})] = p^{s-n}2^t \sum_{\bar{\upsilon}\in\Lambda}[V^+(\bar{\upsilon})]$ *if $n > 2$, where $s = \dim N^{++}_{\bar{0}}$ and $t = \dim N^{++}_{\bar{1}}$, and N^{++} is defined by a decomposition of $B^+ = B_{\mathcal{G}} \oplus N^{++}$. More precisely, when $n = 3$, $N^{++} = \mathcal{G}'_{(1)} + W_2$ for the decomposition $W_1 = \mathcal{G}_{(1)} \oplus \mathcal{G}'_{(1)}$. When $n > 3$, $N^{++} = N^{++}_{\bar{0}} \oplus N^{++}_{\bar{1}}$ for $N^{++}_{\bar{0}} = \sum_{j\geqslant 1} W_{2j}$, and $N^{++}_{\bar{1}} = \mathcal{G}'_{(1)} + \sum_{j\geqslant 2} W_{2j-1}$.*

(2) $[\hat{V}^+(\tau) : \hat{L}(\eta)] = \sum_{\upsilon\in X(\mathfrak{T})}[\hat{V}^0(\tau) : \hat{L}^0(\upsilon)][\hat{K}^+(\upsilon) : \hat{L}(\eta)]$.

(3) *For $\upsilon, \eta \in \Lambda$, $[V^-(\upsilon) : L(\eta)] = p^{s-n}2^t \sum_{\kappa\in\Lambda}[V^+(\kappa) : L(\eta)]$ if $n > 2$. And $[V^-(\upsilon) : L(\eta)] = [V^+(\tilde{\sigma}(\upsilon)) : L(\eta)]$ if $n = 2$.*

Proof. (1) Note that $B^- = B^-_{\mathcal{G}} = \mathfrak{g}_{-1} + \mathfrak{b}^-$. We have

[3] The second formula happens in the $u(\mathfrak{g})$-module category.

$$\begin{aligned}\hat{V}^-(\tau) &= u(\mathfrak{g}) \otimes_{u(B^-)} k_\tau \\ &= u(\mathfrak{g}) \otimes_{u(\mathcal{G})} \big(u(\mathcal{G}) \otimes_{u(B^-_{\mathcal{G}})} k_\tau\big).\end{aligned}$$

By the argument of Lemma 3.4(1),

$$\hat{V}^-(\tau) \cong u(\mathfrak{g}) \otimes_{u(\mathcal{G})} \big(u(\mathcal{G}) \otimes_{u(B^+_{\mathcal{G}})} k_{\tilde{\sigma}(\tau)}\big) \tag{3.4}$$

$$= u(\mathfrak{g}) \otimes_{u(B^+)} \big(u\big(B^+\big) \otimes_{u(B^+_{\mathcal{G}})} k_{\tilde{\sigma}(\tau)}\big). \tag{3.5}$$

Hence $\hat{V}^-(\tau)$ has a $\hat{V}^+(\upsilon)$-filtration. Furthermore, we assert

$$\big[\hat{V}^-(\tau)\big] = \big[\hat{V}^+\big(\tilde{\sigma}(\tau)\big)\big] \quad \text{if } n = 2, \tag{3.6}$$

$$\big[V^-(\bar{\tau})\big] = \sum_{\bar{\upsilon}\in\Lambda} p^{s-n}2^t\big[V^+(\bar{\upsilon})\big] \quad \text{if } n > 2 \tag{3.7}$$

where s, t will be explained below.

The proof for (3.6) and (3.7) will be given below by steps:

Step 1. If $n = 2$, then $B^+ = B^+_{\mathcal{G}}$. So (3.6) follows from Lemma 3.4(1).

Step 2. Now we assume $n > 2$ for (3.7). By (3.4), the multiplicity of $V^+(\upsilon)$ in the V^+-filtration of $V^-(\tau)$ is equal to the multiplicity of the one-dimensional factor k_υ in the composition series of $\mathcal{B} := u(B^+) \otimes_{u(B^+_{\mathcal{G}})} k_{\tilde{\sigma}(\tau)}$. Consider the decomposition $B^+ = B^+_{\mathcal{G}} \oplus N^{++}$, where N^{++} is defined as below. When $n = 3$, $N^{++} = \mathcal{G}'_{(1)} + W_2$ for the decomposition $W_1 = \mathcal{G}_{(1)} \oplus \mathcal{G}'_{(1)}$. When $n > 3$, $N^{++} = N^{++}_{\bar{0}} \oplus N^{++}_{\bar{1}}$ where $N^{++}_{\bar{0}} = \sum_{j=1}^{[\frac{n-1}{2}]} W_{2j}$, and $N^{++}_{\bar{1}} = \mathcal{G}'_{(1)} + \sum_{j=2}^{(\frac{n-1}{2})} W_{2j-1}$. Here $[\frac{n-1}{2}]$ means the greatest positive integer not greater than $\frac{n-1}{2}$, and $(\frac{n-1}{2})$ means the least positive integer not less than $\frac{n-1}{2}$.

We can always take an ordered basis $X_1, \dots, X_s$ in $N^{++}_{\bar{0}}$, and an ordered basis $Y_1, \dots, Y_t$ in $N^{++}_{\bar{1}}$ such that $\{X_i;\ i = 1, \dots, s\}$ and $\{Y_j;\ j = 1, \dots, t\}$ have weights in $X(\mathfrak{T})$ $\{\alpha_i;\ i = 1, \dots, s\}$ and $\{\beta_j;\ j = 1, \dots, t\}$, and the restricted weights $\{\bar{\alpha}_i;\ i = 1, \dots, n\}$ are k-linearly independent. Actually, $\{X_1, \dots, X_n\}$ can be chosen to be $\{\xi_1 \cdots \xi_n \partial_i, \xi_1\xi_2\xi_3\partial_3 \mid i = 1, \dots, n-1\}$ when n is odd, and to be $\{\xi_1 \cdots \hat{\xi}_j \cdots \xi_n \partial_n \mid j = 1, \dots, n\}$ when n is even and $p \nmid n-2$, and to be $\{\xi_1 \cdots \hat{\xi}_j \cdots \xi_n \partial_n, \xi_1\xi_2\xi_n\partial_2, \xi_2 \cdots \xi_{n-1}\xi_n\partial_{n-1} \mid j = 1, \dots, n-2\}$ when n is even and $p \mid n-2$. Set

$$U = k\text{-span}\big\{u_{\underline{a},\underline{b}} := X_1^{a_1} \cdots X_s^{a_s} Y_1^{b_1} \cdots Y_t^{b_t} \bigm| 0 \leqslant a_i \leqslant p-1,\ 0 \leqslant b_j \leqslant 1,\ i = 1, \dots, s;\ j = 1, \dots, t\big\}.$$

As a $u(\mathfrak{h})$-module, $\mathcal{B}$ is isomorphic to U, the latter of which has a direct-sum decomposition of weight spaces $U_{\underline{a},\underline{b}}$ of weights $\{\lambda_{\underline{a},\underline{b}} := \sum_{i=1}^s a_i\alpha_i + \sum_{j=1}^t b_j\beta_j\}$ corresponding to the one-dimensional space $ku_{\underline{a},\underline{b}}$. For each given choice of $\underline{a}' := \{a_i,\ i = n+1, n+2, \dots, s\}$ and $\underline{b} = \{b_j,\ j = 1, \dots, t\}$, the corresponding space $U_{\underline{a}',\underline{b}} := k\text{-span}\{X_1^{a_1} \cdots X_s^{a_s} Y_1^{b_1} \cdots Y_t^{b_t}\}$ is a $u(\mathfrak{h})$-module with all restricted weights occurring and the restricted weight spaces all have dimension 1. Hence, U as $u(\mathfrak{h})$-module admits a weight space decomposition with all possible restricted weights occurring, and with all restricted weight spaces having the same dimension. So the multiplicity in the composition series of $\mathcal{B}$, all irreducible factors are of the same multiplicity. As a result, this multiplicity is equal to $p^{s-n}2^t$.

Step 3. Note that $\hat{V}^+(\tau) = u(\mathfrak{g}) \otimes_{u(B^+)} k_\tau = u(\mathfrak{g}) \otimes_{u(\mathfrak{g}^+)} V_{\mathfrak{g}^+}(\tau)$. Hence $\hat{V}^+(\tau)$ is filtrable by $\hat{\mathcal{K}}^+$. And $[\hat{V}^+(\tau) : \hat{K}^+(\upsilon)] = [\hat{V}_{\mathfrak{g}^+}(\tau) : L^0(\upsilon)] = [\hat{V}^0(\tau) : L^0(\upsilon)]$. The last equation follows from statement (2) of this lemma.

(2) By the argument in Step 3 above, we finally have

$$\begin{aligned}\big[\hat{V}^+(\tau):\hat{L}(\eta)\big] &= \sum_{\upsilon\in X(\mathfrak{T})}\big[\hat{V}^+(\tau):\hat{K}^+(\upsilon)\big]\big[\hat{K}^+(\upsilon):\hat{L}(\eta)\big]\\ &= \sum_{\upsilon\in X(\mathfrak{T})}\big[\hat{V}^0(\tau):\hat{L}^0(\upsilon)\big]\big[\hat{K}^+(\upsilon):\hat{L}(\eta)\big].\end{aligned}$$

(3) This statement follows directly from the first two. □

3.3. Frobenius extensions

In this subsection we need some relation between coinduced modules and induced modules for restricted superrepresentations. We first recall the notion of Frobenius extension on ring extensions. Let R be a ring and S a subring of R, and suppose that θ is an automorphism of S. If M is an S-module, we let ${}_\theta M$ denote the S-module with a new action defined by $s * m = \theta(s)m$. Let $\mathrm{Hom}_S(R, {}_\theta S)$ denote the set of additive maps $f : R \to S$ such that $f(sr) = \theta(s)f(r)$ for all $s \in S$, $r \in R$. This is an (R, S)-bimodule via the action $(r\cdot f\cdot s)(x) = f(xr)s$.

We say R is a θ-Frobenius extension of S if

(i) R is a finitely generated projective S-module, and
(ii) there exists an isomorphism $\Theta : R \to \mathrm{Hom}_S(R, {}_\theta S)$ of (R, S)-bimodules.

Suppose $R : S$ is a θ-Frobenius extension, and let V be an S-module. The theory of Frobenius extensions [NT, p. 96f] provides a natural equivalence

$$R\otimes_S V \cong \mathrm{Hom}_S(R, {}_\theta V). \tag{3.8}$$

We return to the case with $R = U(\mathfrak{g})$ and $S = U(\mathfrak{g}^+)$. Note that $\mathfrak{g}^+$ contains $\mathfrak{g}_{\bar 0}$ with codimension n in $\mathfrak{g}$. Let $f : \mathfrak{g}^+ \to \mathfrak{gl}(\mathfrak{g}/\mathfrak{g}^+) = \mathfrak{gl}(W_{-1})$ be a map defined by $f(a)(y + \mathfrak{g}^+) := [a, y] + \mathfrak{g}^+$ for $a \in \mathfrak{g}^+$ and $y \in \mathfrak{g}$, which is a homomorphism of Lie superalgebras. We then have $\mathfrak{t} : \mathfrak{g}^+ \to k$ defined by $\mathfrak{t}(a) := \mathrm{tr}(f(a)) = -\,\mathrm{str}(f(a))$, which is a linear function on $\mathfrak{g}^+$ vanishing on $[\mathfrak{g}^+, \mathfrak{g}^+] + \mathfrak{g}^+_{\bar 1}$. Then there is a unique automorphism θ in $\mathrm{Aut}(u(\mathfrak{g}^+))$ satisfying

$$\theta(a) = \begin{cases} a + \mathfrak{t}(a)1 & \text{for } a \in \mathfrak{g}^+_{\bar 0},\\ (-1)^n a & \text{for } a \in \mathfrak{g}^+_{\bar 1}.\end{cases} \tag{3.9}$$

According to [BF, Theorem 2.2], the extension $U(\mathfrak{g}) : U(\mathfrak{g}^+)$ is a free θ-Frobenius extension. We can easily see that (3.9) gives rise to a unique automorphism of $u(\mathfrak{g}^+)$, still denoted by θ. This is because $\lambda(a^{[p]}) = \lambda(a)^p$ for $a \in \mathfrak{g}^+_{\bar 0}$. Furthermore, we have the following result.

Lemma 3.6. *Let $R = u(\mathfrak{g})$ and $S = u(\mathfrak{g}^+)$ be the restricted enveloping algebras of restricted Lie superalgebras $\mathfrak{g}$ and $\mathfrak{g}^+$ respectively. The following statements hold.*

(1) *The extension $R : S$ is a free θ-Frobenius extension.*
(2) *Denote by $\overline{K^{\pm}(\lambda)}$ the coinduced module* $\mathrm{Coind}_{u(\mathfrak{g}^{\pm})}(L^0(\lambda)) := \mathrm{Hom}_{u(\mathfrak{g}^{\pm})}(u(\mathfrak{g}), L^0(\lambda))$. *Then $\overline{K^+(\lambda)} \cong K^+(\lambda - \mathcal{E})$, where $\mathcal{E} = \sum_{j=1}^n \epsilon_j$.*

Proof. (1) The proof is the same as that of Theorem 2.2 in [BF].

(2) According to (1) and the formula (3.8), we have

$$\mathrm{Hom}_{u(\mathfrak{g})}\big(u(\mathfrak{g}^+), {}_\theta L^0(\lambda)\big) \cong u(\mathfrak{g})\otimes_{u(\mathfrak{g}^+)} L^0(\lambda).$$

Note that $\mathfrak{g}^+ = \mathfrak{g}_{\bar 0} + \mathfrak{g}_{\bar 1}^+ = \mathfrak{g}_0 + W^1$. Obviously, we have $\theta(x) = x$ for every $x \in W^1 = \mathfrak{g}_{\bar 1}^+ \cup \{W^1 \cap \mathfrak{g}_{\bar 0}\}$. Hence ${}_\theta L^0(\lambda)$ is still an irreducible $\mathfrak{g}^+$-module with trivial W^1-action. We only need to consider the $u(\mathfrak{g}_0)$-module structure on ${}_\theta L^0(\lambda)$, which is still irreducible. The highest weight vector v of module $L^0(\lambda)$ is a highest weight vector of highest weight $\lambda + \mathcal{E}$. This is because each $x \in \mathfrak{n}^+$ still annihilates v, due to $\mathfrak{t}(x) = \operatorname{tr}(\operatorname{ad} x|_{\mathfrak{g}_{-1}}) = 0$. In the meantime, $\mathfrak{t}(h) = \operatorname{tr}(\operatorname{ad} h|_{\mathfrak{g}_{-1}}) = \mathcal{E}(h)$ for $h \in \mathfrak{h}$, and $\mathfrak{t}(y) = 0$ for $y \in \mathfrak{n}^-$. The proof is completed. $\square$

Correspondingly, we can consider Frobenius extensions for $\hat u(\mathfrak{g})$-**mod**. In particular, for $\overline{K^{\pm}(\lambda)} := \operatorname{Hom}_{\hat u(\mathfrak{g}^{\pm})}(\hat u(\mathfrak{g}), \hat L^0(\lambda))$ we have $\overline{\hat K^{+}(\lambda)} \cong \hat K^{+}(\lambda - \mathcal{E})$. Parallel to Nakano's results [N, 1.3.6, 1.3.7] in the Lie algebra case, we have a counterpart in the supercase as follows.

Proposition 3.7.

(1) *Each projective module in* $(u(\mathfrak{g}), \mathfrak{T})$-**mod** *has* $\hat{\mathcal{K}}_Q^{\pm}$*-filtration, and also has* $\hat{\mathcal{K}}^{\pm}$*-filtration.*

(2) *For the projective module* $\hat Q(\tau)$*, we have the following multiplicity formula of* $\hat{\mathcal{K}}_Q^{\pm}$*-filtration*:

$$\big[\hat Q(\tau) : \hat K_Q^{\pm}(\eta)\big] = \big[\overline{\hat K^{\mp}(\eta)} : \hat L(\tau)\big].$$

Proof. (1) Note that for $\upsilon \in X(\mathfrak{T})$,

$$\hat K_Q^{\pm}(\upsilon) = \hat u(\mathfrak{g}) \otimes_{\hat u(\mathfrak{g}^{\pm})} \hat Q^0(\upsilon) = \hat u\big(\mathfrak{g}^{\mp}\big) \otimes_{\hat u(\mathfrak{g}_0)} \hat Q^0(\upsilon), \tag{3.10}$$

as $\hat u(\mathfrak{g}^{\mp})$-module. Hence, $\hat K_Q^{\pm}(\upsilon)$ is a projective $\hat u(\mathfrak{g}^{\mp})$-module. On the other hand,

$$\begin{aligned}\operatorname{Hom}_{\hat u(\mathfrak{g}^{\mp})}\big(\hat K_Q^{\pm}(\upsilon), \hat L^0(\eta)\big) &= \operatorname{Hom}_{\hat u(\mathfrak{g}^{\mp})}\big(u\big(\mathfrak{g}^{\mp}\big) \otimes_{u(\mathfrak{g}_0)} Q^0(\upsilon), \hat L^0(\eta)\big)\\ &= \operatorname{Hom}_{\hat u(\mathfrak{g}_0)}\big(\hat Q^0(\upsilon), \hat L^0(\eta)\big)\\ &= \delta_{\upsilon,\eta} k.\end{aligned}$$

This means that $\hat K_Q^{\pm}(\upsilon)$ is the projective cover of $L^0(\upsilon)$ in the $\hat u(\mathfrak{g}^{\mp})$-module category. From this observation, it follows that in the $\hat u(\mathfrak{g}^{\mp})$-module category, the projective module $\hat Q(\tau)$ for any $\tau \in X(\mathfrak{T})$, must be a direct sum of the $\hat K_Q^{+}(\upsilon)$'s (*resp.* $\hat K_Q^{-}(\upsilon)$'s) for some $\upsilon \in X(\mathfrak{T})$. And $\hat Q(\tau)$ is $\mathcal{K}_Q^{\pm}$-filtrable.

As to the second part, comparing the structure of $\hat K^{\pm}(\eta) = u(\mathfrak{g}) \otimes_{u(\mathfrak{g}^{\pm})} \hat L^0(\eta)$ with $\hat K_Q^{\pm}(\upsilon) = u(\mathfrak{g}) \otimes_{u(\mathfrak{g}^{\pm})} \hat Q^0(\upsilon)$, we easily know $\hat K_Q^{\pm}(\upsilon)$ has a $\hat{\mathcal{K}}^{\mp}$-filtration. Observing that $u(\mathfrak{g}_0)$ is a symmetric algebra (cf. [Sch] and [FP]), we have the further fact which will be used later:

$$\big[\hat K_Q^{\pm}(\upsilon) : \hat K^{\pm}(\eta)\big] = \big[\hat Q^0(\upsilon) : \hat L^0(\eta)\big] = \big[\hat Q^0(\eta) : \hat L^0(\upsilon)\big]. \tag{3.11}$$

(2) We prove this statement in steps, with several assertions and their proofs.

1° *Assertion.* $V \in \operatorname{Ob}((u(\mathfrak{g}), \mathfrak{T})$-**mod**$)$ has a $\hat{\mathcal{K}}_Q^{\pm}$-filtration if and only if V is a projective module in the $\hat u(\mathfrak{g}^{\mp})$-module category.

We have known that $\hat K_Q^{\pm}(\upsilon)$ is a projective cover of simple object $\hat L^0(\upsilon)$ in the $\hat u(\mathfrak{g}^{\mp})$-module category. So each projective module in this category has $\hat{\mathcal{K}}_Q^{\pm}$-filtration. Conversely, assume that $V \in \operatorname{Ob}((u(\mathfrak{g}), \mathfrak{T})$-**mod**$)$ has a $\hat{\mathcal{K}}_Q^{\pm}$-filtration, we show that V is a projective $\hat u(\mathfrak{g}^{\pm})$-module by induction on $l := \mathbf{l}_Q^{\pm}(V)$. When $l = 1$, it's true because $V \cong \hat K_Q^{\pm}(\upsilon)$ as $\hat u(\mathfrak{g}^{\mp})$-module, for some $\upsilon \in X(\mathfrak{T})$. Suppose that the assertion is true for modules admitting such a filtration of length less than l. The inductive hypothesis, and the splitting property of any short exact sequence with the ending projective term ensure that V is a projective $\hat{u}(\mathfrak{g}^{\mp})$-module.

2° *Assertion.* Suppose V has a $\hat{\mathcal{K}}^{\pm}_{Q}$-filtration, then

$$\left[V : \hat{K}^{\pm}_{Q}(\eta)\right] = \dim \operatorname{Hom}_{\hat{u}(\mathfrak{g})}\left(V, \overline{\hat{K}^{\mp}(\eta)}\right). \tag{3.12}$$

According to 1°, we know

$$\left[V : \hat{K}^{\pm}_{Q}(\eta)\right] = \left[V|_{\hat{u}(\mathfrak{g}^{\mp})} : \hat{K}^{\pm}_{Q}(\eta)|_{u(\mathfrak{g}^{\mp})}\right] = \dim \operatorname{Hom}_{\hat{u}(\mathfrak{g}^{\mp})}\left(V, \hat{L}^{0}(\eta)\right).$$

The last term is equal to

$$\dim \operatorname{Hom}_{\hat{u}(\mathfrak{g})}\left(V, \operatorname{Coind}_{\hat{u}(\mathfrak{g}^{\mp})}\left(\hat{L}^{0}(\eta)\right)\right) = \dim \operatorname{Hom}_{\hat{u}(\mathfrak{g})}\left(V, \overline{\hat{K}^{\mp}(\eta)}\right),$$

from which (3.12) follows.

3°. In particular, for $V = \hat{Q}(\tau)$ in 2° we have

$$\left[\hat{Q}(\tau) : \hat{K}^{\pm}_{Q}(\eta)\right] = \dim \operatorname{Hom}_{\hat{u}(\mathfrak{g})}\left(\hat{Q}(\tau), \overline{\hat{K}^{\mp}(\eta)}\right).$$

The right hand side is equal to $[\overline{\hat{K}^{\mp}(\eta)} : \hat{L}(\tau)]$. Hence we have proved

$$\left[\hat{Q}(\tau) : \hat{K}^{\pm}_{Q}(\eta)\right] = \left[\overline{\hat{K}^{\mp}(\eta)} : \hat{L}(\tau)\right]. \qquad \square$$

3.4. Cartan invariants

We turn to the $u(\mathfrak{g})$-module category. We have the following primary result.

Proposition 3.8.

(1) *Assume $n > 2$. The following formula holds*

$$\left[Q(\tau) : L(\eta)\right] = p^{s-n}2^{t} \sum_{\upsilon, \upsilon_1 \in \Lambda} \left[K^{+}(\upsilon_1 - \mathcal{E}) : L(\tau)\right]\left[Q^{0}(\upsilon_1) : V^{0}\left(\tilde{\sigma}(\upsilon) - \mathcal{E}\right)\right]$$
$$\times \sum_{\omega, \omega_1 \in \Lambda} \left[V^{0}(\omega) : L^{0}(\omega_1)\right]\left[K^{+}(\omega_1) : L(\eta)\right]$$

for $\tau, \eta \in \Lambda$.

(2) *Let $n = 2$. Then*

$$\left[Q(\tau) : L(\eta)\right] = \sum_{\upsilon, \upsilon_1, \upsilon_2 \in \Lambda} \left[K^{+}(\upsilon_1 - \mathcal{E}) : L(\tau)\right]\left[\hat{Q}^{0}(\upsilon_1) : V^{0}\left(\tilde{\sigma}(\upsilon) - \mathcal{E}\right)\right]$$
$$\times \left[V^{0}(\upsilon) : L^{0}(\upsilon_2)\right]\left[K^{+}(\upsilon_2) : L(\eta)\right].$$

Proof. By Notations and Facts 3.2(6) we know $\hat{K}^{\pm}_{Q}(\tau)$ is filtrable by $\hat{V}^{\pm}(\upsilon)$. Furthermore, by Proposition 3.7(1), $\hat{Q}(\tau)$ is filtrable by $\mathcal{K}^{\pm}_{Q}$. Hence we know that $\hat{Q}(\tau)$ is filtrable by $V^{\pm}(\upsilon)$, $\upsilon \in X(\mathfrak{T})$. The calculation of multiplicity can be done as

$$\big[\hat{Q}(\tau):\hat{L}(\eta)\big] = \sum_{\upsilon\in X(\mathfrak{T})} \big[\hat{Q}(\tau):\hat{V}^{-}(\upsilon)\big]\big[\hat{V}^{-}(\upsilon):\hat{L}(\eta)\big].$$

On the other hand, we have

$$\begin{aligned}
\big[\hat{Q}(\tau):\hat{V}^{-}(\upsilon)\big] &= \sum_{\upsilon_1\in X(\mathfrak{T})} \big[\hat{Q}(\tau):\hat{K}^{-}_{Q}(\upsilon_1)\big]\big[\hat{K}^{-}_{Q}(\upsilon_1):\hat{V}^{-}(\upsilon)\big] \\
&\overset{\text{Prop. 3.7}}{=} \sum_{\upsilon_1\in X(\mathfrak{T})} \big[\overline{\hat{K}^{+}(\upsilon_1)}:\hat{L}(\tau)\big]\big[\hat{K}^{-}_{Q}(\upsilon_1):\hat{V}^{-}(\upsilon)\big] \\
&\overset{\text{Lem.3.4(2)}}{=} \sum_{\upsilon_1\in X(\mathfrak{T})} \big[\overline{\hat{K}^{+}(\upsilon_1)}:\hat{L}(\tau)\big]\big[\hat{Q}^{0}(\upsilon_1):\hat{V}^{0}_{\mathfrak{g}^-}(\upsilon)\big] \\
&\overset{\text{Lem. 3.4(2)}}{=} \sum_{\upsilon_1\in X(\mathfrak{T})} \big[\overline{\hat{K}^{+}(\upsilon_1)}:\hat{L}(\tau)\big]\big[\hat{Q}^{0}(\upsilon_1):\hat{V}^{0}_{-}(\upsilon)\big] \\
&= \sum_{\upsilon_1\in X(\mathfrak{T})} \big[\overline{\hat{K}^{+}(\upsilon_1)}:\hat{L}(\tau)\big]\big[\hat{Q}^{0}(\upsilon_1):\hat{V}^{0}_{+}\big(-w_0\upsilon+2(p-1)\rho\big)\big] \\
&= \sum_{\upsilon_1\in X(\mathfrak{T})} \big[\overline{\hat{K}^{+}(\upsilon_1)}:\hat{L}(\tau)\big]\big[\hat{Q}^{0}(\upsilon_1):\hat{V}^{0}_{+}\big(\tilde{\sigma}(\upsilon)-\mathcal{E}\big)\big].
\end{aligned}$$

According to Lemma 3.5(3), $[V^{-}(\upsilon):L(\eta)] = p^{s-n}2^t\sum_{\omega\in\Lambda}[V^{+}(\omega):L(\eta)]$ if $n>2$, $[V^{-}(\upsilon):L(\eta)] = [V^{+}(\tilde{\sigma}(\upsilon)):L(\eta)]$ if $n=2$. With those formulas, we continue the arguments in two cases.

(1) For $n>2$, we have

$$\begin{aligned}
\big[Q(\tau):L(\eta)\big] &= \sum_{\upsilon\in\Lambda}\big[Q(\tau):V^{-}(\upsilon)\big]\big[V^{-}(\upsilon):L(\eta)\big] \\
&= \sum_{\upsilon\in\Lambda}\Big(\sum_{\upsilon_1\in\Lambda}\big[\overline{K^{+}(\upsilon_1)}:L(\tau)\big]\big[Q^{0}(\upsilon_1):V^{0}_{+}\big(\tilde{\sigma}(\upsilon)-\mathcal{E}\big)\big]\Big)\big[V^{-}(\upsilon):L(\eta)\big] \\
&\overset{\text{Lem. 3.5(3)}}{=} \sum_{\upsilon,\upsilon_1\in\Lambda}\big(\big[\overline{K^{+}(\upsilon_1)}:L(\tau)\big]\big[Q^{0}(\upsilon_1):V^{0}_{+}\big(\tilde{\sigma}(\upsilon)-\mathcal{E}\big)\big]\big)p^{s-n}2^t\sum_{\omega\in\Lambda}\big[V^{+}(\omega):L(\eta)\big] \\
&\overset{\text{Lem. 3.5(2)}}{=} p^{s-n}2^t\sum_{\upsilon,\upsilon_1\in\Lambda}\big(\big[\overline{K^{+}(\upsilon_1)}:L(\tau)\big]\big[Q^{0}(\upsilon_1):V^{0}_{+}\big(\tilde{\sigma}(\upsilon)-\mathcal{E}\big)\big]\big) \\
&\quad\times\Big(\sum_{\omega,\omega_1\in\Lambda}\big[V^{0}(\omega):L^{0}(\omega_1)\big]\big[K^{+}(\omega_1):L(\eta)\big]\Big) \\
&\overset{\text{Lem. 3.6}}{=} p^{s-n}2^t\sum_{\upsilon,\upsilon_1\in\Lambda}\big(\big[K^{+}(\upsilon_1-\mathcal{E}):L(\tau)\big]\big[Q^{0}(\upsilon_1):V^{0}_{+}\big(\tilde{\sigma}(\upsilon)-\mathcal{E}\big)\big]\big) \\
&\quad\times\sum_{\omega,\omega_1\in\Lambda}\big[V^{0}(\omega):L^{0}(\omega_1)\big]\big[K^{+}(\omega_1):L(\eta)\big].
\end{aligned}$$

(2) For $n=2$, we have

$$\begin{aligned}
\big[Q(\tau):L(\eta)\big] &= \sum_{\upsilon\in\Lambda}\big[Q(\tau):V^-(\upsilon)\big]\big[V^-(\upsilon):L(\eta)\big] \\
&= \sum_{\upsilon\in\Lambda}\Big(\sum_{\upsilon_1\in\Lambda}\big(\big[\overline{K^+(\upsilon_1)}:L(\tau)\big]\big[Q^0(\upsilon_1):V^0_+\big(\tilde\sigma(\upsilon)-\mathcal{E}\big)\big]\big)\big[V^-(\upsilon):L(\eta)\big]\Big) \\
&= \sum_{\upsilon,\upsilon_1\in\Lambda}\big[\overline{K^+(\upsilon_1)}:L(\tau)\big]\big[Q^0(\upsilon_1):V^0_+\big(\tilde\sigma(\upsilon)-\mathcal{E}\big)\big]\big[V^+\big(\tilde\sigma(\upsilon)\big):L(\eta)\big] \\
&\overset{\text{Lem. 3.5(2)}}{=} \sum_{\upsilon,\upsilon_1\in\Lambda}\big(\big[\overline{K^+(\upsilon_1)}:L(\tau)\big]\big[Q^0(\upsilon_1):V^0_+\big(\tilde\sigma(\upsilon)-\mathcal{E}\big)\big]\big) \\
&\qquad\times\Big(\sum_{\upsilon_2\in\Lambda}\big[V^0\big(\tilde\sigma(\upsilon)\big):L^0(\upsilon_2)\big]\big[K^+(\upsilon_2):L(\eta)\big]\Big) \\
&\overset{\text{Lem. 3.6}}{=} \sum_{\upsilon,\upsilon_1,\upsilon_2\in\Lambda}\big[K^+(\upsilon_1-\mathcal{E}):L(\tau)\big]\big[Q^0(\upsilon_1):V^0_+\big(\tilde\sigma(\upsilon)-\mathcal{E}\big)\big] \\
&\qquad\times\big[V^0\big(\tilde\sigma(\upsilon)\big):L^0(\upsilon_2)\big]\big[K^+(\upsilon_2):L(\eta)\big]. \qquad \square
\end{aligned}$$

Suppose $a_{\alpha,\beta}:=[K(\alpha):L(\beta)]$ for $\alpha,\beta\in\Lambda$, which can be computed, according to Corollary 2.16. And suppose $b_{\alpha,\beta}:=[V^0(\alpha):L^0(\beta)]$ for $\alpha,\beta\in\Lambda$, which coincides with $[Q^0(\beta):V^0(\alpha)]$, thanks to the classical modular representation theory of classical Lie algebras.[4] Then we have matrices $A=(a_{\alpha,\beta})^{\mathrm{t}}_{\alpha,\beta\in\Lambda}$ and $B=(b_{\alpha,\beta})^{\mathrm{t}}_{\alpha,\beta\in\Lambda}$. Denote by $A_{\mathcal{E}}$ the column switched matrix of A by $\mathcal{E}$-switching right, this is to say $A_{\mathcal{E}}=(a_{\alpha-\mathcal{E},\beta})^{\mathrm{t}}_{\alpha,\beta\in\Lambda}$. Similarly, we have $B_{\mathcal{E}}$. Denote $\tilde\sigma(B)=(b_{\tilde\sigma(\alpha),\beta})_{\alpha,\beta}$. Then we can state the main result of this section.

Theorem 3.9. *Denote the Cartan invariants by $c_{\tau\eta}=[Q(\tau):L(\eta)]$, $\tau,\eta\in\Lambda$ in the $u(\mathfrak{g})$-module category. Then the Cartan invariants can be given via the following matrix formula for $C=(c_{\tau\eta})_{\tau,\eta\in\Lambda}$:*

(1) *Assume $n>2$. Then $C=A_{\mathcal{E}}\tilde\sigma(B)_{\mathcal{E}}PB^{\mathrm{t}}A^{\mathrm{t}}$, where $P=(p^{s-n}2^t)$ is the square matrix of size $\#\Lambda$, the entries of which are all $p^{s-n}2^t$;*
(2) *Assume $n=2$. Then $C=A_{\mathcal{E}}\tilde\sigma(B)_{\mathcal{E}}\tilde\sigma(B)^{\mathrm{t}}A^{\mathrm{t}}$.*

Appendix A. Odd reflections for $(u(\mathfrak{g}),\mathfrak{T})$-mod

Maintain the notations in §1.3. Especially, $\mathfrak{g}=\mathfrak{h}+\sum_{\alpha\in\Delta}\mathfrak{g}_\alpha$, and

$$\Delta=\{\epsilon_{i_1}+\cdots+\epsilon_{i_t}-\epsilon_j\mid 1\leqslant i_1<\cdots<i_t\leqslant n;\ 1\leqslant t,j\leqslant n\}\cup\{-\epsilon_i\mid i=1,\dots,n\}.$$

A.1. Borel subalgebras and odd reflections

Set $\mathfrak{B}_{\max}:=\mathfrak{b}+W^1$. This is a restricted supersubalgebra. We call $\mathfrak{B}_{\max}$ the maximal Borel subalgebra, which corresponds to the positive root set $\Delta(\mathfrak{B}_{\max})_+=\{\epsilon_i-\epsilon_j\mid 1\leqslant i<j\leqslant n\}\cup\{\epsilon_{i_1}+\cdots+\epsilon_{i_t}-\epsilon_j\mid 1\leqslant i_1<\cdots<i_t\leqslant n,\ 2\leqslant t\leqslant n,\ 1\leqslant j\leqslant n\}$. We simply denote $\Delta(\mathfrak{B}_{\max})_+$ by Δ_+. Set $\Delta_s:=\bigcup_{j=n-s+1}^n\{\alpha+\epsilon_j\mid\alpha\in\Delta_+\cup\{0\}\}$, $s=1,\dots,n$. And set $\mathfrak{B}_0:=\mathfrak{B}_{\max}$, $\mathfrak{B}_s=\mathfrak{b}+\sum_{j=n-s+1}^n k\partial_j+$

[4] When p satisfies some condition relevant to the Coxeter number h $(=n$ here), the classical Cartan invariants are predicted by Lusztig conjecture, which has not been completely proved yet (cf. [J1, II.8.22]).

$\sum_{\alpha\in\Delta_+\backslash\Delta_s}\mathfrak{g}_\alpha$, $s=1,\ldots,n$. Those $\mathfrak{B}_s$ are called (lower) Borel subalgebras, associated with the positive root set $\Delta(\mathfrak{B}_s)_+ := \{\Delta_+\backslash\Delta_s\}\cup\{-\epsilon_{n-s+1},\ldots,-\epsilon_n\}$. Naturally, we can set $\Delta(\mathfrak{B}_s)_- = \Delta\backslash\Delta(\mathfrak{B}_s)_+$. The following meaning is clear for a given Borel subalgebra $\mathfrak{B}$ as above: $\mathfrak{g} = N_{\mathfrak{B}}{}^- \oplus \mathfrak{h} \oplus N_{\mathfrak{B}}{}^+$, and $\mathfrak{B} = \mathfrak{h}\oplus N_{\mathfrak{B}}{}^+$ with $N_{\mathfrak{B}}{}^\pm = \sum_{\alpha\in\Delta(\mathfrak{B})_\pm}\mathfrak{g}_\alpha$. It's easily shown that all $\mathfrak{B}_s$ are restricted Lie supersubalgebras.

Following Serganova [S], we define a sequence of odd reflections r_{ϵ_i}, $i=1,\ldots,n$ so that the corresponding chain of Borel subalgebras can be produced, reflecting the different Kac modules.

Define r_1 by sending $\mathfrak{B}_{\max}$ to $\mathfrak{B}_1$, here $\mathfrak{B}_1$ is actually obtained from $\mathfrak{B}_{\max}$ by removing ϵ_n from Δ_+ and adding $-\epsilon_n$. We can inductively define the ith odd reflection r_i by sending $\mathfrak{B}_{i-1}$ to $\mathfrak{B}_i$ by removing ϵ_{n-i+1} from $\Delta(\mathfrak{B}_{i-1})$ and adding $-\epsilon_{n-i+1}$. Then $\mathfrak{B}_n = r_n\circ\cdots\circ r_1(\mathfrak{B}_{\max}) = \mathfrak{b}+\mathfrak{g}_{-1}$, denoted by $\mathfrak{B}_{\min}$. Those subalgebras $\mathfrak{B}_i$ and $\mathfrak{B}_{i+1}$ are called adjacent Borel subalgebras. We may denote r_i by $r_{\epsilon_{n-i+1}}$ in more general setting (see [S] or see below).

We call a root $\alpha\in\Delta_+$ simple for a given Borel subalgebra $\mathfrak{B}$ if we may obtain a set of positive roots for some other Borel subalgebra $\mathfrak{B}'$ by removing α from Δ_+ and adding $-\alpha$ as long as such a root exists. And denote the relation between Borel subalgebras by $\mathfrak{B}' = r_\alpha(\mathfrak{B})$, which means a relation from $\mathfrak{B}$ to $\mathfrak{B}'$. Call r_α an even (essential) reflection if α is even essential, an odd (essential) reflection if α is odd essential, and a nonessential reflection if α is nonessential.

A.2. $\hat{K}^{\mathfrak{B}}(\lambda)$ and $\hat{L}^{\mathfrak{B}}(\lambda)$

Keep the notations as above. Recall that the set of iso-classes of all simple modules in $G_1\mathfrak{T}$-**mod** have representatives $\{\hat{L}^0(\lambda), \lambda\in X(\mathfrak{T})\}$ for $G=\mathrm{GL}(k,n)$ (cf. [J1] and [J2]). And $u(\mathfrak{g})$, $u(\mathfrak{g}^+)$ naturally become $\mathfrak{T}$-modules, compatible with the restricted $\mathfrak{g}$-module structure. Hence one naturally has $\hat{u}(\mathfrak{g})$-module structure on $\hat{K}^+(\lambda) = \hat{u}(\mathfrak{g})\otimes_{\hat{u}(\mathfrak{g}^+)} L^0(\lambda)$, the projection on $u(\mathfrak{g})$-**mod** of which coincides with $K^+(\mathrm{d}\lambda)$. Note that each submodule in $K^+(\lambda)$ naturally admits $\mathfrak{T}$-module structure, compatible with its $u(\mathfrak{g})$-module structure. Hence we have $\hat{u}(\mathfrak{g})$-module $\hat{L}^+(\lambda)$ which is the unique simple quotient of $\hat{K}^+(\lambda)$ in $\hat{u}(\mathfrak{g})$-**mod**, so that the projection on $u(\mathfrak{g})$-**mod** of $\hat{L}^+(\lambda)$ coincides with $L^+(\mathrm{d}\lambda)$. Similarly, one can work with $\hat{K}^{\mathfrak{B}}(\lambda) := \hat{u}(\mathfrak{g})\otimes_{\hat{u}(\mathfrak{B})}\hat{L}^0(\lambda)$ and its unique simple quotient $\hat{L}^{\mathfrak{B}}(\lambda)$ for a Borel subalgebra $\mathfrak{B}$.

By the same arguments as in [S, 5.1], we have the following properties for the odd reflection, which is helpful to understand Lemma 2.14:

Lemma A.1. *(See [S, 5.1].) Let $\mathfrak{B}$ and $\mathfrak{B}'$ be adjacent Borel subalgebras.*

(1) *If they are related by an odd reflection r_α for a simple α of $\mathfrak{B}$, then $\hat{L}^{\mathfrak{B}}(\tau)\cong\hat{L}^{\mathfrak{B}'}(\tau-\alpha)$ if $\overline{\tau}(\mathfrak{h}_\alpha)\neq 0$.*
(2) *If they are related by an odd reflection r_α for a simple α of $\mathfrak{B}$, then $\hat{L}^{\mathfrak{B}}(\tau)\cong\hat{L}^{\mathfrak{B}'}(\tau)$ if $\overline{\tau}(\mathfrak{h}_\alpha) = 0$.*

Proof. (1) Let α be an old essential root with $\overline{\tau}(\mathfrak{h}_\alpha)\neq 0$. We might as well assume that $\mathfrak{g}_\alpha\in\mathfrak{g}_{-1}$. Then $\dim\mathfrak{g}_{-1}=1$. Let Y be a nonzero element of $\mathfrak{g}_{-\alpha}$. By the assumption, we can find $X\in\mathfrak{g}_\alpha$ with $\overline{\tau}([X,Y])\neq 0$. Let v be a highest vector of $\hat{L}^{\mathfrak{B}}(\tau)$. We have $XYv = [X,Y]v - YXv = \overline{\tau}([X,Y])\neq 0$. Hence $Yv\neq 0$.

Furthermore, the nonzero vector Yv is a highest vector of $\hat{L}^{\mathfrak{B}'}(\tau-\alpha)$. Actually, we can check that Yv is annihilated by $\mathfrak{B}'$. For this, we only need to look at ZYv for $Z\in\mathfrak{g}_\beta\subset\mathfrak{B}'$ with $\beta\neq-\alpha$ because $\mathfrak{g}_{-\alpha}=kY$ and $Y^2=0$, thereby $ZYv\in kY^2v=0$. Hence we consider $Z\in\mathfrak{g}_\beta\subset\mathfrak{B}\cap\mathfrak{B}'$, which implies $Zv=0$, and $[Z,Y]=0$ or $[Z,Y]\in\mathfrak{B}\cap\mathfrak{B}'$. We have

$$ZYv = [Z,Y]v = 0.$$

(2) By the same argument as above, we can prove this statement. $\square$

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