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(Locally soluble)-by-(locally finite) maximal subgroups of $\mathrm{GL}_n(D)$

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ABSTRACT

Let D be a non-commutative division ring and M a maximal subgroup of $\mathrm{GL}_n(D)$ ($n \geqslant 2$). This paper continues the ongoing effort to show that the structure of maximal subgroups of $\mathrm{GL}_n(D)$ is similar, in some sense, to the structure of $\mathrm{GL}_n(D)$. It is known that every locally soluble normal subgroup of $\mathrm{GL}_n(D)$ is abelian. Here, among other results, we prove that if either (i) D is finite-dimensional over its center, or (ii) the center of D contains at least five elements and M is soluble-by-finite, or (iii) $\operatorname{char} D = 0$ and M is (locally soluble)-by-finite, then every locally soluble normal subgroup of M is abelian.

1. Introduction

Throughout this paper D denotes a division ring with center F, n is a natural number, $M_n(D)$ is the full $n \times n$ matrix ring over D and $\mathrm{GL}_n(D)$ is the group of units of $M_n(D)$. The maximal soluble, maximal nilpotent, and maximal locally nilpotent subgroups of general linear groups over algebraically closed fields were extensively studied by Suprunenko; the main results are expounded in [25]. Our object here is to discuss the general skew linear groups whose maximal subgroups are of some special

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types. Some properties of maximal subgroups of $\mathrm{GL}_n(D)$ have been studied in a series of papers, see, e.g., [1,3,5,10,13,18,21].

In all of those papers, authors attempted to show that the structure of maximal subgroups of $\mathrm{GL}_n(D)$ is similar, in some sense, to the structure of $\mathrm{GL}_n(D)$. For instance, if D is an infinite division ring, in [5] it was shown that every nilpotent maximal subgroup of $\mathrm{GL}_n(D)$ is abelian, and in [21] the authors proved that for $n \geqslant 2$, every locally nilpotent maximal subgroup of $\mathrm{GL}_n(D)$ is abelian. Also, if D is non-commutative and $n \geqslant 2$, in [3] it was shown that every soluble maximal subgroup of $\mathrm{GL}_n(D)$ is abelian, and in [21] the authors proved that for $n \geqslant 3$, every locally soluble maximal subgroup of $\mathrm{GL}_n(D)$ is abelian. Note that in [1] and [17] it was proved that $\mathbb{C}^* \cup \mathbb{C}^* j$ is a non-abelian soluble maximal subgroup of the real quaternions division ring. So, $\mathrm{GL}_1(D)$ can have a non-abelian soluble maximal subgroup. Also, note that if $D = F$ is a field, then it is known that the set

$$M = \left\{ \begin{bmatrix} a & c \\ 0 & b \end{bmatrix} \,\middle|\, a, b \in F^*,\ c \in F \right\}$$

is a non-abelian soluble maximal subgroup of $\mathrm{GL}_2(F)$. Thus in the mentioned results, the natural number n was considered bigger than 1 and D was non-commutative. Moreover, in [12] the authors showed that if a maximal subgroup M of $\mathrm{GL}_n(D)$ is an FC-group (i.e., each element of M has only a finite number of conjugates), then M is abelian.

In this paper, we try to generalize above results. It is known that every locally soluble normal subgroup (or FC-normal subgroup) of $\mathrm{GL}_n(D)$ is abelian (see [12, Lemma 1]). Now we ask *is any locally soluble normal subgroup* (*or FC-normal subgroup*) *of a maximal subgroup of* $\mathrm{GL}_n(D)$ *abelian*? In this paper we try to give an affirmative answer to this question. For example, we will show that in the finite-dimensional case the answer is "yes" (Theorem 2). We summarize our results as follows:

Theorem 1. *Let D be a non-commutative division ring, M a maximal subgroup of $\mathrm{GL}_n(D)$, $n \geqslant 2$, and H a normal subgroup of M such that M/H is locally finite.*

(i) *If M is absolutely irreducible, then H is locally soluble* (*or FC-group*) *iff H is abelian.*
(ii) *If M is not absolutely irreducible and* $\mathrm{char}\, D = 0$, *then H is locally soluble* (*or FC-group*) *iff M is abelian.*
(iii) *If the center of D contains at least five elements, then H is soluble iff H is abelian.*

Corollary 1. *Let D be an infinite division ring, M a maximal subgroup of $\mathrm{GL}_n(D)$, $n \geqslant 2$, and H a normal subgroup of M. In each of the following cases H is abelian*:

(i) *D is non-commutative,* $\mathrm{char}\, D = 0$, *M/H is locally finite and H is locally soluble or FC-group.*
(ii) $n \geqslant 3$, $\mathrm{char}\, D = 0$, *M/H is locally finite and H is locally nilpotent.*
(iii) $n \geqslant 3$, *the center of D contains at least five elements, M/H is finite and H is nilpotent.*

Theorem 2. *Let D be a non-commutative division ring and M a maximal subgroup of $\mathrm{GL}_n(D)$. In each of the following cases, every locally soluble normal subgroup of M is abelian.*

(i) *D is finite-dimensional over its center*;
(ii) *the center of D contains at least five elements and M is soluble-by-finite*;
(iii) $\mathrm{char}\, D = 0$ *and M is* (*locally soluble*)*-by-finite.*

A famous result of Tits, known as the Tits' Alternative, asserts that if G is a finitely generated linear group over a (commutative) field, then either G contains a noncyclic free subgroup or G is soluble-by-finite. Let D be a non-commutative division algebra of finite-dimension over its center F, and M a maximal subgroup of $\mathrm{GL}_n(D)$. In [18], by a long discussion, it was proved that either M contains a noncyclic free subgroup or there exists a finite family $\{K_i\}_{i=1}^r$ of fields properly containing F with $K_i^* \subseteq M$ for all $1 \leqslant i \leqslant r$ such that M/A is finite if $\mathrm{char}\, F = 0$ and M/A is locally finite if char $F = p > 0$, where $A = K_1^* \times \cdots \times K_r^*$. In the next corollary, using Theorem 2, we generalize this result with a simple proof.

Corollary 2. *Let D be a non-commutative division ring, finite-dimensional over its center, and M a maximal subgroup of $\mathrm{GL}_n(D)$. Then either M contains a noncyclic free subgroup or there exists a maximal subfield K of $M_n(D)$ such that $K^* \trianglelefteq M$ and M/K^* is finite.*

Since every locally nilpotent group is an Engel group, the next result can be viewed as a generalization of [5, Theorem 6] and [21, Theorem 1.6].

Theorem 3. *Let D be an infinite division ring and M an Engel maximal subgroup of $\mathrm{GL}_n(D)$. Then every nilpotent normal subgroup of M is abelian. Specially, for $n \geqslant 2$, the Hirsch–Plotkin radical of M is abelian.*

Our final result concerns maximal subgroups of subnormal subgroups of $\mathrm{GL}_n(D)$ which are nilpotent; it is a generalization of [20, Proposition 1.1 and Theorem 1.1].

Theorem 4. *Let D be an infinite division ring, N a subnormal subgroup of $\mathrm{GL}_n(D)$, and M a nilpotent maximal subgroup of N.*

(i) *If $n = 1$ and every element of M is algebraic over $Z(D)$, then M is abelian.*
(ii) *If the center of D contains at least five elements, then M is metabelian.*

2. The proofs

Our notation is standard. To be more precise, we shall identify the center FI of $M_n(D)$ with F. Let G be a subgroup of $\mathrm{GL}_n(D)$. We denote by $F[G]$ the F-linear hull of G, i.e., the F-algebra generated in $M_n(D)$ by elements of G over F. If $n = 1$, then $F(G)$ is the division ring generated in D by F and G; note that if each element of G is algebraic over F, then $F(M) = F[M]$. If D^n is the space of row n-vectors over D, then D^n is a D–G bimodule in the obvious manner. We say that G is irreducible, reducible, or completely reducible, whenever D^n has the corresponding property as D–G bimodule. Also, G is called absolutely irreducible if $F[G] = M_n(D)$. G is called FC-group if each element of G has only a finite number of conjugates. Also, G' represents the derived subgroup of G. For a given ring R, the group of units of R is denoted by R^*. Let S be a subset of R, then the centralizer of S in R is denoted by $C_R(S)$.

We begin with a simple lemma which will be used frequently in the proofs.

Lemma 1. *Let D be a division ring such that $\mathrm{GL}_n(D)$ is (locally soluble)-by-(locally finite) or (FC-group)-by-(locally finite). Then D is a field and for $n \geqslant 2$, $\mathrm{GL}_n(D)$ is a locally finite group.*

Proof. Suppose H is a locally soluble normal subgroup of $\mathrm{GL}_n(D)$ and $\mathrm{GL}_n(D)/H$ is locally finite. By [28, Theorem 1.1], H contains an abelian normal subgroup A of $\mathrm{GL}_n(D)$ with H/A locally finite. But A must be central since it is abelian. So $\mathrm{GL}_n(D)/F^*$ is locally finite and thus $(\mathrm{GL}_n(D))'$ is also locally finite. For $n = 1$ use [9, Theorem 8] to conclude that D^* is soluble and thus D is a field, and for $n \geqslant 2$ we conclude that D^* is locally finite and so $\mathrm{GL}_n(D)$ is locally finite.

For the other assertion, use [12, Lemma 1] and similar arguments as above to complete the proof. $\square$

To proceed our study, we shall frequently apply the following results.

Lemma 2. *(See [3].) Given a division ring D, let M be a maximal subgroup of $\mathrm{GL}_n(D)$. Then, either M is primitive or contains a copy of D^*.*

Lemma 3. *(See [3].) Let N be normal in a primitive subgroup M of $\mathrm{GL}_n(D)$. Then, we have:*

(1) *$F[N]$ is a prime ring.*
(2) *$C_{M_n(D)}(N)$ is a simple Artinian ring.*
(3) *If $C_{M_n(D)}(N)$ is a division ring, then N is irreducible.*

Lemma 4. *(See [7].) Let D be an F-central division ring and M be a maximal subgroup of $\mathrm{GL}_n(D)$. If $D \neq F$ or $n \geqslant 2$, then $M/(M \cap F^*)$ cannot be a locally finite group unless $\operatorname{char} F = p > 0$ and either*

(1) *$[D : F] = p^2$, $n = 1$ and $M \cup \{0\}$ is a maximal subfield of D, or*
(2) *$D = F$ and $n = p$ and $M \cup \{0\}$ is a maximal subfield of $M_p(F)$, or*
(3) *$D = F$ and F is a locally finite field.*

Proof of Theorem 1 (i). Let H be locally soluble. By [28, Theorem 1.1], H contains an abelian normal subgroup N of M with H/N locally finite. We use of similar methods to those used in the proof of [3, Theorem 3.6]. Let N be a maximal abelian normal subgroup of M such that M/N is locally finite, and by Lemma 4, we may assume N is noncentral. We claim that $K := N \cup \{0\}$ is a maximal subfield of $M_n(D)$ such that $K^* \trianglelefteq M$. Since $M \subseteq N_{\mathrm{GL}_n(D)}(C_{M_n(D)}(N)^*)$ and N is noncentral, by the Cartan–Brauer–Hua theorem for matrix ring we obtain $C_{M_n(D)}(N)^* \subseteq M$. Since D is non-commutative, Lemma 2 implies that M is primitive. Using Lemma 3 we see that $C_{M_n(D)}(N)$ is a simple Artinian ring. If $C_{M_n(D)}(N)^*$ is a locally finite group, then N is also locally finite; this implies that M is locally finite and hence M is abelian by Lemma 4. Therefore, by Lemma 1 we may suppose that $C_{M_n(D)}(N)$ is a field. Hence, $C_{M_n(D)}(N) = K = C_{M_n(D)}(K)$ is a maximal subfield of $M_n(D)$, as claimed.

We next claim that M/N is simple. To do this, assume that $N \subsetneq A \trianglelefteq M$ is a subgroup of M and set $R := F[A] = K[A]$. Since A/N is locally finite, we may write $R = \bigcup_G K[G]$, where the union is taken over all subgroups G of A such that G/N is finite. As $C_{M_n(D)}(K) = K$, K^* is irreducible by Lemma 3, and so is any subgroup containing it. Thus, by [24, p. 9], $K[G] = F[G]$ is a prime ring that is of finite-dimension over K and hence a simple Artinian ring by [22, Corollary 1.6.30]. Therefore, R is the union of simple Artinian rings. Now, it is clear that $M \subseteq N_{\mathrm{GL}_n(D)}(R^*)$. If $M = N_{\mathrm{GL}_n(D)}(R^*)$, then $R^* \subseteq M$ and hence $K[G]^* \subseteq M$; this leads to the commutativity of $K[G]$ which contradicts the maximality of N (we again used of Lemmas 1 and 4). Consequently $N_{\mathrm{GL}_n(D)}(R^*) = \mathrm{GL}_n(D)$ which implies $R = M_n(D)$, i.e., $F[M] = F[A]$.

Now, to complete the proof of the simplicity of M/N, it is enough to verify that $A = M$. To do so, given $x \in M$, there exists a subgroup G with $K^* \subseteq G \subseteq A$ such that G/K^* is finite and $x \in K[G]$. Also, it is easily seen that $(K[G], K, G, G/K^*)$ is a crossed product central simple algebra with center E, say. Setting $C = K[G]$, the Skolem–Noether theorem gives us

$$G/K^* \subseteq N_{C^*}(K)/K^* \simeq \mathrm{Gal}(K/E).$$

Therefore, by the Centralizer Theorem in [4] we have

$$\left|G/K^*\right| \leqslant \left|\mathrm{Gal}(K/E)\right| = \dim_E K = \dim_K C = \left|G/K^*\right|,$$

which implies that $G/K^* = N_{C^*}(K)/K^*$. But $x \in N_{C^*}(K) = G$ which says that $x \in A$, and hence $A = M$.

Now we have $N \subseteq NH \subseteq M$, and since M/N is a simple group we either have $NH = N$ or $NH = M$. In the former case we conclude that $H = N$ is abelian, and in the second case M is locally soluble which is impossible by [21, Theorem 1.5].

Now assume H is an FC-group. Clearly we have $M \subseteq N_{\mathrm{GL}_n(D)}(F[H])$. If $N_{\mathrm{GL}_n(D)}(F[H]) = \mathrm{GL}_n(D)$, by the Cartan–Brauer–Hua theorem for matrix ring we either have H is central or $F[H] = M_n(D)$. In the later case, clearly $Z(H) = H \cap F^*$. Now H is center-by-(locally finite) by [23, Theorem 15.1.16]; consequently $M/(H \cap F^*)$ is locally finite which is impossible by Lemma 4. Thus by the maximality of M in $\mathrm{GL}_n(D)$ we may assume $N_{\mathrm{GL}_n(D)}(F[H]) = M$, i.e., $F[H]^* \subseteq M$. On the other hand, $F[H]$ is semisimple Artinian by [24, Theorem 1.2.12]. It is also a prime ring by Lemma 3. Thus, there exists a natural number m and a division rings Δ such that $F[H]^* \simeq \mathrm{GL}_m(\Delta)$ as F-algebras. If $m > 1$, then

H (and so M) is locally finite by Lemma 1, contracting Lemma 4. So, $m = 1$ and thus $F[H]^*$ (and hence H) is abelian. This completes the proof of part (i). □

Our next proofs rely on the following four technical results due to Wehrfritz, Lichtman and Faith.

Lemma 5. *(See [29].) Let A be a one-sided Artinian ring. Suppose S is a subring of A such that every prime image of S with the ascending chain condition on right annihilators is right Goldie. Let G be a locally soluble subgroup of the group of units of A normalizing S and set $R = S[G]$. Then R has a semiprime nilpotent ideal $\mathfrak{n}$ such that $R/\mathfrak{n}$ is right Goldie.*

Lemma 6. *(See [16, Lemma 4.3].) Let D be a division ring and K be a subring of $M_n(D)$. Let V be a subgroup of $\mathrm{GL}_n(D)$, U be its normal subgroup. Assume that $K[V]$ is a semiprime ring and $K[U]$ is a (semiprime) Goldie ring, and the quotient group is either abelian or locally finite. Then $K[V]$ is a Goldie ring.*

Lemma 7. *(See [6].) If R is a ring which has a classical right quotient ring $Q = M_n(D)$ (D being a division ring), then there is a right Ore domain S with right quotient ring D such that R contains the matrix ring $M_n(S)$.*

Lemma 8. *(See [16, Corollary 4.6].) Let D be a division ring and K be a subring of $M_n(D)$. Let V be a subgroup of $\mathrm{GL}_n(D)$ such that $K[V]$ is semiprime and assume that V is an extension of an FC-subgroup by a locally finite group. Then $K[V]$ is a Goldie ring.*

Proof of Theorem 1 (ii). Suppose H is a locally soluble normal subgroup of M such that M/H is locally finite. By Lemma 2, M is irreducible and thus $F[M]$ is a prime ring and $F[H]$ is a semiprime ring by [24, 1.1.14]. Now, Lemma 5 yields that $F[H]$ is a Goldie ring, and therefore $F[M]$ is also a Goldie ring by Lemma 6. So by Goldie's theorem, its classical quotient ring Q is simple Artinian, and by [24, Theorem 5.7.8], Q is naturally embedded in $M_n(D)$. Now, $M = Q^*$ implies that M is abelian (see Lemmas 1 and 4). Therefore, let $M \neq Q^*$ and by maximality of M in $\mathrm{GL}_n(D)$ we conclude that $Q = M_n(D)$. Now, by Lemma 7, there exists an Ore domain S with quotient ring D such that $F[M]$ contains the matrix ring $M_n(S)$. So $\mathrm{GL}_n(S) \subseteq F[M]^* = M$ is (locally soluble)-by-(locally finite). If $\operatorname{char} S = 0$, then $\mathrm{GL}_n(\mathbb{Z})$ must be (locally soluble)-by-(locally finite), contracting the fact that it contains a noncyclic free subgroup.

Now let H be an FC-group. As above we can assume $F[M]$ is a prime ring. Now, Lemma 8 yields that $F[M]$ is a Goldie ring. The rest of the proof is now as of the previous paragraph. □

Lemma 9. *(See [15, Theorem 2].) Let R be a prime ring with 1, $Z = Z(R)$ be the center of R containing at least five elements, U the group of units of R, and $\overline{U}$ the Z-subalgebra of R generated by U. Assume that $\overline{U}$ contains a nonzero ideal of R. If N is a soluble normal subgroup of U, then either R is a domain or $N \subseteq Z$.*

Proof of Theorem 1 (iii). Let $R := F[M]$. If $R = M_n(D)$, then H is abelian by part (i) of this theorem. Now, assume that $R^* = M$. As in the proof of part (ii), we know that R is a prime Goldie ring. On the other hand, since the $Z(R)$-subalgebra of R generated by M is R itself, we can use of Lemma 9 to deduce that either R is a domain or $H \subseteq Z(R)$. But, in the first case R is in fact an Ore domain (since it is also prime Goldie). Therefore, the classical quotient ring of R is a division ring D_1 which by [24, Theorem 5.7.8], D_1 is naturally embedded in $M_n(D)$. Since $n \geqslant 2$, maximality of M in $\mathrm{GL}_n(D)$ implies that $M = R^* = D_1^*$, and so M is abelian. □

Proof of Corollary 1. The case (i) follows directly from Theorem 1. We prove (ii) and (iii) simultaneously. By part (i) (or Theorem 1 (iii)) we assume $D = F$. First assume M is not absolutely irreducible. If M is reducible, there exists an invertible matrix P and a natural number $0 < m < n$ such that

$$PMP^{-1} = \left\{ \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \;\middle|\; A \in \mathrm{GL}_m(D),\ C \in \mathrm{GL}_{n-m}(D),\ B \in M_{m\times(n-m)}(D) \right\}.$$

Now, both of $\mathrm{GL}_m(D)$ and $\mathrm{GL}_{n-m}(D)$ are (locally soluble)-by-(locally finite), which Lemma 1 implies $m = n - m = 1$, so $n = 2$, a contradiction. So, M is irreducible and thus $F[M]$ is a prime ring. The rest of the proof for this case is now as of part (ii) (or of part (iii)) of Theorem 1.

Now suppose M is absolutely irreducible. Clearly we have $M \subseteq N_{\mathrm{GL}_n(D)}(F[H])$. If $N_{\mathrm{GL}_n(D)}(F[H]) = \mathrm{GL}_n(D)$, we either have H is central or $F[H] = M_n(D)$. In the later case, clearly $Z(H) = H \cap F^*$. Now H is center-by-(locally finite) by [24, Theorem 5.7.11] (or center-by-finite by [2, p. 114]); consequently $M/(H \cap F^*)$ is locally finite (or finite) which is impossible by Lemma 4 (or by [1, Lemma 9]). Thus by the maximality of M in $\mathrm{GL}_n(D)$ we may assume $N_{\mathrm{GL}_n(D)}(F[H]) = M$, i.e., $F[H]^* \subseteq M$. On the other hand, $F[H]$ is semisimple Artinian by [24, Theorem 1.2.12]. Thus, there exist natural numbers n_i and division rings D_i such that $F[H]^* \simeq \mathrm{GL}_{n_1}(D_1) \times \cdots \times \mathrm{GL}_{n_s}(D_s)$ as F-algebras. If for some i, $n_i > 1$, then $F^* \subseteq D_i^*$ is locally finite by Lemma 1 (or finite: see the proof of Lemma 1), contracting the fact that D is of characteristic zero (or D is infinite). So, $F[H]^*$ (and hence H) is abelian. □

Proof of Theorem 2. (i) Since D is finite-dimensional over its center, we may view M as a linear group. Hence by a result of Zassenhaus, each locally soluble subgroup of M is soluble and M has a unique maximal soluble normal subgroup which we shall denote by $\mathrm{Sol}(M)$ (see [27, Corollary 3.8]). But, the factor group $M/\mathrm{Sol}(M)$ is locally finite by Tits' theorem [26]. Now, Theorem 1 (iii) implies that $\mathrm{Sol}(M)$ is abelian, and thus the result follows.

For (ii) and (iii), by Theorem 1, H is abelian in all cases. So M is abelian-by-finite, and hence by Lemma 1.11 in [19, p. 176], the group ring FM satisfies a polynomial identity. Therefore $F[M]$ as a homomorphic image of FM, satisfies a polynomial identity too. If $F[M] = M_n(D)$, we use Kaplansky's theorem in [22, p. 36] to obtain $[D : F] < \infty$; thus every locally soluble normal subgroup of M is abelian by (i). Now suppose $F[M]^* = M$. Let $F_1 := C_{M_n(D)}(M)$, and recall that M is irreducible. So F_1 is a field (see Lemma 2 of [12] and its proof) which we may assume $F_1^* \subseteq M$ by Lemma 1. Consequently, $F[M]$ is a prime PI-ring whose center F_1 is a field and therefore, by [22, Corollary 1.6.28], it is a simple ring. So, again by Kaplansky's theorem we have $F_1[M] \simeq M_m(\Delta)$ for some natural number m and a division ring Δ. Thus, $M = F_1[M]^* \simeq \mathrm{GL}_m(\Delta)$. Now, use Lemmas 1 and 4 to conclude that M is abelian. This completes the proof. □

Proof of Corollary 2. The case $n = 1$ follows from [17, Theorem 8]. Let $n \geqslant 2$. By Theorem 2 (i), $\mathrm{Sol}(M)$, the unique maximal soluble normal subgroup of M, is abelian. Assume M contains no noncyclic free subgroup. Since D^* contains a noncyclic free subgroup [8], Lemma 2 implies that M is primitive and thus by similar arguments as the first paragraph of the proof of part (i) of Theorem 1 we can show that $K := \mathrm{Sol}(M) \cup \{0\}$ is a maximal subfield of $M_n(D)$ and clearly $K^* \trianglelefteq M$. If $F[M]^* = M$, then as in the proof of Theorem 2 we have $M = F[M]^* \simeq \mathrm{GL}_m(\Delta)$ for some natural number m and a division ring Δ. Thus M contains a noncyclic free subgroup (unless in the case where $m = 1$ and Δ is a field, which implies that $M = K^*$). Now assume $F[M] = M_n(D)$. Then it is easily seen that for every $a \in M$, the mapping $\phi_a : K \to K$ given by $\phi_a(x) = axa^{-1}$, belongs to $\mathrm{Gal}(K/F)$ and that $\mathcal{F}ix(G) = F$ where G is the group of all such automorphisms ϕ_a, $a \in M$. Consider the mapping $f : M \to G$ given by $f(a) = \phi_a$. From this epimorphism we conclude that $G \simeq M/K^*$ (since $K^* = C_{M_n(D)}(K^*)$ by the maximality of the subfield K). Now, from Galois theory we have $|G| \leqslant |\mathrm{Gal}(K/F)| \leqslant [K : F] < \infty$, and we are done. □

In order to prove Theorem 3, we need the following lemma.

Lemma 10. *(See [30, 3.11].) Let G be a locally nilpotent subgroup of the multiplicative group D^* of the division ring D. Suppose also that $H = N_{D^*}(G)$, $E = C_D(G)$, and $D = E(G)$. Denote the maximal 2-subgroup of G by Q. Then one of the following holds:*

(i) *T (the maximal locally finite normal subgroup of G) is abelian and H/GE^* is abelian;*
(ii) *$G = Q \cdot C_G(Q)$ where Q is quaternion of order 8 and $H/GE^* \cong \mathrm{Sym}(3) \times Y$ for Y abelian;*
(iii) *$G \neq Q \cdot C_G(Q)$ where Q is quaternion of order 8 and H/GE^* is abelian;*
(iv) *Q is non-abelian with $|Q| > 8$ and H/GE^* has an abelian subgroup Y with index in H/GE^* at most 2 (1 if Q is infinite).*

Proof of Theorem 3. First we consider $n = 1$. Let G be a nilpotent normal subgroup of M. Since $M \subseteq N_{D^*}(F(G)^*)$, we either have $F(G)^* \subseteq M$ or (by the Cartan–Brauer–Hua theorem) $F(G) = D$. In the former case, G as a nilpotent normal subgroup of $F(G)^*$ is abelian. So assume $F(G) = D$.

If M is absolutely irreducible, then $M/C_M(G)$ is torsion by [24, Theorem 5.7.11]. Since $C_M(G) \subseteq F^*$ (because of $F(G) = D$), we conclude that M is torsion over F and therefore $F[G] = F(G) = D$, i.e., G is absolutely irreducible. Clearly, $Z(G) = G \cap F$; so $G/(G \cap F)$ is locally finite by [24, Theorem 5.7.11]. This implies that D is locally finite-dimensional over F, and hence M is abelian by [21, Corollary 1.7]. Now, suppose M is not absolutely irreducible, so $F[M]^* = M$. Since $F(G) = D$, $F = C_D(G)$. On the other hand, $M \subseteq N_{D^*}(G) \subseteq D^*$. If $N_{D^*}(G) = D$, then G as a nilpotent normal subgroup of D^* is central. So assume $M = N_{D^*}(G)$. Now we can apply Lemma 10. Denote the maximal 2-subgroup of G by Q which is non-abelian by Lemma 10. If Q is finite, then $F[Q]^* \subseteq F[M]^* = M$ implies that the multiplicative group of the division ring $F[Q]$ is Engel which asserts that it is abelian by [21, Theorem 1.3]. This contradiction shows that M/GF^* must be abelian. This gives us $M' \subseteq GF^*$ is nilpotent. Therefore M is soluble; so it is abelian by [3, Theorem 3.7]. This completes the proof for $n = 1$.

Now, assume $n \geqslant 2$. Let H denote the Hirsch–Plotkin radical of M and let $R := F[H]$. Then we have $M \subseteq N_{\mathrm{GL}_n(D)}(R^*)$ and hence either $R = M_n(D)$ or $R^* \subseteq M$. In the first case, clearly $Z(H) = H \cap F$; so $H/(H \cap F)$ is locally finite by [24, Theorem 5.7.11]. This implies that D is locally finite-dimensional over F, and hence M is abelian by [21, Corollary 1.7]. Therefore, assume $R^* \subseteq M$, i.e., R^* is Engel. On the other hand, by Lemmas 2, 3 and 5, R is a prime Goldie ring with a classical quotient ring Q, contained in $M_n(D)$, which is simple Artinian. If $M = N_{\mathrm{GL}_n(D)}(Q^*)$, then we have $H \trianglelefteqslant Q^* \subseteq M$, which implies that H is abelian. If not, $N_{\mathrm{GL}_n(D)}(Q^*) = \mathrm{GL}_n(D)$, i.e., $Q = M_n(D)$. Now, by Lemma 7, there exists an Ore domain S with quotient ring D such that R contains the matrix ring $M_n(S)$. So $\mathrm{GL}_n(S) \subseteq R^* \subseteq M$ is Engel. If $\operatorname{char} S = 0$, then $\mathrm{GL}_n(\mathbb{Z})$ must be Engel, contracting the fact that it contains a noncyclic free subgroup. So $\operatorname{char} S = p > 0$. But then $\mathrm{GL}_n(\mathbb{Z}_p)$ must be Engel, a contradiction. $\square$

Lemma 11. *Let D be an infinite division ring, N a subnormal subgroup of $\mathrm{GL}_n(D)$, and M a maximal subgroup of N. If M is nilpotent, then $F[M'] \neq M_n(D)$.*

Proof. By [13, Corollary 1], we may assume D is infinite-dimensional over F. Suppose on the contrary that $F[M'] = M_n(D)$, thus we also have $F[M] = M_n(D)$. Then, M is center-by-(locally finite) by [24, Theorem 5.7.11], which implies that D is locally finite-dimensional over F and that M' is a locally finite group.

If $\operatorname{char} D = 0$, then M' is abelian-by-finite by [24, 2.5.2]. Consequently, $F[M'] = M_n(D)$ satisfies a polynomial identity which gives us $[D : F] < \infty$, contracting our assumption.

Next let $\operatorname{char} D = p > 0$. If F is a locally finite field, then D is algebraic over a finite field and hence by Jacobson's theorem [14, p. 208], we obtain $D = F$, a contradiction. Let $K \neq F$ be a maximal locally finite subfield of F. By [24, p. 7], $K[M']$ is simple Artinian; thus $K[M'] \simeq M_s(\Delta)$ for some division ring Δ and a positive integer s. If $a \in K[M']^*$, then there exist $n_1, \ldots, n_k$ in M' and $a_1, \ldots, a_k$ in K such that $a = a_1 n_1 + \cdots + a_k n_k$. Since M' is a locally finite group and K is a locally finite field, we conclude that $\mathbb{Z}_p[a_1, \ldots, a_k]$ is a finite field and so $\mathbb{Z}_p[a_1, \ldots, a_k][\langle n_1, \ldots, n_k \rangle]$ is a finite ring. Thus, a must be torsion and hence $K[M']^*$ is a torsion group; consequently Δ is a locally finite field. Now we have $K \subseteq \Delta \subseteq C_{M_n(D)}(F[M']) = F$. By the maximality of K, we conclude that $K = \Delta$. Now $F[M'] = M_n(D)$ implies that $F[K[M']] = M_n(D)$. Since $[K[M'] : K] = s^2$, we conclude that $[M_n(D) : F] \leqslant s^2$ and thus D is of finite-dimension over F. This contradiction shows that $F[M'] \neq M_n(D)$ and completes the proof. $\square$

We close this paper by proving our final theorem using the above lemma.

Proof of Theorem 4. (i) Since M is a maximal subgroup of N, we either have $F(M)^* \cap N = M$ or $F(M)^* \cap N = N$. In the first case, M as a nilpotent subnormal subgroup of the multiplicative group of the division ring $F(M)$ is abelian. In the second case, by [23, 14.3.8], we conclude that $F(M) = D$, i.e., M is absolutely irreducible. Then, M is center-by-(locally finite) by [24, Theorem 5.7.11], which implies that D is locally finite-dimensional over F. Moreover, we may let $M' \subseteq F^*$ by [20, Theorem 1.1].

First we assume $F^* \subseteq M$. Given $x, y \in M$ such that $xy \neq yx$, we have $F^*\langle x, y\rangle \trianglelefteq M$ and hence $M \subseteq N_{D^*}(F[\langle x, y\rangle]^*) \cap N$. By the maximality of M in N, we obtain either $N_{D^*}(F[\langle x, y\rangle]^*) \cap N = M$ or $N_{D^*}(F[\langle x, y\rangle]^*) \cap N = N$. In the first case, $F[\langle x, y\rangle]^* \cap N$ as a nilpotent subnormal subgroup of $F[\langle x, y\rangle]^*$ is abelian, contracting the choose of x and y. In the second case, by [23, 14.3.8], we obtain $F[\langle x, y\rangle]^* = D^*$. Since x and y are algebraic over F, we have $F[\langle x, y\rangle] = F[x, y]$, consequently, $F[x, y] = D$. So $[D : F] < \infty$ since D is locally finite-dimensional over F; this contradicts [13, Corollary 1]. Therefore M is abelian.

Now, if $F^* \nsubseteq M$, let $M_1 = F^*M$ and $N_1 = F^*N$. Then N_1 is a subnormal subgroup of D^* and M_1 is a nilpotent subgroup of N_1. If $M_1 = N_1$, then M_1 is abelian. If $M_1 \neq N_1$, then M_1 is a maximal subgroup of N_1 containing F^*. Therefore, M_1 (and thus M) is abelian by above.

(ii) For $n = 1$ the result follows from [20, Proposition 1.1]. Let $n \geqslant 2$ and $R := F[M]$. Then $M \subseteq R \cap N \subseteq N$; hence by the maximality of M in N we may consider the following two cases:

Case 1. Let $M = R \cap N$. Then M is a subnormal subgroup of R^*. On the other hand, if M is reducible, then it contains an isomorphic copy of D^* by [11, Lemma 1]; so D is a field and therefore M is abelian by [13, Corollary 1]. Assume that M is irreducible; thus R is a prime Goldie ring by [24, 1.1.14] and Lemma 5. As of the proof of Theorem 1 (iii), we may assume that R is an Ore domain. Denote the classical quotient ring of R by Δ; then Δ is a division ring contained in $M_n(D)$. If $N = \Delta \cap N$, then $\mathrm{SL}_n(D) \subseteq N \subseteq \Delta^*$ by [8, Lemma 2.3]; so the Cartan–Brauer–Hua theorem for matrix ring implies that $\Delta = M_n(D)$ which is impossible since $n \geqslant 2$. Therefore $M = \Delta \cap N$, consequently M as a nilpotent subnormal subgroup of Δ^* is abelian.

Case 2. In this case, we consider the case $N = R \cap N$. Thus $\mathrm{SL}_n(D) \subseteq N \subseteq R^*$, therefore $R = M_n(D)$. Then, M is center-by-(locally finite), which implies that D is locally finite-dimensional over F. Let $A := F[M']^* \cap N$. Since $M \subseteq N_N(F[M']^*) \subseteq N$, we either have $N_N(F[M']^*) = N$ or $N_N(F[M']^*) = M$. In the former case, A is a subnormal subgroup of $\mathrm{GL}_n(D)$. Now if A is central, then $M' \subseteq A$ is abelian; if not, $\mathrm{SL}_n(D) \subseteq A \subseteq F[M']^*$ yields that $F[M'] = M_n(D)$, contracting Lemma 11. In the second case, A is a nilpotent subnormal subgroup of $F[M']^*$. But $F[M']$ is a semisimple Artinian ring by [24, p. 7], consequently $M' \subseteq A$ is abelian. $\square$

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