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Injectors with a central socle in a finite solvable group

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ABSTRACT

In response to an Open Question of Doerk and Hawkes (1992) [2, IX §4, p. 628], we shall describe three constructions for the $\mathcal{Z}^\pi$-injectors of a finite solvable group, where $\mathcal{Z}^\pi$ is the Fitting class formed by the finite solvable groups whose π-socle is central (and π is a set of prime numbers).

1. Introduction

Throughout this Introduction, let H be a subgroup of a finite solvable group G, and let $\pi = \{p_1, p_2, \ldots, p_m\}$ be a set of prime numbers. We use the notation of Doerk and Hawkes [2], and in particular we define the *π-socle* of H to be the subgroup $\mathrm{Soc}_\pi H$ generated by the minimal normal π-subgroups of H. If p is a prime number, write $\mathrm{Soc}_p H = \mathrm{Soc}_{\{p\}} H$, and note that

$$\mathrm{Soc}_\pi H = \mathrm{Soc}_{p_1} H \times \mathrm{Soc}_{p_2} H \times \cdots \times \mathrm{Soc}_{p_m} H,$$

where each subgroup $\mathrm{Soc}_{p_i} H$ is an elementary abelian p_i-group [2, A(10.5.a) and (4.4)]. Following Gaschütz [2, IX §2, Construction D and (2.9.a)], define $\mathcal{Z}^\pi$ to be the class of finite solvable groups

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H such that $\mathrm{Soc}_\pi H \leqslant \mathbf{Z}(H)$, and as before write $\mathcal{Z}^p = \mathcal{Z}^{\{p\}}$. Then the above direct decomposition of $\mathrm{Soc}_\pi H$ implies that

$$\mathcal{Z}^\pi = \mathcal{Z}^{p_1} \cap \mathcal{Z}^{p_2} \cap \cdots \cap \mathcal{Z}^{p_m}.$$

These classes were investigated by Frantz and by Lockett [2, IX(4.18)], and also by Blessenohl [1] in his study of dominant Fitting classes. Each class $\mathcal{Z}^\pi$ is closed under taking normal subgroups and normal products, so it is a Fitting class [2, IX(2.8)]. It follows that G has a unique maximal normal $\mathcal{Z}^\pi$-subgroup, called the *$\mathcal{Z}^\pi$-radical* of G [2, II(2.9)]; moreover G has a unique conjugacy class of subgroups J, called the *$\mathcal{Z}^\pi$-injectors* of G, with the property that, for every subnormal subgroup X of G, $J \cap X$ is a maximal $\mathcal{Z}^\pi$-subgroup of X [2, VIII(2.9)].

Now let V_p be the p-socle of the $\mathcal{Z}^p$-radical of G, and let $\mathrm{Syl}_p\, G$ be the set of Sylow p-subgroups of G (where p is a prime number). It was proved by Frantz and by Lockett [2, IX(4.19)] that if $P \in \mathrm{Syl}_p\, G$, then the subgroup

$$\mathbf{D}_p^G(P) = \mathbf{C}_G\big(\mathbf{C}_{V_p}(P)\big)$$

is a $\mathcal{Z}^p$-injector of G. In Theorem 1 we shall rework this proof. Doerk and Hawkes asked [2, IX §4, p. 628] whether an analogous description can be given of the $\mathcal{Z}^\pi$-injectors when π is an arbitrary set of prime numbers. To do this, recall that a *Sylow basis* Σ in G is a set of Sylow subgroups of G, with $|\Sigma \cap \mathrm{Syl}_p\, G| = 1$ for each prime number p, such that every pair of members of Σ permute with each other [2, I(4.7)]. If Σ is a Sylow basis in G, with $\{P\} = \Sigma \cap \mathrm{Syl}_p\, G$, write $\mathbf{D}_p^G(\Sigma) = \mathbf{D}_p^G(P)$, and define

$$\mathbf{D}_\pi^G(\Sigma) = \bigcap_{p\in\pi} \mathbf{D}_p^G(\Sigma).$$

If the set $\Sigma \cap H = \{P \cap H\colon\ P \in \Sigma\}$ is a Sylow basis in H, we say that Σ *reduces into* H, and we write $\Sigma \searrow H$ [2, I(4.15)]. We shall show that $\Sigma \searrow \mathbf{D}_\pi^G(\Sigma)$, which allows us to make the inductive definition

$$\mathbf{D}^i(\Sigma) = \begin{cases} G & \text{when } i = 0,\\ \mathbf{D}_\pi^{\mathbf{D}^{i-1}(\Sigma)}\big(\Sigma \cap \mathbf{D}^{i-1}(\Sigma)\big) & \text{when } i \geqslant 1. \end{cases}$$

There is an integer k such that $\mathbf{D}^0(\Sigma) > \mathbf{D}^1(\Sigma) > \cdots > \mathbf{D}^k(\Sigma) = \mathbf{D}^{k+1}(\Sigma)$, and in Theorem 2 we shall prove that $\mathbf{D}^k(\Sigma)$ is a $\mathcal{Z}^\pi$-injector of G. Moreover in Example 1 we shall exhibit a group with $k = 3$ (based on an example due to Blessenohl); we do not know whether there are groups with $k > 3$. We shall also reprove Blessenohl's result that if H is a maximal $\mathcal{Z}^\pi$-subgroup containing the π-socle of the $\mathcal{Z}^\pi$-radical of G, then H is a $\mathcal{Z}^\pi$-injector; indeed the above construction is suggested by the proof of this fact [2, IX(4.20)].

The class $\mathcal{N}$ of nilpotent groups is closed under taking normal subgroups and normal products [2, A(8.2.a) and (8.8.b)], so $\mathcal{N}$ is a Fitting class. Hence G has an $\mathcal{N}$-radical F (the Fitting subgroup) and $\mathcal{N}$-injectors. Let $q_1, q_2, \ldots, q_n$ be the prime factors of $|F|$, and for each index i, choose $S_i \in \mathrm{Syl}_{q_i} \mathbf{C}_G(\mathbf{O}_{q_i'}(F))$. It has been shown by Dade and by Mann [2, IX(4.12)] that $\langle S_1, S_2, \ldots, S_n\rangle$ is an $\mathcal{N}$-injector of G.

If $H \in \mathcal{N}$, then every chief factor of H is central, which implies that $\mathcal{N} \subseteq \mathcal{Z}^\pi$. This suggests that it may be possible to construct the $\mathcal{Z}^\pi$-injectors by adapting the above characterization of the $\mathcal{N}$-injectors. To do this, let N_π be the $\mathcal{Z}^\pi$-radical of G, and suppose $N_\pi \leqslant H \leqslant G$ with $H \in \mathcal{Z}^\pi$. For each prime number $p_i \in \pi$, take $W_i = \mathbf{O}_{p_i'}(\mathrm{Soc}_\pi H)$. Then $H \leqslant \mathbf{C}_G(W_i)$, so we can choose $S_i^0 \in \mathrm{Syl}_{p_i} \mathbf{C}_G(W_i)$ such that $H \cap S_i^0 \in \mathrm{Syl}_{p_i} H$. Take $H^0 = \langle H, S_1^0, S_2^0, \ldots, S_m^0\rangle$, and put $\mathbf{M}_\pi^G(H) = \mathbf{C}_G(\mathrm{Soc}_\pi H^0)$. We shall show that $N_\pi \leqslant \mathbf{M}_\pi^G(H)$ and $\mathbf{M}_\pi^G(H) \in \mathcal{Z}^\pi$, which allows us to make the inductive definition

$$\mathbf{M}^i(G) = \begin{cases} N_\pi & \text{when } i = 0, \\ \mathbf{M}_\pi^G(\mathbf{M}^{i-1}(G)) & \text{when } i \geqslant 1. \end{cases}$$

There is an index k such that $\mathbf{M}^0(G) < \mathbf{M}^1(G) < \cdots < \mathbf{M}^k(G) = \mathbf{M}^{k+1}(G)$, and in Theorem 3 we shall prove that $\mathbf{M}^k(G)$ is a $\mathcal{Z}^\pi$-injector of G.

This construction can be modified by introducing a Sylow basis Σ. Recall that H is said to be *pronormal* in G if, for every element $g \in G$, H and H^g are conjugate in $\langle H, H^g\rangle$; in this case we write $H \operatorname{pr} G$. Suppose $N_\pi \leqslant H \leqslant G$, with $H \in \mathcal{Z}^\pi$ and $\Sigma \searrow H$. Assume also that $H \operatorname{pr} G$ and $H = \mathbf{C}_G(\operatorname{Soc}_\pi H)$, and put $L = \mathbf{N}_G(H)$. For each prime number $p_i \in \pi$, take $W_i = \mathbf{O}_{p_i'}(\operatorname{Soc}_\pi H)$ as above. We shall show that $\Sigma \searrow \mathbf{C}_L(W_i)$, which allows us to take $\{S_i^*\} = (\Sigma \cap \mathbf{C}_L(W_i)) \cap \operatorname{Syl}_{p_i} \mathbf{C}_L(W_i)$. As before put $H^* = \langle H, S_1^*, S_2^*, \ldots, S_m^*\rangle$, and $\mathbf{E}_\pi^\Sigma(H) = \mathbf{C}_G(\operatorname{Soc}_\pi H^*)$. We shall show that $N_\pi \leqslant \mathbf{E}_\pi^\Sigma(H)$ and $\mathbf{E}_\pi^\Sigma(H) \in \mathcal{Z}^\pi$, with $\Sigma \searrow \mathbf{E}_\pi^\Sigma(H)$ and $\mathbf{E}_\pi^\Sigma(H) \operatorname{pr} G$, and moreover $\mathbf{E}_\pi^\Sigma(H) = \mathbf{C}_G(\operatorname{Soc}_\pi \mathbf{E}_\pi^\Sigma(H))$. This allows us to make the inductive definition

$$\mathbf{E}^i(\Sigma) = \begin{cases} N_\pi & \text{when } i = 0, \\ \mathbf{E}_\pi^\Sigma(\mathbf{E}^{i-1}(\Sigma)) & \text{when } i \geqslant 1. \end{cases}$$

There is an index k such that $\mathbf{E}^0(\Sigma) < \mathbf{E}^1(\Sigma) < \cdots < \mathbf{E}^k(\Sigma) = \mathbf{E}^{k+1}(\Sigma)$, with $\mathbf{E}^i(\Sigma) \operatorname{pr} G$ for all indices i, and in Theorem 4 we shall prove that $\mathbf{E}^k(\Sigma)$ is a $\mathcal{Z}^\pi$-injector of G.

The lay-out of the paper is as follows. In Section 2 we prove some preliminary results which will be used later; in particular, when $H \leqslant G$ with $H \in \mathcal{Z}^\pi$, it is convenient to introduce the subgroup $\mathbf{K}_\pi^G(H) = \mathbf{C}_G(\operatorname{Soc}_\pi H)$. In Section 3 we prove Theorems 1 and 2, and construct Example 1. Then Theorems 3 and 4 are proved in Sections 4 and 5 respectively. Finally in Section 6 we construct Example 2, which shows that the chain $\mathbf{E}^0(\Sigma) < \mathbf{E}^1(\Sigma) < \cdots < \mathbf{E}^k(\Sigma) = \mathbf{E}^{k+1}(\Sigma)$ may be arbitrarily long.

2. The subgroup $\mathbf{K}_\pi^G(H)$

In our first lemma we write down some easy results about subgroups which contain the Fitting subgroup. Then we state a property of $\mathcal{Z}^\pi$-groups proved by Doerk and Hawkes, which will be used implicitly throughout the paper, and in Lemma 3 we record some consequences of this property. Recall that the *socle* of a group H is the subgroup $\operatorname{Soc} H$ generated by all the minimal normal subgroups of H.

Lemma 1. *Let F be the Fitting subgroup of a finite solvable group G.*

(a) *If N_π is the $\mathcal{Z}^\pi$-radical of G (where π is a set of prime numbers), then $F \leqslant N_\pi$.*
(b) *If $F \leqslant N \leqslant G$, then $\mathbf{C}_G(N) \leqslant N$.*
(c) *If $F \leqslant N \leqslant H \leqslant G$, then $\mathbf{Z}(H) \leqslant \mathbf{Z}(N)$.*
(d) *If $F \leqslant N \trianglelefteq H \leqslant G$, then $\operatorname{Soc} H \leqslant \operatorname{Soc} N$.*

Proof. (a) This holds because F is a normal $\mathcal{Z}^\pi$-subgroup of G.

(b) From the inclusion $\mathbf{C}_G(F) \leqslant F$ [2, A(10.6.a)] we get $\mathbf{C}_G(N) \leqslant \mathbf{C}_G(F) \leqslant F \leqslant N$.

(c) Using (b) we deduce that $\mathbf{Z}(H) \leqslant \mathbf{C}_G(N) = \mathbf{C}_N(N) = \mathbf{Z}(N)$.

(d) Suppose U is a minimal normal subgroup of H, and put $V = \operatorname{Soc} N$; we must show that $U \leqslant V$. If $U \cap N = 1$ then $[U, N] \leqslant U \cap N = 1$, so $U \leqslant U \cap \mathbf{C}_G(N) \leqslant U \cap N = 1$ using (b), which contradicts the definition of U. This proves that $U \cap N \neq 1$, so $U \leqslant N$ by the minimality of U. Thus $U \trianglelefteq N$, so U contains a minimal normal subgroup of N. Therefore $U \cap V \neq 1$, and hence $U \leqslant V$. $\square$

Lemma 2. *(See Doerk and Hawkes [2, IX(4.17)].) Suppose p is a prime number, and $\pi = \{p_1, p_2, \ldots, p_m\}$ is a set of prime numbers. Let H be a finite solvable group, with $L \leqslant H$ and V_p, $V_\pi \trianglelefteq H$, where V_p is an elementary abelian p-group, while $V_\pi = V_1 \times V_2 \times \cdots \times V_m$, and V_i is an elementary abelian p_i-group $(1 \leqslant i \leqslant m)$.*

(a) *Suppose* $H \in \mathcal{Z}^p$. *If* $P \in \mathrm{Syl}_p\, H$, *then* $\mathbf{C}_{V_p}(P) \leqslant \mathbf{Z}(H)$.
(b) *Suppose* $H \in \mathcal{Z}^p$. *If* $V_p \geqslant \mathrm{Soc}_p\, H$ *and* $p \nmid |H : L|$, *then* $\mathbf{C}_{V_p}(L) = \mathrm{Soc}_p\, H$.
(c) *Suppose* $H \in \mathcal{Z}^\pi$. *If* $V_\pi \geqslant \mathrm{Soc}_\pi\, H$ *and* $|H : L|$ *is a* π'*-number, then* $\mathbf{C}_{V_\pi}(L) = \mathrm{Soc}_\pi\, H$.

Proof. (a) This is proved in the given reference.

(b) Since $H \in \mathcal{Z}^p$, it is clear that $\mathrm{Soc}_p\, H \leqslant V_p \cap \mathbf{Z}(H) \leqslant \mathbf{C}_{V_p}(L)$. Conversely we can choose $P \in \mathrm{Syl}_p\, L \subseteq \mathrm{Syl}_p\, H$, and it follows from (a) that $\mathbf{C}_{V_p}(L) \leqslant \mathbf{C}_{V_p}(P) \leqslant \mathrm{Soc}_p\, \mathbf{Z}(H) \leqslant \mathrm{Soc}_p\, H$.

(c) We claim first that

$$\mathbf{C}_{V_\pi}(L) = \mathbf{C}_{V_1}(L) \times \mathbf{C}_{V_2}(L) \times \cdots \times \mathbf{C}_{V_m}(L).$$

Suppose $v = v_1 v_2 \ldots v_m \in \mathbf{C}_{V_\pi}(L)$, with $v_i \in V_i$ $(1 \leqslant i \leqslant m)$, and put $w_i = v_1 \ldots v_{i-1} v_{i+1} \ldots v_m$ and $W_i = \mathbf{O}_{p_i'}(V_\pi)$. If $x \in L$, then $1 = [v, x] = [v_i w_i, x] = [v_i, x][w_i, x]$, so $[v_i, x] = [w_i, x]^{-1} \in V_i \cap W_i = 1$; thus $v_i \in \mathbf{C}_{V_i}(L)$, and the claim follows. Since $H \in \mathcal{Z}^\pi = \bigcap_{i=1}^m \mathcal{Z}^{p_i}$, (b) implies that

$$\begin{aligned}\mathrm{Soc}_\pi\, H &= \mathrm{Soc}_{p_1} H \times \mathrm{Soc}_{p_2} H \times \cdots \times \mathrm{Soc}_{p_m} H \\ &= \mathbf{C}_{V_1}(L) \times \mathbf{C}_{V_2}(L) \times \cdots \times \mathbf{C}_{V_m}(L) \\ &= \mathbf{C}_{V_\pi}(L),\end{aligned}$$

as required. □

Lemma 3. *Let F be the Fitting subgroup of a finite solvable group G, and suppose p is a prime number, and π is a set of prime numbers.*

(a) *Suppose* $F \leqslant H \leqslant G$ *with* $H \in \mathcal{Z}^p$. *If P is a p-subgroup of G with* $P \cap H \in \mathrm{Syl}_p\, H$, *then* $\langle P, H \rangle \in \mathcal{Z}^p$.
(b) *Suppose* $F \leqslant N \trianglelefteq G$ *with* $N \in \mathcal{Z}^p$, *and take* $V = \mathrm{Soc}_p\, N$. *If P is a p-subgroup of G, then* $PN \in \mathcal{Z}^p$ *and* $\mathrm{Soc}_p(PN) = \mathbf{C}_V(P)$.
(c) *Suppose* $F \leqslant H \leqslant H^* \leqslant G$ *with* $H, H^* \in \mathcal{Z}^\pi$. *Then* $\mathrm{Soc}_\pi\, H \geqslant \mathrm{Soc}_\pi\, H^*$.
(d) *Suppose* $F \leqslant H \leqslant H^* \leqslant G$ *with* $H, H^* \in \mathcal{Z}^\pi$. *If* $|H^* : H|$ *is a* π'*-number, then* $\mathrm{Soc}_\pi\, H = \mathrm{Soc}_\pi\, H^*$.

Proof. (a) Put $L = \langle P, H \rangle$, and suppose U is a minimal normal p-subgroup of L; we must deduce that $U \leqslant \mathbf{Z}(L)$. Taking $V = \mathrm{Soc}_p\, F$ we get $U \leqslant V$ by Lemma 1(d), and hence $\mathbf{C}_U(P) \leqslant \mathbf{C}_V(P \cap H) \leqslant \mathbf{Z}(H)$ by Lemma 2(a). But $\mathbf{C}_U(P)$ is also centralized by P and therefore $\mathbf{C}_U(P) \leqslant \mathbf{Z}(L)$. Thus $U \cap \mathbf{Z}(L) \geqslant \mathbf{C}_U(P) \neq 1$ [2, A(5.5)], so the minimality of U implies that $U \leqslant \mathbf{Z}(L)$.

(b) Choose $P^* \in \mathrm{Syl}_p(PN)$ with $P^* \geqslant P$, and note that $P^* \cap N \in \mathrm{Syl}_p\, N$ [2, A(6.4.a)]. Applying (a) with $H = N$, we deduce that $PN = P^*N \in \mathcal{Z}^p$. Moreover $p \nmid |PN : P^*|$, so it follows from Lemma 2(b) that $\mathrm{Soc}_p(PN) = \mathbf{C}_V(P^*) = \mathbf{C}_V(P(P^* \cap N)) = \mathbf{C}_V(P)$ (since $P^* \cap N$ centralizes V).

(c) Take $V = \mathrm{Soc}_\pi\, F$, and choose subgroups $L \leqslant L^*$ which are Hall π-subgroups of H and H^* respectively. Then it follows from Lemma 2(c) that $\mathrm{Soc}_\pi\, H = \mathbf{C}_V(L) \geqslant \mathbf{C}_V(L^*) = \mathrm{Soc}_\pi\, H^*$.

(d) Take $V = \mathrm{Soc}_\pi\, F$ and let L be a Hall π-subgroup of H. Then L is also a Hall π-subgroup of H^*, so Lemma 2(b) implies that $\mathrm{Soc}_\pi\, H = \mathbf{C}_V(L) = \mathrm{Soc}_\pi\, H^*$, as in (c). □

We end this section by recalling the definition of $\mathbf{K}_\pi^G(H)$, and by recording some of its properties, which follow from Lemma 3, and which will be used repeatedly in the following sections.

Notation. Let G be a finite solvable group and let π be a set of prime numbers. If $H \leqslant G$ we define

$$\mathbf{K}_\pi^G(H) = \mathbf{C}_G(\mathrm{Soc}_\pi\, H).$$

If $p \in \pi$, we also write $\mathbf{K}_p^G(H) = \mathbf{K}_{\{p\}}^G(H)$ and $\mathbf{K}_{\pi-p}^G(H) = \mathbf{K}_{\pi-\{p\}}^G(H)$.

Remark. Doerk and Hawkes showed that when $H = G$, then $\mathbf{K}_\pi^G(G)$ is the $\mathcal{Z}^\pi$-radical of G [2, IX(2.9.a.2)]. However we shall consider $\mathbf{K}_\pi^G(H)$ when $H \in \mathcal{Z}^\pi$.

Lemma 4. *Let F be the Fitting subgroup of a finite solvable group G, and let π be a set of prime numbers.*

(a) *Suppose $H \leqslant G$. If $\tau \subseteq \pi$, then $\mathbf{K}_\pi^G(H) \leqslant \mathbf{K}_\tau^G(H)$.*
(b) *Suppose $H \leqslant G$ with $H \in \mathcal{Z}^\pi$. Then $H \leqslant \mathbf{K}_\pi^G(H)$.*
(c) *Suppose $F \leqslant H \leqslant H^* \leqslant G$ with $H, H^* \in \mathcal{Z}^\pi$. Then $\mathbf{K}_\pi^G(H) \leqslant \mathbf{K}_\pi^G(H^*)$.*
(d) *Suppose $F \leqslant H \leqslant H^* \leqslant G$ with $H, H^* \in \mathcal{Z}^\pi$. If $|H^* : H|$ is a π'-number, then $\mathbf{K}_\pi^G(H) = \mathbf{K}_\pi^G(H^*)$.*
(e) *Suppose $F \leqslant H \leqslant G$ with $H \in \mathcal{Z}^\pi$. If $H \leqslant L \leqslant \mathbf{K}_\pi^G(H)$, then $L \in \mathcal{Z}^\pi$. In particular $\mathbf{K}_\pi^G(H) \in \mathcal{Z}^\pi$, and if N_π is the $\mathcal{Z}^\pi$-radical of G, then $N_\pi = \mathbf{K}_\pi^G(N_\pi)$.*
(f) *Suppose $F \leqslant H \leqslant G$ with $H \in \mathcal{Z}^\pi$. Then $\operatorname{Soc}_\pi H = \operatorname{Soc}_\pi \mathbf{K}_\pi^G(H)$, and hence $\mathbf{N}_G(\mathbf{K}_\pi^G(H)) = \mathbf{N}_G(\operatorname{Soc}_\pi H)$ and $\mathbf{K}_\pi^G(\mathbf{K}_\pi^G(H)) = \mathbf{K}_\pi^G(H)$.*
(g) *Suppose $F \leqslant H \leqslant G$ with $H = \mathbf{K}_\pi^G(H)$, and take $L = \mathbf{N}_G(H)$ and $p \in \pi$. Then $p \nmid |\mathbf{K}_{\pi-p}^G(H) : H|$ if and only if $p \nmid |\mathbf{K}_{\pi-p}^L(H) : H|$.*

Proof. (a) This holds because $\operatorname{Soc}_\pi H \geqslant \operatorname{Soc}_\tau H$.

(b) This is a consequence of the definitions.

(c) This follows from Lemma 3(c).

(d) Similarly this follows from Lemma 3(d).

(e) Suppose $p \in \pi$ and let U be a minimal normal p-subgroup of L; we must deduce that $U \leqslant \mathbf{Z}(L)$. Take $P \in \operatorname{Syl}_p H$ and $V = \operatorname{Soc}_p F$, and note that $U \leqslant V$ by Lemma 1(d). Using Lemmas 2(b) and 1(c) we get $\mathbf{C}_U(P) \leqslant \mathbf{C}_V(P) = \operatorname{Soc}_p H \leqslant \mathbf{Z}(\mathbf{K}_\pi^G(H)) \leqslant \mathbf{Z}(L)$. Hence $U \cap \mathbf{Z}(L) \geqslant \mathbf{C}_U(P) \neq 1$ [2, A(5.5)], so it follows from the minimality of U that $U \leqslant \mathbf{Z}(L)$. The last equation holds because $\mathbf{K}_\pi^G(N_\pi)$ is a normal $\mathcal{Z}^\pi$-subgroup of G with $N_\pi \leqslant \mathbf{K}_\pi^G(N_\pi)$.

(f) Clearly $\operatorname{Soc}_\pi H \leqslant \operatorname{Soc}_\pi \mathbf{Z}(\mathbf{K}_\pi^G(H)) \leqslant \operatorname{Soc}_\pi \mathbf{K}_\pi^G(H)$. Conversely (b) implies that $H \leqslant \mathbf{K}_\pi^G(H)$, so $\operatorname{Soc}_\pi H \geqslant \operatorname{Soc}_\pi \mathbf{K}_\pi^G(H)$ by Lemma 3(c). This proves the first equation, and hence

$$\mathbf{N}_G\big(\mathbf{K}_\pi^G(H)\big) \leqslant \mathbf{N}_G\big(\operatorname{Soc}_\pi \mathbf{K}_\pi^G(H)\big) = \mathbf{N}_G(\operatorname{Soc}_\pi H) \leqslant \mathbf{N}_G\big(\mathbf{K}_\pi^G(H)\big).$$

It also follows that $\mathbf{K}_\pi^G(\mathbf{K}_\pi^G(H)) = \mathbf{C}_G(\operatorname{Soc}_\pi \mathbf{K}_\pi^G(H)) = \mathbf{C}_G(\operatorname{Soc}_\pi H) = \mathbf{K}_\pi^G(H)$.

(g) Clearly $\mathbf{K}_{\pi-p}^G(H) \geqslant \mathbf{K}_{\pi-p}^L(H)$, and it follows that if $p \nmid |\mathbf{K}_{\pi-p}^G(H) : H|$ then $p \nmid |\mathbf{K}_{\pi-p}^L(H) : H|$. Conversely suppose that p is a factor of $|\mathbf{K}_{\pi-p}^G(H) : H|$, and choose subgroups $P < P^*$ with $P \in \operatorname{Syl}_p H$ and $P^* \in \operatorname{Syl}_p \mathbf{K}_{\pi-p}^G(H)$. Take $P^0 = \mathbf{N}_{P^*}(P)$ and $V = \operatorname{Soc}_p F$, and note that $P^0 > P$ [2, A(8.3.c)]. Also P^0 normalizes $\mathbf{C}_V(P)$ and centralizes $\operatorname{Soc}_{\pi-p} H$. But $\mathbf{C}_V(P) = \operatorname{Soc}_p H$ by Lemma 2(b), so $P^0 \leqslant \mathbf{N}_G(\operatorname{Soc}_\pi H)$. Using (f) we deduce that $P^0 \leqslant \mathbf{N}_G(\mathbf{K}_\pi^G(H)) = \mathbf{N}_G(H) = L$, and hence $P^0 \leqslant \mathbf{K}_{\pi-p}^L(H)$. Thus p is a factor of $|\mathbf{K}_{\pi-p}^L(H) : H|$. $\square$

3. The subgroup $\mathbf{D}_\pi^G(\Sigma)$

Our first aim in this section is to prove Theorem 1 and its corollary, which are due to Blessenohl, Frantz and Lockett. We first give a modified definition of $\mathbf{D}_p^G(P)$, and use Lemma 3 to prove some of its properties.

Notation. Let N_p be the $\mathcal{Z}^p$-radical of a finite solvable group G (where p is a prime number). If P is a p-subgroup of G, define

$$\mathbf{D}_p^G(P) = \mathbf{K}_p^G(PN_p).$$

Remark. We shall show in Lemma 5(b) below that this is equivalent to the definition given by Doerk and Hawkes [2, IX(4.18)].

Lemma 5. *Let N_p be the $\mathcal{Z}^p$-radical of a finite solvable group G, and suppose P is a p-subgroup of G (where p is a prime number).*

(a) *Then $\mathbf{D}_p^G(P) \in \mathcal{Z}^p$.*
(b) *If $V_p = \mathrm{Soc}_p N_p$ and $U = \mathbf{C}_{V_p}(P)$, then $\mathbf{D}_p^G(P) = \mathbf{C}_G(U)$.*
(c) *Suppose $\mathrm{Soc}_p N_p \leqslant H \leqslant G$ with $H \in \mathcal{Z}^p$. If $P \cap H \in \mathrm{Syl}_p H$, then $H \leqslant \mathbf{D}_p^G(P)$.*

Proof. (a) Lemma 3(b) implies that $PN_p \in \mathcal{Z}^p$, and the result follows from Lemma 4(e).
(b) Lemma 3(b) also shows that $\mathrm{Soc}_p(PN_p) = U$, from which the result follows.
(c) As in (b), put $V_p = \mathrm{Soc}_p N_p$ and $U = \mathbf{C}_{V_p}(P)$. Then $U = \mathbf{C}_{V_p}(P) \leqslant \mathbf{C}_{V_p}(P \cap H) \leqslant \mathbf{Z}(H)$ by Lemma 2(a), and hence $H \leqslant \mathbf{C}_G(U) = \mathbf{D}_p^G(P)$, using (b). □

Theorem 1 *(Frantz, Lockett). (See [2, IX(4.19)].) Let G be a finite solvable group, and let p be a prime number. If $P \in \mathrm{Syl}_p G$, then $\mathbf{D}_p^G(P)$ is a $\mathcal{Z}^p$-injector of G.*

Proof. Let J be a $\mathcal{Z}^p$-injector of G, and choose P such that $P \cap J \in \mathrm{Syl}_p J$. Then $N_p \leqslant J \in \mathcal{Z}^p$, and it follows from Lemma 5(c) that $J \leqslant \mathbf{D}_p^G(P)$. But J is a maximal $\mathcal{Z}^p$-subgroup of G, so Lemma 5(a) implies that $J = \mathbf{D}_p^G(P)$. □

Corollary 1. *(See Blessenohl [1, (4.8)].) Let N_p be the $\mathcal{Z}^p$-radical of a finite solvable group G. If $\mathrm{Soc}_p N_p \leqslant H \leqslant G$ with $H \in \mathcal{Z}^p$, then H is contained in a $\mathcal{Z}^p$-injector of G.*

Proof. Choose $P \in \mathrm{Syl}_p G$ such that $P \cap H \in \mathrm{Syl}_p H$. Then Lemma 5(c) shows that $H \leqslant \mathbf{D}_p^G(P)$, so the result follows from Theorem 1. □

We now recall the definition of $\mathbf{D}_\pi^G(\Sigma)$, and in the next 2 lemmas we use Lemma 5 to prove some of its properties. We are then able to deduce Theorem 2.

Notation. Suppose Σ is a Sylow basis in a finite solvable group G. If p is a prime number and $\{P\} = \Sigma \cap \mathrm{Syl}_p G$, write $\mathbf{D}_p^G(\Sigma) = \mathbf{D}_p^G(P)$. If π is a set of prime numbers, define

$$\mathbf{D}_\pi^G(\Sigma) = \bigcap_{p \in \pi} \mathbf{D}_p^G(\Sigma).$$

Lemma 6. *Let Σ be a Sylow basis in a finite solvable group G.*

(a) *If p is a prime number, then $\Sigma \searrow \mathbf{D}_p^G(\Sigma)$.*
(b) *If π is a set of prime numbers, then $\Sigma \searrow \mathbf{D}_\pi^G(\Sigma)$.*

Proof. (a) Take $\{P\} = \Sigma \cap \mathrm{Syl}_p G$, and put $D = \mathbf{D}_p^G(P) = \mathbf{D}_p^G(\Sigma)$. Then there is a conjugate D^g such that $\Sigma \searrow D^g$ [2, I(4.16)], and in particular $P \cap D^g \in \mathrm{Syl}_p D^g$. But $D^g \in \mathcal{Z}^p$ by Lemma 5(a), and clearly $\mathrm{Soc}_p N_p \leqslant D^g$, so $D^g \leqslant D$ by Lemma 5(c). Therefore $\Sigma \searrow D^g = D$.
(b) This follows from (a) [2, I(4.22.a)]. □

Lemma 7. *Let N_π be the $\mathcal{Z}^\pi$-radical of a finite solvable group G (where π is a set of prime numbers), and suppose Σ is a Sylow basis in G.*

(a) *If $G \notin \mathcal{Z}^\pi$, then $\mathbf{D}_\pi^G(\Sigma) < G$.*
(b) *If $\mathrm{Soc}_\pi N_\pi \leqslant H \leqslant G$, with $H \in \mathcal{Z}^\pi$ and $\Sigma \searrow H$, then $H \leqslant \mathbf{D}_\pi^G(\Sigma)$.*

(c) *Suppose $\pi = \{p_1, p_2, \dots, p_m\}$. For each index j, take $\{P_j\} = \Sigma \cap \mathrm{Syl}_{p_j} G$, and suppose $\mathrm{Soc}_{p_j} N_\pi \leqslant W_j \trianglelefteq G$, where W_j is an elementary abelian p_j-group. Put $U_j = \mathbf{C}_{W_j}(P_j)$ $(1 \leqslant j \leqslant m)$, and take $U = U_1 \times U_2 \times \cdots \times U_m$. Then $\mathbf{D}_\pi^G(\Sigma) = \mathbf{C}_G(U)$.*

Proof. (a) Since $G \notin \mathcal{Z}^\pi = \bigcap_{p\in\pi} \mathcal{Z}^p$, there must be a prime number $p \in \pi$ such that $G \notin \mathcal{Z}^p$. Then $\mathbf{D}_\pi^G(\Sigma) \leqslant \mathbf{D}_p^G(\Sigma) < G$ by Lemma 5(a).

(b) Suppose $p \in \pi$, and let N_p be the $\mathcal{Z}^p$-radical of G. Then $N_\pi \leqslant N_p$, so it follows from Lemma 1(d) that $\mathrm{Soc}_p N_p \leqslant \mathrm{Soc}_p N_\pi \leqslant H$. Also $H \in \mathcal{Z}^\pi \subseteq \mathcal{Z}^p$ and $\Sigma \searrow H$, so $H \leqslant \mathbf{D}_p^G(\Sigma)$ by Lemma 5(c). Thus $H \leqslant \bigcap_{p\in\pi} \mathbf{D}_p^G(\Sigma) = \mathbf{D}_\pi^G(\Sigma)$.

(c) For each index j, let N_j be the $\mathcal{Z}^{p_j}$-radical of G, and put $H_j = P_j N_j$. Then $H_j \in \mathcal{Z}^{p_j}$ by Lemma 3(b), and $N_\pi \leqslant N_j \leqslant H_j$. Now Lemma 1(d) implies that $W_j \geqslant \mathrm{Soc}_{p_j} N_\pi \geqslant \mathrm{Soc}_{p_j} H_j$, so $\mathrm{Soc}_{p_j} H_j = U_j$ by Lemma 2(b). Therefore $\mathbf{D}_{p_j}^G(\Sigma) = \mathbf{C}_G(U_j)$ $(1 \leqslant j \leqslant m)$, from which the result follows. $\square$

Construction. Let Σ be a Sylow basis in a finite solvable group G, and let π be a set of prime numbers. Take $\mathbf{D}^0(\Sigma) = G$, and when $i > 0$, assume inductively that a subgroup $\mathbf{D}^{i-1}(\Sigma)$ has been constructed such that $\Sigma \searrow \mathbf{D}^{i-1}(\Sigma)$. Now define

$$\mathbf{D}^i(\Sigma) = \mathbf{D}_\pi^{\mathbf{D}^{i-1}(\Sigma)}\big(\Sigma \cap \mathbf{D}^{i-1}(\Sigma)\big).$$

It follows from Lemma 6(b) that $\Sigma \cap \mathbf{D}^{i-1}(\Sigma) \searrow \mathbf{D}^i(\Sigma)$, and hence $\Sigma \searrow \mathbf{D}^i(\Sigma)$, so the Construction can proceed.

Theorem 2. *Let G be a finite solvable group, and let π be a set of prime numbers. Choose a Sylow basis Σ in G, take $D_i = \mathbf{D}^i(\Sigma)$ as in the above Construction, and let N_i be the $\mathcal{Z}^\pi$-radical of D_i. Then there is an index k such that*

$$G = D_0 > D_1 > \cdots > D_k = N_k \geqslant N_{k-1} \geqslant \cdots \geqslant N_0,$$

and $D_k = \mathbf{D}^k(\Sigma)$ is a $\mathcal{Z}^\pi$-injector of G.

Proof. Let J be a $\mathcal{Z}^\pi$-injector of G, and choose the Sylow basis Σ such that $\Sigma \searrow J$ [2, I(4.16)]. We claim that for all indices $i \geqslant 0$

$$N_i \leqslant J \leqslant D_i.$$

Since $N_0 \leqslant J \leqslant D_0$, we can suppose $i > 0$ and assume inductively that $N_{i-1} \leqslant J \leqslant D_{i-1}$. Then $\mathrm{Soc}_\pi N_{i-1} \leqslant J$, and $\Sigma \cap D_{i-1} \searrow J$, so the hypotheses of Lemma 7(b) hold in D_{i-1}, and therefore $J \leqslant \mathbf{D}_\pi^{D_{i-1}}(\Sigma \cap D_{i-1}) = D_i$. Since J is a $\mathcal{Z}^\pi$-injector of G, it is also a $\mathcal{Z}^\pi$-injector of D_i, and hence $N_i \leqslant J \leqslant D_i$ as required.

By definition $D_i \leqslant D_{i-1}$, and we have proved that $N_{i-1} \leqslant J \leqslant D_i$. Hence N_{i-1} is a normal $\mathcal{Z}^\pi$-subgroup of D_i, and therefore

$$N_{i-1} \leqslant N_i \leqslant D_i \leqslant D_{i-1}.$$

Moreover the strict inclusion $N_i < D_i$ means that $D_i \notin \mathcal{Z}^\pi$, so Lemma 7(a) shows that

$$\text{if } N_i < D_i, \quad \text{then } D_{i+1} < D_i.$$

The result follows from the above inclusions. $\square$

Corollary 2a. *(See Blessenohl [2, IX(4.20)].) Let N_π be the $\mathcal{Z}^\pi$-radical of a finite solvable group G. If $\mathrm{Soc}_\pi N_\pi \leqslant H \leqslant G$ with $H \in \mathcal{Z}^\pi$, then H is contained in a $\mathcal{Z}^\pi$-injector of G.*

Proof. Choose a Sylow basis Σ such that $\Sigma \searrow H$, and using the notation of Theorem 2 put $V_i = \mathrm{Soc}_\pi N_i$. We claim that for all indices $i \geqslant 0$

$$V_i \leqslant H \leqslant D_i.$$

By hypothesis $V_0 \leqslant H \leqslant D_0$, so we can suppose $i > 0$ and assume inductively that $V_{i-1} \leqslant H \leqslant D_{i-1}$. Then $N_{i-1} \leqslant N_i \leqslant D_{i-1}$ by Theorem 2, and using Lemma 1(d) we get $V_i \leqslant V_{i-1} \leqslant H$. Moreover as in Theorem 2, it follows from Lemma 7(b) that $H \leqslant \mathbf{D}_\pi^{D_{i-1}}(\Sigma \cap D_{i-1}) = D_i$. This proves the claim, and therefore $H \leqslant D_k = \mathbf{D}^k(\Sigma)$, so the result follows from Theorem 2. □

Corollary 2b. *With the notation of Theorem 2, suppose $\pi = \{p_1, p_2, \ldots, p_m\}$, and let N_π be the $\mathcal{Z}^\pi$-radical of G. For each index j, take*

$$\{P_j\} = \Sigma \cap \mathrm{Syl}_{p_j} G, \qquad W_j = \mathrm{Soc}_{p_j} N_\pi,$$
$$P_{ij} = P_j \cap D_i, \qquad U_{ij} = \mathbf{C}_{W_j}(P_{ij}), \qquad U_i = U_{i1} \times U_{i2} \times \cdots \times U_{im}.$$

Then $D_{i+1} = \mathbf{C}_G(U_i)$, and if $V_i = \mathrm{Soc}_\pi N_i$ $(0 \leqslant i \leqslant k)$ then

$$U_0 < U_1 < \cdots < U_{k-1} \leqslant U_k = V_k \leqslant V_{k-1} \leqslant \cdots \leqslant V_0 = \mathrm{Soc}_\pi N_\pi.$$

Proof. Note that

$$\mathrm{Soc}_\pi N_\pi = W_1 \times W_2 \times \cdots \times W_m, \qquad P_{0j} \geqslant P_{1j} \geqslant \cdots \geqslant P_{kj},$$
$$U_{0j} \leqslant U_{1j} \leqslant \cdots \leqslant U_{kj} \leqslant W_j, \qquad U_0 \leqslant U_1 \leqslant \cdots \leqslant U_k \leqslant \mathrm{Soc}_\pi N_\pi.$$

Now $N_\pi \leqslant N_i$ as in Theorem 2, so W_j is an elementary abelian normal p_j-subgroup of N_i. Also Lemma 1(d) shows that $W_j \geqslant \mathrm{Soc}_{p_j} N_i$, and moreover $\{P_{ij}\} = (\Sigma \cap D_i) \cap \mathrm{Syl}_{p_j} D_i$. Then Lemma 7(c) implies that

$$D_{i+1} = \mathbf{C}_{D_i}(U_i).$$

In particular $D_1 = \mathbf{C}_{D_0}(U_0) = \mathbf{C}_G(U_0)$, so in proving that $D_{i+1} = \mathbf{C}_G(U_i)$, we may suppose $i > 0$ and assume inductively that $D_i = \mathbf{C}_G(U_{i-1})$. Then $U_{i-1} \leqslant U_i$, so $\mathbf{C}_G(U_i) \leqslant \mathbf{C}_G(U_{i-1}) = D_i$, and it follows that $\mathbf{C}_G(U_i) = \mathbf{C}_{D_i}(U_i) = D_{i+1}$, as required. Now the strict containment $D_i > D_{i+1}$ implies that $U_{i-1} < U_i$ $(1 \leqslant i < k)$.

Finally $N_\pi = N_0 \leqslant D_k$, so $W_j \geqslant \mathrm{Soc}_{p_j} D_k$ by Lemma 1(d). Also $D_k \in \mathcal{Z}^{p_j}$ by Theorem 2, and hence $U_{kj} = \mathbf{C}_{W_j}(P_{kj}) = \mathrm{Soc}_{p_j} D_k$ by Lemma 2(b). Thus $U_k = \mathrm{Soc}_\pi D_k = \mathrm{Soc}_\pi N_k = V_k$, while Lemma 1(d) implies that $V_i \leqslant V_{i-1}$ $(1 \leqslant i \leqslant k)$. □

Corollary 2b gives a procedure for calculating $\mathcal{Z}^\pi$-injectors, and we end this section by using it to show that the chain in Theorem 2 may have length 3. As we remarked in the Introduction, we do not know whether this length can be greater than 3, but we note that a group with this property must have a similar chain of elementary abelian normal π-subgroups.

Example 1. There is a finite solvable group G with a Sylow basis Σ, and a set π of prime numbers, such that

$$G = \mathbf{D}^0(\Sigma) > \mathbf{D}^1(\Sigma) > \mathbf{D}^2(\Sigma) > \mathbf{D}^3(\Sigma) = \mathbf{D}^4(\Sigma).$$

Proof. We can extend an example of Blessenohl [2, §4] as follows. Take the symmetric group $H = \mathbf{S}_4$, and put $P_1 = \langle(12)\rangle$, $P_2 = \langle(12)(34), (13)(24)\rangle$, $P = P_1P_2$ and $Q = \langle(123)\rangle$; then $P \in \mathrm{Syl}_2 H$ and $Q \in \mathrm{Syl}_3 H$. Next take $N = V_2 \times V_3$, where $V_2 = \langle u_1, u_2, u_3, u_4\rangle \cong \mathbf{C}_2^4$ and $V_3 = \langle v_1, v_2, v_3, v_4\rangle \cong \mathbf{C}_3^4$ are elementary abelian groups of order 2^4 and 3^4 respectively. Make H act on N by taking $u_i^\sigma = u_{i\sigma}$ and $v_i^\sigma = v_{i\sigma}$ ($\sigma \in H$), and form the corresponding semidirect product $G = HN$; then G can be regarded as the natural wreath product $\mathbf{C}_6 \wr \mathbf{S}_4$. Finally put $\pi = \{2, 3\}$, and take

$$\Sigma = \{1, PV_2, QV_3\}, \qquad u_0 = u_1u_2, \qquad v_0 = v_1v_2v_3, \qquad u_\infty = u_1u_2u_3u_4.$$

Note that Σ is a Sylow basis in G.

Now N is the $\mathcal{Z}^\pi$-radical of G, with $V_2 = \mathrm{Soc}_2 N$ and $V_3 = \mathrm{Soc}_3 N$. Moreover $\mathbf{C}_{V_2}(PV_2) = \langle u_\infty\rangle$ and $\mathbf{C}_{V_3}(QV_3) = \langle v_0, v_4\rangle$, and as in Corollary 2b we get

$$\mathbf{D}_2^G(\Sigma) = \mathbf{C}_G(u_\infty) = G, \qquad \mathbf{D}_3^G(\Sigma) = \mathbf{C}_G\big(\langle v_0, v_4\rangle\big) = P_1QN.$$

Therefore $D_1 = \mathbf{D}^1(\Sigma) = G \cap P_1QN = P_1QN$, and $\Sigma \cap D_1 = \{1, P_1V_2, QV_3\}$. Repeating the process we get $\mathbf{C}_{V_2}(P_1V_2) = \langle u_0, u_3, u_4\rangle$ and $\mathbf{C}_{V_3}(QV_3) = \langle v_0, v_4\rangle$, and hence

$$\mathbf{D}_2^{D_1}(\Sigma \cap D_1) = \mathbf{C}_{D_1}\big(\langle u_0, u_3, u_4\rangle\big) = P_1N, \qquad \mathbf{D}_3^{D_1}(\Sigma \cap D_1) = \mathbf{C}_{D_1}\big(\langle v_0, v_4\rangle\big) = D_1.$$

Therefore $D_2 = \mathbf{D}^2(\Sigma) = P_1N \cap D_1 = P_1N$, and $\Sigma \cap D_2 = \{1, P_1V_2, V_3\}$. Now $\mathbf{C}_{V_2}(P_1V_2) = \langle u_0, u_3, u_4\rangle$ and $\mathbf{C}_{V_3}(V_3) = V_3$, and hence

$$\mathbf{D}_2^{D_2}(\Sigma \cap D_2) = \mathbf{C}_{D_2}\big(\langle u_0, u_3, u_4\rangle\big) = D_2, \qquad \mathbf{D}_3^{D_2}(\Sigma \cap D_2) = \mathbf{C}_{D_2}(V_3) = N.$$

Therefore $\mathbf{D}^3(\Sigma) = D_2 \cap N = N \in \mathcal{Z}^\pi$, and hence $\mathbf{D}^3(\Sigma) = \mathbf{D}^4(\Sigma)$. $\square$

4. The subgroup $\mathbf{M}_\pi^G(H)$

This section contains the proof of Theorem 3, which follows at once from Lemmas 8 and 9; these lemmas depend on the properties of $\mathbf{K}_\pi^G(H)$ recorded in Lemma 4. We first recall the definition of $\mathbf{M}_\pi^G(H)$.

Construction. Let N_π be the $\mathcal{Z}^\pi$-radical of a finite solvable group G (where π is a set of prime numbers) and suppose $N_\pi \leqslant H \leqslant G$ and $H \in \mathcal{Z}^\pi$. For each prime number $p \in \pi$, choose a subgroup $S_p^* \in \mathrm{Syl}_p \mathbf{K}_{\pi-p}^G(H)$ such that $S_p^* \cap H \in \mathrm{Syl}_p H$. Finally define

$$H^* = \big\langle H, S_p^*\colon\ p \in \pi\big\rangle, \qquad \mathbf{M}_\pi^G(H) = \mathbf{K}_\pi^G\big(H^*\big).$$

Remark. The above Construction may depend on the choice of the Sylow subgroups S_p^*, but we shall see in Lemma 8(b) below that the truth or falsity of the equation $H = \mathbf{M}_\pi^G(H)$ is independent of these choices.

Lemma 8. *Let N_π be the $\mathcal{Z}^\pi$-radical of a finite solvable group G (where π is a set of prime numbers), and suppose $N_\pi \leqslant H \leqslant G$ with $H \in \mathcal{Z}^\pi$.*

(a) *Then* $H \leqslant \mathbf{K}^G_\pi(H) \leqslant \mathbf{M}^G_\pi(H)$ *and* $H^* \leqslant \mathbf{M}^G_\pi(H)$, *with* $H^* \in \mathcal{Z}^\pi$ *and* $\mathbf{M}^G_\pi(H) \in \mathcal{Z}^\pi$.

(b) *Moreover* $H = \mathbf{M}^G_\pi(H)$ *if and only if* H *satisfies the condition* (K_0) *below, together with the conditions* (K_p) *for all prime numbers* $p \in \pi$:

$$(\mathrm{K}_0) \quad H = \mathbf{K}^G_\pi(H); \qquad (\mathrm{K}_p) \quad p \nmid \left|\mathbf{K}^G_{\pi-p}(H) : H\right|.$$

(c) *Suppose* $H \leqslant H^0 \leqslant G$ *with* $H^0 \in \mathcal{Z}^\pi$ *and* $|H^0 : H| = q^k$ *(where* q *is a prime number). Then the Construction can be carried out so that* $H^0 \leqslant \mathbf{M}^G_\pi(H)$.

(d) *Suppose* $\pi = \{p_1, p_2, \ldots, p_m\}$. *For each index* j, *if* $P_j \in \mathrm{Syl}_{p_j} H$, *there are* p_j*-subgroups* $P_j^\infty \geqslant P_j^* \geqslant P_j$, *which can be constructed by taking*

$$V_j = \mathrm{Soc}_{p_j} N_\pi, \qquad U_j = \mathbf{C}_{V_j}(P_j).$$

$$W_j = U_1 \times \cdots \times U_{j-1} \times U_{j+1} \times \cdots \times U_m,$$

$$P_j^* \in \mathrm{Syl}_{p_j} \mathbf{C}_G(W_j), \qquad U_j^* = \mathbf{C}_{V_j}\big(P_j^*\big), \qquad U^* = U_1^* \times U_2^* \times \cdots \times U_m^*,$$

$$P_j^\infty \in \mathrm{Syl}_{p_j} \mathbf{M}^G_\pi(H), \qquad U_j^\infty = \mathbf{C}_{V_j}\big(P_j^\infty\big).$$

These definitions imply that $\mathrm{Soc}_{p_j} H = U_j$ *and* $\mathrm{Soc}_{p_j} \mathbf{M}^G_\pi(H) = U_j^* = U_j^\infty$, *with* $\mathbf{M}^G_\pi(H) = \mathbf{C}_G(U^*)$.

Proof. (a) Suppose $p \in \pi$, take $H_p = \langle H, S_p^* \rangle$, and note that $H_p \in \mathcal{Z}^p$ by Lemma 3(a). If $q \in \pi - p$ then $S_q^* \leqslant \mathbf{K}^G_{\pi-q}(H) \leqslant \mathbf{K}^G_p(H)$ by Lemma 4(a), and $\mathbf{K}^G_p(H) \leqslant \mathbf{K}^G_p(H_p)$ by Lemma 4(c). This proves that $H^* = \langle H_p, S_q^*\colon\ q \in \pi - p \rangle \leqslant \mathbf{K}^G_p(H_p)$, and it follows from Lemma 4(e) that $H^* \in \mathcal{Z}^p$. Therefore $H^* \in \bigcap_{p\in\pi} \mathcal{Z}^p = \mathcal{Z}^\pi$. Applying Lemma 4(c) again we get $\mathbf{K}^G_\pi(H) \leqslant \mathbf{K}^G_\pi(H^*) = \mathbf{M}^G_\pi(H)$, and using Lemma 4(e), we also deduce that $\mathbf{M}^G_\pi(H) = \mathbf{K}^G_\pi(H^*) \in \mathcal{Z}^\pi$.

(b) Assuming that $H = \mathbf{M}^G_\pi(H)$, it follows from (a) that (K_0) holds. Moreover if $p \in \pi$ then $S_p^* \leqslant \mathbf{M}^G_\pi(H) = H$, so $S_p^* \in \mathrm{Syl}_p H$, which proves (K_p). Conversely suppose that (K_p) holds when $p \in \{0\} \cup \pi$. If $p \in \pi$ it follows from (K_p) that $S_p^* \in \mathrm{Syl}_p H$, and hence $H^* = \langle H, S_p^*\colon\ p \in \pi \rangle = H$. Now (K_0) implies that $H = \mathbf{K}^G_\pi(H) = \mathbf{K}^G_\pi(H^*) = \mathbf{M}^G_\pi(H)$.

(c) If $q \notin \pi$, then Lemma 4(d) shows that $H^0 \leqslant \mathbf{K}^G_\pi(H^0) = \mathbf{K}^G_\pi(H) \leqslant \mathbf{M}^G_\pi(H)$, as required. On the other hand, if $q \in \pi$ then $H^0 \leqslant \mathbf{K}^G_{\pi-q}(H^0) = \mathbf{K}^G_{\pi-q}(H)$ by Lemma 4(d). Hence we can choose subgroups $S_q \leqslant S_q^0 \leqslant S_q^*$ with $S_q \in \mathrm{Syl}_q H$, $S_q^0 \in \mathrm{Syl}_q H^0$ and $S_q^* \in \mathrm{Syl}_q \mathbf{K}^G_{\pi-q}(H)$. Then $S_q^0 \leqslant H^* \leqslant \mathbf{M}^G_\pi(H)$, and so $H^0 = HS_q^0 \leqslant \mathbf{M}^G_\pi(H)$.

(d) Note that $V_j \geqslant \mathrm{Soc}_{p_j} H$ by Lemma 1(d), and $p_j \nmid |H : P_j|$, so Lemma 2(b) implies that $\mathrm{Soc}_{p_j} H = U_j$. Also $W_j = \mathrm{Soc}_{\pi-p_j} H \leqslant \mathrm{Soc}_\pi H$, so $H \leqslant \mathbf{C}_G(W_j)$. We can therefore choose a subgroup $P_j^* \in \mathrm{Syl}_{p_j} \mathbf{C}_G(W_j)$ with $P_j^* \geqslant P_j$. Put $H^* = \langle H, P_1^*, P_2^*, \ldots, P_m^* \rangle$ as in the Construction of $\mathbf{M}^G_\pi(H)$, and choose subgroups $P_j^0 \in \mathrm{Syl}_{p_j} H^*$ and $P_j^\infty \in \mathrm{Syl}_{p_j} \mathbf{M}^G_\pi(H)$ such that $P_j^* \leqslant P_j^0 \leqslant P_j^\infty$. Finally put

$$U_j^0 = \mathbf{C}_{V_j}\big(P_j^0\big), \qquad U^0 = U_1^0 \times U_2^0 \times \cdots \times U_m^0.$$

As before, Lemma 2(b) implies that $\mathrm{Soc}_{p_j} H^* = U_j^0$ and $\mathrm{Soc}_{p_j} \mathbf{M}^G_\pi(H) = U_j^\infty$. But $\mathrm{Soc}_{p_j} H^* = \mathrm{Soc}_{p_j} \mathbf{M}^G_\pi(H)$ by Lemma 4(f), so these equations show that $\mathrm{Soc}_{p_j} \mathbf{M}^G_\pi(H) = U_j^0 = U_j^\infty$. Also $\mathbf{M}^G_\pi(H) = \mathbf{C}_G(\mathrm{Soc}_\pi H^*) = \mathbf{C}_G(U^0)$, so it suffices to show that $U_j^0 = U_j^*$.

Clearly $U_j^0 = \mathbf{C}_{V_j}(P_j^0) \leqslant \mathbf{C}_{V_j}(P_j^*) = U_j^*$. To prove the converse, note that $U_j^* = \mathbf{C}_{V_j}(P_j^*) \leqslant \mathbf{C}_{V_j}(P_j) = U_j = \mathrm{Soc}_{p_j} H$, so U_j^* is centralized by H. Also $U_j^* = \mathbf{C}_{V_j}(P_j^*)$ is centralized by P_j^*. Finally if $i \neq j$, then $U_j^* \leqslant U_j \leqslant W_i$, and hence U_j^* is centralized by P_i^*. This proves that $U_j^* \leqslant \mathrm{Soc}_{p_j} \mathbf{Z}(H^*)$, and therefore $U_j^* \leqslant \mathrm{Soc}_{p_j} H^* = U_j^0$. $\square$

Lemma 9. *Let N_π be the $\mathcal{Z}^\pi$-radical of a finite solvable group G (where π is a set of prime numbers), and suppose $N_\pi \leqslant H \leqslant G$ with $H \in \mathcal{Z}^\pi$.*

(a) *Let G^0 be a maximal normal subgroup of G, such that $N_\pi \leqslant G^0 < G$, and take $H^0 = H \cap G^0$. If $H = \mathbf{M}_\pi^G(H)$, then $H^0 = \mathbf{M}_\pi^{G^0}(H^0)$.*
(b) *If $H = \mathbf{M}_\pi^G(H)$, then H is a $\mathcal{Z}^\pi$-injector of G.*

Proof. (a) Note that that N_π is still the $\mathcal{Z}^\pi$-radical of G^0, so by Lemma 8(b), we must show that if $p \in \{0\} \cup \pi$, then (K_p) holds for H^0 in G^0. Now $\mathbf{K}_\pi^G(H^0) \leqslant \mathbf{K}_\pi^G(H)$ by Lemma 4(c), and by applying Lemma 8(a) and (b), we deduce that $H^0 \leqslant \mathbf{K}_\pi^{G^0}(H^0) = G^0 \cap \mathbf{K}_\pi^G(H^0) \leqslant G^0 \cap \mathbf{K}_\pi^G(H) = G^0 \cap H = H^0$. This implies that (K_0) still holds for H^0 in G^0. Next consider a prime number $p \in \pi$, and suppose first that $H \leqslant G^0$. Then $\mathbf{K}_{\pi-p}^{G^0}(H) \leqslant \mathbf{K}_{\pi-p}^G(H)$, so $|\mathbf{K}_{\pi-p}^{G^0}(H) : H|$ is a factor of $|\mathbf{K}_{\pi-p}^G(H) : H|$, and hence (K_p) also continues to hold for H^0 in G^0. We may now assume that $H \nleqslant G^0$.

Put $q = |G/G^0|$ (where q is a prime number), and suppose first that $p \neq q$. Then $\mathbf{K}_{\pi-p}^{G^0}(H^0) \leqslant \mathbf{K}_{\pi-p}^G(H^0) \leqslant \mathbf{K}_{\pi-p}^G(H)$ by Lemma 4(c), while $|H| = q \cdot |H^0|$. Hence $|\mathbf{K}_{\pi-p}^{G^0}(H^0) : H^0|$ is a factor of the product $q \cdot |\mathbf{K}_{\pi-p}^G(H) : H|$, which implies that H^0 satisfies (K_p) in G^0. We may now assume that $p = q$, and it remains to verify (K_q). Now $H \leqslant \mathbf{K}_{\pi-q}^G(H) = \mathbf{K}_{\pi-q}^G(H^0)$ by Lemma 4(d), and hence $\mathbf{K}_{\pi-q}^G(H^0) = H(G^0 \cap \mathbf{K}_{\pi-q}^G(H^0)) = H\mathbf{K}_{\pi-q}^{G^0}(H^0)$. Moreover $H^0 \leqslant H \cap \mathbf{K}_{\pi-q}^{G^0}(H^0) \leqslant H \cap G^0 = H^0$, and hence $H \cap \mathbf{K}_{\pi-q}^{G^0}(H^0) = H^0$. We deduce that

$$\begin{aligned}
\left|\mathbf{K}_{\pi-q}^G(H) : H\right| &= \left|H\mathbf{K}_{\pi-q}^{G^0}\left(H^0\right) : H\right| \\
&= \left|\mathbf{K}_{\pi-q}^{G^0}\left(H^0\right) : H \cap \mathbf{K}_{\pi-q}^{G^0}\left(H^0\right)\right| \\
&= \left|\mathbf{K}_{\pi-q}^{G^0}\left(H^0\right) : H^0\right|,
\end{aligned}$$

which implies that (K_q) still holds for H^0 in G^0.

(b) If $N_\pi = G$ then the result is clear, so we may suppose that $N_\pi < G$ and use induction on $|G/N_\pi|$. Choose a maximal normal subgroup $G^0 \trianglelefteq G$ such that $N_\pi \leqslant G^0 < G$, and suppose first that $H \leqslant G^0$. Then (a) implies that $H = \mathbf{M}_\pi^{G^0}(H)$, so H is a $\mathcal{Z}^\pi$-injector of G^0 by the induction hypothesis, and it suffices to show that H is a maximal $\mathcal{Z}^\pi$-subgroup of G [2, VIII(2.10)]. But if $H < H^* \leqslant G$ with $H^* \in \mathcal{Z}^\pi$, then $H = H^* \cap G^0 \trianglelefteq H^*$; it now follows from Lemma 8(c) that $H^* \leqslant \mathbf{M}_\pi^G(H)$, which contradicts our hypothesis. We may therefore assume that $H \nleqslant G^0$, and put $H^0 = H \cap G^0$. As before we deduce from (a) and the induction hypothesis that H^0 is a $\mathcal{Z}^\pi$-injector of G^0. Since H covers G/G^0, this implies that H is a $\mathcal{Z}^\pi$-injector of G [2, VIII(2.11)]. □

Construction. Let G be a finite solvable group, and take $\mathbf{M}^0(G) = N_\pi$ to be the $\mathcal{Z}^\pi$-radical of G (where π is a set of prime numbers). For each index $i > 0$, assume inductively that a subgroup $\mathbf{M}^{i-1}(G)$ has been constructed such that $N_\pi \leqslant \mathbf{M}^{i-1}(G) \leqslant G$ and $\mathbf{M}^{i-1}(G) \in \mathcal{Z}^\pi$. Now define

$$\mathbf{M}^i(G) = \mathbf{M}_\pi^G\left(\mathbf{M}^{i-1}(G)\right).$$

It follows from Lemma 8(a) that $N_\pi \leqslant \mathbf{M}^i(G) \leqslant G$ and $\mathbf{M}^i(G) \in \mathcal{Z}^\pi$, so the Construction can proceed.

Theorem 3. *Let N_π be the $\mathcal{Z}^\pi$-radical of a finite solvable group G (where π is a set of prime numbers), and take $\mathbf{M}^i(G)$ as in the above Construction. Then there is an index k such that*

$$N_\pi = \mathbf{M}^0(G) < \mathbf{M}^1(G) < \cdots < \mathbf{M}^k(G) = \mathbf{M}^{k+1}(G),$$

and $\mathbf{M}^k(G)$ is a $\mathcal{Z}^\pi$-injector of G.

Proof. This follows from Lemmas 8(a) and 9(b). □

Corollary 3. *With the notation of Theorem* 3*, suppose* $\pi = \{p_1, p_2, \ldots, p_m\}$*. For each index* j*, there are* p_j*-subgroups*

$$P_{0j} \leqslant P^*_{0j} \leqslant P_{1j} \leqslant P^*_{1j} \leqslant \cdots \leqslant P_{k-1,j} \leqslant P^*_{k-1,j} \leqslant P_{kj},$$

which can be constructed by taking

$$P_{ij} \in \mathrm{Syl}_{p_j} \mathbf{M}^i(G), \qquad V_j = \mathrm{Soc}_{p_j} N_\pi, \qquad U_{ij} = \mathbf{C}_{V_j}(P_{ij}),$$
$$W_{ij} = U_{i1} \times \cdots \times U_{i,j-1} \times U_{i,j+1} \times \cdots \times U_{im},$$
$$P^*_{ij} \in \mathrm{Syl}_{p_j} \mathbf{C}_G(W_{ij}), \qquad U^*_{ij} = \mathbf{C}_{V_j}\big(P^*_{ij}\big), \qquad U^*_i = U^*_{i1} \times U^*_{i2} \times \cdots \times U^*_{im}.$$

Then $\mathbf{M}^{i+1}(G) = \mathbf{C}_G(U^*_i)$ *and* $\mathrm{Soc}_{p_j} \mathbf{M}^{i+1}(G) = U^*_{ij} = U_{i+1,j}$*, with*

$$\mathrm{Soc}_\pi N_\pi \geqslant U^*_0 > U^*_1 > \cdots > U^*_{k-1} \geqslant U^*_k.$$

Proof. For each index j, arguing by induction on i, we can use Lemma 8(d) to construct the subgroups $P_{ij} \leqslant P^*_{ij} \leqslant P_{i+1,j}$. Also $\mathbf{M}^{i+1}(G) = \mathbf{C}_G(U^*_i)$, so the strict inclusion $\mathbf{M}^i(G) < \mathbf{M}^{i+1}(G)$ implies that $U^*_{i-1} > U^*_i$ $(1 \leqslant i < k)$. □

5. The subgroup $\mathbf{E}^{\Sigma}_{\pi}(H)$

We now consider Theorem 4. Most of the proof is carried out in Lemma 11, but we first record some properties of pronormality. Recall that a subgroup $H \leqslant G$ is said to be *pronormal* in G if, for every element $g \in G$, H and H^g are conjugate in $\langle H, H^g \rangle$; in this case we write $H \operatorname{pr} G$. Moreover H is *normally embedded* in G if, for every prime number p and every subgroup $P \in \mathrm{Syl}_p H$, P is also a Sylow p-subgroup of its normal closure $\langle P^G \rangle = \langle P^g \colon g \in G \rangle$.

Lemma 10. *Let* G *be a finite solvable group, and suppose* $H \leqslant G$.

- (a) (Lockett [2, I(7.8)]) *Suppose* Σ *is a Sylow basis in* G*. If* $H_1, H_2, \ldots, H_n$ *are normally embedded subgroups of* G *with* $\Sigma \searrow H_i$ *for all indices* i*, then* $H_iH_j = H_jH_i$ *for all indices* i *and* j*, and the product* $H_0 = \prod_{i=1}^n H_i$ *is normally embedded in* G*, with* $\Sigma \searrow H_0$.
- (b) (Chambers [2, I(7.2.b) and (6.14)]) *If* H *is normally embedded in* G*, then* $H \operatorname{pr} G$.
- (c) (Mann [2, I(6.6)]) *Moreover* $H \operatorname{pr} G$ *if and only if the following condition holds*:
  (P) *if* Σ *is a Sylow basis in* G*, and* $g \in G$ *with* $\Sigma, \Sigma^g \searrow H$*, then* $g \in \mathbf{N}_G(H)$.
- (d) (Lockett [2, I(6.8)]) *Let* Σ *be a Sylow basis in* G*, and suppose* $\mathbf{N}_G(H) \leqslant L \leqslant G$*. If* $H \operatorname{pr} G$ *and* $\Sigma \searrow H$*, then* $\Sigma \searrow L$.
- (e) (P. Hall) *If* $H \operatorname{pr} G$ *and* $H \trianglelefteqslant H^* \operatorname{pr} \mathbf{N}_G(H)$*, then* $H^* \operatorname{pr} G$.
- (f) *Let* Σ *be a Sylow basis in* G*, and suppose* $H \in \mathcal{Z}^\pi$ *(where* π *is a set of prime numbers). If* $H \operatorname{pr} G$ *and* $\Sigma \searrow H$*, then* $\mathbf{K}^G_\pi(H) \operatorname{pr} G$ *and* $\Sigma \searrow \mathbf{K}^G_\pi(H)$.

Proof. The statements (a), (b) and (c) are proved in the given references.

(d) By extending $\Sigma \cap H$, we can choose a Sylow basis Σ^g in G such that $\Sigma^g \cap H = \Sigma \cap H$ and $\Sigma^g \searrow L$ [2, I(4.16)]. Then (c) implies that $g \in \mathbf{N}_G(H) \leqslant L$, and hence $\Sigma \searrow L^{g^{-1}} = L$.

(e) Suppose that $g \in G$, and that $\Sigma, \Sigma^g \searrow H^*$; by (c) it suffices to deduce that $g \in N_G(H^*)$. Put $L = \mathbf{N}_G(H)$. Now $H \trianglelefteqslant H^*$, so the hypothesis implies that $\Sigma, \Sigma^g \searrow H$ [2, I(3.2.c)], and hence $g \in L$ by (c). Also $\Sigma, \Sigma^g \searrow L$ by (d), and therefore in L we have Sylow bases $\Sigma \cap L, (\Sigma \cap L)^g \searrow H^*$. But $H^* \operatorname{pr} L$ so it follows from (c) that $g \in \mathbf{N}_L(H^*) \leqslant \mathbf{N}_G(H^*)$.

(f) Suppose $g \in G$; since $H \operatorname{pr} G$, there is an element $x \in \langle H, H^g \rangle$ such that $H^x = H^g$. But $H \in \mathcal{Z}^\pi$, and hence $x \in \langle H, H^g \rangle \leqslant \langle \mathbf{K}_\pi^G(H), \mathbf{K}_\pi^G(H)^g \rangle$ and $\mathbf{K}_\pi^G(H)^x = \mathbf{K}_\pi^G(H^x) = \mathbf{K}_\pi^G(H^g) = \mathbf{K}_\pi^G(H)^g$, which proves that $\mathbf{K}_\pi^G(H) \operatorname{pr} G$. Note next that $\mathbf{N}_G(H) \leqslant \mathbf{N}_G(\operatorname{Soc}_\pi H) \leqslant \mathbf{N}_G(\mathbf{K}_\pi^G(H))$, so it follows from (d) that $\Sigma \searrow \mathbf{N}_G(\mathbf{K}_\pi^G(H))$. This implies that $\Sigma \searrow \mathbf{K}_\pi^G(H)$. $\square$

Construction. Let Σ be a Sylow basis in a finite solvable group G, and let N_π be the $\mathcal{Z}^\pi$-radical of G (where π is a set of prime numbers). Suppose $N_\pi \leqslant H \leqslant G$, with $H \in \mathcal{Z}^\pi$ and $\Sigma \searrow H$, such that $H \operatorname{pr} G$ and $H = \mathbf{K}_\pi^G(H)$. Put $L = \mathbf{N}_G(H)$, and for each prime number $p \in \pi$, take $\{S_p^\infty\} = \Sigma \cap \operatorname{Syl}_p G$, and put $S_p = S_p^\infty \cap H$ and $S_p^0 = S_p^\infty \cap \mathbf{K}_{\pi-p}^L(H)$. Note that $S_p \in \operatorname{Syl}_p H$ and $S_p^0 \in \operatorname{Syl}_p \mathbf{K}_{\pi-p}^L(H)$ by Lemma 10(f) and (d), and define

$$H^0 = \langle H, S_p^0\colon\ p \in \pi \rangle, \qquad \mathbf{E}_\pi^\Sigma(H) = \mathbf{K}_\pi^G\bigl(H^0\bigr).$$

Lemma 11. *Let N_π be the $\mathcal{Z}^\pi$-radical of a finite solvable group G (where π is a set of prime numbers), and let Σ be a Sylow basis in G. Suppose $N_\pi \leqslant H \leqslant G$, with $H \in \mathcal{Z}^\pi$ and $\Sigma \searrow H$, such that $H \operatorname{pr} G$ and $H = \mathbf{K}_\pi^G(H)$. Put $L = \mathbf{N}_G(H)$, and perform the above Construction.*

(a) *Then $H^0 = H \cdot \prod_{p \in \pi} S_p^0$ and $H^0 \in \mathcal{Z}^\pi$, with $\Sigma \searrow H^0$ and $H^0 \operatorname{pr} G$.*
(b) *The Construction of $\mathbf{M}_\pi^G(H)$ can be carried out in such a way that $H \leqslant \mathbf{E}_\pi^\Sigma(H) \leqslant \mathbf{M}_\pi^G(H)$. Also $\mathbf{E}_\pi^\Sigma(H) \in \mathcal{Z}^\pi$ and $\Sigma \searrow \mathbf{E}_\pi^\Sigma(H)$, with $\mathbf{E}_\pi^\Sigma(H) \operatorname{pr} G$ and $\mathbf{E}_\pi^\Sigma(H) = \mathbf{K}_\pi^G(\mathbf{E}_\pi^\Sigma(H))$.*
(c) *Moreover $H = \mathbf{E}_\pi^\Sigma(H)$ if and only if the following condition holds for all prime numbers $p \in \pi$:*

$$(\mathrm{L}_p) \quad p \nmid \bigl|\mathbf{K}_{\pi-p}^L(H) : H\bigr|.$$

(d) *Hence $H = \mathbf{E}_\pi^\Sigma(H)$ if and only if $H = \mathbf{M}_\pi^G(H)$.*
(e) *Suppose $\pi = \{p_1, p_2, \dots, p_m\}$. For each index j, take*

$$\begin{gathered}
\bigl\{P_j^\infty\bigr\} = \Sigma \cap \operatorname{Syl}_{p_j} G, \qquad V_j = \operatorname{Soc}_{p_j} N_\pi, \\
P_j = P_j^\infty \cap H, \qquad U_j = \mathbf{C}_{V_j}(P_j), \\
W_j = U_1 \times \cdots \times U_{j-1} \times U_{j+1} \times \cdots \times U_m, \\
P_j^0 = P_j^\infty \cap \mathbf{C}_L(W_j), \qquad U_j^0 = \mathbf{C}_{V_j}\bigl(P_j^0\bigr), \qquad U^0 = U_1^0 \times U_2^0 \times \cdots \times U_m^0, \\
P_j^* = P_j^\infty \cap \mathbf{E}_\pi^\Sigma(H), \qquad U_j^* = \mathbf{C}_{V_j}\bigl(P_j^*\bigr).
\end{gathered}$$

These definitions imply that $\operatorname{Soc}_{p_j} H = U_j$ and $\operatorname{Soc}_{p_j} \mathbf{E}_\pi^\Sigma(H) = U_j^0 = U_j^*$, with $\mathbf{E}_\pi^\Sigma(H) = \mathbf{C}_G(U^0)$.

Proof. (a) Note that $H \trianglelefteqslant L$ and $\Sigma \searrow H$. Moreover for each prime number $p \in \pi$, $S_p^0 \in \operatorname{Syl}_p \mathbf{K}_{\pi-p}^L(H)$ with $\mathbf{K}_{\pi-p}^L(H) \trianglelefteqslant L$, so S_p^0 is normally embedded in L. Also $\Sigma \searrow S_p^0$, so it follows from Lemma 10(a) that $H^0 = H \cdot \prod_{p \in \pi} S_p^0$ is normally embedded in L, and that $\Sigma \searrow H^0$. Hence $H^0 \operatorname{pr} L$ by Lemma 10(b), and therefore $H^0 \operatorname{pr} G$ by Lemma 10(e).

Finally suppose $p \in \pi$, take $H_p = HS_p^0$, and note that $H_p \in \mathcal{Z}^p$ by Lemma 3(a). If $q \in \pi - p$ then $S_q^0 \leqslant \mathbf{K}_{\pi-q}^L(H) \leqslant \mathbf{K}_p^L(H)$ by Lemma 4(a), and $\mathbf{K}_p^L(H) \leqslant \mathbf{K}_p^L(H_p)$ by Lemma 4(c). Thus $H^0 = H_p \cdot \prod_{q \in \pi - p} S_q^0 \leqslant \mathbf{K}_p^L(H_p)$, and it follows from Lemma 4(e) that $H^0 \in \mathcal{Z}^p$. Therefore $H^0 \in \bigcap_{p \in \pi} \mathcal{Z}^p = \mathcal{Z}^\pi$.

(b) If $p \in \pi$, then $\mathbf{K}_{\pi-p}^L(H) \leqslant \mathbf{K}_{\pi-p}^G(H)$, so we can choose subgroups $S_p^0 \leqslant S_p^*$ with $S_p^0 \in \operatorname{Syl}_p \mathbf{K}_{\pi-p}^L(H)$ and $S_p^* \in \operatorname{Syl}_p \mathbf{K}_{\pi-p}^G(H)$. Then $H^0 = H \cdot \prod_{p \in \pi} S_p^0 \leqslant \langle H, S_p^*\colon\ p \in \pi \rangle = H^*$, and it follows from Lemma 4(c) that $\mathbf{E}_\pi^\Sigma(H) = \mathbf{K}_\pi^G(H^0) \leqslant \mathbf{K}_\pi^G(H^*) = \mathbf{M}_\pi^G(H)$. Moreover $\mathbf{E}_\pi^\Sigma(H) = \mathbf{K}_\pi^G(\mathbf{E}_\pi^\Sigma(H))$ by Lemma 4(f). From (a) we get $\Sigma \searrow H^0$ and $H^0 \operatorname{pr} G$, so Lemma 10(f) implies that $\Sigma \searrow \mathbf{E}_\pi^\Sigma(H)$

and $\mathbf{E}_\pi^\Sigma(H)$ pr G. Finally (a) also shows that $H^0 \in \mathcal{Z}^\pi$, and therefore $\mathbf{E}_\pi^\Sigma(H) = \mathbf{K}_\pi^G(H^0) \in \mathcal{Z}^\pi$ by Lemma 4(e).

(c) If $H = \mathbf{E}_\pi^\Sigma(H)$ and $p \in \pi$, then $S_p^0 \leqslant \mathbf{E}_\pi^\Sigma(H) = H$, so $S_p^0 \in \mathrm{Syl}_p H$ which proves (L_p). Conversely if (L_p) holds for all prime numbers $p \in \pi$, then $S_p = S_p^0$, so $H = H^0$ and hence $H = \mathbf{K}_\pi^G(H) = \mathbf{K}_\pi^G(H^0) = \mathbf{E}_\pi^\Sigma(H)$.

(d) This follows from (b) and (c), together with Lemmas 8(b) and 4(g).

(e) We can copy the proof of Lemma 8(d) as follows. Note that $V_j \geqslant \mathrm{Soc}_{p_j} H$ by Lemma 1(d), and $p_j \nmid |H : P_j|$, so Lemma 2(b) implies that $\mathrm{Soc}_{p_j} H = U_j$. Hence $\mathrm{Soc}_{\pi - p_j} H = W_j$, so $H^0 = \langle H, P_1^0, P_2^0, \ldots, P_m^0 \rangle$ as in the Construction of $\mathbf{E}_\pi^\Sigma(H)$. Now $\Sigma \searrow H^0$ by (a), and we put

$$P_j^{**} = P_j^\infty \cap H^0, \qquad U_j^{**} = \mathbf{C}_{V_j}\big(P_j^{**}\big), \qquad U^{**} = U_1^{**} \times U_2^{**} \times \cdots \times U_m^{**}.$$

As before, Lemma 2(b) implies that $\mathrm{Soc}_{p_j} H^0 = U_j^{**}$ and $\mathrm{Soc}_{p_j} \mathbf{E}_\pi^\Sigma(H) = U_j^*$. But $\mathrm{Soc}_{p_j} H^0 = \mathrm{Soc}_{p_j} \mathbf{E}_\pi^\Sigma(H)$ by Lemma 4(f), so it follows from these equations that $\mathrm{Soc}_{p_j} \mathbf{E}_\pi^\Sigma(H) = U_j^{**} = U_j^*$. Also $\mathbf{E}_\pi^\Sigma(H) = \mathbf{C}_G(\mathrm{Soc}_\pi H^0) = \mathbf{C}_G(U^{**})$, so it suffices to show that $U_j^{**} = U_j^0$.

Clearly $U_j^{**} = \mathbf{C}_{V_j}(P_j^{**}) \leqslant \mathbf{C}_{V_j}(P_j^0) = U_j^0$. To prove the converse, note that $U_j^0 = \mathbf{C}_{V_j}(P_j^0) \leqslant \mathbf{C}_{V_j}(P_j) = U_j = \mathrm{Soc}_{p_j} H$, so U_j^0 is centralized by H. Also $U_j^0 = \mathbf{C}_{V_j}(P_j^0)$ is centralized by P_j^0. Finally if $i \neq j$, then $U_j^0 \leqslant U_j \leqslant W_i$, and hence U_j^0 is centralized by P_i^0. This proves that $U_j^0 \leqslant \mathrm{Soc}_{p_j} \mathbf{Z}(H^0)$, and therefore $U_j^0 \leqslant \mathrm{Soc}_{p_j} H^0 = U_j^*$. □

Construction. Let Σ be a Sylow basis in a finite solvable group G, and take $\mathbf{E}^0(\Sigma) = N_\pi$ to be the $\mathcal{Z}^\pi$-radical of G (where π is a set of prime numbers). For each index $i > 0$, assume inductively that a $\mathcal{Z}^\pi$-subgroup $\mathbf{E}^{i-1}(\Sigma)$ has been constructed, with $N_\pi \leqslant \mathbf{E}^{i-1}(\Sigma) \leqslant G$ and $\Sigma \searrow \mathbf{E}^{i-1}(\Sigma)$, such that $\mathbf{E}^{i-1}(\Sigma) = \mathbf{K}_\pi^G(\mathbf{E}^{i-1}(\Sigma))$ and $\mathbf{E}^{i-1}(\Sigma)$ pr G. Then we define

$$\mathbf{E}^i(\Sigma) = \mathbf{E}_\pi^\Sigma\big(\mathbf{E}^{i-1}(\Sigma)\big).$$

It follows from Lemma 11(b) that $\mathbf{E}^i(\Sigma) \in \mathcal{Z}^\pi$, with $N_\pi \leqslant \mathbf{E}^i(\Sigma) \leqslant G$ and $\Sigma \searrow \mathbf{E}^i(\Sigma)$, and that $\mathbf{E}^i(\Sigma)$ pr G and $\mathbf{E}^i(\Sigma) = \mathbf{K}_\pi^G(\mathbf{E}^i(\Sigma))$, so the Construction can proceed.

Theorem 4. *Let N_π be the $\mathcal{Z}^\pi$-radical of a finite solvable group G (where π is a set of prime numbers). Let Σ be a Sylow basis in G, and take $\mathbf{E}^i(\Sigma)$ as in the above Construction. Then there is an index k such that*

$$N_\pi = \mathbf{E}^0(\Sigma) < \mathbf{E}^1(\Sigma) < \cdots < \mathbf{E}^k(\Sigma) = \mathbf{E}^{k+1}(\Sigma),$$

and $\mathbf{E}^k(\Sigma)$ is a $\mathcal{Z}^\pi$-injector of G, with $\mathbf{E}^i(\Sigma)$ pr G for all indices i.

Proof. This follows from Lemma 11(b) and (d), together with Lemma 9(b). □

Corollary 4. *With the notation of Theorem 4, suppose $\pi = \{p_1, p_2, \ldots, p_m\}$. For each index j, take*

$$\{P_j\} = \Sigma \cap \mathrm{Syl}_{p_j} G, \qquad V_j = \mathrm{Soc}_{p_j} N_\pi, \qquad L_i = \mathbf{N}_G\big(\mathbf{E}^i(\Sigma)\big),$$

$$P_{ij} = P_j \cap \mathbf{E}^i(\Sigma), \qquad U_{ij} = \mathbf{C}_{V_j}(P_{ij}),$$

$$W_{ij} = U_{i1} \times \cdots \times U_{i,j-1} \times U_{i,j+1} \times \cdots \times U_{im},$$

$$P_{ij}^0 = P_j \cap \mathbf{C}_{L_i}(W_{ij}), \qquad U_{ij}^0 = \mathbf{C}_{V_j}\big(P_{ij}^0\big), \qquad U_i^0 = U_{i1}^0 \times U_{i2}^0 \times \cdots \times U_{im}^0.$$

Then $\mathbf{E}^{i+1}(\Sigma) = \mathbf{C}_G(U_i^0)$ *and* $\mathrm{Soc}_{p_j}\mathbf{E}^{i+1}(\Sigma) = U_{ij}^0 = U_{i+1,j}$, *with*

$$\mathrm{Soc}_\pi N_\pi \geqslant U_0^0 > U_1^0 > \cdots > U_{k-1}^0 \geqslant U_k^0.$$

Proof. It follows from Lemma 11(e) that $\mathrm{Soc}_{p_j}\mathbf{E}^{i+1}(\Sigma) = U_{ij}^0 = U_{i+1,j}$. Also $\mathbf{E}^{i+1}(\Sigma) = \mathbf{C}_G(U_i^0)$, so the strict inclusion $\mathbf{E}^i(\Sigma) < \mathbf{E}^{i+1}(\Sigma)$ implies that $U_{i-1}^0 > U_i^0$ $(1 \leqslant i < k)$. □

6. Construction of Example 2

In this section, we first prove Lemma 12, which describes a way to construct a solvable group with a unique chief series of arbitrary length. We then use these groups to obtain Example 2, in which the chain in Theorem 4 is also arbitrarily long.

Notation. Suppose that P and Q are groups, and let $\mathrm{Hom}(P, Q)$ be the set of homomorphisms $\lambda : P \to Q$. If u is an automorphism of P, we can define an action of u on $\mathrm{Hom}(P, Q)$ by the equation $\lambda^u(x) = \lambda(x^{u^{-1}})$, where $\lambda \in \mathrm{Hom}(P, Q)$ and $x \in P$.

Write $\mathbf{F}_{p^n}$ for the field of order p^n (where p is a prime number and n is a positive integer), and let $\mathbf{F}_{p^n}^+$ and $\mathbf{F}_{p^n}^\times$ be the additive and multiplicative groups respectively; then $\mathbf{F}_{p^n}^+$ is elementary abelian of order p^n, and $\mathbf{F}_{p^n}^\times$ is cyclic of order $p^n - 1$. In particular let $\mathbf{F}_4 = \{0, 1, \theta, \theta^2\}$ be the field of order 4, with $\theta^3 = \theta + \theta^2 = 1$. Identify $\mathbf{F}_4^+$ with a multiplicative group $\langle y, y'\rangle$, where y and y' correspond to θ and θ^2 respectively (and so yy' corresponds to the element $\theta + \theta^2 = 1 \in \mathbf{F}_4^+$). Make the elements of $\mathbf{F}_4$ act 'multiplicatively' on $\langle y, y'\rangle$ by taking

$$\begin{array}{llll} y^0 = 1, & y^1 = y, & y^\theta = y', & y^{\theta^2} = yy', \\ y'^0 = 1, & y'^1 = y', & y'^\theta = yy', & y'^{\theta^2} = y; \end{array}$$

then $\mathbf{F}_4^\times$ is identified with a group of automorphisms of $\langle y, y'\rangle$.

Lemma 12. *For each integer* $n \geqslant 0$, *there is a group*

$$K_n = P_1 Q_1 P_2 Q_2 \ldots P_{n+1} Q_{n+1}$$

and a subgroup $H_n = P_1 Q_1 P_2 Q_2 \ldots P_n Q_n P_{n+1}$, *with the following properties*:

(a) *for each* $r \geqslant 0$, $K_r = H_r Q_{r+1}$ *is a semidirect product, with* $Q_{r+1} \trianglelefteq K_r$ *and* $H_r \cap Q_{r+1} = 1$; *similarly when* $r \geqslant 1$, $H_r = K_{r-1} P_{r+1}$ *with* $P_{r+1} \trianglelefteq H_r$ *and* $K_{r-1} \cap P_{r+1} = 1$;

(b) *for each* $r \geqslant 0$, P_{r+1} *and* Q_{r+1} *are the unique minimal normal subgroups of* H_r *and* K_r *respectively*; *moreover*

$$|P_r| = \begin{cases} 2 & \text{when } r = 1, \\ 2^2 & \text{when } r = 2, \\ 2^{2|P_{r-1}|} & \text{when } r \geqslant 3, \end{cases} \qquad |Q_r| = \begin{cases} 3 & \text{when } r = 1, \\ 3^{|Q_{r-1}|} & \text{when } r \geqslant 2; \end{cases}$$

(c) *there are generators* a *and* b *of* P_1 *and* Q_1 *respectively, and a basis* $\{c, c'\}$ *of* P_2, *such that* $b^a = b^{-1}$, $c^a = c^b = c'$, $c'^a = c$ *and* $c'^b = c^*$, *where* $c^* = cc'$; *hence* $\mathbf{C}_{P_2}(H_0) = \mathbf{C}_{P_2}(P_1) = \langle c^*\rangle$;

(d) *for each* $n \geqslant 1$, *there is a basis* $\{x_u\colon u \in Q_n\}$ *of* Q_{n+1}, *and a homomorphism* $\lambda \in \mathrm{Hom}(P_{n+1}, \mathbf{F}_3^\times)$, *such that when* $h \in H_{n-1}$, $u, u' \in Q_n$ *and* $y \in P_{n+1}$, *then* $x_u^h = x_{u^h}$, $x_u^{u'} = x_{uu'}$ *and* $x_u^y = x_u^{\lambda^u(y)}$ (*where* u

acts on $\mathrm{Hom}(P_{n+1}, \mathbf{F}_3^\times)$ *as in the Notation above*); *hence* $\mathbf{C}_{Q_{n+1}}(K_{n-1}) = \mathbf{C}_{Q_{n+1}}(Q_n) = \langle x^* \rangle$, *where* $x^* = \prod_{u \in Q_n} x_u$;

(e) *for each* $n \geqslant 2$, *there is a basis* $\{y_v, y'_v\colon\ v \in P_n\}$ *of* P_{n+1}, *and a homomorphism* $\mu \in \mathrm{Hom}(Q_n, \mathbf{F}_4^\times)$, *such that when* $k \in K_{n-2}$, $v, v' \in P_n$ *and* $x \in Q_n$, *then* $y_v^k = y_{v^k}$, $y_v^{v'} = y_{vv'}$, $y_v^x = y_v^{\mu^v(x)}$ *and* ${y'_v}^k = y'_{v^k}$, ${y'_v}^{v'} = y'_{vv'}$, ${y'_v}^x = {y'_v}^{\mu^v(x)}$ (*where* $\mu^v(x)$ *acts on* $\langle y_v, y'_v \rangle$ *as in the Notation above*); *hence* $\mathbf{C}_{P_{n+1}}(H_{n-1}) = \mathbf{C}_{P_{n+1}}(P_n) = \langle y^*, y'^* \rangle$, *where* $y^* = \prod_{v \in P_n} y_v$ *and* $y'^* = \prod_{v \in P_n} y'_v$.

Proof. This is based on a well-known construction [2, B(9.15)]. As in (c), take $H_1 = P_1 Q_1 P_2 \cong \mathbf{S}_4$, with $P_1 = \langle a \rangle \cong \mathbf{C}_2$, $Q_1 = \langle b \rangle \cong \mathbf{C}_3$ and $P_2 = \langle c, c' \rangle \cong \mathbf{C}_2 \times \mathbf{C}_2$. Then P_1, Q_1 and P_2 are the unique minimal normal subgroups of H_0, K_0 and H_1 respectively, as in (b), and $H_0 \cap Q_1 = K_0 \cap P_2 = 1$, as in (a).

We next construct the group $K_1 = H_1 Q_2$, using a method which will be generalized to continue the proof. Since $\mathbf{F}_3^\times = \{\pm 1\} \cong \mathbf{C}_2$, we can define a homomorphism $\lambda \in \mathrm{Hom}(P_2, \mathbf{F}_3^\times)$ by taking

$$\lambda\big(c^\alpha {c'}^{\alpha'}\big) = (-1)^{\alpha+\alpha'} \quad \big(\alpha, \alpha' \in \{0,1\} = \mathbf{F}_2\big).$$

Then the kernel of λ is $\mathrm{Ker}\,\lambda = \langle c^* \rangle = [P_2, P_1]$, where $c^* = cc'$. Extend λ to a map $\lambda^* : P_1 P_2 \to \mathbf{F}_3^\times$ by defining $\lambda^*(hy) = \lambda(y)$ ($h \in P_1$, $y \in P_2$). Since $P_1[P_2, P_1] \trianglelefteq P_1 P_2$, it follows that $\lambda^* \in \mathrm{Hom}(P_1 P_2, \mathbf{F}_3^\times)$ with $\mathrm{Ker}\,\lambda^* = P_1[P_2, P_1]$. Let $X = \mathbf{F}_3 x$ be the corresponding 1-dimensional $\mathbf{F}_3(P_1 P_2)$-module, where $xg = \lambda^*(g)x$ ($g \in P_1 P_2$), and let $Q_2 = X^{H_1} = X \otimes_{\mathbf{F}_3(P_1 P_2)} \mathbf{F}_3 H_1$ be the induced module [2, B(6.1)]. Now Q_1 is a transversal to $P_1 P_2$ in H_1, so the set $\{x \otimes u\colon\ u \in Q_1\}$ is an $\mathbf{F}_3$-basis of Q_2. Suppose $h \in P_1$, $u, u' \in Q_1$ and $y \in P_2$; then the action of $H_1 = P_1 Q_1 P_2$ on Q_2 is determined by the equations

$$(x \otimes u)h = x \otimes hu^h = x \otimes u^h, \qquad (x \otimes u)u' = x \otimes uu',$$
$$(x \otimes u)y = x \otimes y^{u^{-1}} u = \lambda\big(y^{u^{-1}}\big) x \otimes u = \lambda^u(y) x \otimes u.$$

Thus P_1 permutes the given basis of Q_2, Q_1 permutes it regularly, and P_2 acts diagonally (with the basis elements as eigenvectors). Now $\mathrm{Ker}\,\lambda = \langle c^* \rangle$, $\mathrm{Ker}\,\lambda^b = \langle c \rangle$ and $\mathrm{Ker}\,\lambda^{b^2} = \langle c' \rangle$, so the linear characters λ^u of P_2 ($u \in Q_1$) are distinct, and hence Q_2 is $\mathbf{F}_3 H_1$-irreducible [2, B(7.8)]. Also $\mathbf{C}_{P_2}(Q_2) < P_2$, and therefore $\mathbf{C}_{H_1}(Q_2) = 1$, because P_2 is the unique minimal normal subgroup of H_1. Thus Q_2 is a faithful irreducible $\mathbf{F}_3 H_1$-module. Now write Q_2 multiplicatively, with $x_u = x \otimes u$, so $Q_2 = \langle x_u\colon\ u \in Q_1 \rangle$. We form the semidirect product $K_1 = H_1 Q_2$ as in (a), and we deduce that Q_2 is the unique minimal normal subgroup of K_1, with $|Q_2| = 3^{|Q_1|}$, as in (b). Moreover if $h \in P_1$, $u, u' \in Q_1$ and $y \in P_2$, then the above relations show that $x_u^h = x_{u^h}$, $x_u^{u'} = x_{uu'}$ and $x_u^y = x_u^{\lambda^u(y)}$, as in (d). We now suppose that $n \geqslant 2$ and assume inductively the existence of the group K_{n-1}, and aim to construct K_n.

By (d), there is a basis $\{x_u\colon\ u \in Q_{n-1}\}$ of Q_n, so we can define a homomorphism $\mu \in \mathrm{Hom}(Q_n, \mathbf{F}_4^\times)$ by taking

$$\mu\Bigg(\prod_{u \in Q_{n-1}} x_u^{\gamma_u}\Bigg) = \theta^{\sum_{u \in Q_{n-1}} \gamma_u},$$

where $\gamma_u \in \{0, \pm 1\} = \mathbf{F}_3$ ($u \in Q_{n-1}$). Moreover this basis of Q_n is permuted transitively by K_{n-2}, and therefore

$$\mathrm{Ker}\,\mu = \Bigg\{\prod_{u \in Q_{n-1}} x_u^{\gamma_u}\colon \sum_{u \in Q_{n-1}} \gamma_u = 0\Bigg\} = [Q_n, K_{n-2}].$$

As before, we extend μ to a map $\mu^* : K_{n-2}Q_n \to \mathbf{F}_4^\times$ by defining $\mu^*(kx) = \mu(x)$ ($k \in K_{n-2}$, $x \in Q_n$). Since $K_{n-2}[Q_n, K_{n-2}] \trianglelefteq K_{n-2}Q_n$, it follows that $\mu^* \in \mathrm{Hom}(K_{n-2}Q_n, \mathbf{F}_4^\times)$ with $\mathrm{Ker}\,\mu^* = K_{n-2}[Q_n, K_{n-2}]$. Let $Y = \mathbf{F}_4 y^*$ be the corresponding 1-dimensional $\mathbf{F}_4(K_{n-2}Q_n)$-module, where $y^*g = \mu^*(g)y^*$ ($g \in K_{n-2}Q_n$), and put $y = \theta y^*$, $y' = \theta^2 y^*$. Then $\{y, y'\}$ is an $\mathbf{F}_2$-basis of Y, so that Y can also be regarded as a 2-dimensional $\mathbf{F}_2(K_{n-2}Q_n)$-module. Next form the induced module $P_{n+1} = Y^{K_{n-1}} = Y \otimes_{\mathbf{F}_2(K_{n-2}Q_n)} \mathbf{F}_2 K_{n-1}$. Now P_n is a transversal to $K_{n-2}Q_n$ in K_{n-1}, so the set $\{y \otimes v,\ y' \otimes v\colon\ v \in P_n\}$ is an $\mathbf{F}_2$-basis of P_{n+1}. Suppose $k \in K_{n-2}$, $v, v' \in P_n$ and $x \in Q_n$; then the action of $K_{n-1} = K_{n-2}P_nQ_n$ on P_{n+1} is determined by the equations

$$(y \otimes v)k = y \otimes kv^k = y \otimes v^k, \qquad (y \otimes v)v' = y \otimes vv',$$
$$(y \otimes v)x = y \otimes x^{v^{-1}} v = \mu\big(x^{v^{-1}}\big) y \otimes v = \mu^v(x) y \otimes v,$$
$$\big(y' \otimes v\big)k = y' \otimes v^k, \qquad \big(y' \otimes v\big)v' = y' \otimes vv', \qquad \big(y' \otimes v\big)x = \mu^v(x) y' \otimes v.$$

Here $\mu^v(x) \in \mathbf{F}_4^\times = \{1, \theta, \theta^2\} \cong \mathbf{C}_3$, and its action on $Y = \mathbf{F}_2 y \oplus \mathbf{F}_2 y'$ is determined by the rules $\theta y = y'$ and $\theta y' = y + y'$. Thus the subspaces $Y \otimes v = \mathbf{F}_2(y \otimes v) \oplus \mathbf{F}_2(y' \otimes v)$ ($v \in P_n$) are permuted by K_{n-2}, and permuted regularly by P_n, and stabilized by Q_n.

Now suppose $v \in P_n$ with $v \neq 1$; we claim that

$$\mathrm{Ker}\,\mu^v \neq \mathrm{Ker}\,\mu.$$

To prove this, note first that (b) implies that $\mathbf{C}_{Q_{n-1}}(P_n) = 1$. Hence there is an element $u' \in Q_{n-1}$ such that $v^{-1}u' \neq u'v^{-1}$. Now (b) also implies that H_{n-1} acts faithfully on Q_n, and $\{x_u\colon\ u \in Q_{n-1}\}$ is a basis of Q_n. Hence there is an element $u \in Q_{n-1}$ such that $x_u^{v^{-1}u'} \neq x_u^{u'v^{-1}}$. Now (d) shows that

$$x_u^{v^{-1}u'} = x_u^{\lambda^u(v^{-1})u'} = x_{uu'}^{\lambda^u(v^{-1})}, \qquad x_u^{u'v^{-1}} = x_{uu'}^{v^{-1}} = x_{uu'}^{\lambda^{uu'}(v^{-1})},$$

and therefore $\lambda^u(v^{-1}) \neq \lambda^{uu'}(v^{-1})$. Take $x = x_u x_{uu'}^{-1}$, and note that $\mu(x) = 1$, using the definition of μ. However $x^{v^{-1}} = x_u^{\lambda^u(v^{-1})} x_{uu'}^{-\lambda^{uu'}(v^{-1})}$, and hence $\mu^v(x) = \mu(x^{v^{-1}}) \neq 1$, which proves the claim.

The claim shows that the 2-dimensional representations μ^v of Q_n over $\mathbf{F}_2$ ($v \in P_n$) are inequivalent, which implies that P_{n+1} is $\mathbf{F}_2K_{n-1}$-irreducible. Moreover $\mathbf{C}_{Q_n}(P_{n+1}) < Q_n$, and hence $\mathbf{C}_{K_{n-1}}(P_{n+1}) = 1$, because Q_n is the unique minimal normal subgroup of K_{n-1}. Thus P_{n+1} is a faithful irreducible $\mathbf{F}_2K_{n-1}$-module. Now write P_{n+1} multiplicatively, with $y_v = y \otimes v$ and $y'_v = y' \otimes v$, so $P_{n+1} = \langle y_v, y'_v\colon\ v \in P_n\rangle$. We form the semidirect product $H_n = K_{n-1}P_{n+1}$ as in (a), and we deduce that P_{n+1} is the unique minimal normal subgroup of H_n, with $|P_{n+1}| = 2^{2|P_n|}$, as in (b). If $k \in K_{n-2}$, $v, v' \in P_n$ and $x \in Q_n$, then the above relations show that $y_v^k = y_{v^k}$, $y_v^{v'} = y_{vv'}$, $y_v^x = y_v^{\mu^v(x)}$ and ${y'_v}^k = y'_{v^k}$, ${y'_v}^{v'} = y'_{vv'}$, ${y'_v}^x = {y'_v}^{\mu^v(x)}$. This proves that (e) holds for P_{n+1}.

Now $\mathbf{F}_4$ acts on $Y_v = \langle y_v, y'_v\rangle$ with $y_v^0 = 1$, $y_v^1 = y_v$, $y_v^\theta = y'_v$, $y_v^{\theta^2} = y_v y'_v$ and ${y'_v}^0 = 1$, ${y'_v}^1 = y'_v$, ${y'_v}^\theta = y_v y'_v$, ${y'_v}^{\theta^2} = y_v$, as in the Notation above. Put $y_v^* = y_v y'_v$, and note that $y_v = y_v^{*\theta}$ and $y'_v = y_v^{*\theta^2}$. Hence $Y_v = \{y_v^{*\delta}\colon\ \delta \in \mathbf{F}_4\}$, so the elements of P_{n+1} can be written as $\prod_{v \in P_n} y_v^{*\delta_v}$ with $\delta_v \in \mathbf{F}_4$ ($v \in P_n$). Let $\tau : \mathbf{F}_4^+ \to \mathbf{F}_2^+$ be the trace homomorphism, with $\tau(\delta) = \delta + \delta^2$ ($\delta \in \mathbf{F}_4$), and note that $\tau(\theta) = \tau(\theta^2) = 1$ and $\tau(1) = 0$. Define a homomorphism $\nu \in \mathrm{Hom}(P_{n+1}, \mathbf{F}_3^\times)$ by taking

$$\nu\Bigg(\prod_{v \in P_n} y_v^{*\delta_v}\Bigg) = (-1)^{\tau(\sum_{v \in P_n} \delta_v)},$$

where $\delta_v \in \mathbf{F}_4$ ($v \in P_n$). Writing $\delta_v = \alpha_v\theta + \beta_v\theta^2$ with $\alpha_v, \beta_v \in \mathbf{F}_2$, we deduce that

$$\begin{aligned}
\operatorname{Ker}\nu &= \left\{ \prod_{v\in P_n} y_v^{*\,\delta_v} \colon \tau\left(\sum_{v\in P_n} \delta_v\right) = 0 \right\} \\
&= \left\{ \prod_{v\in P_n} y_v^{\alpha_v} {y'_v}^{\beta_v} \colon \sum_{v\in P_n} (\alpha_v + \beta_v) = 0 \right\} \\
&\geqslant \left\{ \prod_{v\in P_n} y_v^{\alpha_v} {y'_v}^{\beta_v} \colon \sum_{v\in P_n} \alpha_v = \sum_{v\in P_n} \beta_v = 0 \right\} \\
&= [P_{n+1}, H_{n-1}],
\end{aligned}$$

because the sets $\{y_v\colon\ v \in P_n\}$ and $\{y'_v\colon\ v \in P_n\}$ are permuted transitively by H_{n-1}. Extend ν to a map $\nu^* : H_{n-1}P_{n+1} \to \mathbf{F}_3^\times$ by defining $\nu^*(hy) = \nu(y)$ $(h \in H_{n-1},\ y \in P_{n+1})$. Then $H_{n-1}[P_{n+1}, H_{n-1}] \leqslant H_{n-1}.\operatorname{Ker}\nu \leqslant H_{n-1}P_{n+1}$, with $H_{n-1}[P_{n+1}, H_{n-1}] \trianglelefteq H_{n-1}P_{n+1}$ and

$$H_{n-1}P_{n+1}/H_{n-1}[P_{n+1}, H_{n-1}] \cong \mathbf{F}_4^+ \cong \mathbf{C}_2 \times \mathbf{C}_2.$$

Hence $H_{n-1}.\operatorname{Ker}\nu \trianglelefteq H_{n-1}P_{n+1}$, and therefore $\nu^* \in \operatorname{Hom}(H_{n-1}P_{n+1}, \mathbf{F}_3^\times)$ with $\operatorname{Ker}\nu^* = H_{n-1}.\operatorname{Ker}\nu$. Let $Z = \mathbf{F}_3 z$ be the corresponding 1-dimensional $\mathbf{F}_3(H_{n-1}P_{n+1})$-module, where $zg = \nu^*(g)z$ $(g \in H_{n-1}P_{n+1})$. Next form the induced module $Q_{n+1} = Z^{H_n} = Z \otimes_{\mathbf{F}_3(H_{n-1}P_{n+1})} \mathbf{F}_3 H_n$. Now Q_n is a transversal to $H_{n-1}P_{n+1}$ in H_n, so the set $\{z \otimes w\colon\ w \in Q_n\}$ is an $\mathbf{F}_3$-basis of Q_{n+1}. Suppose $h \in H_{n-1}$, $w, w' \in Q_n$ and $y \in P_{n+1}$; then the action of $H_n = H_{n-1}Q_nP_{n+1}$ on Q_{n+1} is determined by the equations

$$\begin{gathered}
(z \otimes w)h = z \otimes hw^h = z \otimes w^h, \qquad (z \otimes w)w' = z \otimes ww', \\
(z \otimes w)y = z \otimes y^{w^{-1}} w = \nu\big(y^{w^{-1}}\big) z \otimes w = \nu^w(y) z \otimes w.
\end{gathered}$$

Thus H_{n-1} permutes the above basis of Q_{n+1}, Q_n permutes it regularly, and P_{n+1} acts diagonally (with the basis elements as eigenvectors).

Now suppose $w \in Q_n$ with $w \neq 1$; we claim that

$$\operatorname{Ker}\nu^w \neq \operatorname{Ker}\nu.$$

To prove this, note first that (b) implies that $\mathbf{C}_{P_n}(Q_n) = 1$. Hence there is an element $v' \in P_n$ such that $w^{-1}v' \neq v'w^{-1}$. Now (b) also implies that K_{n-1} acts faithfully on P_{n+1}, and P_{n+1} is generated by the subgroups $Y_v = \langle y_v, y'_v\colon\ v \in P_n\rangle$. Hence there is an element $v \in P_n$ such that either $y_v^{w^{-1}v'} \neq y_v^{v'w^{-1}}$ or ${y'_v}^{w^{-1}v'} \neq {y'_v}^{v'w^{-1}}$. We deduce from (e) that

$$\begin{gathered}
y_v^{w^{-1}v'} = y_v^{\mu^v(w^{-1})v'} = y_{vv'}^{\mu^v(w^{-1})}, \qquad y_v^{v'w^{-1}} = y_{vv'}^{w^{-1}} = y_{vv'}^{\mu^{vv'}(w^{-1})}, \\
{y'_v}^{w^{-1}v'} = {y'_{vv'}}^{\mu^v(w^{-1})}, \qquad {y'_v}^{v'w^{-1}} = {y'_{vv'}}^{\mu^{vv'}(w^{-1})},
\end{gathered}$$

and hence $\mu^v(w^{-1}) \neq \mu^{vv'}(w^{-1})$. To complete the proof, suppose first that $\tau(\mu^v(w^{-1})) \neq \tau(\mu^{vv'}(w^{-1}))$, and take $y = y_v^* y_{vv'}^*$. Then

$$y^{w^{-1}} = y_v^{*\,\mu^v(w^{-1})} y_{vv'}^{*\,\mu^{vv'}(w^{-1})},$$

and it follows from the definition of ν that $\nu(y) = 1$ and $\nu^w(y) = \nu(y^{w^{-1}}) \neq 1$. This proves the claim in this case, so we may assume that $\mu^v(w^{-1}) \neq \mu^{vv'}(w^{-1})$ but $\tau(\mu^v(w^{-1})) = \tau(\mu^{vv'}(w^{-1}))$.

It follows that either $\mu^{\nu}(w^{-1}) = \theta$ and $\mu^{\nu\nu'}(w^{-1}) = \theta^2$, or else $\mu^{\nu}(w^{-1}) = \theta^2$ and $\mu^{\nu\nu'}(w^{-1}) = \theta$. In both cases we can take $y = y_\nu y_{\nu\nu'} = y_\nu^{*\theta} y_{\nu\nu'}^{*\theta}$, and note that

$$y^{w^{-1}} = y_\nu^{*\theta\mu^{\nu}(w^{-1})} y_{\nu\nu'}^{*\theta\mu^{\nu\nu'}(w^{-1})} = y_\nu^{*\theta^2} y_{\nu\nu'}^{*} a \text{ or } y_\nu^{*} y_{\nu\nu'}^{*\theta^2}.$$

Again it follows from the definition of ν that $\nu(y) = 1$ and $\nu^w(y) = \nu(y^{w^{-1}}) \neq 1$, which completes the proof of the claim.

The claim shows that the linear characters ν^w of P_{n+1} $(w \in Q_n)$ are distinct, which implies that Q_{n+1} is $\mathbf{F}_3 H_n$-irreducible. Also $\mathbf{C}_{P_{n+1}}(Q_{n+1}) < P_{n+1}$, and hence $\mathbf{C}_{H_n}(Q_{n+1}) = 1$, because P_{n+1} is the unique minimal normal subgroup of H_n. Thus Q_{n+1} is a faithful irreducible $\mathbf{F}_3 H_n$-module. Now write Q_{n+1} multiplicatively, with $z_w = z \otimes w$, so $Q_{n+1} = \langle z_w\colon\ w \in Q_n\rangle$. We form the semidirect product $K_n = H_n Q_{n+1}$ as in (a), and we deduce that Q_{n+1} is the unique minimal normal subgroup of K_n, with $|Q_{n+1}| = 3^{|Q_n|}$, as in (b). If $h \in H_{n-1}$, $w, w' \in Q_n$ and $y \in P_{n+1}$, then the above relations show that $z_w^h = z_{w^h}$, $z_w^{w'} = z_{ww'}$ and $z_w^y = z_w^{\nu^w(y)}$. This proves that (d) holds for Q_{n+1}. $\square$

Example 2. For each integer $n \geqslant 1$, there is a finite solvable group G with a Sylow basis Σ, and a set π of prime numbers, such that

$$\mathbf{E}^0(\Sigma) < \mathbf{E}^1(\Sigma) < \mathbf{E}^2(\Sigma) < \cdots < \mathbf{E}^{2n}(\Sigma) = \mathbf{E}^{2n+1}(\Sigma).$$

Proof. Consider groups $H_1, K_1, H_2, K_2, \ldots, H_n, K_n$ which are isomorphic to the groups constructed in Lemma 12, but are not regarded as subgroups of each other. More explicitly, take

$$\begin{aligned} H_r &= P_{1r} Q_{1r} P_{2r} Q_{2r} \ldots P_{rr} Q_{rr} P_{r+1,r}, \\ K_r &= P_{1r}^0 Q_{1r}^0 P_{2r}^0 Q_{2r}^0 \ldots P_{r+1,r}^0 Q_{r+1,r}^0, \end{aligned}$$

with isomorphisms $P_i \cong P_{ir} \cong P_{ir}^0$ $(1 \leqslant i \leqslant r+1)$, $Q_j \cong Q_{jr} \cong Q_{jr}^0$ $(1 \leqslant j \leqslant r)$ and $Q_{r+1} \cong Q_{r+1,r}^0$. Suppose further that if $x \in P_i$ and $y \in Q_j$, then x_r and y_r are the corresponding elements of P_{ir} and Q_{jr} respectively, and similarly x_r' and y_r' are the corresponding elements of P_{ir}^0 and Q_{jr}^0 respectively. Now form the direct product

$$G_0 = H_1 \times K_1 \times H_2 \times K_2 \times \cdots \times H_n \times K_n,$$

and take the 'diagonal' subgroups

$$\begin{aligned} P_r^* &= \{x_r x_r' x_{r+1} x_{r+1}' \ldots x_n x_n'\colon\ x \in P_r\} \\ &\leqslant H_r \times K_r \times H_{r+1} \times K_{r+1} \times \cdots \times H_n \times K_n, \\ Q_r^* &= \{y_r' y_{r+1} y_{r+1}' y_{r+2} y_{r+2}' \ldots y_n y_n'\colon\ y \in Q_r\} \\ &\leqslant K_r \times H_{r+1} \times K_{r+1} \times H_{r+2} \times K_{r+2} \times \cdots \times H_n \times K_n, \end{aligned}$$

with $P_r^* \cong P_r$ and $Q_r^* \cong Q_r$ $(1 \leqslant r \leqslant n)$. We also put

$$\begin{aligned} S_r &= Q_{rr} \leqslant H_r, \qquad T_r = P_{r+1,r}^0 \leqslant K_r, \\ U_r &= P_{r+1,r} \leqslant H_r, \qquad V_r = Q_{r+1,r}^0 \leqslant K_r, \\ U_0 &= U_1 \times U_2 \times \cdots \times U_n \leqslant H_1 \times H_2 \times \cdots \times H_n, \\ V_0 &= V_1 \times V_2 \times \cdots \times V_n \leqslant K_1 \times K_2 \times \cdots \times K_n. \end{aligned}$$

Clearly $S_rU_r, T_rV_r \trianglelefteq G_0$, and P_r^* normalizes Q_r^* $(1 \leqslant r \leqslant n)$; moreover the subgroup $P_r^*S_rU_r \cdot Q_r^*T_rV_r$ normalizes P_s^* and Q_s^* $(1 \leqslant r < s \leqslant n)$, and we can define

$$G = P_1^*S_1U_1 \cdot Q_1^*T_1V_1 \cdot P_2^*S_2U_2 \cdot Q_2^*T_2V_2 \cdot \ldots \cdot P_n^*S_nU_n \cdot Q_n^*T_nV_n.$$

There are Sylow subgroups

$$T_0 = P_1^*U_1 \cdot T_1 \cdot P_2^*U_2 \cdot T_2 \cdot \ldots \cdot P_n^*U_n \cdot T_n \in \operatorname{Syl}_2 G,$$
$$S_0 = S_1 \cdot Q_1^*V_1 \cdot S_2 \cdot Q_2^*V_2 \cdot \ldots \cdot S_n \cdot Q_n^*V_n \in \operatorname{Syl}_3 G,$$

so we put $\pi = \{2, 3\}$, and take the Sylow basis $\Sigma = \{1, T_0, S_0\}$. We also put $U_r^* = \mathbf{C}_{U_r}(P_{rr}) = \mathbf{C}_{U_r}(P_r^*)$ and $V_r^* = \mathbf{C}_{V_r}(Q_{rr}^0) = \mathbf{C}_{V_r}(Q_r^*)$, and we note that Lemma 12(c), (e) and (d) show that

$$U_r^* = \mathbf{C}_{U_r}\left(P_1^*Q_1^*P_2^*Q_2^* \ldots P_n^*Q_n^*\right) \cong \begin{cases} \mathbf{C}_2 & \text{when } r = 1, \\ \mathbf{C}_2 \times \mathbf{C}_2 & \text{when } r \geqslant 2, \end{cases}$$
$$V_r^* = \mathbf{C}_{V_r}\left(P_1^*Q_1^*P_2^*Q_2^* \ldots P_n^*Q_n^*\right) \cong \mathbf{C}_3 \quad \text{when } r \geqslant 1.$$

We take

$$\begin{aligned} E_{2r} &= P_{n-r+1}^*Q_{n-r+1}^*P_{n-r+2}^*Q_{n-r+2}^* \ldots P_n^*Q_n^*U_0V_0 \\ &= U_1 \times V_1 \times \cdots \times U_{n-r} \times V_{n-r} \times P_{n-r+1}^*U_{n-r+1} \cdot Q_{n-r+1}^*V_{n-r+1} \cdot \ldots \cdot P_n^*U_n \cdot Q_n^*V_n, \end{aligned}$$

and we claim that

$$\mathbf{E}^{2r}(\Sigma) = E_{2r} \quad (0 \leqslant r \leqslant n).$$

To prove this, note that $U_0V_0 = \mathbf{C}_{G_0}(U_0V_0) = \mathbf{C}_G(U_0V_0)$ is the $\mathscr{Z}^\pi$-radical of G, and therefore $\mathbf{E}^0(\Sigma) = U_0V_0 = E_0$, with

$$\operatorname{Soc}_2 E_0 = U_0, \qquad \operatorname{Soc}_3 E_0 = V_0.$$

Arguing by induction on r, we may now suppose that $r < n$, and assume that $\mathbf{E}^{2r}(\Sigma) = E_{2r}$, and aim to prove the corresponding formula for $\mathbf{E}^{2r+2}(\Sigma)$.

We first calculate $L_{2r} = \mathbf{N}_G(E_{2r})$. Now $\mathbf{N}_{P_i^*S_iU_i}(P_i^*U_i) = P_i^*\mathbf{N}_{S_i}(P_i^*)U_i$, and

$$\mathbf{N}_{S_i}\left(P_i^*\right) = \mathbf{N}_{Q_{ii}}(P_{ii}) = \mathbf{C}_{Q_{ii}}(P_{ii}) = 1,$$

because Q_{ii} is a faithful irreducible module over $\mathbf{F}_3$ for the group

$$P_{1i}Q_{1i}P_{2i}Q_{2i} \ldots P_{i-1,i}Q_{i-1,i}P_{ii}.$$

Thus $\mathbf{N}_{P_i^*S_iU_i}(P_i^*U_i) = P_i^*U_i$, and similarly $\mathbf{N}_{Q_i^*T_iV_i}(Q_i^*V_i) = Q_i^*V_i$. Hence

$$\begin{aligned} L_{2r} &= \mathbf{N}_G(E_{2r}) \\ &= P_1^*S_1U_1 \cdot Q_1^*T_1V_1 \cdot \ldots \cdot P_{n-r}^*S_{n-r}U_{n-r} \cdot Q_{n-r}^*T_{n-r}V_{n-r} \\ &\quad \cdot P_{n-r+1}^*U_{n-r+1} \cdot Q_{n-r+1}^*V_{n-r+1} \cdot \ldots \cdot P_n^*U_n \cdot Q_n^*V_n. \end{aligned}$$

Using the procedure of Corollary 4, take

$$\begin{aligned}
S_0 \cap E_{2r} &= V_1 \times V_2 \times \cdots \times V_{n-r} \times Q^*_{n-r+1} V_{n-r+1} \cdot Q^*_{n-r+2} V_{n-r+2} \cdot \ldots \cdot Q^*_n V_n \\
&\in \mathrm{Syl}_3 E_{2r}, \\
W_{2r,2} &= \mathbf{C}_{V_0}(S_0 \cap E_{2r}) \\
&= V_1 \times V_2 \times \cdots \times V_{n-r} \times V^*_{n-r+1} \times V^*_{n-r+2} \times \cdots \times V^*_n, \\
\mathbf{C}_{L_{2r}}(W_{2r,2}) &= S_1 U_1 \times V_1 \times \cdots \times S_{n-r} U_{n-r} \times V_{n-r} \times P^*_{n-r+1} U_{n-r+1} \\
&\quad \cdot Q^*_{n-r+1} V_{n-r+1} \cdot \ldots \cdot P^*_n U_n \cdot Q^*_n V_n,
\end{aligned}$$

so

$$\begin{aligned}
T_0 \cap \mathbf{C}_{L_{2r}}(W_{2r,2}) &= U_1 \times U_2 \times \cdots \times U_{n-r} \times P^*_{n-r+1} U_{n-r+1} \cdot P^*_{n-r+2} U_{n-r+2} \cdot \ldots \cdot P^*_n U_n, \\
W^0_{2r,2} &= \mathbf{C}_{U_0}\big(T_0 \cap \mathbf{C}_{L_{2r}}(W_{2r,2})\big) \\
&= U_1 \times U_2 \times \cdots \times U_{n-r} \times U^*_{n-r+1} \times U^*_{n-r+2} \times \cdots \times U^*_n.
\end{aligned}$$

Similarly

$$\begin{aligned}
T_0 \cap E_{2r} &= U_1 \times U_2 \times \cdots \times U_{n-r} \times P^*_{n-r+1} U_{n-r+1} \cdot P^*_{n-r+2} U_{n-r+2} \cdot \ldots \cdot P^*_n U_n \\
&\in \mathrm{Syl}_2 E_{2r}, \\
W_{2r,3} &= \mathbf{C}_{U_0}(T_0 \cap E_{2r}) \\
&= U_1 \times U_2 \times \cdots \times U_{n-r} \times U^*_{n-r+1} \times U^*_{n-r+2} \times \cdots \times U^*_n, \\
\mathbf{C}_{L_{2r}}(W_{2r,3}) &= U_1 \times T_1 V_1 \times \cdots \times U_{n-r-1} \times T_{n-r-1} V_{n-r-1} \times U_{n-r} \times Q^*_{n-r} T_{n-r} V_{n-r} \\
&\quad \cdot P^*_{n-r+1} U_{n-r+1} \cdot Q^*_{n-r+1} V_{n-r+1} \cdot \ldots \cdot P^*_n U_n \cdot Q^*_n V_n,
\end{aligned}$$

with an extra factor Q^*_{n-r}, so

$$\begin{aligned}
S_0 \cap \mathbf{C}_{L_{2r}}(W_{2r,3}) &= V_1 \times V_2 \times \cdots \times V_{n-r-1} \times Q^*_{n-r} V_{n-r} \cdot Q^*_{n-r+1} V_{n-r+1} \cdot \ldots \cdot Q^*_n V_n, \\
W^0_{2r,3} &= \mathbf{C}_{V_0}\big(S_0 \cap \mathbf{C}_{L_{2r}}(W_{2r,3})\big) \\
&= V_1 \times V_2 \times \cdots \times V_{n-r-1} \times V^*_{n-r} \times V^*_{n-r+1} \times \cdots \times V^*_n.
\end{aligned}$$

We therefore take

$$\begin{aligned}
W^0_{2r} &= W^0_{2r,2} \times W^0_{2r,3} \\
&= U_1 \times V_1 \times \cdots \times U_{n-r-1} \times V_{n-r-1} \times U_{n-r} \times V^*_{n-r} \times U^*_{n-r+1} \times V^*_{n-r+1} \times \cdots \times U^*_n \times V^*_n, \\
E_{2r+1} &= \mathbf{E}^{2r+1}(\Sigma) = \mathbf{C}_G\big(W^0_{2r}\big) \\
&= U_1 \times V_1 \times \cdots \times U_{n-r-1} \times V_{n-r-1} \times U_{n-r} \times Q^*_{n-r} V_{n-r} \\
&\quad \cdot P^*_{n-r+1} U_{n-r+1} \cdot Q^*_{n-r+1} V_{n-r+1} \cdot \ldots \cdot P^*_n U_n \cdot Q^*_n V_n \\
&= Q^*_{n-r} E_{2r},
\end{aligned}$$

$$\begin{aligned}L_{2r+1} &= \mathbf{N}_G(E_{2r+1})\\ &= P_1^*S_1U_1 \cdot Q_1^*T_1V_1 \cdot \ldots \cdot P_{n-r-1}^*S_{n-r-1}U_{n-r-1} \cdot Q_{n-r-1}^*T_{n-r-1}V_{n-r-1}\\ &\quad \cdot P_{n-r}^*S_{n-r}U_{n-r} \cdot Q_{n-r}^*V_{n-r} \cdot P_{n-r+1}^*U_{n-r+1} \cdot Q_{n-r-1}^*V_{n-r-1} \cdot \ldots \cdot P_n^*U_n \cdot Q_n^*V_n.\end{aligned}$$

Repeating the process, take

$$\begin{aligned}S_0 \cap E_{2r+1} &= V_1 \times V_2 \times \cdots \times V_{n-r-1} \times Q_{n-r}^*V_{n-r} \cdot Q_{n-r+1}^*V_{n-r+1} \cdot \ldots \cdot Q_n^*V_n\\ &\in \operatorname{Syl}_3 E_{2r+1},\\ W_{2r+1,2} &= \mathbf{C}_{V_0}(S_0 \cap E_{2r+1})\\ &= V_1 \times V_2 \times \cdots \times V_{n-r-1} \times V_{n-r}^* \times V_{n-r+1}^* \times \cdots \times V_n^*,\\ \mathbf{C}_{L_{2r+1}}(W_{2r+1,2}) &= S_1U_1 \times V_1 \times \cdots \times S_{n-r-1}U_{n-r-1} \times V_{n-r-1} \times P_{n-r}^*S_{n-r}U_{n-r} \cdot Q_{n-r}^*V_{n-r}\\ &\quad \cdot P_{n-r+1}^*U_{n-r+1} \cdot Q_{n-r+1}^*V_{n-r+1} \ldots \cdot P_n^*U_n \cdot Q_n^*V_n,\end{aligned}$$

with an extra factor P_{n-r}^*, so

$$\begin{aligned}T_0 \cap \mathbf{C}_{L_{2r+1}}(W_{2r+1,2}) &= U_1 \times U_2 \times \cdots \times U_{n-r-1} \times P_{n-r}^*U_{n-r} \cdot P_{n-r+1}^*U_{n-r+1} \cdot \ldots \cdot P_n^*U_n,\\ W_{2r+1,2}^0 &= \mathbf{C}_{U_0}\big(T_0 \cap \mathbf{C}_{L_{2r+1}}(W_{2r+1,2})\big)\\ &= U_1 \times U_2 \times \cdots \times U_{n-r-1} \times U_{n-r}^* \times U_{n-r+1}^* \times \cdots \times U_n^*.\end{aligned}$$

Similarly

$$\begin{aligned}T_0 \cap E_{2r+1} &= U_1 \times U_2 \times \cdots \times U_{n-r} \times P_{n-r+1}^*U_{n-r+1} \cdot P_{n-r+2}^*U_{n-r+2} \cdot \ldots \cdot P_n^*U_n\\ &\in \operatorname{Syl}_2 E_{2r+1},\\ W_{2r+1,3} &= \mathbf{C}_{U_0}(T_0 \cap E_{2r+1})\\ &= U_1 \times U_2 \times \cdots \times U_{n-r} \times U_{n-r+1}^* \times U_{n-r+2}^* \times \cdots \times U_n^*,\\ \mathbf{C}_{L_{2r+1}}(W_{2r+1,3}) &= U_1 \times T_1V_1 \times \cdots \times U_{n-r-1} \times T_{n-r-1}V_{n-r-1} \times U_{n-r} \cdot Q_{n-r}^*V_{n-r}\\ &\quad \cdot P_{n-r+1}^*U_{n-r+1} \cdot Q_{n-r+1}^*V_{n-r+1} \cdot \ldots \cdot P_n^*U_n \cdot Q_n^*V_n,\end{aligned}$$

so

$$\begin{aligned}S_0 \cap \mathbf{C}_{L_{2r+1}}(W_{2r+1,3}) &= V_1 \times V_2 \times \cdots \times V_{n-r-1} \times Q_{n-r}^*V_{n-r} \cdot Q_{n-r+1}^*V_{n-r+1} \cdot \ldots \cdot Q_n^*V_n,\\ W_{2r+1,3}^0 &= \mathbf{C}_{V_0}\big(S_0 \cap \mathbf{C}_{L_{2r+1}}(W_{2r+1,3})\big)\\ &= V_1 \times V_2 \times \cdots \times V_{n-r-1} \times V_{n-r}^* \times V_{n-r+1}^* \times \cdots \times V_n^*.\end{aligned}$$

We therefore take

$$\begin{aligned}W_{2r+1}^0 &= W_{2r+1,2}^0 \times W_{2r+1,3}^0\\ &= U_1 \times V_1 \times \cdots \times U_{n-r-1} \times V_{n-r-1} \times U_{n-r}^* \times V_{n-r}^* \times \cdots \times U_n^* \times V_n^*,\end{aligned}$$

$$\begin{aligned}\mathbf{E}^{2r+2}(\Sigma) &= \mathbf{C}_G\big(W^0_{2r+1}\big)\\ &= U_1 \times V_1 \times \cdots \times U_{n-r-1} \times V_{n-r-1} \times P^*_{n-r}U_{n-r} \cdot Q^*_{n-r}V_{n-r} \cdot \ldots \cdot P^*_n U_n \cdot Q^*_n V_n\\ &= P^*_{n-r}E_{2r+1} = E_{2r+2},\end{aligned}$$

which completes the proof of the claim.

The claim shows that

$$\begin{aligned}\mathbf{E}^{2n}(\Sigma) = E_{2n} &= P^*_1 Q^*_1 P^*_2 Q^*_2 \ldots P^*_n Q^*_n U_0 V_0\\ &= P^*_1 U_1 \cdot Q^*_1 V_1 \cdot P^*_2 U_2 \cdot Q^*_2 V_2 \cdot \ldots \cdot P^*_n U_n \cdot Q^*_n V_n,\end{aligned}$$

and it remains to deduce that $\mathbf{E}^{2n+1}(\Sigma) = E_{2n}$. By Lemma 11(c) it suffices to verify the conditions (L_2) and (L_3) with $H = E_{2n}$. Now

$$\begin{gathered}L_{2n} = \mathbf{N}_G(E_{2n}) = E_{2n},\\ \operatorname{Soc} E_{2n} = U^*_1 \times V^*_1 \times U^*_2 \times V^*_2 \times \cdots \times U^*_n \times V^*_n,\\ \operatorname{Soc}_{\pi-2} E_{2n} = \operatorname{Soc}_3 E_{2n} = V^*_1 \times V^*_2 \times \cdots \times V^*_n,\\ \mathbf{K}^{L_{2n}}_{\pi-2}(E_{2n}) = \mathbf{C}_{E_{2n}}\big(V^*_1 \times V^*_2 \times \cdots \times V^*_n\big) = E_{2n},\end{gathered}$$

which shows that (L_2) holds. Similarly

$$\begin{gathered}\operatorname{Soc}_{\pi-3} E_{2n} = \operatorname{Soc}_2 E_{2n} = U^*_1 \times U^*_2 \times \cdots \times U^*_n,\\ \mathbf{K}^{L_{2n}}_{\pi-3}(E_{2n}) = \mathbf{C}_{E_{2n}}\big(U^*_1 \times U^*_2 \times \cdots \times U^*_n\big) = E_{2n},\end{gathered}$$

which proves (L_3). $\square$

Remark. We have found it convenient to describe a group

$$G \leqslant G_0 = H_1 \times K_1 \times H_2 \times K_2 \times \cdots \times H_n \times K_n,$$

with a Sylow basis Σ such that $\mathbf{E}^{2n-1}(\Sigma) < \mathbf{E}^{2n}(\Sigma) = \mathbf{E}^{2n+1}(\Sigma)$, where $2n$ is even. A similar method can be used to construct a group

$$G^* \leqslant G^*_0 = H_1 \times K_1 \times H_2 \times K_2 \times \cdots \times H_{n-1} \times K_{n-1} \times H_n,$$

with a Sylow basis Σ^* such that $\mathbf{E}^{2n-2}(\Sigma^*) < \mathbf{E}^{2n-1}(\Sigma^*) = \mathbf{E}^{2n}(\Sigma^*)$, where $2n-1$ is odd.

Remark. It may be checked that in Example 2, $\mathbf{E}^i(\Sigma) = \mathbf{M}^i(G)$ for all $i \geqslant 0$, i.e., the chains

$$\mathbf{E}^0(\Sigma) < \mathbf{E}^1(\Sigma) < \mathbf{E}^2(\Sigma) < \cdots < \mathbf{E}^{2n}(\Sigma) = \mathbf{E}^{2n+1}(\Sigma)$$

and

$$\mathbf{M}^0(G) < \mathbf{M}^1(G) < \mathbf{M}^2(G) < \cdots < \mathbf{M}^{2n}(G) = \mathbf{M}^{2n+1}(G)$$

coincide.

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References

[1] D. Blessenohl, Über ordentliche Fittingklassen, Habilitationsschrift, Universität Kiel, 1976.
[2] K. Doerk, T. Hawkes, Finite Soluble Groups, de Gruyter, 1992.