Journal of Algebra 399 (2014) 269–276

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Nilpotent and polycyclic-by-finite maximal subgroups of skew linear groups

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ARTICLE INFO

Article history:
Received 10 April 2013
Available online 31 October 2013
Communicated by Louis Rowen

MSC:
12E15
16K40
16R20
20E28

Keywords:
Division ring
Maximal subgroup
Nilpotent group
Polycyclic-by-finite group

ABSTRACT

Let D be an infinite division ring, n a natural number and N a subnormal subgroup of $\mathrm{GL}_n(D)$ such that $n = 1$ or the center of D contains at least five elements. This paper contains two main results. In the first one we prove that each nilpotent maximal subgroup of N is abelian; this generalizes the result in Ebrahimian (2004) [3] (which asserts that each maximal subgroup of $\mathrm{GL}_n(D)$ is abelian) and a result in Ramezan-Nassab and Kiani (2013) [12]. In the second one we show that a maximal subgroup of $\mathrm{GL}_n(D)$ cannot be polycyclic-by-finite.

1. Introduction

Throughout this paper D denotes a division ring, n is a natural number, $M_n(D)$ is the full $n \times n$ matrix ring over D and $\mathrm{GL}_n(D)$ is the group of units of $M_n(D)$. The maximal soluble, maximal nilpotent, and maximal locally nilpotent subgroups of general linear groups (over fields) were extensively studied by Suprunenko; the main results are expounded in [19].

Our object here is to discuss the general skew linear groups whose maximal subgroups are of some special types. Some properties of maximal subgroups of $\mathrm{GL}_n(D)$ have been studied in a series of papers, see, e.g., [1–3,7,12,13]. In all of those papers, authors attempted to show that the structure of maximal subgroups of $\mathrm{GL}_n(D)$ is similar, in some sense, to the structure of $\mathrm{GL}_n(D)$. For instance, if

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0021-8693/$ – see front matter
http://dx.doi.org/10.1016/j.jalgebra.2013.09.042

D is an infinite division ring, in [3] it was shown that every nilpotent maximal subgroup of $\mathrm{GL}_n(D)$ is abelian, and in [13] the authors proved that for $n \geqslant 2$, every locally nilpotent maximal subgroup of $\mathrm{GL}_n(D)$ is abelian. Also, if D is non-commutative and $n \geqslant 2$, in [2] it was shown that every soluble maximal subgroup of $\mathrm{GL}_n(D)$ is abelian, and in [13] the authors proved that for $n \geqslant 3$, every locally soluble maximal subgroup of $\mathrm{GL}_n(D)$ is abelian. For some recent results see [12].

This paper contains two main results. In Section 2, instead of maximal subgroups of $\mathrm{GL}_n(D)$, we consider maximal subgroups of subnormal subgroups of $\mathrm{GL}_n(D)$. The structure of such groups have been investigated in various papers, see, e.g., [6,8,12,14]. As mentioned earlier, if D is an infinite division ring, in [3] it was shown that every nilpotent maximal subgroup of $\mathrm{GL}_n(D)$ is abelian. In [8, Corollary 1], the authors proved that if D is a finite-dimensional division algebra over its center, then every nilpotent maximal subgroup of a subnormal subgroup of $\mathrm{GL}_n(D)$ is abelian. In [12, Theorem 4], the authors showed that (without any condition on dimension) every nilpotent maximal subgroup of a subnormal subgroup of $\mathrm{GL}_n(D)$ is metabelian. Here, we generalize those results and show that every nilpotent maximal subgroup of a subnormal subgroup of $\mathrm{GL}_n(D)$ is abelian. More precisely, in Section 2 we prove the following theorem.

Theorem A. *Let D be an infinite division ring, n a natural number, N a subnormal subgroup of $\mathrm{GL}_n(D)$ and M a nilpotent maximal subgroup of N. If $n = 1$ or the center of D contains at least five elements, then M is abelian.*

In Section 3, we consider polycyclic-by-finite skew linear groups. For the reasons for doing this, see [18, Chapter 4] and references therein. We will see that $\mathrm{GL}_n(D)$ cannot be polycyclic-by-finite (Lemma 3.1). In this direction, we show that maximal subgroups of $\mathrm{GL}_n(D)$ have the same property, i.e., maximal subgroups of $\mathrm{GL}_n(D)$ are not polycyclic-by-finite. In fact we have:

Theorem B. *Let D be an infinite division ring, n a natural number and M a maximal subgroup of $\mathrm{GL}_n(D)$. If $n = 1$ or the center of D contains at least five elements, then M cannot be polycyclic-by-finite.*

Note that all division rings which are of characteristic zero, or algebraic over their centers have at least five elements in their centers. However, it seems that in Theorems A and B (for $n \geqslant 2$) the condition that the center of D contains at least five elements is not necessary (see also the remarks after Theorem 3.7).

Our notation is standard. To be more precise, F always denotes the center of the division ring D unless stated otherwise. We shall identify the center FI_n of $M_n(D)$ with F. If D has at least four elements, for $n \geqslant 2$ we denote by $\mathrm{SL}_n(D)$ the derived subgroup of $\mathrm{GL}_n(D)$. Let G be a subgroup of $\mathrm{GL}_n(D)$. We denote by $F[G]$ the F-linear hull of G, i.e., the F-algebra generated in $M_n(D)$ by elements of G over F. If $n = 1$, then $F(G)$ is the division ring generated in D by F and G; note that if each element of G is algebraic over F, then $F(G) = F[G]$. If D^n is the space of row n-vectors over D, then D^n is a D–G bimodule in the obvious manner. We say that G is irreducible, reducible, or completely reducible, whenever D^n has the corresponding property as D–G bimodule. Also, G is called absolutely irreducible if $F[G] = M_n(D)$. The derived subgroup of G is denoted by G'. For a given ring R, the group of units of R is denoted by R^*. Let S be a subset of R, then the centralizer of S in R is denoted by $C_R(S)$.

2. Nilpotent maximal subgroups

In this section we prove Theorem A. First we assert some useful lemmas which are also used in the next section.

Lemma 2.1. *(See [23, Corollary 24].) Let A be a one-sided Artinian ring. Suppose S is a right Goldie subring of A and G is a locally soluble subgroup of the group of units of A normalizing S. Set $R = S[G] \leqslant A$ and assume R is prime. Then R too is right Goldie.*

Lemma 2.2. *Let D be an infinite-dimensional division algebra over its center, N a subnormal subgroup of D^*, and M a maximal subgroup of N. If M is metabelian, then it is abelian.*

Proof. Since M' is abelian, we can find a maximal normal abelian subgroup A of M containing M'. Suppose on the contrary that $A \neq M$. If T is a subgroup of M such that $A \subsetneqq T$, we claim that $F(T) = D$. In fact, we have $M \subseteq N_N(F(T)^*) \subseteq N$. If $M = N_N(F(T)^*)$, then $F(T)^* \cap N \leqslant M$, so $F(T)^* \cap N$ as a metabelian subnormal subgroup of $F(T)^*$ is abelian, so T is also abelian, this contradicts the choice of T. Therefore, by the maximality of M in N we may assume $N_N(F(T)^*) = N$. Then $N \subseteq N_{D^*}(F(T)^*)$, which by [17, 13.3.8, 14.3.8] implies $F(T) = D$, as claimed.

Setting $K = F(A)$, clearly $M \nsubseteq K$. Suppose there is some $a \in M \setminus K$ which is transcendental over K and set $T = A\langle a^2\rangle$. By Lemma 2.1, $F[T]$ is a Goldie ring; since it is also a domain, it is an Ore domain. On the other hand, by the fact that $a^2 \notin A$ we conclude that T is a subgroup of M properly containing A; hence by what we proved before we conclude that $F(T) = D$. Hence the division ring generated by $F[T]$, which is exactly its classical ring of quotients, coincides with D. Thus there exist two elements $s_1, s_2 \in F[T]$ such that $a = s_1 s_2^{-1}$. Write $s_1 = \sum_{i=l}^{m} k_i a^{2i}$ and $s_2 = \sum_{i=l}^{m} k_i' a^{2i}$, where $k_i, k_i' \in K$, for any $l \leqslant i \leqslant m$. Hence

$$\sum_{i=l}^{m} a k_i' a^{2i} = \sum_{i=l}^{m} k_i a^{2i}.$$

If we set $l_i = a k_i' a^{-1}$, for any $1 \leqslant i \leqslant m$, then l_i's are elements of K and we have

$$\sum_{i=l}^{m} l_i a^{2i+1} = \sum_{i=l}^{m} k_i a^{2i},$$

which shows that a is algebraic over K, a contradiction.

Now let $x \in M \setminus K$ be algebraic over K. Assume that x satisfies an equation of the form $\sum_{i=0}^{n} k_i x^i = 0$, where $k_i \in K$ for any $0 \leqslant i \leqslant n$ and $k_n = 1$. Using the fact that x normalizes K and the above equality one can easily show that $R = \sum_{i=0}^{n} K x^i$ is a ring that is of finite dimension as a left vector space over K. Therefore it is a division ring. If we set $T = A\langle x\rangle$, by what we proved before $F(T) = D$. On the other hand obviously we have $F(T) = R$. Therefore $[D:K]_l < \infty$. Thus D is a finite-dimensional division algebra over its center. This contradiction shows that M is abelian. □

In [1] it was proved that $\mathbb{C}^* \cup \mathbb{C}^* j$ is a maximal subgroup of the real quaternion division algebra. Clearly $(\mathbb{C}^* \cup \mathbb{C}^* j)' \subseteq \mathbb{C}^*$ and so $\mathbb{C}^* \cup \mathbb{C}^* j$ is metabelian but not abelian. Thus in Lemma 2.2, the condition that D is of infinite dimension cannot be removed.

Now, using Lemma 2.2, we can prove Theorem A for $n = 1$.

Theorem 2.3. *Let D be a division ring and N a subnormal subgroup of D^*. Then every nilpotent maximal subgroup of N is abelian.*

Proof. Let M be a nilpotent maximal subgroup of N. By [14, Proposition 1.1] M' is an abelian group. Thus by Lemma 2.2, we may assume that D is a finite-dimensional division ring, which by [8, Corollary 1] we conclude the result. □

The proof of Theorem A, for $n \geqslant 2$, needs some different approaches. To this end, we have the following two lemmas.

Lemma 2.4. *Let D be a non-commutative division ring with center F, N a subnormal subgroup of $\mathrm{GL}_n(D)$, and M an absolutely irreducible and maximal subgroup of N. If $M/(M \cap F^*)$ is locally finite, then for any normal* *subgroup H of M we either have $H \subseteq F^*$, $F[H] = M_n(D)$, or $F[H] \simeq F_1 \times \cdots \times F_s$ for some natural number s and fields $F_i \supseteq F$.*

Proof. Since $M \subseteq N_N(F[H]^*) \subseteq N$, the maximality of M in N implies that, either $N_N(F[H]^*) = N$ or $N_N(F[H]^*) = M$. Let $G := F[H]^* \cap N$. In the first case, G is a subnormal subgroup of $\mathrm{GL}_n(D)$. Consequently, if H is not central, $\mathrm{SL}_n(D) \subseteq G \subseteq F[H]^*$ by [4, Lemma 2.3], and thus the Cartan–Brauer–Hua theorem for matrix ring implies that $F[H] = M_n(D)$.

Next assume $N_N(F[H]^*) = M$. Then $G \leqslant M$ and G is a normal subgroup of $F[H]^*$. On the other hand, since $F[M] = M_n(D)$ and $M/(M \cap F^*)$ is locally finite, D is a locally finite-dimensional division algebra over F. Also, by Clifford's theorem H is completely reducible; therefore $F[H]$ is semisimple Artinian by [18, p. 7]. Thus, by the Wedderburn–Artin theorem, there exist natural numbers n_i and division rings D_i such that

$$F[H] \simeq M_{n_1}(D_1) \times \cdots \times M_{n_s}(D_s),$$

as F-algebras. Now,

$$G \trianglelefteq F[H]^* \simeq \mathrm{GL}_{n_1}(D_1) \times \cdots \times \mathrm{GL}_{n_s}(D_s).$$

Define $\pi_i : G \to \mathrm{GL}_{n_i}(D_i)$ by $\pi_i((g_1, \ldots, g_s)) = g_i$, for $1 \leqslant i \leqslant s$. Clearly, π_i is a group homomorphism. Let $G_i := \pi_i(G)$. Thus, for every i, G_i is a normal subgroup of $\mathrm{GL}_{n_i}(D_i)$ which is locally finite over $F_i^* := Z(D_i)^* \supseteq F^*$. If there exists some i such that G_i is non-abelian, then $n_i \geqslant 2$: for if $n_i = 1$, as $G_i/Z(G_i)$ is locally finite, G_i' is a torsion subnormal subgroup of D_i^*, which by [5, Theorem 8] $G_i' \subseteq F_i^*$, which implies that G_i is soluble, and so G_i is also central by [17, 14.4.4], a contradiction. Then, by [18, p. 154], D_i is a locally finite field and hence F is also locally finite. Thus D is algebraic over a finite field and hence by Jacobson's theorem in [9, p. 208], we obtain $D = F$, which is a contradiction. Therefore, for every i, G_i is abelian and so G (thus H) is an abelian group. This implies that $F[H] \simeq F_1 \times \cdots \times F_s$, and completes the proof. □

Lemma 2.5. *(See [10, Theorem 2].) Let R be a prime ring with 1, $Z = Z(R)$ be the center of R containing at least five elements, and $\bar{U}$ the Z-subalgebra of R generated by R^*. Assume that $\bar{U}$ contains a nonzero ideal of R. If R^* has a soluble normal subgroup which is not central, then R is a domain.*

We are now in a position to complete the proof of Theorem A as follows.

Theorem 2.6. *Let D be an infinite division ring, N a subnormal subgroup of $\mathrm{GL}_n(D)$, $n \geqslant 2$, and M a nilpotent maximal subgroup of N. If the center of D contains at least five elements, then M is abelian.*

Proof. By [8, Corollary 1] we may assume D is infinite-dimensional over F. Let $R := F[M]$. Since $M \subseteq R \cap N \subseteq N$, by the maximality of M in N we consider the following two cases:

Case 1. Suppose $M = R \cap N$. Then M is a normal subgroup of R^*. On the other hand, if M is reducible, then it contains an isomorphic copy of D^* by [6, Lemma 1]; so D is a field, a contradiction. Assume that M is irreducible; thus R is a prime ring by [18, 1.1.14] and Goldie by Lemma 2.1. Moreover, since the $Z(R)$-subalgebra of R generated by R^* is R itself, we can use Lemma 2.5 to deduce that either R is a domain or $M \subseteq Z(R)$. But, in the first case R is in fact an Ore domain. Denote the classical quotient ring of R by Δ; then Δ is a division ring contained in $M_n(D)$ by [18, Theorem 5.7.8]. If $N = \Delta \cap N$, then $N \subseteq \Delta^*$, so the Cartan–Brauer–Hua theorem for matrix ring implies that $\Delta = M_n(D)$ which is impossible since $n \geqslant 2$. Therefore $M = \Delta \cap N$; thus M as a nilpotent normal subgroup of Δ^* is abelian.

Case 2. In this case, we consider the case $N = R \cap N$. Thus $\mathrm{SL}_n(D) \subseteq N \subseteq R^*$, so $R = M_n(D)$. Therefore, M is center-by-(locally finite) by [18, Theorem 5.7.11]. Clearly $Z(M) = M \cap F^*$, so $M/(M \cap F^*)$ is locally finite.

First we assume $M' \subseteq F^*$. Given $x, y \in M$ such that $xy \neq yx$, we have $F^*\langle x, y\rangle \trianglelefteq M$ (note that we may assume $F^* \subseteq M$, since otherwise, we can replace M by F^*M and N by F^*N). Hence, by Lemma 2.4, $F[\langle x, y\rangle] = M_n(D)$. Since x and y are algebraic over F, we have $F[\langle x, y\rangle] = F[x, y]$, consequently, $F[x, y] = M_n(D)$. So $[D : F] < \infty$ since D is locally finite-dimensional over F; this contradicts our assumption and proves that M is abelian.

Next we assume $M' \nsubseteq F^*$. Let $K = F[M']$. Then by Lemma 2.4 and [12, Lemma 11], $K = F_1 \times \cdots \times F_s$ for some natural number s and fields $F_i \supseteq F$. Suppose $x \in M' \setminus F^*$ and let $f(t) \in F[t]$ be the minimal polynomial of x over F. Since $M \subseteq N_{\mathrm{GL}_n(D)}(K)$, every conjugate of x with respect to M is in K and as well is a root of $f(t)$. Since $f(t)$ has a finite number of roots in K we have $[M : C_M(x)] < \infty$. Therefore, there is a normal subgroup H of M such that $H \subseteq C_M(x)$, and $|M/H| < \infty$. If $F[H] = M_n(D)$, since $H \subseteq C_M(x)$, every element of $M_n(D)$ commutes with x. So $x \in F$ which conflicts with the choice of x. Thus by Lemma 2.4 we may assume H is abelian. Therefore, M is abelian-by-finite and so, by Lemma 1.11 of [15, p. 176], the group ring FM satisfies a polynomial identity. Therefore $F[M] = M_n(D)$ as a homomorphic image of FM, satisfies a polynomial identity too. So by Kaplansky's theorem in [16, p. 36] we conclude that $[D : F] < \infty$, a contradiction. This finishes the proof. □

3. Polycyclic-by-finite maximal subgroups

The principal aim of this section is to prove Theorem B. The main step in the proof is to show that the result holds in the case $n = 1$. For our purposes, we need several lemmas as follows.

Lemma 3.1. *Let D be an infinite division ring. Then $\mathrm{GL}_n(D)$ cannot be a polycyclic-by-finite group.*

Proof. Suppose H is a polycyclic normal subgroup of $\mathrm{GL}_n(D)$ and $\mathrm{GL}_n(D)/H$ is finite. It is known that H must be central, so $\mathrm{GL}_n(D)/F^*$ is finite and therefore D^*/F^* is a torsion group. Consequently $D = F$ is an infinite field. On the other hand, since every polycyclic-by-finite group is finitely generated, $\mathrm{GL}_n(D)$ is finitely generated. This is impossible: for $n \geqslant 2$ use [11, Corollary 1], and for $n = 1$ use the fact that the multiplicative group of a field cannot be finitely generated unless the field is finite. □

Lemma 3.2. *Let D be an infinite division ring and M be a polycyclic-by-finite maximal subgroup of $\mathrm{GL}_n(D)$. Then M cannot be abelian-by-finite.*

Proof. By [11, Corollary 3], we may assume D is of infinite-dimensional division algebra over its center. Suppose on the contrary that M is abelian-by-finite. Then, by Lemma 1.11 in [15, p. 176], $F[M]$ satisfies a polynomial identity. If $F[M] = M_n(D)$, we use Kaplansky's theorem in [16, p. 36] to obtain $[D : F] < \infty$, a contradiction.

Now suppose $F[M]^* = M$. By [6, Lemma 1], M is irreducible and so $F[M]$ is a prime ring. Let $F_1 := C_{M_n(D)}(M)$ and recall that F_1 is a division ring by [1, Lemma 8]. We claim that F_1 is a field. Let $x \in F_1'$. Now, by the maximality of M in $\mathrm{GL}_n(D)$, either $\langle x, M\rangle = M$ or $\langle x, M\rangle = \mathrm{GL}_n(D)$. In the first case we have $x \in M \cap F_1$ and so $x \in Z(M)$. In the second case we obtain $x \in F^*$. Hence in any case we have $x \in F^*Z(M)$ and so $F_1' \subseteq F^*Z(M)$. This means that F_1' is abelian. So, F_1^* is soluble and hence F_1 is a field and the claim is established. Also, by the maximality of M in $\mathrm{GL}_n(D)$ (and by similar argument as in the proof of Lemma 3.1), we may assume $F_1^* \subseteq M$. Consequently, $F[M]$ is a prime PI-ring whose center F_1 is a field and therefore, by [16, Corollary 1.6.28], it is a simple ring. So, again by Kaplansky's theorem we have $F_1[M] \simeq M_m(\Delta)$ for some natural number m and a division ring Δ. Thus $M = F_1[M]^* \simeq \mathrm{GL}_m(\Delta)$. If Δ is finite, M is also finite; this is impossible by [1, Lemma 9]. Thus Δ is an infinite division ring; but this contradicts Lemma 3.1 and completes the proof. □

Lemma 3.3. *(See [21, 3.11].) Let G be a locally nilpotent subgroup of the multiplicative group D^* of the division ring D. Suppose also that $H = N_{D^*}(G)$, $E = C_D(G)$, and $D = E(G)$. Denote the maximal 2-subgroup of G by Q. Then one of the following holds*:

(i) *T (the maximal locally finite normal subgroup of G) is abelian and H/GE^* is abelian*;
(ii) *$G = Q \cdot C_G(Q)$ where Q is quaternion of order 8 and $H/GE^* \simeq \mathrm{Sym}(3) \times Y$ for Y abelian*;

(iii) *$G \neq Q \cdot C_G(Q)$ where Q is quaternion of order 8 and H/GE^* is abelian;*
(iv) *Q is non-abelian with $|Q| > 8$ and H/GE^* has an abelian subgroup Y with index in H/GE^* at most 2 (1 if Q is infinite).*

Lemma 3.4. *Let D be an infinite division ring, M a polycyclic-by-finite maximal subgroup of D^* and G a nilpotent normal subgroup of M. Then G is abelian and $F(G)^* \trianglelefteq M$.*

Proof. Since $M \subseteq N_{D^*}(F(G)^*)$, we either have $F(G)^* \trianglelefteq M$ or (by the Cartan–Brauer–Hua theorem) $F(G) = D$. In the former case, as the multiplicative group of the division ring $F(G)$ is polycyclic-by-finite, G is abelian.

We claim that $F(G) \neq D$. Assume on the contrary $F(G) = D$. If M is absolutely irreducible, then $M/C_M(G)$ is torsion by [18, Theorem 5.7.11]. Since $C_M(G) \subseteq F^*$ (because of $F(G) = D$), we conclude that M is torsion over F and therefore $F[G] = F(G) = D$, i.e., G is absolutely irreducible. Clearly, $Z(G) = G \cap F$; so $G/(G \cap F)$ is locally finite by [18, Theorem 5.7.11]. This implies that D is locally finite-dimensional over F. Since M is finitely generated, we may assume $M = \langle m_1, \dots, m_s \rangle$. So, $F[m_1, \dots, m_s] = F[\langle m_1, \dots, m_s \rangle] = D$ implies that $[D : F] < \infty$. This conflicts [11, Corollary 3].

Now, suppose M is not absolutely irreducible, so $F[M]^* = M$. Since $F(G) = D$, $F = C_D(G)$. On the other hand, $M \subseteq N_{D^*}(G) \subseteq D^*$. If $N_{D^*}(G) = D^*$, then G as a nilpotent normal subgroup of D^* is central, so $D = F$ which is impossible. So assume $M = N_{D^*}(G)$. Now we can apply Lemma 3.3. Denote the maximal 2-subgroup of G by Q. If Q is finite, then $F[Q]^* \subseteq F[M]^* = M$ implies that the multiplicative group of the division ring $F[Q]$ is polycyclic-by-finite which asserts that Q is abelian. Thus by Lemma 3.3 we may assume that M/GF^* is abelian. This gives us $M' \subseteq GF^*$ is nilpotent. Therefore M is soluble, so it is abelian by [2, Theorem 3.7]; this cannot happen by Lemma 3.2. Therefore our claim, and so the statement of the lemma, holds. □

Lemma 3.5. *(See [22, Proposition 4.1].) Let $D = E(M)$ be a division ring generated as such by its metabelian subgroup M and its division subring E such that $E \subseteq C_D(M)$. Set $K = N_{D^*}(M)$, $G = C_M(M')$, T to be the maximal periodic normal subgroup of G, $F = E(Z(G))$, $L = N_{F^*}(M) = K \cap F$. Then*

(i) *if M has a quaternion subgroup Q of order 8 with $M = QC_M(Q)$, then $K = Q^+ML$;*
(ii) *if T is abelian and contains an element x of order 4 not in the center of G, then $K = \langle 1 + x \rangle ML$;*
(iii) *in all other cases $K = ML$.*

We are now ready to prove Theorem B in the case $n = 1$.

Theorem 3.6. *Let D be an infinite division ring and M a maximal subgroup of D^*. Then M cannot be polycyclic-by-finite.*

Proof. Let M be a polycyclic-by-finite group. We have a series of the form

$$1 = H^{(s)} \lhd \cdots \lhd H' \lhd H \lhd M,$$

where M/H is a finite group. By Lemma 3.2, H is non-abelian. Set $H^{(0)} = H$, and let r be the largest integer such that $H^{(r)} \nsubseteq F$, and so $H^{(r+1)} \subseteq F$. Note that $H^{(r)}$ is a nilpotent normal subgroup of M, so $H^{(r)}$ is abelian by Lemma 3.4. Let $M_1 := H^{(r-1)}$ which is a (non-abelian) metabelian normal subgroup of M. Since $M \subseteq N_{D^*}(F(M_1))$, we have $F(M_1) = D$. If we set $G = C_{M_1}(M_1')$, then clearly G is a nilpotent normal subgroup of M; thus by Lemma 3.4, G is abelian and $F(G)^* \trianglelefteq M$. Now, by Lemma 3.5, we have the following three cases to consider:

(i) $M_1 = QC_{M_1}(Q)$. Then $C_{M_1}(Q) \lhd M_1$ and hence we conclude that $M_1/C_{M_1}(Q) \simeq Q/(Q \cap C_{M_1}(Q)) = Q/Z(Q)$ is abelian. Thus, $M_1' \subseteq C_{M_1}(Q)$ and so $Q \subseteq C_{M_1}(M_1') = G$, which is a contradiction since G is abelian.

(ii) The case (ii) of Lemma 3.5 cannot occur since G is abelian.
(iii) $M = M_1 F(G)^*$. In this case $M/F(G)^* \simeq M_1/(F(G)^* \cap M_1)$ is abelian because $M_1' \subseteq F(G)^* \cap M_1$, and hence $M' \subseteq F(G)^*$ is abelian; consequently M is abelian by Lemma 2.2. Since M_1 was non-abelian, we arrive at a contradiction.

The proof of the theorem is completed. □

Finally we assert a theorem which completes the proof of Theorem B.

Theorem 3.7. *Let D be an infinite division ring and M a maximal subgroup of $\mathrm{GL}_n(D)$, $n \geqslant 2$. If the center of D contains at least five elements, then M cannot be polycyclic-by-finite.*

Proof. Let M be a polycyclic-by-finite group. If M is absolutely irreducible, M is abelian-by-finite by [12, Theorem 1(i)]. This cannot happen by Lemma 3.2. Now let $F[M]^* = M$. Since M is polycyclic-by-finite, $F[M]$ is a Noetherian ring by a result of Hall (see, e.g., [20, p. 35]). On the other hand, by Lemmas 2.5 and 3.2, $F[M]$ is a domain. Then, $F[M]$ is in fact an Ore domain. Therefore, the classical quotient ring of $F[M]$ is a division ring Δ which by [18, Theorem 5.7.8] is contained in $M_n(D)$. Since $n \geqslant 2$, maximality of M in $\mathrm{GL}_n(D)$ implies that $M = F[M]^* = \Delta^*$, and so M is abelian. This, again, contradicts Lemma 3.2. □

We close this paper by some remarks. Firstly, from our proofs of Theorems 2.6 and 3.7, we see that in those theorems it is enough that (instead of the center of D) the center of $F[M]$ contains at least five elements. Secondly, by the work in Section 3 we see that the statement of Theorem B (instead of polycyclic-by-finite groups) holds also for finitely generated soluble-by-finite groups:

Theorem B′. *Let D be an infinite division ring, n a natural number and M a maximal subgroup of $\mathrm{GL}_n(D)$. If $n = 1$ or the center of $F[M]$ contains at least five elements, then M cannot be finitely generated soluble-by-finite.*

Finally, an example of a finitely generated soluble skew linear group is given in [18, p. 128]. Thus the condition that M is a maximal subgroup of $\mathrm{GL}_n(D)$ in Theorem B′ is essential.

Acknowledgments

The research of the second author was in part supported by a grant from IPM (Grant No. 92050220). The second author also thanks National Foundation of Elites for financial support.

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