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Relative commutator calculus in Chevalley groups [☆]

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ABSTRACT

We revisit localisation and patching method in the setting of Chevalley groups. Introducing certain subgroups of relative elementary Chevalley groups, we develop relative versions of the conjugation calculus and the commutator calculus in Chevalley groups $G(\Phi, R)$, $\text{rk}(\Phi) \geq 2$, which are both more general, and substantially easier than the ones available in the literature. For classical groups such relative commutator calculus has been recently developed by the authors in Hazrat, Zhang (2011) [34], Hazrat et al. (2011) [33]. As an application we prove the mixed commutator formula,

$$[E(\Phi, R, \mathfrak{a}), G(\Phi, R, \mathfrak{b})] = [E(\Phi, R, \mathfrak{a}), E(\Phi, R, \mathfrak{b})],$$

for two ideals $\mathfrak{a}, \mathfrak{b} \trianglelefteq R$. This answers a problem posed in a paper by Alexei Stepanov and the second author.

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بر رهگذرم هزار جا دام نهی
گویی که بگیرمت اگر گام نهی
خیام

*O Life, you put thousand traps in my way
Dare to try, is what you clearly say*
Omar Khayam

1. Introduction

One of the most powerful ideas in the study of groups of points of reductive groups over rings is localisation. It allows to reduce many important problems over arbitrary commutative rings, to similar problems for semi-local rings. Localisation comes in a number of versions. The two most familiar ones are *localisation and patching*, proposed by Daniel Quillen [55] and Andrei Suslin [65], and *localisation-completion*, proposed by Anthony Bak [8].

Originally, the above papers addressed the case of the general linear group $GL(n, R)$. Soon thereafter, Suslin himself, Vyacheslav Kopeiko, Marat Tulenbaev, Giovanni Taddei, Leonid Vaserstein, Li Fuan, Eiichi Abe, You Hong, and others proposed working versions of localisation and patching for other classical groups, such as symplectic and orthogonal ones, as well as exceptional Chevalley groups, see, for example, [35,70,66,68,37,38,67,74] and further references in [76,13,62,32]. Recently, these methods were further generalised to unitary groups, and isotropic reductive groups, by Tony Bak, Alexei Stepanov, ourselves, Victor Petrov, Anastasia Stavrova, Ravi Rao, Rabeya Basu, and others [9,16,10–13, 17,26,27,31,50–53,56,63].

As a matter of fact, both methods rely on a large body of common calculations, and technical facts, known as *conjugation calculus* and *commutator calculus*. Their objective is to obtain explicit estimates of the modulus of continuity in s -adic topology for conjugation by a specific matrix, in terms of the powers of s occurring in the denominators of its entries, and similar estimates for commutators of two matrices.

These calculations are *elementary*, in the strict technical sense of [87]. But being elementary, they are by no means easy. Sometimes these calculations are even called the *yoga of conjugation*, and the *yoga of commutators*, to stress the overwhelming feeling of technical strain and exertion.

A specific motivation for the present work was the desire to create tools to prove *relative* versions of structure results for Chevalley groups. Here we list three such immediate applications, in which we were particularly interested.

- Description of subnormal subgroups and subgroups normalised by the relative elementary subgroup. In full generality such description is only available for classical groups [97–99,96], but, apart from the case of $GL(n, R)$ [91,7,71,39,75,73], sharp bounds are not obtained even in this case.
- Results on description of intermediate subgroups, such as, for example, overgroups of regularly embedded semi-simple subgroups, overgroups of exceptional Chevalley groups in an appropriate $GL(n, R)$, etc., see, for example, [41,42,64] and [40,77,89] for a survey and further references.
- Generalisation of the mixed commutator formula

$$[E(n, R, \alpha), GL(n, R, b)] = [E(n, R, \alpha), E(n, R, b)],$$

to exceptional Chevalley groups.

The first two problems are discussed in somewhat more detail in the last section, complete proofs are relegated to subsequent papers by the authors. Here we discuss only the third one, relative standard commutator formulae, another major objective of the present paper, apart from developing the localisation machinery itself.

The above formula was proved in the setting of general linear groups by Alexei Stepanov and the second author [88]. This formula is a common generalisation of both absolute standard commutator formulae. At the stable level, absolute commutator formulae were first established in the foundational work of Hyman Bass [14]. In another decade, Andrei Suslin, Leonid Vaserstein, Zenon Borewicz, and the second author [65,70,19,62] discovered that for commutative rings similar formulae hold for all $n \geq 3$. For two relative subgroups such formulae were proven only at the stable level, by Alec Mason [45–48].

However, the proof in [88] relied on a very strong and precise form of decomposition of unipotents [62], and was not likely to easily generalise to groups of other types. Stepanov and the second-named author raised the following problems.

- Establish the relative standard commutator formula via localisation method [88, Problem 2].
- Generalise the relative standard commutator formula to Bak's unitary groups and to Chevalley groups [88, Problem 1].

In the paper [34] the first and the third authors developed relative versions of conjugation calculus and commutator calculus in the general linear group $GL(n, R)$, thus solving [88, Problem 2]. In [33] we developed a similar relative conjugation calculus in Bak's unitary groups, thus accounting for all even classical groups.

In the present paper, which is a direct sequel of [34,33], we in a similar way evolve relative conjugation calculus and commutator calculus in arbitrary Chevalley groups. Actually, the present paper does not depend on the calculations from [31,63]. Instead, here we develop *relative* versions of the yoga of conjugation, and the yoga of commutators *from scratch*, in a more general setting. The reason is that in the relative setting it is not enough to prove the continuity of conjugation by g . What we now need, is its *equi-continuity* on all congruence subgroups $G(\Phi, R, I)$. In other words, we need explicit bounds for the modulus of continuity, *uniform* in the ideal I . The resulting versions of conjugation calculus and commutator calculus are both *substantially* more powerful, and *easier* than the ones available in the literature.

The overall scheme is always the same as devised by the first and the second authors in [31] (which, in turn, was a further elaboration of [8,26,27]), and as later implemented by Alexei Stepanov and the second author [63] in a slightly more precise version, with length bounds. However, we propose several major technical innovations, and simplifications. Most importantly, following [34] and [33] we construct another base of s -adic neighbourhoods of 1, consisting of *partially* relativised elementary groups, and prove all results not at the absolute, but at the relative level.

As an immediate application of our methods we prove the following result which, together with [33], solves [88, Problem 1] and [10, Problem 4]. Specifically, for Chevalley groups the same question was reiterated as [33, Problem 6]. Definitions of the elementary subgroup $E(\Phi, R, \alpha)$ and the full congruence subgroup $C(\Phi, R, \alpha)$ of level $\alpha \trianglelefteq R$ are recalled in Sections 3, 4.

Theorem 1. *Let Φ be a reduced irreducible root system, $\text{rk}(\Phi) \geq 2$. Further, let R be a commutative ring, and $\alpha, \mathfrak{b} \trianglelefteq R$ be two ideals of R . In the cases $\Phi = C_2, G_2$ assume that R does not have residue fields \mathbb{F}_2 of 2 elements and in the case $\Phi = C_l, l \geq 2$, assume additionally that any $c \in R$ is contained in the ideal $c^2 R + 2cR$. Then*

$$[E(\Phi, R, \alpha), C(\Phi, R, \mathfrak{b})] = [E(\Phi, R, \alpha), E(\Phi, R, \mathfrak{b})].$$

Actually, as we shall see in Section 10, this commutator formula is equivalent to a slightly weaker formula

$$[E(\Phi, R, \alpha), G(\Phi, R, \mathfrak{b})] = [E(\Phi, R, \alpha), E(\Phi, R, \mathfrak{b})].$$

Before, for exceptional groups this theorem was known in the two special cases,¹ where $R = \mathfrak{a}$ or where $R = \mathfrak{b}$, see [68,72].

With the above precise condition Theorem 1 is proven in Section 10. Our localisation proof in Section 9 requires a somewhat stronger condition $2 \in R^*$, in the cases $\Phi = C_2, G_2$. Strictly speaking, this stronger condition is not necessary, we only use it to simplify the proof of the induction base of the relative commutator calculus in Section 8.

This small compromise allows us to spare some 5–6 pages of calculations, and to eventually develop a more technical and powerful version of relative localisation with two denominators. As a matter of fact, the main result of the present paper is not the above Theorem 1 itself, but rather Theorem 2 established in Section 9. Theorem 2 looks too technical to stand well on its own, but actually it is *terribly* much stronger and more general than Theorem 1. It is devised to be used in our subsequent publications to derive *multiple* commutator formulae, which are simultaneous generalisations of Theorem 1 and nilpotency of K_1 .

However, not to further complicate things, we decided to relegate the detailed analysis of the rank 2 cases to a subsequent publication, especially that it should be carried in a more general setting, much more technically demanding. In the meantime, let us explain, why the rank 2 case, namely the types C_2 and G_2 , require some serious extra care. This is due to the following circumstances.

- In these cases, the elementary group $E(\Phi, R)$ is not perfect when R has residue field \mathbb{F}_2 , which accounts for the first assumption in Theorem 1.
- There is *substantially* less freedom in the Chevalley commutator formula, especially for groups of type C_2 , which accounts for the additional assumption in this case.
- There is somewhat less freedom also in the choice of semi-simple factors.
- Most importantly, in these cases it is natural to define relative subgroups not in terms of ideals, but in terms of form ideals, or even more general structures, such as radices [21,22].

As in [33], in the present paper we concentrate on actual calculations. The history of localisation methods, the philosophy behind them, and their possible applications are extensively discussed in our mini-survey with Alexei Stepanov [29]. There, we also describe another remarkable recent advance, *universal* localisation developed by Stepanov [61]. For *algebraic* groups, universal localisation allows – among other things – to remove dependence on the dimension of the ground ring R in the results of [63]. Unfortunately, generalised unitary groups are not always algebraic, so that our width bounds for commutators in unitary groups [30] still depend on $\dim(\text{Max}(R))$.

The paper is organised as follows. In Sections 2–4 we recall basic notation, and some background facts, used in the sequel. In Section 5 we discuss injectivity of localisation homomorphism and in Section 6 we calculate levels of mixed commutator subgroups. The next two sections constitute the technical core of the paper. Namely, in Section 7, and in Section 8 we develop relative conjugation calculus, and relative commutator calculus in Chevalley groups, respectively. After that we are in a position to give a localisation proof of Theorem 1 – and in fact of a much stronger Theorem 2 – in Section 9. On the other hand, using level calculations in Section 10 we give another proof of Theorem 1, deducing it from the *absolute* standard commutator formula. There we also obtain slightly more precise results in some special situations, such as Theorem 3, which completely calculates the relative commutator subgroup in the important case, where \mathfrak{a} and \mathfrak{b} are comaximal, $\mathfrak{a} + \mathfrak{b} = R$. Finally, in Section 11 we state and briefly review some further related problems.

¹ Actually, after submitting the present paper we learned a very important paper by You Hong [92], which contains essentially the same result, with a proof very close in spirit to our *second* proof here. A slight technical difference is that [92] relies on straightforward commutator identities for individual elements, whereas we invoke the three subgroup lemma, which makes the argument slightly shorter and more transparent. Also, as too many other publications, [92] contains a minor inaccuracy – one of the *hazar* = *thousand* traps, of which Omar speaks! – in that an extra condition is imposed only in the case C_2 , whereas it is requisite for all C_l , $l \geq 2$. Fortunately, we were not aware of [92], when writing [34,33] and the present paper. Otherwise, we would had been much less eager to develop a *localisation* approach towards the proof of Theorem 1. We are convinced that the main contribution of the present paper are the relative versions of conjugation calculus, commutator calculus, and patching, developed in Sections 7–9. They already have several further important applications, which go well beyond Theorem 1 or the main results of [31,10,63].

2. Chevalley groups

In this and the next two sections we recall some preliminaries pertaining to Chevalley groups over rings, and their relative subgroups, see [18,49,57,59,76,87,54,78] for details and further references. As above, let Φ be a reduced irreducible root system of rank $l = \text{rk}(\Phi)$, and $P, Q(\Phi) \leq P \leq P(\Phi)$ be a lattice between the root lattice $Q(\Phi)$ and the weight lattice $P(\Phi)$. Usually, we fix an order on Φ and denote by $\Pi = \{\alpha_1, \dots, \alpha_l\}$, Φ^+, Φ^- the corresponding sets of fundamental, positive, and negative roots, respectively. Recall, that $Q(\Phi) = \mathbb{Z}\alpha_1 \oplus \dots \oplus \mathbb{Z}\alpha_l$ and $P(\Phi) = \mathbb{Z}\varpi_1 \oplus \dots \oplus \mathbb{Z}\varpi_l$, where $\varpi_1, \dots, \varpi_l$ are the corresponding fundamental weights. Finally, $W = W(\Phi)$ denotes the Weyl group of Φ .

Further, let R be a commutative ring. We denote by $G = G_P(\Phi, R)$ the Chevalley group of type (Φ, P) over R , by $T = T_P(\Phi, R)$ a split maximal torus of G and by $E = E_P(\Phi, R)$ the corresponding (absolute) elementary subgroup. Usually P does not play role in our calculations and we suppress it in the notation.

The elementary group $E(\Phi, R)$ is generated by all root unipotents $x_\alpha(a)$, $\alpha \in \Phi$, $a \in R$, elementary with respect to T . The fact that E is normal in G means exactly that E does not depend on the choice of T .

Let G be a group. For any $x, y \in G$, ${}^x y = xyx^{-1}$ denotes the left x -conjugate of y . Let $[x, y] = xyx^{-1}y^{-1}$ denote the commutator of x and y . We will make frequent use of the following formulae,

$$(C1) \quad [x, yz] = [x, y] \cdot {}^y [x, z],$$

$$(C2) \quad [xy, z] = {}^x [y, z] \cdot [x, z],$$

$$(C3) \quad \text{Hall–Witt identity}$$

$${}^x [[x^{-1}, y], z] = {}^x [[y, x^{-1}]^{-1}, z] = {}^y [x, [y^{-1}, z]] \cdot {}^z [y, [z^{-1}, x]],$$

$$(C4) \quad [x, {}^y z] = {}^y [{}^{y^{-1}} x, z],$$

$$(C5) \quad [{}^y x, z] = {}^y [x, {}^{y^{-1}} z].$$

Most of the calculations in the present paper are based on the Steinberg relations

$$(R1) \quad \text{Additivity of } x_\alpha,$$

$$x_\alpha(a+b) = x_\alpha(a)x_\alpha(b).$$

$$(R2) \quad \text{Chevalley commutator formula}$$

$$[x_\alpha(a), x_\beta(b)] = \prod_{i\alpha+j\beta \in \Phi} x_{i\alpha+j\beta}(N_{\alpha\beta ij} a^i b^j),$$

where $\alpha \neq -\beta$ and $N_{\alpha\beta ij}$ are the structure constants which do not depend on a and b . Notice, though, that for $\Phi = G_2$ they may depend on the order of the roots in the product on the right hand side. The following observation was made by Chevalley himself: let $\alpha - p\beta, \dots, \alpha - \beta, \alpha, \alpha + \beta, \dots, \alpha + q\beta$ be the α -series of roots through β , then $N_{\alpha\beta 11} = \pm(p+1)$ and $N_{\alpha\beta 12} = \pm(p+1)(p+2)/2$.

Let i_Φ be the largest integer which may appear as i in a root $i\alpha + j\beta \in \Phi$ for all $\alpha, \beta \in \Phi$. Obviously $i_\Phi = 1, 2$ or 3 , depending on whether Φ is simply laced, doubly laced or triply laced. The following result makes the proof for $\Phi \neq C_l$ slightly easier than for the symplectic case. Recall that $A_1 = C_1$ and $B_2 = C_2$ so that root systems of types A_1 and B_2 are symplectic. All roots of A_1 are long.

Our calculations in Section 7 and Section 8 rely on the following result, which is Lemma 2.12 in [31].

Lemma 1. Let $\beta \in \Phi$ and either $\Phi \neq C_l$ or β is short. Then there exist two roots $\gamma, \delta \in \Phi$ such that $\beta = \gamma + \delta$ and $N_{\gamma\delta 11} = 1$.

If $\Phi = C_l$, $l \geq 2$, and β is long, then there exist two roots $\gamma, \delta \in \Phi$ such that either $\beta = \gamma + 2\delta$ and $N_{\gamma\delta 12} = 1$, or $\beta = 2\gamma + \delta$ and $N_{\gamma\delta 21} = 1$.

In the sequel we also use semi-simple root elements. Namely, for $\alpha \in \Phi$ and $\varepsilon \in R^*$ we set

$$w_\alpha(\varepsilon) = x_\alpha(\varepsilon)x_{-\alpha}(-\varepsilon^{-1})x_\alpha(\varepsilon), \quad h_\alpha(\varepsilon) = w_\alpha(\varepsilon)w_\alpha(1)^{-1}.$$

Let $H(\Phi, R)$ be the subgroup of $T(\Phi, R)$, generated by all $h_\alpha(\varepsilon)$, $\alpha \in \Phi$, $\varepsilon \in R^*$.

Clearly, $H(\Phi, R) \leq E(\Phi, R)$, and in fact, $H(\Phi, R) = T(\Phi, R) \cap E(\Phi, R)$. In particular, for simply connected group one has

$$H_{sc}(\Phi, R) = T_{sc}(\Phi, R) = \text{Hom}(P(\Phi), R^*).$$

For non-simply connected groups, specifically, for the adjoint ones, $T(\Phi, R)$ is usually somewhat larger, than $H(\Phi, R)$. For the proof of our main theorem we have to understand, what the generators of $T(\Phi, R)$ look like in this case, see [80] for explicit constructions and many further references.

Let $\omega \in P(\Phi^\vee)$, by definition $(\alpha, \omega) \in \mathbb{Z}$ for all $\alpha \in \Phi$. The adjoint torus contains weight elements $h_\omega(\varepsilon)$, which commute with all elements from T and satisfy the following commutator relation:

$$h_\omega(\varepsilon)x_\alpha(\xi)h_\omega(\varepsilon)^{-1} = x_\alpha(\varepsilon^{(\alpha, \omega)}\xi),$$

for all $\alpha \in \Phi$ and all $\xi \in R$. For $\Phi = E_8, F_4$ and G_2 , one has $P(\Phi) = Q(\Phi)$, in particular, in these cases $T_{ad}(\Phi, R) = H_{ad}(\Phi, R)$. For other cases $T_{ad}(\Phi, R)$ is generated by $H_{ad}(\Phi, R)$ and some weight elements.

In Section 9 we can refer to either one of the following lemmas. The first one follows from [80], Proposition 1, while the second one is well known and obvious.

Lemma 2. The torus $T_{ad}(\Phi, R)$ is generated by $H_{ad}(\Phi, R)$ and weight elements $h_\omega(\varepsilon)$, where $\varepsilon \in R^*$, and ω are the following weights

- $\omega = \varpi_1$, for $\Phi = A_l, B_l$ and E_6 ,
- $\omega = \varpi_l$, for $\Phi = C_l$,
- $\omega = \varpi_1, \varpi_l$, for $\Phi = D_l$,
- $\omega = \varpi_7$, for $\Phi = E_7$.

Lemma 3. Assume that either $\Phi \neq C_l$, or $\alpha \in \Phi$ is short. Then for any $\varepsilon \in R^*$ there exists an $h \in H(\Phi, R)$ such that $hx_\alpha(\xi)h^{-1} = x_\alpha(\varepsilon\xi)$, for all $\xi \in R$.

In the exceptional case, where $\Phi = C_l$ and $\alpha \in \Phi$ is long, $hx_\alpha(\xi)h^{-1} = x_\alpha(\varepsilon^2\xi)$, for all $h \in H(\Phi, R)$. On the other hand, if $\alpha \in \Phi^+$ is a positive long root,

$$h_{\varpi_l}(\varepsilon)x_\alpha(\xi)h_{\varpi_l}(\varepsilon)^{-1} = x_\alpha(\varepsilon\xi).$$

Clearly, in the last case for a negative long root one has $h_{\varpi_l}(\varepsilon)x_\alpha(\xi)h_{\varpi_l}(\varepsilon)^{-1} = x_\alpha(\varepsilon^{-1}\xi)$. In the vector representation of the extended simply connected Chevalley group $\overline{G}(C_l, R) = \text{GSp}(2l, R)$ this weight element has the form

$$h_{\varpi_l}(\varepsilon) = \text{diag}(\varepsilon, \dots, \varepsilon, 1, \dots, 1).$$

It follows that – with the only possible exception when $\Phi = C_l$ and α is long – for any $\alpha \in \Phi$ and any $h \in T(\Phi, R)$ there exists a $g \in H(\Phi, R)$ such that $gx_\alpha(\xi)g^{-1} = hx_\alpha(\xi)h^{-1}$. In particular, $g^{-1}h$ commutes with $x_\alpha(\xi)$. However, in the exceptional case, where $\Phi = C_l$ and α is long, no such g exists in general. One can only ensure the existence of such a $g \in H(\Phi, R)$ that $g^{-1}h = h_{\varpi_l}(\varepsilon)$ for some $\varepsilon \in R^*$.

3. Relative elementary subgroups

In this section we recall the definitions of relative subgroups, and some basic facts used in the sequel. The usual one-parameter relative subgroups are well known. However, for multiply laced systems one should consider two-parameter relative subgroups, with one parameter corresponding to short roots, and another one to long roots. Such two-parameter relative subgroups were introduced and studied by Eiichi Abe [1–5] and Michael Stein [57].

Let \mathfrak{a} be an additive subgroup of R . Then $E(\Phi, \mathfrak{a})$ denotes the subgroup of E generated by all elementary root unipotents $x_\alpha(t)$ where $\alpha \in \Phi$ and $t \in \mathfrak{a}$. Further, let L denote a nonnegative integer and let $E^L(\Phi, \mathfrak{a})$ denote the subset of $E(\Phi, \mathfrak{a})$ consisting of all products of L or fewer elementary root unipotents $x_\alpha(t)$, where $\alpha \in \Phi$ and $t \in \mathfrak{a}$. In particular, $E^1(\Phi, \mathfrak{a})$ is the set of all $x_\alpha(t)$, $\alpha \in \Phi$, $t \in \mathfrak{a}$.

When $\mathfrak{a} \leq R$ is an ideal of R , the elementary group $E(\Phi, \mathfrak{a})$ of level \mathfrak{a} should be distinguished from the relative elementary subgroup $E(\Phi, R, \mathfrak{a})$ of level \mathfrak{a} . By definition $E(\Phi, R, \mathfrak{a})$ is the normal closure of $E(\Phi, \mathfrak{a})$ in the absolute elementary subgroup $E(\Phi, R)$. In general $E(\Phi, R, \mathfrak{a})$ is not generated by elementary transvections of level \mathfrak{a} . Below we describe its generators for $\text{rk}(\Phi) \geq 2$. The following result can be found in [57,69].

Lemma 4. *In the case $\Phi \neq C_l$ one has $E(\Phi, \mathfrak{a}) \geq E(\Phi, R, \mathfrak{a}^2)$. In the exceptional case $\Phi = C_l$ one has $E(\Phi, \mathfrak{a}) \geq E(\Phi, R, (2R + \mathfrak{a})\mathfrak{a}^2)$.*

Let \mathfrak{a} be an ideal of R . Denote by \mathfrak{a}_2 the ideal, generated by 2ξ and ξ^2 for all $\xi \in \mathfrak{a}$. The first component \mathfrak{a} of an admissible pair $(\mathfrak{a}, \mathfrak{b})$ is an ideal of R , parametrising short roots. When $\Phi \neq C_l$ the second component \mathfrak{b} , $\mathfrak{a}_2 \leq \mathfrak{b} \leq \mathfrak{a}$, is also an ideal, parametrising long roots. In the exceptional case $\Phi = C_l$ the second component \mathfrak{b} is an additive subgroup stable under multiplication by ξ^2 , $\xi \in R$ (in other words, it is a relative form parameter in the sense of Bak [13,25,32]). A similar notion can be introduced for the type G_2 as well, but in this case one should replace 2 by 3 everywhere in the above definition.

Now the relative elementary subgroup, corresponding to an admissible pair $(\mathfrak{a}, \mathfrak{b})$, is defined as follows:

$$E(\Phi, R, \mathfrak{a}, \mathfrak{b}) = \langle x_\alpha(\xi), \alpha \in \Phi_s, \xi \in \mathfrak{a}; x_\beta(\zeta), \beta \in \Phi_l, \zeta \in \mathfrak{b} \rangle^{E(\Phi, R)},$$

where Φ_s and Φ_l are the sets of short and long roots in Φ , respectively. The following results can be found in [57,3,4].

Lemma 5. *Let $\text{rk}(\Phi) \geq 2$. When $\Phi = B_2$ or $\Phi = G_2$ assume moreover that R has no residue fields \mathbb{F}_2 of 2 elements. Then the elementary subgroup $E(\Phi, R, \mathfrak{a}, \mathfrak{b})$ is $E(\Phi, R)$ -perfect, in other words,*

$$[E(\Phi, R), E(\Phi, R, \mathfrak{a}, \mathfrak{b})] = E(\Phi, R, \mathfrak{a}, \mathfrak{b}).$$

In particular, $E(\Phi, R)$ is perfect.

Lemma 6. *As a subgroup $E(\Phi, R, \mathfrak{a}, \mathfrak{b})$ is generated by the elements*

$$z_\alpha(\xi, \zeta) = x_{-\alpha}(\zeta)x_\alpha(\xi)x_{-\alpha}(-\zeta),$$

where $\xi \in \mathfrak{a}$ for $\alpha \in \Phi_s$ and $\xi \in \mathfrak{b}$ for $\alpha \in \Phi_l$, while $\zeta \in R$.

Actually, in the sequel we mostly use these results in the special case, where $\mathfrak{a} = \mathfrak{b}$.

4. Congruence subgroups

Usually, one defines congruence subgroups as follows. An ideal $\mathfrak{a} \trianglelefteq R$ determines the reduction homomorphism $\rho_{\mathfrak{a}} : R \longrightarrow R/\mathfrak{a}$. Since $G(\Phi, _)$ is a functor from rings to groups, this homomorphism induces reduction homomorphism $\rho_{\mathfrak{a}} : G(\Phi, R) \longrightarrow G(\Phi, R/\mathfrak{a})$.

- The kernel of the reduction homomorphism $\rho_{\mathfrak{a}}$ modulo \mathfrak{a} is called the principal congruence subgroup of level \mathfrak{a} and is denoted by $G(\Phi, R, \mathfrak{a})$.
- The full pre-image of the centre of $G(\Phi, R/\mathfrak{a})$ with respect to the reduction homomorphism $\rho_{\mathfrak{a}}$ modulo \mathfrak{a} is called the full congruence subgroup of level \mathfrak{a} , and is denoted by $C(\Phi, R, \mathfrak{a})$.

A more general notion of congruence subgroup was introduced in [28]. Namely, consider a linear action of G on a right R -module V and let $U \leq V$ be a G -submodule. Then we can define a set

$$G(V, U) = \{g \in G \mid \forall v \in V, \quad gv - v \in U\}.$$

This set is in fact a *normal* subgroup of G .

An application of this construction to a Chevalley group $G = G(\Phi, R)$ and its rational module V allows us to recover the usual subgroups. For any module V and any ideal $\mathfrak{a} \trianglelefteq R$ the product $U = V\mathfrak{a}$ is a G -submodule. The following result is [28, Lemma 6].

Lemma 7. *When V is a faithful rational module, $G(V, V\mathfrak{a}) = G(\Phi, R, \mathfrak{a})$ is the usual principal congruence subgroup of level \mathfrak{a} .*

In matrix language, this lemma means that the principal congruence subgroup of level \mathfrak{a} can be defined as

$$G(\Phi, R, \mathfrak{a}) = G(\Phi, R) \cap \mathrm{GL}(n, R, \mathfrak{a}),$$

for any *faithful* rational representation $G(\Phi, R) \leq \mathrm{GL}(n, R)$.

Clearly, for any rational representation $\phi : G(\Phi, R) \longrightarrow \mathrm{GL}(n, R)$, one has the inclusions

$$\phi^{-1}(G(\Phi, R) \cap \mathrm{GL}(n, R, \mathfrak{a})) \leq C(\Phi, R, \mathfrak{a}) \leq \phi^{-1}(G(\Phi, R) \cap C(n, R, \mathfrak{a})),$$

for the full congruence subgroup. In the general case there is no reason, why either of these inclusions should be an equality. However, there is one important special case, where the left inclusion becomes an equality [28, Lemma 7].

Lemma 8. *When $V = L$ is the Lie algebra of $G(\Phi, R)$, considered as the adjoint module, then $G(L, L\mathfrak{a}) = C(\Phi, R, \mathfrak{a})$ is the usual full congruence subgroup of level \mathfrak{a} .*

The following result, Theorem 2 of [28], asserts that three possible definitions of the full congruence subgroup coincide.

Lemma 9. *Let Φ be a reduced irreducible root system of rank ≥ 2 , R be a commutative ring, $(\mathfrak{a}, \mathfrak{b})$ an admissible pair. Then the following four subgroups coincide:*

$$\begin{aligned}
C(\Phi, R, \alpha, b) &= \{g \in G(\Phi, R) \mid [g, E(\Phi, R)] \leq E(\Phi, R, \alpha, b)\} \\
&= \{g \in G(\Phi, R) \mid [g, E(\Phi, R)] \leq C(\Phi, R, \alpha, b)\} \\
&= \{g \in G(\Phi, R) \mid [g, G(\Phi, R)] \leq C(\Phi, R, \alpha, b)\}.
\end{aligned}$$

In fact, in [28] we established standard commutator formulae for the case, where one argument is an *absolute* subgroup, whereas the second argument is a relative subgroup with *two* parameters. In particular, the following result is Theorem 1 of [28]. Of course, in all cases, except Chevalley groups of type F_4 , it was known before [13,51,22].

Lemma 10. *Let Φ be a reduced irreducible root system of rank ≥ 2 , R be a commutative ring, (α, b) an admissible pair. In the case, where $\Phi = C_2$ or $\Phi = G_2$ assume moreover that R has no residue fields \mathbb{F}_2 of 2 elements. Then the following standard commutator formulae holds*

$$[G(\Phi, R), E(\Phi, R, \alpha, b)] = [E(\Phi, R), C(\Phi, R, \alpha, b)] = E(\Phi, R, \alpha, b).$$

We will use the following form of Gauß decomposition, stated by Eiichi Abe [1,5]. Namely, let $\alpha \leq R$ be an ideal of R . We denote by $T(\Phi, R, \alpha)$ the subgroup of the split maximal torus $T(\Phi, R)$, consisting of all elements congruent to e modulo α ,

$$T(\Phi, R, \alpha) = T(\Phi, R) \cap G(\Phi, R, \alpha).$$

As usual, we set

$$U(\Phi, \alpha) = \langle x_\alpha(a), \alpha \in \Phi^+, a \in \alpha \rangle, \quad U^-(\Phi, \alpha) = \langle x_\alpha(a), \alpha \in \Phi^-, a \in \alpha \rangle.$$

Obviously, $U(\Phi, \alpha), U^-(\Phi, \alpha) \leq E(\Phi, \alpha)$.

Lemma 11. *Let α be an ideal of R contained in the Jacobson radical $\text{Rad}(R)$. Then*

$$G(\Phi, R, \alpha) = U(\Phi, \alpha)T(\Phi, R, \alpha)U^-(\Phi, \alpha).$$

We will mostly use this lemma in the following form, see [31], Lemma 2.10.

Lemma 12. *If α is an ideal of local ring R then*

$$G(\Phi, R, \alpha) = E(\Phi, \alpha)T(\Phi, R, \alpha).$$

5. Injectivity of localisation homomorphism

Let us fix some notation. Let R be a commutative ring with 1, S be a multiplicative system in R and $S^{-1}R$ be the corresponding localisation. We will mostly use localisation with respect to the two following types of multiplicative systems.

- If $s \in R$ and the multiplicative system S coincides with $\langle s \rangle = \{1, s, s^2, \dots\}$ we usually write $\langle s \rangle^{-1}R = R_s$.
- If $\mathfrak{m} \in \text{Max}(R)$ is a maximal ideal in R , and $S = R \setminus \mathfrak{m}$, we usually write $(R \setminus \mathfrak{m})^{-1}R = R_{\mathfrak{m}}$.

We denote by $F_S : R \longrightarrow S^{-1}R$ the canonical ring homomorphism called the localisation homomorphism. For the two special cases mentioned above, we write $F_S : R \longrightarrow R_S$ and $F_a : R \longrightarrow R_a$, respectively.

When we write an element as a fraction, like a/s or $\frac{a}{s}$, we always think of it as an element of some localisation $S^{-1}R$, where $s \in S$. If s were actually invertible in R , we would have written as^{-1} instead.

5.1. The property of these functors which will be crucial for what follows is that they *commute with direct limits*. In other words, if $R = \varinjlim R_i$, where $\{R_i\}_{i \in I}$ is an inductive system of rings, then $X(\Phi, \varinjlim R_i) = \varinjlim X(\Phi, R_i)$. We will use this property in the following two situations.

- First, let R_i be the inductive system of all finitely generated subrings of R with respect to inclusion. Then $X = \varinjlim X(\Phi, R_i)$, which reduces most of the proofs to the case of Noetherian rings.
- Second, let S be a multiplicative system in R and $R_s, s \in S$, the inductive system with respect to the localisation homomorphisms: $F_t : R_s \longrightarrow R_{st}$. Then $X(\Phi, S^{-1}R) = \varinjlim X(\Phi, R_s)$, which allows to reduce localisation in any multiplicative system to principal localisations.

Our proofs rely on the injectivity of localisation homomorphism F_S . On the group $G(\Phi, R)$ itself it is seldom injective, but its restrictions to appropriate congruence subgroups often are, see the discussion in [29]. Below we cite two important typical cases, Noetherian rings [8] and semi-simple rings [63].

Lemma 13. Suppose R is Noetherian and $s \in R$. Then there exists a natural number k such that the homomorphism $F_s : G(\Phi, R, s^k R) \longrightarrow G(\Phi, R_s)$ is injective.

Proof. The homomorphism $F_s : G(\Phi, R, s^k R) \longrightarrow G(\Phi, R_s)$ is injective whenever $F_s : s^k R \longrightarrow R_s$ is injective. Let $\alpha_i = \text{Ann}_R(s^i)$ be the annihilator of s^i in R . Since R is Noetherian, there exists k such that $\alpha_k = \alpha_{k+1} = \dots$. If $s^k a$ vanishes in R_s , then $s^i s^k a = 0$ for some i . But since $\alpha_{k+i} = \alpha_k$, already $s^k a = 0$ and thus $s^k R$ injects in R_s . \square

Lemma 14. If $\text{Rad}(R) = 0$, then $F_s : G(\Phi, R, sR) \longrightarrow G(\Phi, R_s)$ is injective for all $s \in R, s \neq 0$.

Proof. It suffices to prove that $F_s : sR \longrightarrow R_s$ is injective. Suppose that $s\xi \in sR$ goes to 0 in R_s . Then there exists an $m \in \mathbb{N}$ such that $s^m s\xi = 0$. It follows that $(s\xi)^{m+1} = 0$ and since R is semi-simple, $s\xi = 0$. \square

In [31] we used reduction to Noetherian rings, whereas in [63] reduction to semi-simple rings was used instead.

6. Levels of mixed commutators

In this section we establish some obvious facts, concerning the lower and the upper levels of mixed commutators

$$[E(\Phi, R, a), E(\Phi, R, b)] \leq [G(\Phi, R, a), G(\Phi, R, b)].$$

Unlike most other results of the present paper, the next lemma also holds for $\text{rk}(\Phi) = 1$.

Lemma 15. Then for any two ideals a and b of the ring R one has

$$E(\Phi, R, a, c)E(\Phi, R, b, d) = E(\Phi, R, a + b, c + d).$$

Proof. Additivity of the elementary root unipotents $x_\alpha(a+b) = x_\alpha(a)x_\alpha(b)$, where $\alpha \in \Phi$, while $a \in \mathfrak{a}$, $b \in \mathfrak{b}$ for $\alpha \in \Phi_s$, and $a \in \mathfrak{c}$, $b \in \mathfrak{d}$ for $\alpha \in \Phi_l$, implies that the left hand side contains generators of the right hand side. The product of two normal subgroups is normal in $E(\Phi, R)$. \square

As a preparation to the calculation of lower level, we generalise Lemma 4. It is a toy version of the main results of the present paper, whose proof heavily depends on Lemma 6. There one considers

$$z_\alpha(ab, \zeta) = x_{-\alpha}(\zeta)x_\alpha(ab).$$

The idea is to express $x_\alpha(ab)$ as the commutator of two root elements $[x_\beta(a), x_\gamma(b)]$, where $\beta + \gamma = \alpha$, plus, possibly, some tail. Now, neither the roots β, γ , nor the roots appearing in the tail, are opposite to $-\alpha$, and thus we can distribute conjugation by $x_{-\alpha}(\zeta)$ and apply the Chevalley commutator formula to each occurring factor. The first explicit appearance of this idea, which we were able to trace in the literature, was in Bass–Milnor–Serre foundational work [15].

In the following lemma, we should distinguish the ideal \mathfrak{a}^2 , generated by the products ab , where $a, b \in \mathfrak{a}$, from the ideal $\mathfrak{a}^{\square 2}$, generated by a^2 , where $a \in \mathfrak{a}$. Clearly, when $2 \in R^*$ these ideals coincide, but this case is trivial anyway.

Lemma 16. Let $\text{rk}(\Phi) \geq 2$ and further let \mathfrak{a} and \mathfrak{b} be two ideals of R . Assume that either $\Phi \neq C_l$, or $2 \in R^*$. Then one has

$$E(\Phi, R, \mathfrak{a}\mathfrak{b}) \leq E(\Phi, \mathfrak{a} + \mathfrak{b}).$$

In the exceptional case, where $\Phi = C_l$ and $2 \notin R^*$ one has

$$E(\Phi, R, \mathfrak{a}\mathfrak{b}, \mathfrak{a}\mathfrak{b}^{\square 2} + 2\mathfrak{a}\mathfrak{b} + \mathfrak{a}^{\square 2}\mathfrak{b}) \leq E(\Phi, \mathfrak{a} + \mathfrak{b}).$$

Proof. By Lemma 6 it suffices to find conditions on ξ which imply that $z_\alpha(\xi, \zeta) \in E(\Phi, \mathfrak{a} + \mathfrak{b})$ for each root $\alpha \in \Phi$ and $\zeta \in R$.

General case. First, assume that α is short or $\Phi \neq C_l$. By Lemma 1 there exist roots β and γ such that $\beta + \gamma = \alpha$ and $N_{\beta\gamma 11} = 1$. In this case we prove that $z_\alpha(ab, \zeta) \in E(\Phi, \mathfrak{a} + \mathfrak{b})$ for each root $\alpha \in \Phi$ and all $a \in \mathfrak{a}$, $b \in \mathfrak{b}$ and $\zeta \in R$. With this end we decompose $x_\alpha(ab)$ as follows:

$$x_\alpha(ab) = [x_\beta(a), x_\gamma(b)] \prod x_{i\beta+j\gamma}(-N_{\gamma\delta ij}a^i b^j),$$

where the product on the right hand side is taken over all roots $i\beta + j\gamma \neq \alpha$. Conjugating this equality by $x_{-\alpha}(\zeta)$, we obtain an expression of $z_\alpha(ab, \zeta)$ as a product of elementary root unipotents belonging either to $E(\Phi, \mathfrak{a})$ or to $E(\Phi, \mathfrak{b})$, or, as in the case of factors occurring in the tail, even to $E(\Phi, \mathfrak{a}\mathfrak{b})$.

Case $\Phi = C_l$. This leaves us with the analysis of the exceptional case, where $\Phi = C_l$ and the root α is long. We will have to use several instances of the Chevalley commutator formula.

First of all, there exist short roots β and γ such that $\beta + \gamma = \alpha$ and $N_{\beta\gamma 11} = 2$. Thus,

$$x_\alpha(2ab) = [x_\beta(a), x_\gamma(b)],$$

for all $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$. Now, exactly the same argument, as in the general case, shows that $z_\alpha(2ab, \zeta) \in E(\Phi, \mathfrak{a} + \mathfrak{b})$. This shows that when $2 \in R^*$ we again recover the general answer.

By Lemma 1 there exist a long root β and a short root γ such that $\beta\gamma = \alpha$ and $N_{\beta\gamma 12} = \pm 1$. Without loss of generality we can assume that $N_{\beta\gamma 12} = \pm 1$, otherwise we would just replace the $x_\beta(a)$ in the following formula by $x_\beta(-a)$. We decompose $x_\alpha(ab^2)$ as follows:

$$x_{\alpha}(ab^2) = [x_{\beta}(a), x_{\gamma}(b)]x_{\beta\gamma}(-N_{\beta\gamma 11}ab)$$

for all $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$. Now, exactly the same argument, as in the general case, shows that $z_{\alpha}(ab^2, \zeta) \in E(\Phi, \mathfrak{a} + \mathfrak{b})$.

Interchanging a and b in the above formula, we see, that $z_{\alpha}(ab^2, \zeta) \in E(\Phi, \mathfrak{a} + \mathfrak{b})$. To finish the proof, it remains only to refer to the preceding lemma. \square

In the next lemma we calculate the lower level of the mixed commutator subgroup.

Lemma 17. *Let $\text{rk}(\Phi) \geq 2$. In the cases $\Phi = C_2, G_2$ assume that R does not have residue fields \mathbb{F}_2 of 2 elements and in the case $\Phi = C_l, l \geq 2$, assume additionally that any $c \in R$ is contained in the ideal $c^2R + 2cR$.*

Then for any two ideals \mathfrak{a} and \mathfrak{b} of the ring R one has the following inclusion

$$E(\Phi, R, \mathfrak{a}\mathfrak{b}) \leq [E(\Phi, R, \mathfrak{a}), E(\Phi, R, \mathfrak{b})].$$

Proof. The mixed commutator of two normal subgroups is normal. Thus, it suffices to prove that

$$E(\Phi, \mathfrak{a}\mathfrak{b}) \leq [E(\Phi, R, \mathfrak{a}), E(\Phi, R, \mathfrak{b})],$$

and the result will automatically follow. Actually, we prove a slightly stronger inclusion $E(\Phi, \mathfrak{a}\mathfrak{b}) \leq [E(\Phi, \mathfrak{a}), E(\Phi, \mathfrak{b})]$. This more precise formula is not used in the present paper, but it still might be useful to record this for future applications. Denote $H = [E(\Phi, \mathfrak{a}), E(\Phi, \mathfrak{b})]$. Then our claim amounts to the following. Let $\alpha \in \Phi$, $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$. Then $x_{\alpha}(ab) \in H$.

• First, assume that α can be embedded in a root system of type A_2 . Then there exist roots $\beta, \gamma \in \Phi$, of the same length as α such that $\alpha = \beta + \gamma$, and $N_{\beta\gamma 11} = 1$. Then

$$[x_{\beta}(a), x_{\gamma}(b)] = x_{\alpha}(ab) \in H.$$

This proves the lemma for simply laced Chevalley groups, and for the Chevalley group of type F_4 . It also proves necessary inclusions for short roots in Chevalley groups of type $C_l, l \geq 3$, and for long roots in Chevalley groups of type $B_l, l \geq 3$, and of type G_2 .

• Next, assume that α can be embedded in a root system of type C_2 as a long root. We wish to prove that $x_{\alpha}(ab) \in H$, where $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$. As the first approximation, we prove that $x_{\alpha}(a^2b), x_{\alpha}(ab^2) \in H$. There exist roots $\beta, \gamma \in \Phi$, such that $\alpha = \beta + 2\gamma$ and $N_{\beta\gamma 11} = 1$. Clearly, in this case β is long and γ is short. Take an arbitrary $c \in R$. Then

$$[x_{\beta}(ca), x_{\gamma}(b)] = x_{\beta+\gamma}(cab)x_{\alpha}(\pm cab^2) \in H,$$

whereas

$$[x_{\beta}(a), x_{\gamma}(cb)] = x_{\beta+\gamma}(cab)x_{\alpha}(\pm c^2ab^2) \in H.$$

Comparing these two inclusions we can conclude that $x_{\alpha}(\pm(c^2 - c)ab^2) \in H$. Since by assumption R does not have residue field of 2 elements, the ideal generated by $c^2 - c$, where $c \in R$, is not contained in any maximal ideal, and thus coincides with R . It follows that $x_{\alpha}(ab^2) \in H$. Interchanging a and b we see that $x_{\alpha}(a^2b) \in H$.

• Now, assume that α can be embedded in a root system of type C_2 as a short root. Choose the same β and γ as in the preceding item. In other words, $\alpha = \beta + \gamma$, β is long, γ is short, and $N_{\beta\gamma 11} = 1$. Then

$$[x_{\beta}(a), x_{\gamma}(b)] = x_{\alpha}(ab)x_{\alpha+\gamma}(\pm ab^2).$$

From the previous item we already know that the second factor belongs to H provided that R does not have residue field of 2 elements. Actually, from the first item, we know that for $\Phi = B_l$, $l \geq 3$, even the stronger inclusion $x_{\alpha+\gamma}(\pm ab) \in H$ holds without any such assumption.

Thus, in both cases we can conclude that $x_\alpha(ab) \in H$, for a short root α . Again, already from the first item we know that for $\Phi = C_l$, $l \geq 3$, this inclusion holds without any assumptions on R .

On the other hand, a long root α of $\Phi = C_l$, $l \geq 3$, cannot be embedded in an irreducible rank 2 subsystem other than C_2 . This leaves us with the analysis of exactly two rank 2 cases: $\Phi = C_2$ and α is long and $\Phi = G_2$ and α is short. This is where one needs the additional assumptions on R .

• $\Phi = C_2$ and α is long. Then α can be expressed as $\alpha = \beta + \gamma$ for two short roots β and γ . Interchanging β and γ we can assume that $N_{\beta\gamma 11} = 2$. Then one has

$$[x_\beta(a), x_\gamma(b)] = x_\alpha(2ab) \in H.$$

On the other hand, we already know that $x_\alpha(a^2b) \in H$. Since by assumption the ideal generated by $2a$ and a^2 contains a , we can conclude that $x_\alpha(ab) \in H$.

• Finally, let $\Phi = G_2$ and α is short. We wish to prove that $x_\alpha(ab) \in H$, where $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$. With this end we argue in the same way as for the case of $\Phi = C_2$. Actually, now it is even easier, since we already have necessary inclusions for long roots.

Again, as the first approximation, we prove that $x_\alpha(a^2b), x_\alpha(ab^2) \in H$. With this end, express α as $\alpha = \beta\gamma$, where β is short, γ is long, and $N_{\beta\gamma 11} = 1$. Take an arbitrary $c \in R$. Then

$$[x_\beta(ca), x_\gamma(b)] = x_\alpha(cab)x_{\alpha+\beta}(\pm c^2a^2b)x_{3\beta+\gamma}(\pm c^3a^3b)x_{3\beta+2\gamma}(\pm c^3a^3b^2) \in H,$$

whereas

$$[x_\beta(a), x_\gamma(cb)] = x_\alpha(cab)x_{\alpha+\beta}(\pm ca^2b)x_{3\beta+\gamma}(\pm ca^3b)x_{3\beta+2\gamma}(\pm c^2a^3b^2) \in H.$$

Since the roots $3\beta + \gamma$ and $3\beta + 2\gamma$ are long, from the first item we already know that the corresponding root elements belong to H . Thus,

$$x_\alpha(cab)x_{\alpha+\beta}(\pm c^2a^2b), x_\alpha(cab)x_{\alpha+\beta}(\pm ca^2b) \in H.$$

Comparing these inclusions, we conclude that $x_{\alpha+\beta}(\pm(c^2 - c)a^2b) \in H$ for all $c \in R$. Again, since R does not have residue field of two elements, it follows that $x_{\alpha+\beta}(\pm a^2b) \in H$. Interchanging a and b , we see that $x_{\alpha+\beta}(ab^2) \in H$.

It only remains to look at the commutator

$$[x_\beta(a), x_\gamma(b)] = x_\alpha(ab)x_{\alpha+\beta}(\pm a^2b)x_{3\beta+\gamma}(\pm a^3b)x_{3\beta+2\gamma}(\pm a^3b^2) \in H.$$

Since all elementary factors on the right hand side, apart from the first one, already belong to H , we can conclude that this first factor also belongs to H , in other words, $x_\alpha(ab) \in H$, as claimed. \square

Not to overburden the present paper with technical details, here we only consider the usual relative subgroups depending on one parameter. To illustrate, why we do this, let us state a general version of Lemma 17, with form parameters, which can be easily derived from the proof of Lemma 17.

Lemma 18. Let $\text{rk}(\Phi) \geq 2$. Then for any two ideals \mathfrak{a} and \mathfrak{b} of the ring R one has the following inclusions

$$E(\Phi, R, \mathfrak{a}\mathfrak{b}, i_\Phi \mathfrak{a}\mathfrak{b} + \mathfrak{a}^2\mathfrak{d} + \mathfrak{b}^2\mathfrak{c}) \leq [E(\Phi, R, \mathfrak{a}, \mathfrak{c}), E(\Phi, R, \mathfrak{b}, \mathfrak{d})].$$

Without the additional assumption in the case C_l , $l \geq 2$, the upper and lower levels of the commutator of two relative elementary subgroups do not coincide, and $E(\Phi, R, ab)$ in the statement of Lemma 17 should be replaced² by $E(\Phi, R, a^2b + 2ab + ab^2)$. Nevertheless, when a and b are co-maximal, $a + b = R$, these levels do coincide, so that no additional assumption is necessary in the statement of Theorem 3.

Next lemma bounds the upper level of mixed commutator subgroups. Observe, that it also holds for $\text{rk}(\Phi) = 1$.

Lemma 19. *Let $\text{rk}(\Phi) \geq 1$. Then for any two ideals a and b of the ring R one has the following inclusion*

$$[G(\Phi, R, a), C(\Phi, R, b)] \leq G(\Phi, R, ab).$$

Proof. Consider a faithful rational representation $G(\Phi, R) \leq \text{GL}(n, R)$. Then

$$G(\Phi, R, a) \leq \text{GL}(n, R, a), \quad C(\Phi, R, b) \leq C(n, R, b).$$

Now, by Lemma 5 of [90] one has

$$[G(\Phi, R, a), C(\Phi, R, b)] \leq [\text{GL}(n, R, a), C(n, R, b)] \leq \text{GL}(n, R, ab).$$

Since the left hand side is a subgroup of $G(\Phi, R)$, by lemma 7 we get

$$[G(\Phi, R, a), C(\Phi, R, b)] \leq G(\Phi, R) \cap \text{GL}(n, R, ab) = G(\Phi, R, ab). \quad \square$$

7. Relative conjugation calculus

This section, and the next one constitute the technical core of the paper. Here, we develop a relative version of the conjugation calculus in Chevalley groups, whereas in the next section we evolve a relative version of the commutator calculus. Throughout this section we assume $\text{rk}(\Phi) \geq 2$.

In our survey [32] we explain the essence of Bak's method in non-technical terms, and in our conference paper [29], joint with Alexei Stepanov, we discuss the general philosophy of our versions of that method, their interpretation in terms of s -adic topologies, and their precise relation with other localisation methods. Not to repeat ourselves, we simply refer the reader to these two sources, and the references therein.

For future applications we allow two localisation parameters. Strictly speaking, this is not necessary for the proof of Theorem 1. However, this is essential in the proof of Theorem 2 and is an absolute must for the more advanced applications we ultimately have in mind, such as *general* multiple commutator formulas, where *none* of the factors is elementary. With this end we fix two elements $s, t \in R$ and look at the localisation

$$R_{st} = (R_s)_t = (R_t)_s.$$

All calculations in this and the next sections take place in $E(\Phi, R_{st})$. Thus, when we write something like $E(\Phi, s^p t^q R)$, or $x_\alpha(s^p a)$, what we *really* mean, is $E(\Phi, F_s(s^p t^q R))$, or $x_\alpha(F_s(s^p a))$, respectively, but we suppress F_s in our notation. This shouldn't lead to a confusion, since here we *never* refer to elements or subgroups of $G(\Phi, R)$.

² After the submission of the present paper, Himanee Apte and Alexei Stepanov [6] addressed similar problems from a slightly different viewpoint. Their approach depends on similar level calculations, and in particular, they indicate missing assumptions in previous publications, and provide detailed proofs for the case of $\Phi = C_l$, without such additional assumptions, see in particular, [6], Lemma 5.2.

The overall strategy in this and the next sections is *exactly* the same, as in the proofs of Lemmas 3.1 and 4.1 of [31] and in the proofs of Lemmas 8–10 of [63]. In turn, as we have already mentioned in the introduction, both [31] and [63] followed the general scheme proposed by Anthony Bak [8] for the general linear group, and developed by the first author [26,27] for unitary groups. Ideologically closely cognate, the actual calculations in [31,63] were technically noticeably different from those in [8,26,27] in some respects, due to the two contrasting factors: some important technical simplifications, and the fancier forms of the Chevalley commutator formula.

However, now we wish to do the same at the *relative*, rather than absolute level. In other words, we have to introduce another parameter belonging to an ideal $\mathfrak{a} \trianglelefteq R$. The difference with the existing versions of localisation is that whereas powers of localising elements s and t are at our disposal, and can be distributed among the factors, the ideal \mathfrak{a} is fixed, and cannot be distributed.

The first main objective of the conjugation calculus is to establish that conjugation by a fixed matrix $g \in G(\Phi, R_s)$ is continuous in s -adic topology. In the proof one uses a base of neighbourhoods of e and establishes that for any such neighbourhood V there exists another neighbourhood U such that ${}^gU \subseteq V$. Usually, one takes either elementary subgroups $E(\Phi, s^k\mathfrak{a})$ of level $s^k\mathfrak{a}$, or *relative* elementary subgroups $E(\Phi, R, s^k\mathfrak{a})$ of level $s^k\mathfrak{a}$, as a base.

However, both choices are not fully satisfactory in that they lead to extremely onerous calculations. The reason is that the first of these choices is too small as the neighbourhood on the right hand side, while the second of these choices is too large as the neighbourhood on the left hand side. The solution proposed for $GL(n, R)$ in [34] and later applied to unitary groups in [33] consists in selecting another base of neighbourhoods

$$E(\Phi, s^k\mathfrak{a}) \leq E(\Phi, s^kR, s^k\mathfrak{a}) \leq E(\Phi, R, s^k\mathfrak{a}),$$

which is much better balanced with respect to conjugation. The following definition embodies the gist of this method.

Definition 7.1. Let R be a commutative ring, \mathfrak{a} an ideal of R and $s \in R$. For a positive integer k , define

$$E(\Phi, s^kR, s^k\mathfrak{a}) = E(\Phi, s^k\mathfrak{a})^{E(\Phi, s^kR)}$$

as the normal closure of $E(\Phi, s^k\mathfrak{a})$ in $E(\Phi, s^kR)$, i.e., the group generated by ${}^e\chi_\alpha(s^k\mathfrak{a})$ where $e \in E^K(\Phi, s^kR)$, for some positive integer K , $a \in \mathfrak{a}$ and $\alpha \in \Phi$.

The following lemma is a relative version of Lemma 3.1 of [31] and of Lemma 8 of [63]. Observe, that we could not simply put $E(\Phi, s^pt^q\mathfrak{a})$ on the right hand side. While the powers of s and t can be distributed among the factors on the right hand side in the calculations below, this is not the case for the parameter $a \in \mathfrak{a}$. This is why we need conjugates by elements of $E(\Phi, s^pt^qR)$.

Observe, that the proof works in terms of roots alone, and thus one gets *uniform* estimates for the powers of s and t , which do not depend on the ideal \mathfrak{a} . This circumstance, the *equi-continuity* of conjugation by $g \in G(\Phi, R_s)$ on congruence subgroups, is extremely important, and will be repeatedly used in the sequel.

Lemma 20. *If p, q and k are given, there exist h and m such that*

$$E^1(\Phi, \frac{1}{s^k}R) E(\Phi, s^ht^m\mathfrak{a}) \subseteq E(\Phi, s^pt^qR, s^pt^q\mathfrak{a}).$$

Such h and m depend on Φ, k, p and q alone, but do not depend on the ideal \mathfrak{a} .

Proof. Since by definition $E(\Phi, s^ht^m\mathfrak{a})$ is generated by $\chi_\beta(s^ht^m\mathfrak{a})$, where $\beta \in \Phi$ and $a \in \mathfrak{a}$, it suffices to show that there exist h and m such that

$$x_\alpha\left(\frac{r}{s^k}\right)x_\beta(s^h t^m a) \in E(\Phi, s^p t^q R, s^p t^q a),$$

for any $x_\alpha(r/s^k) \in E^1(\Phi, \frac{1}{s^k} R)$ and any $x_\beta(s^h t^m a) \in E(\Phi, s^h t^m a)$.

Case 1. Let $\alpha \neq -\beta$ and set $h \geq i_\phi k + p + 1$, $m \geq q$. By the Chevalley commutator formula,

$$x_\alpha\left(\frac{r}{s^k}\right)x_\beta(s^h t^m a)x_\alpha\left(-\frac{r}{s^k}\right) = \prod_{i\alpha+j\beta \in \Phi} x_{i\alpha+j\beta} \left(N_{\alpha\beta ij} \left(\frac{r}{s^k} \right)^i (s^h t^m a)^j \right) x_\beta(s^h t^m a)$$

and a quick inspection shows that the right hand side of the above equality is in $E^L(\Phi, s^p t^q a)$, where $L = 2, 3$ or 5 , depending on whether Φ is simply laced, doubly laced or triply laced. Clearly,

$$E^L(\Phi, s^p t^q a) \subseteq E(\Phi, s^p t^q a) \leq E(\Phi, s^p t^q R, s^p t^q a).$$

Case 2. Let $\alpha = -\beta$ and one of the following holds: β is short or $\Phi \neq C_l$. By Lemma 1 there exist roots γ and δ such that $\gamma + \delta = \beta$ and $N_{\gamma\delta 11} = 1$. We set $h = 2(i_\phi k + p + 1)$, $m = 2q$, and decompose $x_\beta(s^h t^m a)$ as follows:

$$x_\beta(s^h t^m a) = [x_\gamma(s^{h/2} t^{m/2}), x_\delta(s^{h/2} t^{m/2} a)] \prod x_{i\gamma+j\delta} (-N_{\gamma\delta ij} (s^{h/2} t^{m/2})^i (s^{h/2} t^{m/2} a)^j),$$

where the product on the right hand side is taken over all roots $i\gamma + j\delta \neq \beta$.

Conjugating this expression by $x_\alpha(\frac{r}{s^k})$ we get

$$\begin{aligned} x_\alpha\left(\frac{r}{s^k}\right)x_\beta(s^h t^m a) &= [x_\alpha\left(\frac{r}{s^k}\right)x_\gamma(s^{h/2} t^{m/2}), x_\alpha\left(\frac{r}{s^k}\right)x_\delta(s^{h/2} t^{m/2} a)] \\ &\quad \cdot \prod x_{i\gamma+j\delta} (-N_{\gamma\delta ij} (s^{h/2} t^{m/2})^i (s^{h/2} t^{m/2} a)^j). \end{aligned}$$

Obviously, γ, δ and all the roots $i\gamma + j\delta \neq \beta$, occurring in the product, are distinct from $-\alpha$. Now, by Case 1 the first element of the commutator belongs to $E(\Phi, s^p t^q R)$, while the second element of the commutator, and all factors of the product belong to $E(\Phi, s^p t^q a)$. Since $E(\Phi, s^p t^q R, s^p t^q a)$ is normalised by $E(\Phi, s^p t^q R)$, it follows that each term on right hand side sits in $E(\Phi, s^p t^q R, s^p t^q a)$.

Case 3. Let $\Phi = C_l$ and $\alpha = -\beta$ be a long root. By Lemma 1 there exist roots γ and δ such that either $\gamma + 2\delta = \beta$ and $N_{\gamma\delta 12} = 1$, or $2\gamma + \delta = \beta$ and $N_{\gamma\delta 21} = 1$. We look at the first case, the second case is similar. Alternatively, if $N_{\gamma\delta 12} = -1$, one could change the sign of $x_\gamma(a)$ in the following formula by $x_\gamma(-a)$. We set $h = 3(i_\phi k + p + 1)$ and $m = 3q$, and decompose $x_\beta(s^h t^m a)$ as follows:

$$x_\beta(s^h t^m a) = [x_\gamma(s^{h/3} t^{m/3} a), x_\delta(s^{h/3} t^{m/3} a)] \cdot x_{\gamma+\delta} (-N_{\gamma\delta 11} s^{2h/3} t^{2m/3} a).$$

Conjugating this expression by $x_\alpha(\frac{r}{s^k})$ we get

$$\begin{aligned} x_\alpha\left(\frac{r}{s^k}\right)x_\beta(s^h t^m a) &= [x_\alpha\left(\frac{r}{s^k}\right)x_\gamma(s^{h/3} t^{m/3} a), x_\alpha\left(\frac{r}{s^k}\right)x_\delta(s^{h/3} t^{m/3} a)] \\ &\quad \cdot x_{\gamma+\delta} (-N_{\gamma\delta 11} s^{2h/3} t^{2m/3} a). \end{aligned}$$

As in Case 2, we can apply Case 1 to each conjugate on the right hand side. The first element of the commutator, and the last factor belong to $E(\Phi, s^p t^q a)$, while the second element of the commutator belongs to $E(\Phi, s^p t^q R)$. Again, it remains only to recall that $E(\Phi, s^p t^q R, s^p t^q a)$ is normalised by $E(\Phi, s^p t^q R)$. \square

Now, the trick is that the elementary group $E(\Phi, s^h t^m \mathfrak{a})$ on the left hand side can be *effortlessly* replaced by $E(\Phi, s^h t^m R, s^h t^m \mathfrak{a})$. Notice, that this step does not work like that for the *usual* relative group $E(\Phi, R, s^h t^m \mathfrak{a})$. The reason are the obstinate denominators in the exponent, which force to reiterate the procedure several times, according to the length of the conjugating element.

Lemma 21. *If p, q and k are given, there exist h and m such that*

$$E^1(\Phi, \frac{1}{s^k R}) E(\Phi, s^h t^m R, s^h t^m \mathfrak{a}) \subseteq E(\Phi, s^p t^q R, s^p t^q \mathfrak{a}).$$

Proof. Indeed, one has ${}^h(g\chi) = (hgh^{-1})h\chi$. Thus,

$$\begin{aligned} E^1(\Phi, \frac{R}{s^k}) E(\Phi, s^h t^m R, s^h t^m \mathfrak{a}) &= E^1(\Phi, \frac{R}{s^k}) (E(\Phi, s^h t^m R) E(\Phi, s^h t^m R)) \\ &= E^1(\Phi, \frac{R}{s^k}) E(\Phi, s^h t^m R) (E^1(\Phi, \frac{R}{s^k}) E(\Phi, s^h t^m R)). \end{aligned}$$

Now, by the preceding lemma, for any given p and q there exist sufficiently large h and m such that the exponent is contained in $E(\Phi, s^p t^q R)$, while the base is contained in $E(\Phi, s^p t^q R, s^p t^q \mathfrak{a})$. It remains to recall that by the very definition $E(\Phi, s^p t^q R, s^p t^q \mathfrak{a})$ is normalised by $E(\Phi, s^p t^q R)$. \square

Now, since h and m in Lemma 20 do not depend on the ideal \mathfrak{a} , the preceding lemma immediately implies the following fact.

Lemma 22. *If p, k are given, then there is a q such that*

$$E^1(\Phi, \frac{R}{s^k}) [E(\Phi, s^q R, s^q \mathfrak{a}), E(\Phi, s^q R, s^q \mathfrak{b})] \subseteq [E(\Phi, s^p R, s^p \mathfrak{a}), E(\Phi, s^p R, s^p \mathfrak{b})].$$

Iterated application of the above lemma, gives the following result.

Lemma 23. *If p, k and L are given, then there is a q such that*

$$E^L(\Phi, \frac{R}{s^k}) [E(\Phi, s^q R, s^q \mathfrak{a}), E(\Phi, s^q R, s^q \mathfrak{b})] \subseteq [E(\Phi, s^p R, s^p \mathfrak{a}), E(\Phi, s^p R, s^p \mathfrak{b})].$$

Now, we are all set for the next round of calculations. Namely, it is our intention to obtain similar formulae, admitting denominators not only in the exponent, but also on the ground level.

8. Relative commutator calculus

To implement second localisation, we will have to be able to fight powers of *two* elements in the denominator. The relative commutator calculus turns out to be much more technically demanding, than the relative conjugation calculus. Not only that the first step of induction is *by far* the hardest one. Actually, even the usually trivial first *substep* of the first step, the case of two non-opposite roots, turns out to be a real challenge. As always, it is extremely important for the sequel that the resulting power estimates do not depend on the ideals \mathfrak{a} and \mathfrak{b} .

Throughout we continue to assume $\text{rk}(\Phi) \geq 2$. In the cases $\Phi = B_2 = C_2$ and $\Phi = G_2$ we additionally assume that $2 \in R^*$. These standing assumptions will not be repeated in the statements of lemmas.

Lemma 24. If p, q, k, m are given, then there exist l and n such that

$$\left[E^1\left(\Phi, \frac{t^l}{s^k}a\right), E^1\left(\Phi, \frac{s^n}{t^m}b\right) \right] \subseteq [E(\Phi, s^p t^q R, s^p t^q a), E(\Phi, s^p t^q R, s^p t^q b)].$$

These l and n depend on Φ, p, q, k, m alone, and do not depend on the choice of ideals a and b .

Proof. Let $\alpha, \beta \in \Phi$, $a \in a$ and $b \in b$. We have to prove that

$$\left[x_\alpha\left(\frac{t^l}{s^k}a\right), x_\beta\left(\frac{s^n}{t^m}b\right) \right] \in [E(\Phi, s^p t^q R, s^p t^q a), E(\Phi, s^p t^q R, s^p t^q b)].$$

The partition into cases is exactly the same as in the proof of Lemma 20, but the calculations themselves – and the resulting length bounds, should we attempt to record them – are now much fancier.

Case 1. Let $\alpha \neq -\beta$. Then using the Chevalley commutator formula we get

$$\begin{aligned} \left[x_\alpha\left(\frac{t^l}{s^k}a\right), x_\beta\left(\frac{s^n}{t^m}b\right) \right] &= \prod_{i,j>0} x_{i\alpha+j\beta} \left(N_{\alpha\beta ij} \left(\frac{t^l}{s^k}a\right)^i \left(\frac{s^n}{t^m}b\right)^j \right) \\ &= \prod_{i\alpha+j\beta \in \Phi} x_{i\alpha+j\beta} (N_{\alpha\beta ij} s^{jn-ik} t^{il-jm} a^i b^j). \end{aligned}$$

Clearly, one can take sufficiently large l and n . It suffices to show, that taking large enough l and n we can redistribute powers of s and t between the first and the second parameters in each factor on the right hand side in such a way, that the resulting product can be expressed as a *product* of commutators without denominators.

Warning. This is one of the key new points in the whole argument, where we cannot thoughtlessly imitate [31] or [63]. Namely, expressing an element as a product of commutators without denominators, with parameters sitting where they should, is not quite the same as just observing that taking large enough l and n we can kill all the denominators in each factor on the right hand side of the Chevalley commutator formula. This is precisely the point, where the cases $\Phi = C_2, G_2$ require *substantial* extra care.

• First, assume that the right hand side of the Chevalley commutator formula consists of one factor. In this case

$$\left[x_\alpha\left(\frac{t^l}{s^k}a\right), x_\beta\left(\frac{s^n}{t^m}b\right) \right] = x_{\alpha+\beta} (N_{\alpha\beta} s^{n-k} t^{l-m} ab).$$

Taking $n \geq 2p + k$ and $l \geq 2q + m$ we can rewrite this commutator as a commutator without denominators as follows:

$$\left[x_\alpha\left(\frac{t^l}{s^k}a\right), x_\beta\left(\frac{s^n}{t^m}b\right) \right] = [x_\alpha(s^p t^q a), x_\beta(s^{n-k-p} t^{l-m-q} b)].$$

Observe, that the right hand side belongs to $[E(\Phi, s^p t^q R, s^p t^q a), E(\Phi, s^p t^q R, s^p t^q b)]$.

The assumption of this item amounts to saying that $|\alpha| = |\beta|$, with the sole exception of two short roots in G_2 , whose sum is a short root, where the right hand side of the Chevalley commutator formula consists of *three* factors, rather than one.

Thus, in fact, we have established somewhat more, than claimed. Namely, assume that if $\gamma = \alpha + \beta$, $|\alpha| = |\beta|$, and, moreover, the mutual angle of α and β is not $2\pi/3$ if α, β are short roots of $\Phi = G_2$. Then for any $h \geq 2p$, any $r \geq 2q$, any $a \in \mathfrak{a}$ and any $b \in \mathfrak{b}$ one has

$$x_{\alpha+\beta}(N_{\alpha\beta}s^h t^r ab) \in [E(\Phi, s^p t^q R, s^p t^q \mathfrak{a}), E(\Phi, s^p t^q R, s^p t^q \mathfrak{b})].$$

In particular, this proves Case 1 for *simply* laced systems.

• Actually, with the use of the above argument it is easy to completely settle also the case of *doubly* laced systems, except for $\Phi = C_2$. Indeed, for doubly laced systems it accounts for the case, where $|\alpha| = |\beta|$. Now, let α and β have distinct lengths. If necessary, replacing $[x, y]$ by $[y, x] = [x, y]^{-1}$, we can assume that α is long, and β is short. In this case

$$y = \left[x_{\alpha} \left(\frac{t^l}{s^k} a \right), x_{\beta} \left(\frac{s^n}{t^m} b \right) \right] = x_{\alpha+\beta}(N_{\alpha\beta 11} s^{n-k} t^{l-m} ab) x_{\alpha+2\beta}(N_{\alpha\beta 12} s^{2n-k} t^{l-2m} ab^2).$$

Now, if $\Phi = F_4$, every root embeds in a subsystem of type A_2 . In other words, the root $\alpha + \beta$ is a sum of two *short* roots, whereas $\alpha + 2\beta$ is a sum of two *long* roots. Thus, taking $2n \geq 2p + k$ and $l \geq 2q + 2m$, we see that each elementary unipotent on the right hand side of the above formula is itself a single commutator in $[E(\Phi, s^p t^q R, s^p t^q \mathfrak{a}), E(\Phi, s^p t^q R, s^p t^q \mathfrak{b})]$.

The cases B_l , $l \geq 3$ and C_l , $l \geq 3$, are treated in a similar way, and are only marginally trickier.

• First, let $\Phi = B_l$, $l \geq 3$. In this case every *long* root is a sum of two long roots. Clearly, the first elementary factor in expression of the commutator

$$\begin{aligned} z &= [x_{\alpha}(s^p t^{l-m-q} a), x_{\beta}(s^{n-k-p} t^q b)] \\ &= x_{\alpha+\beta}(N_{\alpha\beta 11} s^{n-k} t^{l-m} ab) x_{\alpha+2\beta}(N_{\alpha\beta 12} s^{2n-2k-p} t^{l-m+q} ab^2), \end{aligned}$$

coincides with the first elementary factor of the above commutator. If $l \geq 2q + m$ and $n \geq 2p + k$, one has $z \in [E(\Phi, s^p t^q R, s^p t^q \mathfrak{a}), E(\Phi, s^p t^q R, s^p t^q \mathfrak{b})]$. On the other hand, if, moreover, $l \geq 2q + 2m$, then the long root unipotent

$$yz^{-1} = x_{\alpha+2\beta}(N_{\alpha\beta 12}(s^{2n-k} t^{l-2m} - s^{2n-2k-p} t^{l-m+q}) ab^2)$$

is also a single commutator in $[E(\Phi, s^p t^q R, s^p t^q \mathfrak{a}), E(\Phi, s^p t^q R, s^p t^q \mathfrak{b})]$, by the first item.

• Now, let $\Phi = C_l$, $l \geq 3$. In this case every *short* root is a sum of two short roots. Set $p' = p$ if $p \equiv k \pmod{2}$, and $p' = p + 1$ otherwise. Then *second* elementary factor in expression of the commutator

$$\begin{aligned} z &= [x_{\alpha}(s^{p'} t^{l-2m-2q} a), x_{\beta}(s^{(2n-k-p')/2} t^q b)] \\ &= x_{\alpha+\beta}(N_{\alpha\beta 11} s^{(2n-k+p')/2} t^{l-2m-q} ab) x_{\alpha+2\beta}(N_{\alpha\beta 12} s^{2n-k} t^{l-2m} ab^2), \end{aligned}$$

coincides with the second elementary factor of the commutator y . If $l \geq 3q + 2m$ and $n \geq (2p + k + 1)/2$, one has $z \in [E(\Phi, s^p t^q R, s^p t^q \mathfrak{a}), E(\Phi, s^p t^q R, s^p t^q \mathfrak{b})]$. On the other hand, if, moreover, $n \geq (5p + k + 1)/2$, then the short root unipotent

$$yz^{-1} = x_{\alpha+\beta}(N_{\alpha\beta 11}(s^{n-k} t^{l-m} - s^{(2n-k+p')/2} t^{l-2m-q}) ab)$$

is also a single commutator in $[E(\Phi, s^p t^q R, s^p t^q \mathfrak{a}), E(\Phi, s^p t^q R, s^p t^q \mathfrak{b})]$, by the first item.

• Finally, let $\Phi = C_2$ or G_2 . We will see that under assumption $2 \in R^*$ the proof is essentially the same, as in the above cases. First, let $\Phi = C_2$, and let $\alpha, \beta, \alpha \neq \pm\beta$, be two short roots. Then by the first item one has

$$x_{\alpha+\beta}(2s^h t^r ab) \in [E(\Phi, s^p t^q R, s^p t^q a), E(\Phi, s^p t^q R, s^p t^q b)].$$

Since $2 \in R^*$, it follows that $x_{\alpha+\beta}(2s^h t^r ab)$ is a single commutator of requested shape, whenever $h \geq 2p$ and $r \geq 2q$. Now, we can repeat exactly the same argument, as in the case $\Phi = B_l, l \geq 3$.

Next, let $\Phi = G_2$. First, observe that by the first item $x_\alpha(s^h t^r ab)$ is already a single commutator of the required shape for any $h \geq 2p$ and any $r \geq 2q$. Now, let α, β be two short roots, whose sum is a short root. Then the Chevalley commutator formula takes the following form

$$[x_\alpha(\xi), x_\beta(\zeta)] = x_{\alpha+\beta}(\pm 2\xi\zeta)x_{2\alpha+\beta}(\pm 3\xi^2\zeta)x_{\alpha+2\beta}(\pm 3\xi\zeta^2),$$

see, for example, [60], [20] or [87].

Now, setting here $\xi = s^p t^{r-q}a$ and $\zeta = s^{h-p}t^q b$, for some $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$, we see that

$$\begin{aligned} x_{\alpha+\beta}(\pm 2s^h t^r ab)x_{2\alpha+\beta}(\pm 3s^{h+p}t^{2r-q}a^2b)x_{\alpha+2\beta}(\pm 3s^{2h-p}t^{r+q}ab^2) \\ \in [E(\Phi, s^p t^q R, s^p t^q a), E(\Phi, s^p t^q R, s^p t^q b)], \end{aligned}$$

for any $h \geq 2p$ and $r \geq 2q$. Since each of the resulting long root elements is already a single commutator of requested shape, and $2 \in R^*$, one sees that $x_{\alpha+\beta}(s^h t^r ab)$ is a product of at most three commutators of requested shape. Now, we conclude the analysis of this case by exactly the same argument, as in the case of $\Phi = G_2$, and conclude that for any two linearly independent roots, any $\alpha \in \mathfrak{a}$, $b \in \mathfrak{b}$, and any $n \geq 2p + 3k, l \geq 2q + 3m$, one has

$$\left[x_\alpha\left(\frac{t^l}{s^k}a\right), x_\beta\left(\frac{s^n}{t^m}b\right) \right] \in [E(\Phi, s^p t^q R, s^p t^q a), E(\Phi, s^p t^q R, s^p t^q b)],$$

in fact, the commutator on the left hand side is the product of not more than eight commutators of two elementary unipotents, belonging to $E(\Phi, s^p t^q R, s^p t^q a)$ and $E(\Phi, s^p t^q R, s^p t^q b)$, respectively.

Case 2. Let $\alpha = -\beta$, and one of the following holds, α is short or $\Phi \neq C_l$. By Lemma 1, there are roots γ and δ such that $\gamma + \delta = \alpha$ and $N_{\gamma\delta 11} = 1$. We can assume that k and l are even and decompose $x_\alpha(\frac{t^l}{s^k}a)$ as follows

$$x_\alpha\left(\frac{t^l}{s^k}a\right) = \left[x_\gamma\left(\frac{t^{l/2}}{s^{k/2}}\right), x_\delta\left(\frac{t^{l/2}}{s^{k/2}}\right) \right] \prod_{i\gamma+j\delta \in \Phi} x_{i\gamma+j\delta} \left(-N_{\gamma\delta ij} \frac{t^{l(i+j)/2}}{s^{k(i+j)/2}} a^j \right), \quad (1)$$

where the product on the right hand side is taken over all roots $i\gamma + j\delta \neq \alpha$. Consider the commutator formula

$$\left[[y, z] \prod_{i=1}^t u_i, x \right] = \prod_{i=1}^t [y, z] \prod_{j=1}^{i-1} u_j [u_i, x] [[y, z], x] \quad (2)$$

where by convention $\prod_{j=1}^0 u_j = 1$. Now let $y = x_\gamma(\frac{t^{l/2}}{s^{k/2}})$, $z = x_\delta(\frac{t^{l/2}}{s^{k/2}}a)$ and u_i 's stand for the terms $x_{i\gamma+j\delta}(\ast)$ in Eq. (1). Let $x = x_\beta(\frac{s^n}{t^m}b)$ and plug these in to Eq. (2). The terms $[u_i, x]$ are all of the form considered in Case 1, and thus for suitable l and n they belong to $[E(\Phi, s^p t^q R, s^p t^q a),$

$E(\Phi, s^p t^q R, s^p t^q \mathfrak{b})$. Thus, it immediately follows that $\prod_{i=1}^t [y, z] \prod_{j=1}^{i-1} u_j [u_i, x]$ belongs to this commutator group.

We are left to show that for a suitable q ,

$$[y, z], x = \left[\left[x_\gamma \left(\frac{t^{l/2}}{s^{k/2}} \right), x_\delta \left(\frac{t^{l/2}}{s^{k/2}} a \right) \right], x_\beta \left(\frac{s^n}{t^m} b \right) \right]$$

is also in $[E(\Phi, s^p t^q R, s^p t^q \mathfrak{a}), E(\Phi, s^p t^q R, s^p t^q \mathfrak{b})]$. Consider the conjugate

$$\begin{aligned} & x_\gamma \left(-\frac{t^{l/2}}{s^{k/2}} \right) \left[\left[x_\gamma \left(\frac{t^{l/2}}{s^{k/2}} \right), x_\delta \left(\frac{t^{l/2}}{s^{k/2}} a \right) \right], x_\beta \left(\frac{s^n}{t^m} b \right) \right] \\ &= x_\gamma \left(-\frac{t^{l/2}}{s^{k/2}} \right) \left[\left[x_\delta \left(\frac{t^{l/2}}{s^{k/2}} a \right), x_\gamma \left(\frac{t^{l/2}}{s^{k/2}} \right) \right]^{-1}, x_\beta \left(\frac{s^n}{t^m} b \right) \right]. \end{aligned}$$

By the Hall–Witt identity it can be rewritten as

$$\begin{aligned} uv &= x_\delta \left(\frac{t^{l/2}}{s^{k/2}} a \right) \left[x_\gamma \left(-\frac{t^{l/2}}{s^{k/2}} \right), \left[x_\beta \left(\frac{s^n}{t^m} b \right), x_\delta \left(-\frac{t^{l/2}}{s^{k/2}} a \right) \right]^{-1} \right] \\ &\quad \cdot x_\beta \left(\frac{s^n}{t^m} b \right) \left[x_\delta \left(\frac{t^{l/2}}{s^{k/2}} a \right), \left[x_\gamma \left(-\frac{t^{l/2}}{s^{k/2}} \right), x_\beta \left(-\frac{s^n}{t^m} b \right) \right]^{-1} \right]. \end{aligned}$$

Let us consider the factors separately.

Since $\gamma, \delta \neq -\beta$, by Case 1 one can find suitable l and n such that the commutator $[x_\beta \left(\frac{s^n}{t^m} b \right), x_\delta \left(-\frac{t^{l/2}}{s^{k/2}} a \right)]$ belongs to $[E(\Phi, s^p t^q R, s^p t^q \mathfrak{a}), E(\Phi, s^p t^q R, s^p t^q \mathfrak{b})]$, and it immediately follows that u belongs to this group.

Applying Chevalley commutator formula to the internal commutator in v , we have

$$v = x_\beta \left(\frac{s^n}{t^m} b \right) \left[x_\delta \left(\frac{t^{l/2}}{s^{k/2}} a \right), \prod_{i\gamma+j\beta \in \Phi} x_{i\gamma+j\beta} \left(-N_{\gamma\beta ij} \left(-\frac{t^{l/2}}{s^{k/2}} \right)^i \left(-\frac{s^n}{t^m} b \right)^j \right) \right].$$

Now, for suitable l and n all $x_{i\gamma+j\beta} \left(-N_{\gamma\beta ij} \left(-\frac{t^{l/2}}{s^{k/2}} \right)^i \left(-\frac{s^n}{t^m} b \right)^j \right)$ belong to $E(\Phi, s^{p'} t^{q'} \mathfrak{b})$ for any prescribed p' and q' . Now employing Lemma 20 twice, we can secure that for suitable l and n the second factor v also belongs to the commutator group $[E(\Phi, s^p t^q R, s^p t^q \mathfrak{a}), E(\Phi, s^p t^q R, s^p t^q \mathfrak{b})]$, and we are done.

Case 3. Let $\Phi = C_l$ and $\alpha = -\beta$ be a long root. Let $\Phi = C_l$ and $\alpha = -\beta$ be a long root. By Lemma 1 there exist roots γ and δ such that either $\gamma + 2\delta = \beta$ and $N_{\gamma\delta 12} = 1$, or $2\gamma + \delta = \beta$ and $N_{\gamma\delta 21} = 1$. Like in the proof of Lemma 20, we lose nothing by looking at the second case. Increasing k and l , in necessary, we can assume that k and l are divisible by 3 and decompose $x_\alpha \left(\frac{t^l}{s^k} a \right)$ as follows

$$x_\alpha \left(\frac{t^l}{s^k} a \right) = \left[x_\gamma \left(\frac{t^{l/3}}{s^{k/3}} \right), x_\delta \left(\frac{t^{l/3}}{s^{k/3}} a \right) \right] \prod_{i\gamma+j\delta \in \Phi} x_{i\gamma+j\delta} \left(-N_{\gamma\delta ij} \frac{t^{l(i+j)/3}}{s^{k(i+j)/3}} a^j \right), \quad (3)$$

where the product is taken over all $(i, j) \neq (1, 2)$. Now, repeating the same argument as in Case 2, one can find suitable l and n such that

$$\left[x_\alpha \left(\frac{t^l}{s^k} a \right), x_\beta \left(\frac{s^n}{t^m} b \right) \right] \in [E(\Phi, s^p t^q R, s^p t^q a), E(\Phi, s^p t^q R, s^p t^q b)],$$

as claimed. \square

Lemma 25. *If p, q, k, m and L are given, there exist l and n , independent of L , such that*

$$\left[E^L \left(\Phi, \frac{t^l}{s^k} a \right), E^1 \left(\Phi, \frac{s^n}{t^m} b \right) \right] \subseteq [E(\Phi, s^p t^q R, s^p t^q a), E(\Phi, s^p t^q R, s^p t^q b)].$$

Proof. An easy induction, using identity (C2), shows that

$$\left[\prod_{i=1}^K u_i, x \right] = \prod_{i=1}^K \prod_{j=1}^{K-i} u_j [u_{K-i+1}, x],$$

where by convention $\prod_{j=1}^0 u_j = 1$. This, with the fact that $E(\Phi, s^p t^q R, s^p t^q a)$ and $E(\Phi, s^p t^q R, s^p t^q b)$ are both normalised by $E(\Phi, s^p t^Q R)$, where $P \geq p$, $Q \geq q$, show that this lemma immediately follows from the previous one. \square

Recall that $E(\Phi, \frac{t^l}{s^k} R, \frac{t^l}{s^k} a)$ is generated by all elements of the form $u x_\alpha (\frac{t^l}{s^k} a)$, where $u \in E^L(\Phi, \frac{t^l}{s^k} R)$, for some L , and $a \in \mathfrak{a}$.

Lemma 26. *If p, q, k, m are given, there exist l and n such that*

$$\left[E \left(\Phi, \frac{t^l}{s^k} R, \frac{t^l}{s^k} a \right), E^1 \left(\Phi, \frac{s^n}{t^m} b \right) \right] \subseteq [E(\Phi, s^p t^q R, s^p t^q a), E(\Phi, s^p t^q R, s^p t^q b)].$$

Proof. Obviously, it suffices to prove that for any given p, q, k, m and any L , there exist l and n independent of L such that

$$\left[E^L(\Phi, \frac{t^l}{s^k} R) E^1 \left(\Phi, \frac{t^l}{s^k} a \right), E^1 \left(\Phi, \frac{s^n}{t^m} b \right) \right] \subseteq [E(\Phi, s^p t^q R, s^p t^q a), E(\Phi, s^p t^q R, s^p t^q b)], \quad (4)$$

after that the lemma follows from (4) and identity (C2).

Let $x \in E^L(\Phi, \frac{t^l}{s^k} R)$, $y \in E^1(\Phi, \frac{t^l}{s^k} a)$ and $z \in E^1(\Phi, \frac{s^n}{t^m} b)$. Using (C2) and the Hall–Witt identity we can write

$$\begin{aligned} [{}^x y, z] &= [y[y^{-1}, x], z] = {}^y [[y^{-1}, x], z] \cdot [y, z] = {}^{yx^{-1}} ({}^x [[y^{-1}, x], z]) \cdot [y, z] \\ &= {}^{yx^{-1}} (y[x^{-1}, [z, y]] \cdot {}^{z^{-1}} [y^{-1}, [x^{-1}, z^{-1}]]) \cdot [y, z]. \end{aligned}$$

Now Lemma 24, along with the fact that $E(\Phi, s^p t^q R, s^p t^q a)$ and $E(\Phi, s^p t^q R, s^p t^q b)$ are both normal in $E(\Phi, s^p t^q R)$, imply that for suitable l and n all three commutators $[y, z]$, ${}^y [x^{-1}, [z, y]]$ and $[y^{-1}, [x^{-1}, z^{-1}]]$ are in

$$[E(\Phi, s^p t^q R, s^p t^q a), E(\Phi, s^p t^q R, s^p t^q b)].$$

Now, we can invoke Lemma 22 to ensure that there are suitable l and n such that the conjugate $z^{-1} [y^{-1}, [x^{-1}, z^{-1}]]$, and therefore the whole commutator $[{}^x y, z]$, is in $[E(\Phi, s^p t^q R, s^p t^q a), E(\Phi, s^p t^q R, s^p t^q b)]$. \square

9. Mixed commutator formula: localisation proof

Now we are all set to complete a localisation proof of Theorem 1. In fact, we will prove a much more powerful result, in the spirit of Theorem 5.3 of [31]. We start with the following lemma, whose proof mimics the proof of [31], Lemma 5.2, modulo replacing elementary factors by commutators, and correcting some misprints.

Lemma 27. Fix an element $s \in R$, $s \neq 0$. Then for any k and p there exists an r such that for any $a \in \mathfrak{a}$, any $g \in G(\Phi, R, s^l \mathfrak{b})$ and any maximal ideal \mathfrak{m} of R , there exists an element $t \in R \setminus \mathfrak{m}$, and an integer l such that

$$\left[x_\alpha \left(\frac{t^l}{s^k} a \right), F_s(g) \right] \in \left[E(\Phi, F_s(s^p R), F_s(s^p \mathfrak{a})), E(\Phi, F_s(s^p R), F_s(s^p \mathfrak{b})) \right]. \quad (5)$$

Note that here l will depend on the choice of x_α .

Proof. By 5.1 one has $G(\Phi, R) = \varinjlim G(\Phi, R_i)$, where the limit is taken over all finitely generated subrings of R . Thus, without loss of generality we may assume that R is Noetherian. To be specific, we can replace R by the ring generated by a , s and the matrix entries of g in a faithful polynomial representation.

Since $R_{\mathfrak{m}}$ is a local ring, by Lemma 12 we have the decomposition

$$G(\Phi, R_{\mathfrak{m}}, \mathfrak{b}_{\mathfrak{m}}) = E(\Phi, R_{\mathfrak{m}}, \mathfrak{b}_{\mathfrak{m}}) T(\Phi, R_{\mathfrak{m}}, \mathfrak{b}_{\mathfrak{m}}).$$

Thus, one can decompose $F_{\mathfrak{m}}(g)$ as $F_{\mathfrak{m}}(g) = uh$ where $u \in E(\Phi, R_{\mathfrak{m}}, \mathfrak{b}_{\mathfrak{m}}) \leq G(\Phi, R_{\mathfrak{m}})$ and $h \in T(\Phi, R_{\mathfrak{m}}, \mathfrak{b}_{\mathfrak{m}})$.

Since $G(\Phi, R_{\mathfrak{m}}) = \varinjlim G(\Phi, R_t)$, over all $t \in R \setminus \mathfrak{m}$, and the same holds for $E(\Phi, s^q R_{\mathfrak{m}})$, $T(\Phi, R_{\mathfrak{m}}, s^q R_{\mathfrak{m}})$, etc., we can find an element $t \in R \setminus \mathfrak{m}$ such that already $F_t(g)$ can be factored as $F_t(g) = uh$, where $u \in E(\Phi, R_t, s^q R_t)$ and $h \in T(\Phi, R_t, s^q R_t)$.

On the other hand, since R is assumed to be Noetherian, R_s is also Noetherian and by Lemma 13 there exists an n such that the canonical homomorphism

$$F_t : G(\Phi, R_s, t^n R_s) \longrightarrow G(\Phi, R_{st})$$

is injective. Next, we take any $l > n$. Since $x_\alpha \left(\frac{t^l}{s^k} a \right) \in G(\Phi, R_s, t^n \mathfrak{a}_s)$, and the principal congruence subgroup $G(\Phi, R_s, t^n \mathfrak{a}_s)$ is normal in $G(\Phi, R_s)$, one has

$$x = \left[x_\alpha \left(\frac{t^l}{s^k} a \right), F_s(g) \right] \in G(\Phi, R_s, t^n \mathfrak{a}_s) \leq G(\Phi, R_s, t^n R_s).$$

Consider the image $F_t(x) \in G(\Phi, R_{st})$ of x under localisation with respect to t . Since F_t is a homomorphism, one has

$$F_t(x) = \left[F_t \left(x_\alpha \left(\frac{t^l}{s^k} a \right) \right), F_{st}(g) \right].$$

Now $F_{st}(g)$ can be factored as $F_{st}(g) = F_s(u)F_s(h) \in G(\Phi, R_{st})$. It follows that

$$\begin{aligned} F_t(x) &= \left[F_t \left(x_\alpha \left(\frac{t^l}{s^k} a \right) \right), F_s(u)F_s(h) \right] \\ &= \left[F_t \left(x_\alpha \left(\frac{t^l}{s^k} a \right) \right), F_t(u) \right] \cdot^{F_t(u)} \left[F_s \left(x_\alpha \left(\frac{t^l}{s^k} a \right) \right), F_s(h) \right]. \end{aligned}$$

Now, for all cases apart from the case, where $G(\Phi, R) = G_{\text{ad}}(C_l, R)$, and α is a long root, by Lemmas 2 or 3 one can choose a decomposition $F_t(g) = uh$, where h commutes with $x_\alpha(*)$. Therefore,

$$F_t(x) = \left[F_t \left(x_\alpha \left(\frac{t^l}{s^k} a \right) \right), F_t(u) \right].$$

Now, by Lemma 26 one can choose such l and n that

$$F_t(x) \in \left[E(\Phi, F_{st}(s^p t^q R), F_{st}(s^p t^q \mathfrak{a})), E(\Phi, F_{st}(s^p t^q R), F_{st}(s^p t^q \mathfrak{b})) \right], \quad (6)$$

considered as a subgroup of $G(\Phi, R_{st})$. In general, this is the first factor of the above expression for $F_t(x)$.

In the exceptional case we can choose $h = h_{\overline{\alpha}_l}(\varepsilon)$, for some $\varepsilon \equiv 1 \pmod{s^r \mathfrak{b}}$. Clearly, also $\varepsilon^{-1} \equiv 1 \pmod{s^r \mathfrak{b}}$, and thus

$$\left[F_s \left(x_\alpha \left(\frac{t^l}{s^k} a \right) \right), F_s(h) \right] = x_\alpha(t^l s^{r-k} F_{st}(ab)),$$

for some $b \in \mathfrak{b}$. Now a reference to Lemmas 17 and 21 shows that one can choose such l and n that the second factor

$$F_t(u) \left[F_s \left(x_\alpha \left(\frac{t^l}{s^k} a \right) \right), F_s(h) \right] = {}^{F_t(u)} x_\alpha(t^l s^{r-k} F_{st}(ab))$$

sits in the same commutator subgroup, as the first factor. Thus, in all cases we get inclusion (6). In other words, $F_t(x)$ can be expressed as

$$F_t(x) = \prod_{i=1}^L \left[{}^{x_{-\beta_i}(F_{st}(s^p t^q c_i))}} x_{\beta_i}(F_{st}(s^p t^q a_i)), {}^{x_{-\gamma_i}(F_{st}(s^p t^q d_i))}} x_{\gamma_i}(F_{st}(s^p t^q b_i)) \right],$$

for some $\beta_i, \gamma_i \in \Phi$, some $a_i \in \mathfrak{a}$, $b_i \in \mathfrak{b}$ and some $c_i, d_i \in R$.

Form the following product of commutators in $G(\Phi, R_s)$,

$$y = \prod_{i=1}^L \left[{}^{x_{-\beta_i}(F_s(s^p t^q c_i))}} x_{\beta_i a}(F_s(s^p t^q a_i)), {}^{x_{-\gamma_i}(F_s(s^p t^q d_i))}} x_{\gamma_i}(F_s(s^p t^q b_i)) \right],$$

by the very construction, $F_t(x) = F_t(y)$. On the other hand, $x, y \in G(\Phi, R_s, t^n R_s)$ and the restriction of F_t to $G(\Phi, R_s, t^n R_s)$ is injective by Lemma 13, it follows $x = y$ and thus we established (5). \square

Now we are in a position to finish the proof of Theorem 1 and, in fact, of the following much stronger result. Morally, it culminates all calculations of Sections 7, 8 and 9, and asserts that for any elements $g_1, \dots, g_K \in E(\Phi, R_s, \mathfrak{a}_s)$, in finite number, and any s -adic neighbourhoods Y and Z of e in the elementary subgroups $E(\Phi, R, \mathfrak{a})$ and $E(\Phi, R, \mathfrak{b})$, respectively, there exists a small s -adic neighbourhood X of e in the principal congruence subgroup $G(\Phi, R, \mathfrak{b})$ such that $[g_i, F_s(X)] \subseteq F_s([Y, Z])$, for all i . This is a very powerful result, which will be used in this form in the proposed description of some classes of intermediate subgroups. See [85,64] for a clarification, why one needs commutator formulae in this stronger form. In turn, this result can be easily deduced from Lemma 27 by a standard patching argument using partitions of 1.

Theorem 2. Let Φ be a reduced irreducible root system, $\text{rk}(\Phi) \geq 2$. In the cases $\Phi = C_2, G_2$ assume additionally that $2 \in R^*$. Then for any $s \in R$, $s \neq 0$, any p, k and L , there exists an r such that for any two ideals \mathfrak{a} and \mathfrak{b} of a commutative ring R , one has

$$\begin{aligned} & \left[E^L \left(\Phi, \frac{1}{s^k} R, \frac{1}{s^k} \mathfrak{a} \right), F_s(G(\Phi, R, s^r \mathfrak{b})) \right] \\ & \subseteq [E(\Phi, F_s(s^p R), F_s(s^p \mathfrak{a})), E(\Phi, F_s(s^p R), F_s(s^p \mathfrak{b}))]. \end{aligned} \quad (7)$$

Proof. First we claim that for the same k and L and any q there exists an r such that

$$\begin{aligned} & \left[E^1 \left(\Phi, \frac{1}{s^k} \mathfrak{a} \right), F_s(G(\Phi, R, s^r \mathfrak{b})) \right] \\ & \subseteq [E(\Phi, F_s(s^q R), F_s(s^q \mathfrak{a})), E(\Phi, F_s(s^q R), F_s(s^q \mathfrak{b}))]. \end{aligned} \quad (8)$$

Indeed, let $x_\alpha(\frac{1}{s^k} \mathfrak{a}) \in E^1(\Phi, \frac{1}{s^k} \mathfrak{a})$, and $g \in G(\Phi, R, s^r \mathfrak{b})$. For any maximal ideal $\mathfrak{m} \triangleleft R$, choose a $t_{\mathfrak{m}} \in R \setminus \mathfrak{m}$ and a positive integer $l_{\mathfrak{m}}$ according to (5). Since the collection of all $t_{\mathfrak{m}}^{l_{\mathfrak{m}}}$ is not contained in any maximal ideal, we may find a finite number of them, $t_1^{l_1}, \dots, t_K^{l_K}$ and such $c_1, \dots, c_K \in R$ that

$$t_1^{l_1} c_1 + \dots + t_K^{l_K} c_K = 1.$$

It follows that

$$x_\alpha \left(\frac{1}{s^k} \mathfrak{a} \right) = x_\alpha \left(a \sum_{i=1}^K \frac{t_i^{l_i}}{s^k} c_i \right) = \prod_{i=1}^K x_\alpha \left(\frac{t_i^{l_i}}{s^k} c_i a \right).$$

Since there are only finitely many factors, it follows from (5) that for any h there exists an r such that

$$\left[x_\alpha \left(\frac{t_i^{l_i}}{s^k} c_i a \right), F_s(g) \right] \in [E(\Phi, F_s(s^h R), F_s(s^h \mathfrak{a})), E(\Phi, F_s(s^h R), F_s(s^h \mathfrak{b}))]. \quad (9)$$

A direct computation using (9), Formula (C2) and Lemma 23, shows that if h was large enough, we get

$$\begin{aligned} \left[x_\alpha \left(\frac{1}{s^k} \mathfrak{a} \right), F_s(g) \right] &= \left[\prod_{i=1}^K x_\alpha \left(\frac{t_i^{l_i}}{s^k} c_i a \right), F_s(g) \right] \\ &\in [E(\Phi, F_s(s^q R), F_s(s^q \mathfrak{a})), E(\Phi, F_s(s^q R), F_s(s^q \mathfrak{b}))]. \end{aligned}$$

This proves our claim.

Now, applying to (8) the commutator formula (C2), we see that if q was large enough, we get

$$\begin{aligned} & \left[E^L \left(\Phi, \frac{1}{s^k} R, \frac{1}{s^k} \mathfrak{a} \right), F_s(G(\Phi, R, s^r \mathfrak{b})) \right] \\ & \subseteq E^{L-1} \left(\Phi, \frac{1}{s^k} R, \frac{1}{s^k} \mathfrak{a} \right) [E(\Phi, F_s(s^p R), F_s(s^p \mathfrak{a})), E(\Phi, F_s(s^p R), F_s(s^p \mathfrak{b}))]. \end{aligned}$$

To finish the proof it only remains to once more invoke Lemma 23. \square

Now we are in a position to prove a slightly weaker statement of Theorem 1. Namely,

$$[E(\Phi, R, \alpha), G(\Phi, R, \mathfrak{b})] = [E(\Phi, R, \alpha), E(\Phi, R, \mathfrak{b})].$$

To get the inclusion of the left hand side into the right hand side, set $s = 1$ in Theorem 2. Inclusion in the other direction is obvious.

10. Relative versus absolute, and variations

Using the absolute standard commutator formula and calculations of Sections 4 and 6 we can give a proof of Theorem 1 – but not of the stronger Theorem 2.

Proof of Theorem 1. By Lemma 5 one has

$$[E(\Phi, R, \alpha), C(\Phi, R, \mathfrak{b})] = [[E(\Phi, R), E(\Phi, R, \alpha)], C(\Phi, R, \mathfrak{b})].$$

Since all subgroups here are normal in $G(\Phi, R)$, Lemma 5 implies

$$\begin{aligned} & [E(\Phi, R, \alpha), C(\Phi, R, \mathfrak{b})] \\ & \leq [E(\Phi, R, \alpha), [E(\Phi, R), C(\Phi, R, \mathfrak{b})]] \cdot [E(\Phi, R), [E(\Phi, R, \alpha), C(\Phi, R, \mathfrak{b})]]. \end{aligned}$$

Applying the *absolute* standard commutator formula [28, Theorem 1] = Lemma 10 above, to the first factor on the right hand side, we immediately see that it *coincides* with $[E(\Phi, R, \alpha), E(\Phi, R, \mathfrak{b})]$.

On the other hand, applying to the second factor on the right hand side Lemma 19 followed by Lemma 10 and Lemma 17, we can conclude that it is *contained* in

$$[E(\Phi, R), G(\Phi, R, \alpha\mathfrak{b})] = E(\Phi, R, \alpha\mathfrak{b}) \leq [E(\Phi, R, \alpha), E(\Phi, R, \mathfrak{b})].$$

Thus, the left hand side is contained in the right hand side, the inverse inclusion being obvious. \square

Lemma 5 asserts that the commutator of two elementary subgroups, one of which is absolute, is itself an elementary subgroup. One can ask, whether one always has

$$[E(\Phi, R, \alpha), E(\Phi, R, \mathfrak{b})] = E(\Phi, R, \alpha\mathfrak{b})?$$

Easy examples show that in general this equality may fail quite spectacularly. In fact, when $\alpha = \mathfrak{b}$, one can only conclude that

$$E(\Phi, R, \alpha^2) \leq [E(\Phi, R, \alpha), E(\Phi, R, \alpha)] \leq E(\Phi, R, \alpha).$$

with right bound attained for some *proper* ideals, such as an ideal α generated by an idempotent.

Nevertheless, the true reason, why the equality in Lemma 5 holds, is not the fact that one of the ideals α or \mathfrak{b} coincides with R , but only the fact that α and \mathfrak{b} are comaximal.

Theorem 3. Let Φ be a reduced irreducible root system, $\text{rk}(\Phi) \geq 2$. When $\Phi = B_2$ or $\Phi = G_2$, assume moreover that R has no residue fields \mathbb{F}_2 of 2 elements. Further, let R be a commutative ring and $\alpha, \mathfrak{b} \triangleleft R$ be two comaximal ideals of R , i.e., $\alpha + \mathfrak{b} = R$. Then one has the following equality

$$[E(\Phi, R, \alpha), E(\Phi, R, \mathfrak{b})] = E(\Phi, R, \alpha\mathfrak{b}).$$

Proof. First of all, observe that by Lemmas 5 and 16 one has

$$E(\Phi, R, a) = [E(\Phi, R, a), E(\Phi, R)] = [E(\Phi, R, a), E(\Phi, R, a) \cdot E(\Phi, R, b)].$$

Thus,

$$\begin{aligned} E(\Phi, R, a) &\leq [E(\Phi, R, a), E(\Phi, R, a)] \cdot [E(\Phi, R, a), E(\Phi, R, b)] \\ &\leq [E(\Phi, R, a), E(\Phi, R, a)] \cdot E(\Phi, R, ab). \end{aligned}$$

Commuting this inclusion with $E(\Phi, R, b)$, we see that

$$\begin{aligned} [E(\Phi, R, a), E(\Phi, R, b)] &\leq [[E(\Phi, R, a), E(\Phi, R, a)], E(\Phi, R, b)] \\ &\quad \cdot [E(\Phi, R, ab), E(\Phi, R, b)]. \end{aligned}$$

The absolute standard commutator formula, applied to the second factor, shows that it is contained in

$$[G(\Phi, R, ab), E(\Phi, R, b)] \leq [G(\Phi, R, ab), E(\Phi, R)] = E(\Phi, R, ab).$$

On the other hand, applying to the first factor formula (C3), and then again the absolute standard commutator formula, we see that it is contained in

$$\begin{aligned} [[E(\Phi, R, a), E(\Phi, R, b)], E(\Phi, R, a)] &\leq [G(\Phi, R, ab), E(\Phi, R, a)] \\ &\leq [G(\Phi, R, ab), E(\Phi, R)] = E(\Phi, R, ab). \end{aligned}$$

Together with Lemma 16 this finishes the proof. \square

11. Where next?

In this section we state and very briefly discuss some further relativisation problems, related to the results of the present paper. We are convinced that these problems can be successfully addressed with our methods. Throughout we assume that $\text{rk}(\Phi) \geq 2$.

Outside of some initial observations in Sections 3, 4, and 6, in the present paper we consider only the usual relative subgroups depending on one ideal of the ground ring, rather than relative subgroups defined in terms of admissible pairs. In fact, calculations necessary to unwind the relative commutator calculus are already awkward enough with one parameter, especially in rank 2. After some thought, we decided not to overcharge the first exposition of our method in this setting with unwieldy technical details. Actually, most of these details are immaterial for the method itself. This suggests the following problems.

Problem 1. Develop working versions of relative conjugation calculus and relative commutator calculus, for relative subgroups corresponding to admissible pairs.

Problem 2. Prove the relative standard commutator formula

$$[E(\Phi, R, a, c), C(\Phi, R, b, d)] = [E(\Phi, R, a, c), E(\Phi, R, b, d)].$$

There is little doubt that what one needs to solve these problems is a stubborn combination of the methods of the present paper with those developed by Michael Stein in [58]. Solution of the following problem is also in sight, and would require mostly technical efforts.

Problem 3. Obtain explicit length estimates in the relative conjugation calculus and relative commutator calculus.

Let us mention some further problems, where we hope to apply methods of the present paper. Firstly, we have in mind description of subnormal subgroups of Chevalley groups.

Problem 4. Describe subnormal subgroups of a Chevalley group $G(\Phi, R)$.

It is well known that this problem is essentially a special case of the following more general problem.

Problem 5. Describe subgroups of a Chevalley group $G(\Phi, R)$, normalised by the relative elementary subgroup $E(\Phi, R, q)$, for an ideal $q \trianglelefteq R$.

Conjectural answer may be stated as follows: there exists an integer $m = m(\Phi)$, depending only on Φ , with the following property. For any subgroup $H \leq G(\Phi, R)$ normalised by $E(\Phi, R, q)$ there exists an ideal $\alpha \trianglelefteq R$ such that

$$E(\Phi, R, q^m \alpha) \leq H \leq C(\Phi, R, \alpha).$$

The ideal α is unique up to equivalence relation \trianglelefteq_q .

The real challenge is to find the smallest possible value of m . For instance, for the case of $GL(n, R)$, $n \geq 3$, it has taken the following values:

- $m = 7$ for $n \geq 4$, John Wilson, 1972 [91],
- $m = 24$ (under some stability conditions), Anthony Bak, 1982 [7],
- $m = 6$, Leonid Vaserstein, 1986 [71],
- $m = 48$, Li Fuan and Liu Mulan, 1987 [39],
- $m = 5$, the second author 1990 [75],
- $m = 4$, Vaserstein 1990 [73].

An exposition of these results with detailed proofs may be found in [97]. Clearly, [7] and [39] drop out of the mainstream. The reason is that [7] was published some 15 years after completion, and [39] relied upon [7]. Nevertheless, these papers are very pertinent in what concerns discussion of equivalence relation \trianglelefteq_q .

For other classical groups the best known results are due to Günter Habdank [23,24] and the third author [97–99], under assumption $2 \in R^*$, and to You Hong, in general, see the discussion in [33].

For exceptional groups there are no published results. Recently, the second and the third authors have modified the third generation proof of the main structure theorems [81–83,79], and obtained the following values: $m = 7$ for Chevalley groups of types E_6 and E_7 . This result will be published in a separate paper. But to get results with the same bound for groups of type E_8 one will have to use localisation.

Other problems we intend to address with relative concern description of various classes of intermediate subgroups, see [77,89,40] for a survey. In [64] we specifically discuss how localisation comes into play. Let us mention two of the most immediate such problems.

Problem 6. Describe the following classes of subgroups

- subgroups in $GL(27, R)$, containing $E(E_6, R)$,
- subgroups in $Sp(56, R)$, containing $E(E_7, R)$.

These problems are discussed by Alexander Luzgarev in [41], where one can find conjectural answers. Before that the second author and Victor Petrov [84–86,50,52], and independently and simultaneously You Hong [93–95] described overgroups of classical groups, in the corresponding $GL(n, R)$. The proofs of these results partly relied on localisation. Immediately thereafter Alexander Luzgarev described subgroups of $G(E_6, R)$, containing $E(F_4, R)$, in his splendid paper [42], also using localisation, see also [43].

Also, we propose to apply the methods of the present paper to describe overgroups of subsystem subgroups in exceptional groups.

Problem 7. Describe subgroups in $G(\Phi, R)$, containing $E(\Delta, R)$, under assumption that $\Delta^\perp = \emptyset$ and all irreducible components of Δ except maybe one have rank ≥ 2 .

The following problem appeared as Problem 9 in [29]. It seems to be extremely challenging, and would certainly require the full force of localisation–completion.³ Its solution would be a simultaneous generalisation of the results in [31,10], as also of our Theorem 1.

Problem 8. Let R be a ring of finite Bass–Serre dimension $\delta(R) = d < \infty$, and let (I_i, Γ_i) , $1 \leq i \leq m$, be form ideals of (R, Λ) . Prove that for any $m > d$ one has

$$\begin{aligned} & [[\dots [G(\Phi, R, I_1), G(\Phi, R, I_2)], \dots], G(\Phi, R, I_m)] \\ &= [[\dots [E(\Phi, R, I_1), E(\Phi, R, I_2)], \dots], E(\Phi, R, I_m)]. \end{aligned}$$

Let us also reiterate very ambitious Problems 7 and 8 posed in [33]. The first of these problems refers to the context of odd unitary groups, as created by Victor Petrov [50–52].

Problem 9. Generalise results of the present paper to odd unitary groups.

One of the first steps towards a solution of this problem, and other related problems for odd unitary groups was recently done by Rabeya Basu [16].

The next problem refers to the recent context of isotropic reductive groups. Of course, it only makes sense over commutative rings, but on the other hand, a lot of new complications occur, due to the fact that relative roots do not form a root system, and the interrelations of the elementary subgroup with the group itself are abstruse even over fields (the Kneser–Tits problem). Still, we are convinced that after the recent breakthrough by Victor Petrov and Anastasia Stavrova [53,56] most necessary tools are already there. See also their subsequent papers with Alexander Luzgarev and Ekaterina Kulikova [44,36].

Problem 10. Obtain results similar to those of the present paper for [groups of points of] isotropic reductive groups.

³ After the submission of the present paper, jointly with Alexei Stepanov we succeeded in solving this problem in the special case of $SL(n, R)$. It did in fact require both the full force of the relative commutator calculus, with two parameters, and new birelative and trirelative versions of Bak’s completion theorem [8], to avoid relativisation with several parameters. We are positive that the same strategy works for all Chevalley groups, but it may take quite a while to supply actual details, primarily because many of the fundamental results, classical for $GL(n, R)$, are simply not there in this larger generality.

Of course, here one shall have to develop the whole conjugation and commutator calculus almost from scratch.

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References

- [1] E. Abe, Chevalley groups over local rings, *Tôhoku Math. J.* 21 (1969) 474–494.
- [2] E. Abe, Chevalley groups over commutative rings, in: *Proc. Conf. Radical Theory*, Sendai, 1988, Uchida Rokakuho, Tokyo, 1989, pp. 1–23.
- [3] E. Abe, Normal subgroups of Chevalley groups over commutative rings, in: *Algebraic K-Theory and Algebraic Number Theory*, Honolulu, HI, 1987, in: *Contemp. Math.*, vol. 83, Amer. Math. Soc., Providence, RI, 1989, pp. 1–17.
- [4] E. Abe, Chevalley groups over commutative rings. Normal subgroups and automorphisms, in: *Second International Conference on Algebra*, Barnaul, 1991, in: *Contemp. Math.*, vol. 184, Amer. Math. Soc., Providence, RI, 1995, pp. 13–23.
- [5] E. Abe, K. Suzuki, On normal subgroups of Chevalley groups over commutative rings, *Tôhoku Math. J.* 28 (1) (1976) 185–198.
- [6] H. Apte, A. Stepanov, Local-global principle for congruence subgroups of Chevalley groups, arXiv:1211.3575v1 [math:RA], 15 November 2012, pp. 1–9.
- [7] A. Bak, Subgroups of the general linear group normalized by relative elementary subgroup, in: *Algebraic K-Theory*, Part II, Oberwolfach, 1980, in: *Lecture Notes in Math.*, vol. 967, Springer, Berlin, 1982, pp. 1–22.
- [8] A. Bak, Nonabelian K -theory: the nilpotent class of K_1 and general stability, *K-Theory* 4 (1991) 363–397.
- [9] A. Bak, R. Basu, R.A. Rao, Local-global principle for transvection groups, *Proc. Amer. Math. Soc.* 138 (4) (2010) 1191–1204.
- [10] A. Bak, R. Hazrat, N. Vavilov, Localization completion strikes again: relative K_1 is nilpotent by abelian, *J. Pure Appl. Algebra* 213 (2009) 1075–1085.
- [11] A. Bak, A. Stepanov, Dimension theory and nonstable K -theory for net groups, *Rend. Semin. Mat. Univ. Padova* 106 (2001) 207–253.
- [12] A. Bak, N. Vavilov, Normality for elementary subgroup functors, *Math. Proc. Cambridge Philos. Soc.* 118 (1) (1995) 35–47.
- [13] A. Bak, N. Vavilov, Structure of hyperbolic unitary groups I: elementary subgroups, *Algebra Colloq.* 7 (2) (2000) 159–196.
- [14] H. Bass, K -theory and stable algebra, *Inst. Hautes Etudes Sci., Publ. Math.* 22 (1964) 5–60.
- [15] H. Bass, J. Milnor, J.-P. Serre, Solution of the congruence subgroup problem for SL_n ($n \geq 3$) and Sp_{2n} ($n \geq 2$), *Publ. Math. Inst. Hautes Etudes Sci.* 33 (1967) 59–137.
- [16] R. Basu, Local-global principle for quadratic and hermitian groups and the nilpotency of K_1 , in press.
- [17] R. Basu, R.A. Rao, R. Khanna, On Quillen's local global principle, in: *Commutative Algebra and Algebraic Geometry*, in: *Contemp. Math.*, vol. 390, Amer. Math. Soc., Providence, RI, 2005, pp. 17–30.
- [18] A. Borel, Properties and linear representations of Chevalley groups, in: *Seminar on Algebraic Groups and Related Finite Groups*, Springer-Verlag, Berlin, 1970, pp. 1–55.
- [19] Z.I. Borewicz, N.A. Vavilov, The distribution of subgroups in the general linear group over a commutative ring, *Proc. Steklov Inst. Math.* (3) (1985) 27–46.
- [20] R. Carter, *Simple Groups of Lie Type*, Wiley, London, 1972.
- [21] D.L. Costa, G.E. Keller, Radix redux: normal subgroups of symplectic groups, *J. Reine Angew. Math.* 427 (1992) 51–105.
- [22] D.L. Costa, G.E. Keller, On the normal subgroups of $G_2(A)$, *Trans. Amer. Math. Soc.* 351 (12) (1999) 5051–5088.
- [23] G. Habdank, Mixed commutator groups in classical groups and a classification of subgroups of classical groups normalized by relative elementary groups, *Doktorarbeit*, Bielefeld Univ., 1987, pp. 1–71.
- [24] G. Habdank, A classification of subgroups of Λ -quadratic groups normalized by relative elementary groups, *Adv. Math.* 110 (1995) 191–233.
- [25] A.J. Hahn, O.T. O'Meara, *The Classical Groups and K-Theory*, Springer, 1989.
- [26] R. Hazrat, Dimension theory and nonstable K_1 of quadratic modules, *K-Theory* 27 (2002) 293–328.
- [27] R. Hazrat, On K -theory of classical-like groups, *Doktorarbeit*, Bielefeld Univ., 2002, pp. 1–62.
- [28] R. Hazrat, V. Petrov, N. Vavilov, Relative subgroups in Chevalley groups, *K-Theory* 5 (2010) 603–618.
- [29] R. Hazrat, A. Stepanov, N. Vavilov, Z. Zhang, The yoga of commutators, *J. Math. Sci.* 387 (2011) 53–82.
- [30] R. Hazrat, A. Stepanov, N. Vavilov, Z. Zhang, On the length of commutators in unitary groups, in press.
- [31] R. Hazrat, N. Vavilov, K_1 of Chevalley groups are nilpotent, *J. Pure Appl. Algebra* 179 (2003) 99–116.
- [32] R. Hazrat, N. Vavilov, Bak's work on the K -theory of rings, with an appendix by Max Karoubi, *K-Theory* 4 (2009) 1–65.
- [33] R. Hazrat, N. Vavilov, Z. Zhang, Relative unitary commutator calculus, and applications, *J. Algebra* 343 (2011) 107–137.
- [34] R. Hazrat, Z. Zhang, Generalized commutator formulas, *Comm. Algebra* 39 (2011) 1441–1454.
- [35] V.I. Kopeiko, The stabilization of symplectic groups over a polynomial ring, *Math. USSR Sb.* 34 (1978) 655–669.
- [36] E. Kulikova, A. Stavrova, Centralizer of the elementary subgroup of an isotropic reductive group, *Vestn. St. Petersburg. Univ., Ser. 1. Mat. Mekh. Astronom.* 1 (2013) 35–43.
- [37] Li Fuan, The structure of symplectic group over arbitrary commutative rings, *Acta Math. Sin. (N. S.)* 3 (3) (1987) 247–255.
- [38] Li Fuan, The structure of orthogonal groups over arbitrary commutative rings, *Chinese Ann. Math. Ser. B* 10 (3) (1989) 341–350.
- [39] Li Fuan, Liu Mulan, Generalized sandwich theorem, *K-Theory* 1 (1987) 171–184.
- [40] Li Shangzhi, N. Vavilov, Large subgroup classification project, in press.

- [41] A.Yu. Luzgarev, On overgroups of $E(E_6, R)$ and $E(E_7, R)$ in their minimal representations, *J. Math. Sci.* 134 (6) (2006) 2558–2571.
- [42] A.Yu. Luzgarev, Overgroups of $E(F_4, R)$ in $G(E_6, R)$, *St. Petersburg Math. J.* 20 (6) (2009) 955–981.
- [43] Alexander Luzgarev, Overgroups of exceptional groups, *Doktorarbeit, St.-Petersburg Univ.*, 2008, pp. 1–106 (in Russian).
- [44] A.Yu. Luzgarev, A.K. Stavrova, Elementary subgroup of an isotropic reductive group is perfect, *St. Petersburg Math. J.* 24 (5) (2012) 881–890.
- [45] A.W. Mason, A note on subgroups of $GL(n, A)$ which are generated by commutators, *J. Lond. Math. Soc.* 11 (1974) 509–512.
- [46] A.W. Mason, On subgroup of $GL(n, A)$ which are generated by commutators, II, *J. Reine Angew. Math.* 322 (1981) 118–135.
- [47] A.W. Mason, A further note on subgroups of $GL(n, A)$ which are generated by commutators, *Arch. Math.* 37 (5) (1981) 401–405.
- [48] A.W. Mason, W.W. Stothers, On subgroup of $GL(n, A)$ which are generated by commutators, *Invent. Math.* 23 (1974) 327–346.
- [49] H. Matsumoto, Sur les sous-groupes arithmétiques des groupes semi-simples déployés, *Ann. Sci. École Norm. Super.* (4) 2 (1969) 1–62.
- [50] V. Petrov, Overgroups of unitary groups, *K-Theory* 29 (2003) 147–174.
- [51] V.A. Petrov, Odd unitary groups, *J. Math. Sci.* 130 (3) (2003) 4752–4766.
- [52] V.A. Petrov, Overgroups of classical groups, *Doktorarbeit, St.-Petersburg Univ.*, 2005, pp. 1–129 (in Russian).
- [53] V.A. Petrov, A.K. Stavrova, Elementary subgroups of isotropic reductive groups, *St. Petersburg Math. J.* 20 (3) (2008) 160–188.
- [54] E. Plotkin, On the stability of the K_1 -functor for Chevalley groups of type E_7 , *J. Algebra* 210 (1) (1998) 67–95.
- [55] D. Quillen, Projective modules over polynomial rings, *Invent. Math.* 36 (1976) 167–171.
- [56] A.K. Stavrova, Structure of isotropic reductive groups, *Doktorarbeit, St.-Petersburg Univ.*, 2009, pp. 1–158 (in Russian).
- [57] M.R. Stein, Generators, relations and coverings of Chevalley groups over commutative rings, *Amer. J. Math.* 93 (4) (1971) 965–1004.
- [58] M.R. Stein, Relativising functors on rings and algebraic K-theory, *J. Algebra* 19 (1) (1971) 140–152.
- [59] M.R. Stein, Stability theorems for K_1 , K_2 and related functors modeled on Chevalley groups, *Jpn. J. Math.* 4 (1) (1978) 77–108.
- [60] R. Steinberg, *Lectures on Chevalley Groups*, Yale University, 1967.
- [61] A. Stepanov, Universal localisation in algebraic groups, in press, <http://alexei.stepanov.spb.ru/publicat.html>, 2010.
- [62] A. Stepanov, N. Vavilov, Decomposition of transvections: a theme with variations, *K-Theory* 19 (2000) 109–153.
- [63] A. Stepanov, N. Vavilov, On the length of commutators in Chevalley groups, *Israel J. Math.* 185 (2011) 253–276.
- [64] A. Stepanov, N. Vavilov, Hong You, Overgroups of semi-simple subgroups via localisation-completion, in press.
- [65] A.A. Suslin, On the structure of the special linear group over the ring of polynomials, *Izv. Akad. Nauk SSSR Mat.* 141 (2) (1977) 235–253.
- [66] A.A. Suslin, V.I. Kopeiko, Quadratic modules and orthogonal groups over polynomial rings, *J. Sov. Math.* 20 (6) (1985) 2665–2691.
- [67] K. Suzuki, Normality of the elementary subgroups of twisted Chevalley groups over commutative rings, *J. Algebra* 175 (2) (1995) 526–536.
- [68] G. Taddei, Normalité des groupes élémentaires dans les groupes de Chevalley sur un anneau, *Contemp. Math.* 55 (II) (1986) 693–710.
- [69] J. Tits, Systèmes générateurs de groupes de congruence, *C. R. Acad. Sci. Paris Sér. A* 283 (1976) 693–695.
- [70] L.N. Vaserstein, On the Normal Subgroups of GL_n Over a Ring, *Lecture Notes in Math.*, vol. 854, 1981, pp. 456–465.
- [71] L.N. Vaserstein, The subnormal structure of general linear groups, *Math. Proc. Cambridge Philos. Soc.* 99 (3) (1986) 425–431.
- [72] L.N. Vaserstein, On normal subgroups of Chevalley groups over commutative rings, *Tôhoku Math. J.* 36 (5) (1986) 219–230.
- [73] L.N. Vaserstein, The subnormal structure of general linear groups over rings, *Math. Proc. Cambridge Philos. Soc.* 108 (2) (1990) 219–229.
- [74] L.N. Vaserstein, You Hong, Normal subgroups of classical groups over rings, *J. Pure Appl. Algebra* 105 (1995) 93–105.
- [75] N. Vavilov, Subnormal structure of general linear group, *Math. Proc. Cambridge Philos. Soc.* 107 (1990) 103–106.
- [76] N. Vavilov, Structure of Chevalley groups over commutative rings, in: *Proc. Conf. Nonassociative Algebras and Related Topics*, Hiroshima, 1990, World Sci. Publ., London, 1991, pp. 219–335.
- [77] N. Vavilov, Intermediate subgroups in Chevalley groups, in: *Proc. Conf. Groups of Lie Type and their Geometries*, Como, 1993, Cambridge Univ. Press, 1995, pp. 233–280.
- [78] N. Vavilov, A third look at weight diagrams, *Rend. Semin. Mat. Univ. Padova* 104 (2) (2000) 1–50.
- [79] N. Vavilov, An A_3 -proof of structure theorems for Chevalley groups of types E_6 and E_7 , I, *Internat. J. Algebra Comput.* 17 (5–6) (2007) 1283–1298;
- [80] N. Vavilov, An A_3 -proof of structure theorems for Chevalley groups of types E_6 and E_7 , II, *St. Petersburg Math. J.* 23 (6) (2011).
- [81] N.A. Vavilov, Weight elements of Chevalley groups, *St. Petersburg Math. J.* 20 (1) (2009) 23–57.
- [82] N.A. Vavilov, M.R. Gavrilovich, An A_2 -proof of the structure theorems for Chevalley groups of types E_6 and E_7 , *St. Petersburg Math. J.* 16 (4) (2005) 649–672.
- [83] N.A. Vavilov, M.R. Gavrilovich, S.I. Nikolenko, Structure of Chevalley groups: the Proof from the Book, *J. Math. Sci.* 140 (5) (2007) 626–645.
- [84] N.A. Vavilov, S.I. Nikolenko, An A_2 -proof of the structure theorems for Chevalley groups of types F_4 , *St. Petersburg Math. J.* 20 (4) (2009) 27–63.
- [85] N.A. Vavilov, V.A. Petrov, On overgroups of $EO(2l, R)$, *J. Math. Sci.* 116 (1) (2003) 2917–2925.

- [85] N.A. Vavilov, V.A. Petrov, On overgroups of $\text{Ep}(2l, R)$, St. Petersburg Math. J. 15 (4) (2004) 515–543.
- [86] N.A. Vavilov, V.A. Petrov, On overgroups of $\text{EO}(n, R)$, St. Petersburg Math. J. 19 (2) (2008) 167–195.
- [87] N. Vavilov, E. Plotkin, Chevalley groups over commutative rings I: Elementary calculations, Acta Appl. Math. 45 (1996) 73–113.
- [88] N.A. Vavilov, A.V. Stepanov, Standard commutator formula, Vestnik St. Petersburg Univ. Math. 41 (1) (2008) 5–8.
- [89] N.A. Vavilov, A.V. Stepanov, Overgroups of semisimple subgroups, Vestnik Samar. Univ. Nat. Sci. (3 (62)) (2008) 51–94 (in Russian).
- [90] N.A. Vavilov, A.V. Stepanov, Standard commutator formulae, revisited, Vestnik St. Petersburg Univ. Math. 43 (1) (2010) 12–17.
- [91] J.S. Wilson, The normal and subnormal structure of general linear groups, Math. Proc. Cambridge Philos. Soc. 71 (1972) 163–177.
- [92] You Hong, On subgroups of Chevalley groups which are generated by commutators, J. Northeast. Normal Univ. 2 (1992) 9–13.
- [93] You Hong, Overgroups of symplectic group in linear group over commutative rings, J. Algebra 282 (1) (2004) 23–32.
- [94] You Hong, Overgroups of classical groups in linear group over Banach algebras, J. Algebra 304 (2006) 1004–1013.
- [95] You Hong, Overgroups of classical groups over commutative group in linear group, Sci. China Ser. A 49 (5) (2006) 626–638.
- [96] You Hong, Subgroups of classical groups normalised by relative elementary groups, J. Pure Appl. Algebra 216 (2011) 1040–1051.
- [97] Z. Zhang, Lower K -theory of unitary groups, Doktorarbeit, Belfast Univ., 2007, pp. 1–67.
- [98] Z. Zhang, Stable sandwich classification theorem for classical-like groups, Math. Proc. Cambridge Philos. Soc. 143 (2007) 607–619.
- [99] Z. Zhang, Subnormal structure of non-stable unitary groups over rings, J. Pure Appl. Algebra 214 (2010) 622–628.