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The clean property is not a Morita invariant



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ABSTRACT

For each $n \geq 2$, we construct a ring R such that the matrix ring $M_n(R)$ is clean and $M_k(R)$ is not clean for $k < n$. This answers in negative the question posed by Han and Nicholson [11] whether the clean property is Morita invariant.

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1. Introduction

The notion of a clean ring was introduced in 1977 by Nicholson [15], as a ring in which every element can be expressed as the sum of an idempotent and a unit. Nicholson proved that clean rings are a subclass of exchange rings, and that the converse also holds in the case when idempotents in the ring are central.

In the past, the study of clean rings was mostly focused on verifying which known exchange rings are clean. Thus, it was proved that semiperfect rings [7], unit-regular rings [5], and endomorphism rings of vector spaces [16] are all clean. In addition, endomorphism rings of continuous modules [6] and exchange rings with primitive factors

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artinian [8,13] are clean. These examples show that clean rings are quite a large subclass of exchange rings.

The first example of a non-clean exchange ring was found by Camillo and Yu [7] in 1994. Camillo and Yu observed that Bergman's example [12, Example 1], of a regular ring that is not generated by its units, is not clean. Therefore, this example proved that regular rings (which are always exchange) need not be clean.

One advantage of clean rings over exchange rings is that the group K_0 of clean rings with a “large” ideal behaves nicer than in the case of exchange rings. In particular, in [21] it was proved that, if R is a clean ring with an ideal I such that R/I is local, then units in R/I can be lifted modulo I and we have $K_0(R) \cong K_0(R/I) \oplus K_0(I)$. Also, if R is clean, R/I is semiperfect and $K_0(I)$ is torsion-free, then, under mild conditions, units lift modulo I and we have $K_0(R) \cong K_0(R/I) \oplus K_0(I)$. These properties can easily fail in exchange rings, as Bergman's example demonstrates (see [21]).

In 2001, Han and Nicholson [11] proved that clean rings are closed under matrix extensions. More generally, if e is an idempotent in a ring R such that the corner rings eRe and $(1-e)R(1-e)$ are clean, then R is also clean. This implies that the matrix ring $M_n(R)$ over a clean ring R is clean, for every n . Han and Nicholson also asked if the converse of this proposition holds: if $M_n(R)$ is clean, is then R also clean. More generally, they asked if the clean property is a Morita invariant, i.e. if eRe is a clean ring whenever R is clean and e is a *full* idempotent in R (an idempotent $e \in R$ with $R = ReR$).

In [20] an example was given showing that, in general, corners of clean rings need not be clean. However, the example given there did not answer the question about the Morita invariance since the corner rings in that example were not full. The main purpose of this paper is to give an example of a ring such that the matrix ring is clean but the ring itself is not. In fact, for every n we give an example of a ring R such that $M_k(R)$ is clean if and only if k is a multiple of n . These examples therefore show that the clean property is not Morita invariant.

The idea behind our construction comes from the above mentioned K -theoretic property of clean rings. In particular, we can find a clean ring R with an ideal I such that R/I is semiperfect and $K_0(I)$ has torsion, but units do not lift modulo I . This indeed turns out to be the key idea to finding the example.

The paper is organized as follows. In Section 2 we give some preliminary results, and introduce terminology used throughout the paper. In Section 3 we construct an example of a clean ring R such that the group $K_0(R)$ is cyclic of order n , generated by the module R_R . This ring is obtained from Goodearl's example of a regular ring with the group K_0 cyclic of order n .

In Section 4 we define a generalization of Bergman's example of a non-clean exchange ring. We also prove some basic properties of the obtained ring. In Section 5 we prove that any non-scalar matrix (in a suitable sense) over generalized Bergman's ring is a sum of an idempotent and a unit. In the last section we put together all the previous results to obtain the desired example.

All rings and algebras in the paper will be associative and unital, unless stated otherwise. The set of all units in the ring R will be denoted by $U(R)$, the set of idempotents by $\text{Id}(R)$, and the ring of $n \times n$ matrices over R by $M_n(R)$. The set of units in $M_n(R)$ will be denoted by $GL_n(R)$.

The set of integers is denoted by \mathbb{Z} , the set of integers modulo n by \mathbb{Z}_n , and the set of positive integers by \mathbb{N} . For a ring R , we denote by $M_{\mathbb{N}}(R)$ the ring of all countably infinite matrices over R with finite columns.

2. Preliminaries

A ring R is called an *exchange* ring if for every $a \in R$ there exists an idempotent $e \in Ra$ such that $1 - e \in R(1 - a)$. The class of exchange rings is closed under taking corners and matrix extensions, meaning that if R is an exchange ring then $M_n(R)$ and eRe are exchange rings, for every $n \in \mathbb{N}$ and $e \in \text{Id}(R)$.

The definition of an exchange ring can be generalized to non-unital rings. Namely, a (possibly non-unital) ring I is said to be an *exchange* ring if for each $a \in I$ there exist an idempotent $e \in I$ and elements $r, s \in I$ such that $e = ra = a + s - sa$ (see [1]). This definition agrees with the definition for unital rings if I has a unity. In [1], Ara proved that if I is a two-sided ideal of a ring R then R is an exchange ring if and only if I and R/I are exchange and idempotents in R/I can be lifted modulo I .

An element a of a ring R is called *clean* if a is a sum of an idempotent and a unit in R . A ring R is called *clean* if every $a \in R$ is clean in R (see [15]). Every clean ring is exchange and if idempotents in the ring are central, then the converse also holds [15, Proposition 1.8]. In [12, Example 1], Bergman constructed a ring which later turned out to be an example of a non-clean exchange ring (see [7]).

In [11], Han and Nicholson proved that if R is a ring with an idempotent $e \in \text{Id}(R)$ such that eRe and $(1-e)R(1-e)$ are clean rings, then R is also clean. This, in particular, implies that the matrix ring $M_n(R)$ over a clean ring R is clean for every $n \in \mathbb{N}$. Moreover, if we look at the proof of this fact in [11] we also see that the following holds:

Lemma 2.1. *Let R be a ring and $a \in R$. Suppose that there exists an idempotent $e \in R$, with $f = 1 - e$, such that eRe is a clean ring and faf is clean in fRf . Then a is clean in R .*

Proof. See the proof of [11, Lemma on p. 2590]. \square

For a ring R , we denote by $FP(R)$ the monoid of all isomorphism classes of finitely generated projective right modules over R , endowed with the direct sum operation. The group $K_0(R)$ is defined as the Grothendieck group of the monoid $FP(R)$ (see [19]). Thus, every element in $K_0(R)$ is of the form $[P] - [Q]$, where P and Q are finitely generated projective right modules over R .

There is another definition of $K_0(R)$ which comes from idempotents. Let $\text{Idem}(R)$ denote the monoid of all equivalence classes of idempotent matrices over R , where two matrices $E \in \text{Id}(M_m(R))$ and $F \in \text{Id}(M_n(R))$ are equivalent provided there exist $k \geq m, n$ and $U \in GL_k(R)$ such that $\begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix} = U \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix} U^{-1}$. The operation on $\text{Idem}(R)$ is given by $[E] + [F] = [\begin{pmatrix} E & 0 \\ 0 & F \end{pmatrix}]$. Now $K_0(R)$ can be defined as the Grothendieck group of the monoid $\text{Idem}(R)$ (see [19, Theorem 1.2.3]). According to this definition, elements in $K_0(R)$ are of the form $[E] - [F]$, with E and F idempotent matrices (of possibly different sizes) over R .

For a non-unital ring I we denote by I^1 the unitization $I^1 = \mathbb{Z} \oplus I$. Let $p : I^1 \rightarrow \mathbb{Z}$ denote the canonical projection, and $K_0(p) : K_0(I^1) \rightarrow K_0(\mathbb{Z})$ the induced homomorphism of K_0 -groups. The group $K_0(I)$ is defined as $K_0(I) = \ker(K_0(p))$. According to this definition, every element in $K_0(I)$ can be written as $[E] - [F]$, with E, F idempotent matrices over I^1 such that $[p(E)] = [p(F)]$ in $K_0(\mathbb{Z})$. Note that idempotent matrices over I^1 can be also written as pairs (M, X) , with $M \in \text{Id}(M_n(\mathbb{Z}))$ and $X \in M_n(I)$ satisfying $MX + XM + X^2 = X$. Thus, elements in $K_0(I)$ are of the form $[(M, X)] - [(N, Y)]$, with $[M] = [N]$ in $K_0(\mathbb{Z})$.

If R is a ring, then $GL(R)$ is defined to be the direct limit of the directed system

$$U(R) \rightarrow GL_2(R) \rightarrow GL_3(R) \rightarrow \cdots$$

where each $a \in GL_n(R)$ is mapped to $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \in GL_{n+1}(R)$. A matrix $P \in GL_n(R)$ is called elementary if all its diagonal entries are 1 and at most one off-diagonal entry is non-zero. Let $E(R)$ denote the subgroup of $GL(R)$ generated by all elementary matrices. Then $E(R) = [GL(R), GL(R)]$ and $K_1(R)$ is defined to be the quotient group $K_1(R) = GL(R)/E(R) = GL(R)_{\text{ab}}$.

Recall that if I is a two-sided ideal of a ring R , then there is an exact sequence of abelian groups

$$K_1(R) \xrightarrow{K_1(q)} K_1(R/I) \xrightarrow{\partial} K_0(I) \xrightarrow{\iota} K_0(R) \xrightarrow{K_0(q)} K_0(R/I),$$

where the maps $K_i(q) : K_i(R) \rightarrow K_i(R/I)$ are induced by the canonical projection $q : R \rightarrow R/I$ and ι is induced by the natural homomorphism $I^1 \rightarrow R$. The homomorphism $\partial : K_1(R/I) \rightarrow K_0(I)$ is called the *connecting homomorphism*. We refer the reader to [19] for more details.

Let I be a ring (possibly non-unital), embedded into a unital ring R as a two-sided ideal. Then $V(I)$ is defined to be the monoid of all isomorphism classes of finitely generated projective right R -modules P such that $PI = P$, endowed with the direct sum operation. It turns out that the monoid $V(I)$ does not depend on the involving ring R (see [14], or also [19]). When I is a unital ring, we have just $V(I) = FP(I)$.

An abelian monoid $(A, +)$ is called *separative* provided that $a + a = a + b = b + b$ implies $a = b$ for every $a, b \in A$. A (possibly non-unital) ring I is *separative* if $V(I)$ is a separative monoid.

It turns out that separativity is the key condition for many problems related to exchange rings (see [2,3,18]). The most interesting to us is the following result due to Perera [18], on lifting units in exchange rings. Call an ideal I of a ring R an *exchange ideal* if I is an exchange non-unital ring.

Theorem 2.2. (See [18, Theorem 2.4].) *Let R be a ring with a separative exchange ideal I . Then a unit $\alpha \in U(R/I)$ can be lifted modulo I if and only if $\partial([\alpha]) = 0$, where $\partial : K_1(R/I) \rightarrow K_0(I)$ is the connecting map in K -theory.*

We will also need the following lemma which follows from the fact that, if R is an endomorphism ring of a vector space, then the monoid $FP(R)$ possesses uniqueness of n -th roots, meaning that $A^n \cong B^n$ implies $A \cong B$ for any two finitely generated projective right R -modules A and B (see [2]). Clearly, uniqueness of n -th roots is stronger than separativity. Thus we have:

Lemma 2.3. *The endomorphism ring of any vector space over a field is separative. \square*

3. A clean ring R with $K_0(R) = \langle R_R \rangle \cong \mathbb{Z}_n$

In this section, we construct, for each $n \geq 2$, a clean ring R such that the group $K_0(R)$ is isomorphic to \mathbb{Z}_n and generated by the free right R -module R_R . This ring will be needed later for the construction of our counter-example.

Note that there are known examples, for each n , of (von Neumann) regular rings R such that $K_0(R)$ is isomorphic to \mathbb{Z}_n (see [10, Example 6.13 and Example 15.1]). It is not difficult to see that those rings are clean. However, the group $K_0(R)$ in those examples is not generated by the module R_R .

There are also known examples of purely infinite simple regular rings R with $K_0(R) \cong \mathbb{Z}_n$ and $K_0(R)$ generated by R_R (see [4, Theorem 5.6]), but we do not know if those rings are clean or not. However, we will obtain the desired example by making a slight modification of [10, Example 6.13]. The first step will be to prove that the rings defined in that example are actually clean.

Following [10, Example 6.13], we fix any $n \geq 2$, and let F be any field embedded in a larger field $F \subseteq F'$ such that $\dim_F(F') = n$. Let V be an infinite-dimensional vector space over F' . Note that V is also a vector space over F . Denote the endomorphism rings by $S = \text{End}_F(V)$ and $S' = \text{End}_{F'}(V)$, and note that S' is a subring of S . Let $I = \{a \in S \mid \dim_F(\text{im}(a)) < \infty\}$ and $I' = \{a \in S' \mid \dim_{F'}(\text{im}(a)) < \infty\}$, so that I is a two-sided ideal in S and I' is a two-sided ideal in S' . Note that $I' = I \cap S'$.

Now define

$$R = S' + I.$$

Then, by [10, Example 6.13 and Example 15.1], R is a regular ring with $K_0(R) \cong \mathbb{Z}_n$.

Proposition 3.1. *The ring R is clean.*

Proof. Take any element $a = b + c \in R$, with $b \in S'$ and $c \in I$. Let e be an idempotent in S' such that $\text{im}(c) \subseteq \text{im}(e)$ and $\dim_{F'}(\text{im}(e)) = k < \infty$. Note that $\dim_F(\text{im}(e)) = kn$, and let $f = 1 - e$.

To prove that a is clean in R , by Lemma 2.1 it suffices to prove that eRe is a clean ring and faf is clean in fRf . The former is clear since $eRe = eSe \cong M_{kn}(F)$.

To prove the latter, first observe that $fc = 0$, so that $faf = fbf$. Hence faf is an element of $fS'f$. Since $fS'f$ is isomorphic to S' and S' is clean (the endomorphism ring of a vector space is always clean by [17, Proposition 3]), it follows that faf is clean in $fS'f$. Clearly this implies that faf is also clean in fRf , which finishes the proof. \square

Now let us proceed with our construction. Fix any idempotent $e \in S'$ such that $\dim_{F'}(\text{im}(e)) = 1$, and write $f = 1 - e \in S'$. Note that $\dim_F(\text{im}(e)) = n$, hence there exist pair-wise orthogonal idempotents $e_1, \dots, e_n \in I$ with $\dim_F(\text{im}(e_i)) = 1$ and $e = e_1 + \dots + e_n$. Since $f_1 = e_n + f$ is an idempotent in R , we may define the corner ring

$$R_1 = f_1 R f_1.$$

We will prove that the ring R_1 satisfies the desired properties, i.e. R_1 is clean and $K_0(R_1)$ is isomorphic to \mathbb{Z}_n and generated by the module $(R_1)_{R_1}$.

Proposition 3.2. *The ring R_1 is clean.*

Proof. First observe that $e_n \in R_1$, hence by [11, Lemma on p. 2590] it suffices to prove that $e_n R_1 e_n$ and $(f_1 - e_n) R_1 (f_1 - e_n)$ are clean rings. The cleanness of $e_n R_1 e_n$ follows from an obvious isomorphism $e_n R_1 e_n \cong F$. It is also easy to see that the ring $(f_1 - e_n) R_1 (f_1 - e_n) = fRf$ is actually isomorphic to R . The ring R is clean by Proposition 3.1, which completes the proof. \square

Now let us investigate the structure of the group $K_0(R_1)$. First we prove:

Proposition 3.3. *The ring $M_n(R_1)$ is isomorphic to R .*

Proof. Throughout the proof, we will identify matrices in $M_n(S)$ with endomorphisms in $\text{End}_F(W)$, where $W = V^n$ viewed as columns, and $M_n(S)$ will be acting on W from the left.

First, choose idempotents $g_2, g_3, \dots, g_n \in S'$ such that $\dim_{F'}(\text{im}(g_i)) = 1$ for each i and e, g_2, g_3, \dots, g_n are pair-wise orthogonal. For each $i = 2, \dots, n$, we have $\dim_F(\text{im}(g_i)) = n$, hence there exist pair-wise orthogonal idempotents $g_{i1}, \dots, g_{in} \in I$ with $\dim_F(\text{im}(g_{ij})) = 1$ for each j , and $g_i = g_{i1} + \dots + g_{in}$. To simplify the notation, we denote $g_1 = e$ and $g_{1j} = e_j$ for $j = 1, \dots, n$, so that we have $g_i = g_{i1} + \dots + g_{in}$ for each $i = 1, \dots, n$.

Let $g = g_1 + \dots + g_n \in S'$ and $h = 1 - g$, and take vectors $x_{ij} \in V$, for $i, j = 1, \dots, n$, such that $\text{im}(g_{ij}) = Fx_{ij}$ for each i, j . Then, considering $f = 1 - e$, we have

$$V = \left(\bigoplus_{j=1}^n Fx_{1j} \right) \oplus \text{im}(f) = \left(\bigoplus_{i,j=1}^n Fx_{ij} \right) \oplus \text{im}(h).$$

Now, observe that $\bigoplus_{i=1}^n \text{im}(f) \leq W$ and $\text{im}(h)$ are isomorphic as vector spaces over F' . Let $\varphi : \bigoplus_{i=1}^n \text{im}(f) \rightarrow \text{im}(h)$ be an F' -isomorphism. We extend φ to an F -linear map $\varphi : W \rightarrow V$ by setting

$$\varphi(0, \dots, 0, x_{1j}, 0, \dots, 0) = x_{jk}$$

where x_{1j} on the left side stands at k -th position. Observe that $\varphi : W \rightarrow V$ is then an isomorphism of F -spaces.

The isomorphism φ induces a ring isomorphism $\psi : M_n(S) \rightarrow S$, defined by $\psi(A) = \varphi A \varphi^{-1}$. We will prove that ψ maps $M_n(R_1) \leq M_n(S)$ onto $(g_n + h)R(g_n + h)$. Since $(g_n + h)R(g_n + h)$ is clearly isomorphic to R , this will conclude the proof.

Let $F_1 = \text{diag}(f_1, \dots, f_1) \in M_n(S)$, so that $M_n(R_1) = F_1 M_n(R) F_1$. We can directly verify that $\varphi F_1 = (g_n + h)\varphi$. Thus $\psi(F_1) = \varphi F_1 \varphi^{-1} = g_n + h$, which yields

$$\psi(M_n(R_1)) = \psi(F_1 M_n(R) F_1) = (g_n + h)\psi(M_n(R))(g_n + h).$$

Therefore we only need to prove that $\psi(M_n(R)) = R$.

First we prove the inclusion $\psi(M_n(R)) \subseteq R$. Observe that, since $1 - f \in I$, we have $R = S' + I = fS'f + I$. Denoting $F_0 = \text{diag}(f, \dots, f) \in M_n(S')$ we have $M_n(R) = F_0 M_n(S') F_0 + M_n(I)$. Note that $\varphi|_{\text{im}(F_0)}$ is F' -linear, hence $\varphi F_0 : W \rightarrow V$ is F' -linear. Also observe that $F_0 \varphi^{-1} : V \rightarrow W$ is F' -linear. Therefore $\psi(F_0 M_n(S') F_0) = \varphi F_0 M_n(S') F_0 \varphi^{-1} \subseteq S'$. Clearly, we also have $\psi(M_n(I)) \subseteq I$, hence

$$\psi(M_n(R)) = \psi(F_0 M_n(S') F_0 + M_n(I)) \subseteq S' + I = R.$$

Conversely, we have $1 - h \in I$ and hence $R = hS'h + I$. Observe that the maps $h\varphi$ and $\varphi^{-1}h$ are F' -linear, hence $\varphi^{-1}hS'h\varphi \subseteq M_n(S')$. Clearly we also have $\varphi^{-1}I\varphi \subseteq M_n(I)$. This yields $\varphi^{-1}R\varphi = \varphi^{-1}hS'h\varphi + \varphi^{-1}I\varphi \subseteq M_n(S') + M_n(I) = M_n(R)$ and therefore $R \subseteq \varphi M_n(R) \varphi^{-1}$, as desired. This completes the proof. \square

Proposition 3.4. *The group $K_0(R_1)$ is isomorphic to \mathbb{Z}_n and generated by the module $(R_1)_{R_1}$.*

Proof. By Proposition 3.3 we have $M_n(R_1) \cong R$, and by [10, Example 15.1] we have $K_0(R) \cong \mathbb{Z}_n$. Hence $K_0(R_1) \cong K_0(M_n(R_1)) \cong K_0(R) \cong \mathbb{Z}_n$.

Let us prove that $K_0(R_1)$ is generated by the equivalence class of the free right module $(R_1)_{R_1}$. In the idempotent notation, we need to prove that $K_0(R_1)$ is generated by $[(f_1)]$, where (f_1) is treated as a 1×1 idempotent matrix over R_1 .

First observe that the idempotents $\text{diag}(f, f)$ and $\text{diag}(0, f)$ are conjugate in $M_2(fS'f)$ as they both have infinite ranks. This implies that $\text{diag}(f, f)$ and $\text{diag}(0, f)$ are also conjugate in $M_2(f_1Rf_1) = M_2(R_1)$. It follows that $[(f)] = 0$ in $K_0(R_1)$. Hence $[(f_1)] = [(e_n)] + [(f)] = [(e_n)]$ in $K_0(R_1)$, thus we need to prove that $[(e_n)]$ is a generator in $K_0(R_1)$.

Denote the canonical isomorphism $\alpha : K_0(M_n(R_1)) \rightarrow K_0(R_1)$, and let $\psi : M_n(R_1) \rightarrow (g_n + h)R(g_n + h)$ be as in the proof of Proposition 3.3. Denote the canonical isomorphism $\beta : (g_n + h)R(g_n + h) \rightarrow R$, so that $\psi' = \beta\psi$ is an isomorphism $M_n(R_1) \rightarrow R$. Let $K_0(\psi') : K_0(M_n(R_1)) \rightarrow K_0(R)$ denote the induced isomorphism. We can easily verify that $\psi'(\text{diag}(e_n, 0, \dots, 0)) = \beta(g_{n1}) = e_1$, hence $K_0(\psi')(\alpha^{-1}([(e_n)])) = K_0(\psi')([\text{diag}(e_n, 0, \dots, 0)]) = [(e_1)]$. By [10, Example 15.1], $[(e_1)]$ is a generator of $K_0(R)$, hence $[(e_n)]$ is a generator of $K_0(R_1)$. \square

We will also need the following:

Proposition 3.5. *The ring R_1 is separative.*

Proof. Since $M_n(R_1) \cong R$, by [2, Proposition 2.2] it suffices to prove that R is separative. Furthermore, by [2, Theorem 4.2] it suffices to prove that I and R/I are separative rings. Since $R/I \cong S'/I'$, this clearly follows from Lemma 2.3 and [2, Theorem 4.2]. \square

4. A generalization of Bergman's ring

Bergman's example is the first known example of a non-clean exchange ring. It was defined in [12, Example 1]. In [7] it was observed that this ring is exchange but not clean. An alternative definition of Bergman's ring was given in [20, Example 3.1].

In this section we will generalize Bergman's construction, and prove some basic properties of the ring that we will obtain. These properties will be needed later in our counter-example.

The construction we perform is similar to the one in [20, Example 3.1]. We begin with an algebra R over a field F . Denote by $M_{\mathbb{N}}(R)$ the ring of all countably-infinite matrices over R with finite columns. Matrices in $M_{\mathbb{N}}(R)$ are denoted, as usual, by $A = (a_{ij})_{i,j=1}^{\infty}$ where $(a_{ij})_i$ are columns and $(a_{ij})_j$ are rows in A .

Define

$$S(R, F) = \{A = (a_{ij}) \in M_{\mathbb{N}}(R) \mid \text{there exists } h(A) \geq 0 \\ \text{such that } a_{ij} = a_{i+1, j+1} \in F \text{ for all } i \geq h(A) + 1, j \geq 1\}.$$

Then $S(R, F)$ is a subring of $M_{\mathbb{N}}(R)$, which can be easily verified. Note that, when $R = F$, the above definition coincides with Bergman's ring as given in [20, Example 3.1].

We will abbreviate $S = S(R, F)$, while keeping in mind that S actually depends on R and F .

Also note that the integer $h(A)$ in the above definition is not unique. So, by $h \geq h(A)$ or $h = h(A)$ we always mean that $A = (a_{ij}) \in S$ satisfies $a_{ij} = a_{i+1, j+1} \in F$ for all $i \geq h + 1, j \geq 1$.

As in [20, Example 3.1], denote by $F((X))$ the field of all formal Laurent series over F , and let $\psi : S \rightarrow F((X))$ be the epimorphism defined by

$$\psi(A) = \sum_{j=1}^{\infty} a_{hj} X^{j-h},$$

where $A = (a_{ij}) \in S$, and $h \geq 0$ is any integer satisfying $h \geq h(A) + 1$ and $a_{i1} = 0$ for all $i \geq h + 1$. It can be easily verified that ψ is indeed a well-defined surjective ring homomorphism. As in [20], set

$$I = \{A = (a_{ij}) \in S \mid \text{there exists } h(A) \geq 0 \\ \text{such that } a_{ij} = 0 \text{ for all } i \geq h(A) + 1, j \geq 1\}.$$

Note that I is precisely the kernel of the homomorphism ψ , so that $S/I \cong F((X))$.

Later we will prove that the ring S , with F and R suitably defined, is the desired counter-example for the Morita invariance problem for clean rings. In this section we prove some general properties of the ring S and the ideal I . Our first goal is to compute the group $K_0(I)$.

Recall that, according to the definition, $K_0(I)$ consists precisely of elements in $K_0(I^1)$ of the form $[(E_1, X_1)] - [(E_2, X_2)] \in K_0(I^1)$, where E_1, E_2 are idempotent matrices over \mathbb{Z} satisfying $[E_1] = [E_2]$ in $K_0(\mathbb{Z})$, and X_1, X_2 are matrices over I such that $E_i X_i + X_i E_i + X_i^2 = X_i$ for $i = 1, 2$. Thus, $K_0(I)$ also includes all elements $[(0, X_1)] \in K_0(I^1)$, where X_1 is an idempotent matrix over I .

For any matrix $A \in M_k(R)$, we denote by $i(A)$ the canonical image of A in I , i.e. $i(A)$ is the matrix with the block form $i(A) = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \in I$.

Proposition 4.1. *The group $K_0(I)$ is isomorphic to $K_0(R)$. The isomorphism $K_0(R) \rightarrow K_0(I)$ is given by $[E] \mapsto [(0, i(E))]$, where E is an idempotent matrix over R , and $(0, i(E))$ is an idempotent 1×1 matrix over I^1 .*

The proof of this proposition is a matter of routine K -theoretic computation and is therefore left to the reader.

Let $\partial : K_1(S/I) \rightarrow K_0(I)$ denote the connecting homomorphism in K -theory. This homomorphism is defined in a quite abstract way, but, in our case, we are able to provide another definition of ∂ which is more concrete and therefore more convenient to use.

To write down this definition, first observe that $S/I \cong F((X))$ is a field, hence $K_1(S/I)$ is isomorphic to the multiplicative group $F((X))^* = F((X)) \setminus \{0\}$. Denote by

$\xi : K_1(S/I) \rightarrow F((X))^*$ the canonical isomorphism. Note that ξ is induced by the determinant map $\det : GL(S/I) \rightarrow F((X))^*$ (see [19, Theorem 2.2.5 and Corollary 2.2.6]).

Further, let $\zeta : K_0(I) \rightarrow K_0(R)$ be the isomorphism as in Proposition 4.1. Define

$$\sigma = \zeta \partial \xi^{-1} : F((X))^* \rightarrow K_0(R).$$

We can easily verify that σ satisfies the following nice formula:

Proposition 4.2. *For every $f(X) = \sum_{i=n_0}^{\infty} \lambda_i X^i \in F((X))^*$, we have*

$$\sigma(f(X)) = \min\{i \mid \lambda_i \neq 0\} \cdot [R_R].$$

The proof of this formula is left to the reader, as it follows directly from the definition of the homomorphisms ∂ , ξ and ζ (for the explicit definition of ∂ , see, for example, [19, Proof of Theorem 2.5.4] or [9, (18)]).

We will also need the following property of the ideal I :

Proposition 4.3. *If R is a separative exchange ring then I is a separative exchange non-unital ring.*

Proof. Suppose that R is separative and exchange. First we prove that this implies that I is exchange (in the sense of Ara [1]).

Take $A \in I$, so that A is of the form $A = \begin{pmatrix} A_0 & Y \\ 0 & 0 \end{pmatrix}$, where $A_0 \in M_k(R)$ and Y is a block with k rows and infinitely many columns. Since R is an exchange ring, $M_k(R)$ is exchange, hence we may write $E_0 = R_0 A_0$ and $1 - E_0 = S_0(1 - A_0)$ where $E_0 = E_0^2 \in M_k(R)$ and $R_0, S_0 \in M_k(R)$. Let $E = \begin{pmatrix} E_0 & E_0 R_0 Y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} E_0 R_0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A_0 & Y \\ 0 & 0 \end{pmatrix} \in I$ and $T = \begin{pmatrix} 1 - S_0 & (E_0 R_0 - S_0) Y \\ 0 & 0 \end{pmatrix} \in I$. Then E is clearly an idempotent in IA , and, considering $S_0 A_0 = S_0 + E_0 - 1$, we can also verify that $E = T + A - TA$. This proves that I is an exchange ring.

To prove the separativity of I , first observe that, since I is an exchange ring and $S/I \cong F((X))$ is a field, by [1, Theorem 2.2] S is an exchange ring. Hence by [2, Lemma 4.1] it suffices to prove that ESE is a separative ring for every $E = E^2 \in I$.

Take any idempotent $E \in I$. Then E is of the form $E = \begin{pmatrix} E_0 & E_0 Y \\ 0 & 0 \end{pmatrix}$, where $E_0 = E_0^2 \in M_k(R)$ and Y is a block with k rows and infinitely many columns. A straightforward computation shows that ESE is just

$$ESE = \left\{ \begin{pmatrix} E_0 A_0 E_0 & E_0 A_0 E_0 Y \\ 0 & 0 \end{pmatrix} \mid A_0 \in M_k(R) \right\},$$

and that this ring is isomorphic to $E_0 M_k(R) E_0$. Now the conclusion follows from [2, Proposition 2.2]. \square

We conclude this section with one more proposition which is an application of Perera's theorem, Theorem 2.2, on lifting units modulo separative exchange ideals.

For every $k \geq 1$ and $A \in M_k(S)$, we denote by $\psi(A)$ the homomorphic image of A in $M_k(F((X)))$. We say that an invertible matrix $\alpha \in GL_k(F((X)))$ can be *lifted* to an invertible matrix in $GL_k(S)$ if there exists $U \in GL_k(S)$ such that $\psi(U) = \alpha$.

Proposition 4.4. *Suppose that the ring R is separative and exchange, and let $k \geq 1$ and $\alpha \in GL_k(F((X)))$. Then α can be lifted to $GL_k(S)$ if and only if $\sigma(\det(\alpha)) = 0$.*

Proof. Let $\alpha \in GL_k(F((X)))$, and take $A \in M_k(S)$ such that $\psi(A) = \alpha$. Note that $A + M_k(I)$ is invertible in $M_k(S)/M_k(I)$, and that α can be lifted to $GL_k(S)$ if and only if $A + M_k(I)$ can be lifted to $GL_k(S)$.

We will apply Perera's theorem for the ring $T = M_k(S)$ with the ideal $J = M_k(I)$. Note that by Proposition 4.3 I is a separative exchange ring, hence by [1, Theorem 2.2] and [2, Theorem 4.2] S is also separative exchange. Thus by [2, Proposition 2.2] T is separative exchange, and hence J is separative exchange.

Now, by Theorem 2.2, a unit $A + J \in T/J$ can be lifted modulo J if and only if $\tilde{\partial}([A + J]) = 0$, where $\tilde{\partial}$ denotes the connecting map $\tilde{\partial} : K_1(T/J) \rightarrow K_0(J)$. Thus α can be lifted to $GL_k(S)$ if and only if $\tilde{\partial}([A + J]) = 0$.

Now recall that there is a commutative diagram

$$\begin{array}{ccc} K_1(T/J) & \xrightarrow{\tilde{\partial}} & K_0(J) \\ \cong \downarrow & & \downarrow \cong \\ K_1(S/I) & \xrightarrow{\partial} & K_0(I) \end{array}$$

with columns the natural isomorphisms. Thus $\tilde{\partial}([A + J]) = 0$ is equivalent to $\partial([\bar{A}]) = 0$, where \bar{A} here denotes the homomorphic image of A in $GL_k(S/I)$. We have $\partial = \zeta^{-1}\sigma\xi$, so $\partial([\bar{A}]) = 0$ is equivalent to $\sigma(\xi([\bar{A}])) = 0$. Since $\xi([\bar{A}]) = \det(\psi(A)) = \det(\alpha)$, this is further equivalent to $\sigma(\det(\alpha)) = 0$, as desired. \square

5. The clean property for non-scalar matrices

Let F , R and $S = S(R, F)$ be as in the previous section. In this section our objective is to prove the following fact: if the ring R is separative and clean, then every matrix $A \in M_k(S)$ with a non-scalar homomorphic image $\psi(A) \in M_k(F((X)))$ is clean in $M_k(S)$. This fact will be used later in the construction of our counter-example.

We wish to remark that if $S = S(F, F)$ is Bergman's ring then, as shown in [21], the matrix ring $M_k(S)$ is not clean, for every k . In fact, if we follow the proof of this result in [21], we see that scalar matrices in $M_k(S)$ are typically not clean in $M_k(S)$. So, the main result of this section will show that scalar matrices (more precisely, matrices A with $\psi(A)$ scalar) are the *only* non-clean elements of $M_k(S)$.

We begin with a few technical lemmas that will be needed to prove our result. These lemmas hold for arbitrary rings (not necessarily clean or exchange), so we will state them in the general form.

Lemma 5.1. *Let R be any ring and $k \geq 2$. Let $a_1, \dots, a_k, b \in R$ be elements such that b is right invertible in R . Then the matrix*

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & a_1 \\ 1 & 0 & \cdots & 0 & a_2 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & 1 & 0 & a_{k-1} \\ 0 & \cdots & 0 & b & a_k \end{pmatrix} \in M_k(R)$$

is clean in $M_k(R)$.

Proof. Let A be the matrix as in the lemma. Denote by c the right inverse of b , and set $d = \sum_{i=1}^{k-1} a_i + ca_k + c \in R$. Define

$$E = \begin{pmatrix} 1 & \cdots & 1 & d \\ 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix} \in M_k(R).$$

Note that E is an idempotent in $M_k(R)$. Let

$$P = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & 0 \\ b & \cdots & b & 1 \end{pmatrix} \in GL_k(R).$$

Then we can easily verify that $P(A - E)$ is upper triangular with -1 on the diagonal, thus $P(A - E) \in GL_k(R)$. It follows that $X = A - E \in GL_k(R)$, and hence $A = E + X$ is clean in $M_k(R)$. \square

Lemma 5.2. *Let R be any ring and $k \geq 2$. Let $a_1, \dots, a_k, b \in R$ be elements such that b is right invertible in R . Then the matrix*

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & a_1 \\ b & 0 & \cdots & 0 & a_2 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & a_{k-1} \\ 0 & \cdots & 0 & 1 & a_k \end{pmatrix} \in M_k(R)$$

is clean in $M_k(R)$.

Proof. Let A be the matrix as in the lemma. Denote by c the right inverse of b , and set $d = a_1 + c \sum_{i=2}^k a_i + c \in R$. Let

$$E = \begin{pmatrix} 1 & c & \cdots & c & d \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \in M_k(R)$$

(with $E = \begin{pmatrix} 1 & d \\ 0 & 0 \end{pmatrix}$ if $k = 2$). Note that E is an idempotent in $M_k(R)$. Write

$$P = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ b & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ b & 1 & \cdots & 1 \end{pmatrix} \in GL_k(R).$$

Then we can verify that $P(A - E)$ is upper triangular with -1 on the diagonal, thus $P(A - E) \in GL_k(R)$. It follows that $X = A - E \in GL_k(R)$, and hence $A = E + X$ is clean in $M_k(R)$. \square

Lemma 5.3. Let R be a ring, $r \geq 2$, $s \geq 1$, and $k = r + s$. Take elements $a_1, \dots, a_r, a \in R$ such that $a - 1$ is left invertible in R . Let $A \in M_k(R)$ be the matrix with the block form $A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$, where A_1 is the matrix

$$A_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 & a_1 \\ 1 & 0 & \cdots & 0 & a_2 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & a_{r-1} \\ 0 & \cdots & 0 & 1 & a_r \end{pmatrix} \in M_r(R),$$

A_2 is arbitrary with r rows and s columns, and A_3 is a scalar matrix $A_3 = \text{diag}(a, a, \dots, a) \in M_s(R)$. Then A is clean in $M_k(R)$.

Proof. Let A be the matrix as in the lemma. Denote by c the left inverse of $b = a - 1$, and set $d = \sum_{i=1}^r a_i - c^s \in R$. Let E_1 be the matrix

$$E_1 = \begin{pmatrix} 1 & \cdots & 1 & d \\ 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix} \in M_r(R).$$

Then E_1 is an idempotent in $M_r(R)$. Let $e = bc$ and $f = 1 - e$, so that e and f are orthogonal idempotents in R . Let Y_1 be the matrix over R with r columns and s rows

$$Y_1 = \begin{pmatrix} 0 & \cdots & 0 & -f \\ 0 & \cdots & 0 & -fc \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & -fc^{s-1} \end{pmatrix}.$$

We have $Y_1 E_1 = 0$, hence $E = \begin{pmatrix} E_1 & 0 \\ Y_1 & 1 \end{pmatrix} \in M_k(R)$ is an idempotent in $M_k(R)$.

Let us prove that $X = A - E$ is invertible in $M_k(R)$. Define

$$P_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 1 & \cdots & 1 & 1 \end{pmatrix} \in GL_r(R).$$

Then we can verify that $P_1(A_1 - E_1) \in M_r(R)$ is an upper triangular matrix with the diagonal entries $-1, \dots, -1, c^s$. Thus, denoting $P = \begin{pmatrix} P_1 & 0 \\ 0 & 1 \end{pmatrix} \in GL_k(R)$, we see that the matrix $PX = \begin{pmatrix} P_1(A_1 - E_1) & P_1 A_2 \\ -Y_1 & A_3 - 1 \end{pmatrix}$ can be written in the block form

$$PX = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$$

where $T_1 \in GL_{r-1}(R)$ is an upper triangular matrix with -1 on the diagonal, T_2 is a matrix with $r - 1$ rows and $s + 1$ columns, and $T_3 \in M_{s+1}(R)$ is the matrix of the form

$$T_3 = \begin{pmatrix} c^s & x_1 & x_2 & \cdots & x_s \\ f & b & 0 & \cdots & 0 \\ fc & 0 & b & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ fc^{s-1} & 0 & \cdots & 0 & b \end{pmatrix},$$

with $x_i \in R$. To prove that X is invertible, we only need to prove that T_3 is invertible. Let

$$Q_3 = \begin{pmatrix} 1 & -x_1 c & -x_2 c & \cdots & -x_s c \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \in GL_{s+1}(R).$$

Considering $cf = 0$, we see that

$$Q_3T_3 = \begin{pmatrix} c^s & 0 & 0 & \cdots & 0 \\ f & b & 0 & \cdots & 0 \\ fc & 0 & b & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ fc^{s-1} & 0 & \cdots & 0 & b \end{pmatrix}.$$

Considering $cf = 0$ and $fb = 0$, we can also verify that Q_3T_3 has the inverse

$$\begin{pmatrix} b^s & f & bf & \cdots & b^{s-1}f \\ 0 & c & 0 & \cdots & 0 \\ 0 & 0 & c & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & c \end{pmatrix} \in M_{s+1}(R).$$

Hence $Q_3T_3 \in GL_{s+1}(R)$, and therefore $T_3 \in GL_{s+1}(R)$. This proves that $X \in GL_k(R)$, and hence $A = E + X$ is clean in $M_k(R)$. \square

Now we are in position to prove the main result of this section. As in the previous section, let again R be an algebra over a field F , and $S = S(R, F)$. We will also use freely some other definitions from the previous section.

For any $i, j \geq 1$, let $e_{ij} \in S$ denote the matrix with the (i, j) -th entry equal to 1 and zeros elsewhere. Recall that, for any matrix $A \in M_k(S)$, $\psi(A)$ denotes the homomorphic image of A in $M_k(F((X)))$.

Theorem 5.4. *Let $k \geq 1$. If the ring R is clean and separative then every $A \in M_k(S)$, with $\psi(A)$ a non-scalar matrix, is clean in $M_k(S)$.*

Proof. We will prove the statement by induction on k . For $k = 1$, there is nothing to prove. So, let $k \geq 2$ and assume that the theorem holds for all matrices $A' \in M_{k'}(S)$ where $k' < k$. Let $A \in M_k(S)$ be such that $\alpha = \psi(A) \in M_k(F((X)))$ is not scalar.

Let $V = F((X))^k$ denote a k -dimensional vector space over $F((X))$. We will identify the ring $M_k(F((X)))$ with $\text{End}_{F((X))}(V)$, so that matrices in $M_k(F((X)))$ will be also treated as endomorphisms $V \rightarrow V$, acting on V from the left.

Since $\alpha : V \rightarrow V$ is not scalar, there exists $v \in V$ such that the vectors v and $\alpha(v)$ are linearly independent. Let $r \geq 2$ denote the maximal integer such that $v, \alpha(v), \alpha^2(v), \dots, \alpha^{r-1}(v) \in V$ are linearly independent. We will consider two possible cases, $r = k$ and $r \neq k$.

Case 1: $r = k$. In this case $\{v, \alpha(v), \alpha^2(v), \dots, \alpha^{k-1}(v)\}$ forms a basis of V . Let $\mu \in GL_k(F((X)))$ denote the transition matrix corresponding to that basis, so that $\alpha = \mu\beta\mu^{-1}$, where $\beta \in M_k(F((X)))$ is the matrix of the form

$$\beta = \begin{pmatrix} & & \nu_1 \\ 1 & & \nu_2 \\ & \ddots & \vdots \\ & & 1 & \nu_k \end{pmatrix}$$

(with zeros elsewhere). Let $F[[X]] \subset F((X))$ denote the ring of formal power series over F . We will consider two possible cases, $\det(\mu) \in F[[X]]$ and $\det(\mu) \notin F[[X]]$.

Case 1.1: $\det(\mu) \in F[[X]]$. Denote the matrix

$$\pi = \text{diag}(1, 1, \dots, 1, \det(\mu)^{-1}) \in GL_k(F((X))),$$

and write $\mu' = \mu\pi$ and $\beta' = \pi^{-1}\beta\pi$, so that $\alpha = \mu'\beta'\mu'^{-1}$. We have $\det(\mu') = \det(\mu)\det(\pi) = 1$, hence by Proposition 4.4 there exists $U \in GL_k(S)$ such that $\psi(U) = \mu'$. Writing $B = U^{-1}AU \in M_k(S)$, we have $\psi(B) = \mu'^{-1}\alpha\mu' = \beta'$. If we prove that B is clean in $M_k(S)$, it will clearly follow that $A = UBU^{-1}$ is clean in $M_k(S)$.

Observe that the matrix B has the homomorphic image $\psi(B) = \beta' = \pi^{-1}\beta\pi$ of the form

$$\psi(B) = \begin{pmatrix} & & \nu'_1 \\ 1 & & \nu'_2 \\ & \ddots & \vdots \\ & & 1 & \nu'_{k-1} \\ & & \det(\mu) & \nu'_k \end{pmatrix}.$$

Denote $B = (b_{ij})$, with $b_{ij} \in S$, and let $h \geq 1$ be an integer satisfying $h \geq h(b_{ij})$ for all i, j . Write $e = \sum_{i=1}^h e_{ii} \in \text{Id}(S)$, $f = 1 - e$, $G = \text{diag}(e, e, \dots, e) \in \text{Id}(M_k(S))$ and $H = 1 - G$. We have $eSe \cong M_h(R)$, hence $GM_k(S)G = M_k(eSe) \cong M_{kh}(R)$ is a clean ring. Hence by Lemma 2.1 it suffices to prove that $HBH = (fb_{ij}f)$ is clean in $HM_k(S)H = M_k(fSf)$.

Note that for each i, j , with $j \leq k-1$ and $i \neq j+1$, we have $\psi(b_{ij}) = 0$. This, together with $h(b_{ij}) \leq h$, yields $b_{ij} \in eS$, hence $fb_{ij}f = 0$. Similarly, for $j \leq k-2$ and $i = j+1$ we have $\psi(b_{ij}) = 1$, which, together with $h(b_{ij}) \leq h$, yields $b_{ij} - 1 \in eS$ and therefore $fb_{ij}f = f$. Moreover, considering $\psi(b_{k-1}) = \det(\mu) \in F[[X]]$, we see that $fb_{k-1}f$ is right invertible in fSf .

Putting these facts together, we conclude that HBH is of the form

$$HBH = \begin{pmatrix} & & \kappa_1 \\ f & & \kappa_2 \\ & \ddots & \vdots \\ & & f & \kappa_{k-1} \\ & & b & \kappa_k \end{pmatrix} \in M_k(fSf)$$

(with $\kappa_i \in fSf$), where $b = fb_{k-k-1}f$ is right invertible in fSf . By [Lemma 5.1](#), HBH is clean in $M_k(fSf)$. This proves that B is clean in $M_k(S)$, as desired.

Case 1.2: $\det(\mu) \notin F[[X]]$. We perform similar computations as for the case $\det(\mu) \in F[[X]]$, except that this time we take

$$\pi = \text{diag}(\det(\mu)^{-1}, 1, 1, \dots, 1) \in GL_k(F((X))).$$

As before, let $\mu' = \mu\pi$ and take $U \in GL_k(S)$ such that $\psi(U) = \mu'$. Again, note that it suffices to prove that $B = U^{-1}AU$ is clean in $M_k(S)$.

Observe that the matrix $B = (b_{ij})$ has the homomorphic image $\psi(B) = \pi^{-1}\beta\pi$ of the form

$$\psi(B) = \begin{pmatrix} & & \nu'_1 \\ \det(\mu)^{-1} & & \nu'_2 \\ & 1 & \nu'_3 \\ & & \ddots & \vdots \\ & & & 1 & \nu'_k \end{pmatrix}.$$

We define e, f, G, H as previously, and note again that it suffices to prove that the corner matrix HBH is clean in $HM_k(S)H = M_k(fSf)$. This time, we see that the matrix HBH is of the form

$$HBH = \begin{pmatrix} & & \kappa_1 \\ b & & \kappa_2 \\ & f & \kappa_3 \\ & & \ddots & \vdots \\ & & & f & \kappa_k \end{pmatrix} \in M_k(fSf),$$

with $\kappa_i \in fSf$ and $b = fb_{21}f$. Observe that, since $\psi(b) = \det(\mu)^{-1} \in F[[X]]$, b is right invertible in fSf . Hence by [Lemma 5.2](#) HBH is clean in $M_k(fSf)$, and therefore B is clean in $M_k(S)$. This completes the proof when $r = k$.

Case 2: $r \neq k$. Set $s = k - r \geq 1$, and choose vectors v_1, \dots, v_s such that $\{v, \alpha(v), \dots, \alpha^{r-1}(v), v_1, \dots, v_s\}$ forms a basis of V . Denote by $\mu \in GL_k(F((X)))$ the corresponding transition matrix and write $\alpha = \mu\beta\mu^{-1}$, so that $\beta \in M_k(F((X)))$ is the matrix of the form $\beta = \begin{pmatrix} \beta_1 & \beta_2 \\ 0 & \beta_3 \end{pmatrix}$, where $\beta_1 \in M_r(F((X)))$ is of the form

$$\beta_1 = \begin{pmatrix} & \nu_1 \\ 1 & \nu_2 \\ & \ddots & \vdots \\ & & 1 & \nu_r \end{pmatrix},$$

β_2 is a block with r rows and s columns, and $\beta_3 \in M_s(F((X)))$.

Denote $\pi = \text{diag}(1, 1, \dots, 1, \det(\mu)^{-1}) \in GL_k(F((X)))$, $\mu' = \mu\pi$ and $\beta' = \pi^{-1}\beta\pi$, so that $\alpha = \mu'\beta'\mu'^{-1}$. We have $\det(\mu') = \det(\mu)\det(\pi) = 1$, so by [Proposition 4.4](#) there exists $U \in GL_k(S)$ such that $\psi(U) = \mu'$. Writing $B = U^{-1}AU$ we have $\psi(B) = \mu'^{-1}\alpha\mu' = \beta'$. As before, it suffices to prove that B is clean in $M_k(S)$.

Observe that $\psi(B) = \pi^{-1}\beta\pi$ is of the form $\psi(B) = \begin{pmatrix} \beta'_1 & \beta'_2 \\ 0 & \beta'_3 \end{pmatrix}$, with β'_2 a block with r rows and s columns, and $\beta'_3 \in M_s(F((X)))$. We will consider two possible cases, the first when β'_3 is a scalar matrix and the second when β'_3 is a non-scalar matrix.

Case 2.1: β'_3 is scalar. Write $B = (b_{ij})$, and let $h \geq 1$ be an integer satisfying $h \geq h(b_{ij})$ for all i, j . As before, denote the idempotents $e = \sum_{i=1}^h e_{ii} \in S$, $f = 1 - e$, $G = \text{diag}(e, \dots, e) \in M_k(S)$ and $H = 1 - G$. Since $GM_k(S)G = M_k(eSe)$ is a clean ring, by [Lemma 2.1](#) it suffices to prove that $HBH = (fb_{ij}f)$ is clean in $HM_k(S)H = M_k(fSf)$.

As in the previous cases, we see that $fb_{ij}f = 0$ for $i \geq r + 1$ and $j \leq r$, and that HBH is actually the matrix of the form $HBH = \begin{pmatrix} B_1 & B_2 \\ 0 & B_3 \end{pmatrix}$, where $B_1 \in M_r(fSf)$ is of the form

$$B_1 = \begin{pmatrix} f & & \kappa_1 \\ & \ddots & \kappa_2 \\ & & \vdots \\ & f & \kappa_r \end{pmatrix} \in M_r(fSf),$$

B_2 is a block with r rows and s columns, and $B_3 = \text{diag}(t, \dots, t) \in M_s(fSf)$ is a scalar matrix.

If $\psi(t) \in F[[X]]$ then $t \in fSf$ is upper triangular and thus clearly clean in fSf . It follows that B_3 is clean in $M_s(fSf)$. Furthermore, the matrix B_1 is clean in $M_r(fSf)$ by [Lemma 5.1](#). We conclude that, in this case, HBH is clean in $M_k(fSf)$, as desired. Hence we may assume that $\psi(t) \notin F[[X]]$. But this assumption clearly implies that t is left invertible in fSf . Hence we may apply [Lemma 5.3](#) to conclude that HBH is clean in $M_k(fSf)$. This concludes the proof when β'_3 is a scalar matrix.

Case 2.2: β'_3 is not scalar. Write $B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$, where $B_1 \in M_r(S)$, B_2 is a block with r rows and s columns, B_3 is a block with s rows and r columns, and $B_4 \in M_s(S)$.

To prove the cleanness of B , we will apply the same argument as in [\[11, Proof of Lemma on p. 2590\]](#). First observe that, by the induction hypothesis, B_1 is clean in $M_r(S)$. Write $B_1 = E_1 + U_1$, with $E_1 \in \text{Id}(M_r(S))$ and $U_1 \in GL_r(S)$. We have $\psi(B_3) = 0$, hence $\psi(B_4 - B_3U_1^{-1}B_2) = \psi(B_4) = \beta'_3$. Since β'_3 is a non-scalar matrix, by the induction hypothesis it follows that $B_4 - B_3U_1^{-1}B_2$ is clean in $M_s(S)$. Now, by [\[11, Proof of Lemma on p. 2590\]](#), it follows that $B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$ is clean in $M_k(S)$. This completes the proof of the theorem. \square

6. The construction of the example

In this section we construct, for each $n \geq 2$, a non-clean ring S such that the matrix ring $M_n(S)$ is clean. In fact, in our case we will see that, for every $k \geq 1$, the ring $M_k(S)$ will be clean if and only if k is a multiple of n .

Let $n \geq 2$. We choose an algebra R over a field F such that R is separative, clean, and the element $[R_R] \in K_0(R)$ is of order n in $K_0(R)$. One example of a ring with these properties was given in Section 3.

As in Section 4, we set $S = S(R, F)$. Throughout the rest of the paper, we will use freely notations and definitions introduced in that section.

Now consider the element $X^{-n} \in F((X))$. By Proposition 4.2 we have $\sigma(X^{-n}) = -n[R_R] = 0$, hence by Proposition 4.4 there exists $u_0 \in U(S)$ such that $\psi(u_0) = X^{-n}$.

First let us prove the following:

Proposition 6.1. *We may assume that u_0 satisfies $h(u_0) = n + 1$.*

Proof. Suppose that $h(u_0) = k \geq n + 2$. Take any integer d such that n and d are coprime and $d \geq k - n$. We can easily verify that $h(u_0^i) \leq k + (i - 1)n$ for every $i \geq 1$, hence $h(u_0^d) \leq k + (d - 1)n \leq (n + 1)d$.

Let $R' = M_d(R)$. Note that R' is a separative clean algebra over F . Furthermore, the order of the element $[R'_{R'}]$ in $K_0(R') \cong K_0(R)$ is equal to the order of the element $d \cdot [R_R]$ in $K_0(R)$, which is precisely n . Thus R' satisfies all the properties that have been assumed for R .

Denote by $\phi : M_{\mathbb{N}}(R') \rightarrow M_{\mathbb{N}}(R)$ the natural isomorphism, and let $S' = S(R', F) \leq M_{\mathbb{N}}(R')$. Observe that the image of S' under ϕ is precisely $\phi(S') = \{A \in S \mid \psi(A) \in F((X^d))\}$. We have $\psi(u_0^d) = X^{-nd}$ and hence $u_0^d \in \phi(S')$. Denote $u'_0 = \phi^{-1}(u_0^d) \in S'$. Observe that $\psi(u'_0) = X^{-n}$ and, since u_0 is invertible in S , u'_0 is invertible in S' . Furthermore, since $h(\phi(u'_0)) = h(u_0^d) \leq (n + 1)d$, we also have $h(u'_0) \leq n + 1$. Therefore, replacing R , S and u_0 by R' , S' and u'_0 , the conclusion follows. \square

Now we are ready to give the final result:

Theorem 6.2. *Let $k \geq 1$. Then the ring $M_k(S)$ is clean if and only if k is a multiple of n .*

Proof. First let us assume that $M_k(S)$ is a clean ring. To prove that k is a multiple of n , we perform a similar argument as in [21, Proof of Theorem 3.5]. Let

$$\alpha = \text{diag}(X^{-1}, \dots, X^{-1}) \in M_k(F((X))),$$

and take any matrix $A \in M_k(S)$ such that $\psi(A) = \alpha$. By the assumption we have $A = E + U$ with $E \in \text{Id}(M_k(S))$ and $U \in GL_k(S)$. Denote $\epsilon = \psi(E)$ and $\mu = \psi(U)$, so that $\alpha = \epsilon + \mu$. Since $\epsilon \in M_k(F((X)))$ is an idempotent matrix over a field, there exists $\pi \in GL_k(F((X)))$ such that $\delta = \pi\epsilon\pi^{-1}$ is a diagonal matrix with the entries 1 and 0 on the diagonal. Note that α is a central matrix, hence $\alpha = \pi\alpha\pi^{-1} = \delta + \pi\mu\pi^{-1}$, and therefore $\det(\alpha - \delta) = \det(\pi\mu\pi^{-1}) = \det(\mu)$. By Proposition 4.4 we have $\sigma(\det(\mu)) = 0$, hence $\sigma(\det(\alpha - \delta)) = 0$. On the other hand, we see that $\alpha - \delta$ is a diagonal matrix

with the diagonal entries $X^{-1} - 1$ and X^{-1} , hence $\det(\alpha - \delta) = (X^{-1} - 1)^r (X^{-1})^{k-r} = (1 - X)^r X^{-k}$, where r denotes the rank of δ . Thus $\sigma(\det(\alpha - \delta)) = \sigma((1 - X)^r X^{-k}) = -k \cdot [R_R]$. We conclude that $k \cdot [R_R] = 0$ in $K_0(R)$, and therefore k is a multiple of n , as desired.

Conversely, let us prove that if k is a multiple of n then $M_k(S)$ is clean. Clearly, it suffices to prove that $M_n(S)$ is clean.

Let $A \in M_n(S)$. If $\psi(A)$ is a non-scalar matrix then A is clean by [Theorem 5.4](#), hence we may assume that $\psi(A)$ is scalar.

In the first step we perform the same argument as in the proof of [Theorem 5.4](#). We write $A = (a_{ij})$, with $a_{ij} \in S$, and take $h \geq 1$ such that $h \geq h(a_{ij})$ for all i, j . For every $i, j \geq 1$, let $e_{ij} \in S$ denote the matrix with the (i, j) -th entry equal to 1 and zeros elsewhere, and define $e = \sum_{i=1}^h e_{ii} \in \text{Id}(S)$, $f = 1 - e$, $G = \text{diag}(e, e, \dots, e) \in M_n(S)$ and $H = 1 - G$. We have $eSe \cong M_h(R)$, hence $GM_n(S)G = M_n(eSe) \cong M_{hn}(R)$ is a clean ring, and hence by [Lemma 2.1](#) it suffices to prove that $HAH = (fa_{ij}f)$ is clean in $HM_n(S)H = M_n(fSf)$.

Observe that fSf is isomorphic to S , and that the canonical isomorphism $\varphi : fSf \rightarrow S$ maps HAH to a scalar matrix $A_1 = \varphi(HAH) = \text{diag}(a, a, \dots, a) \in M_n(S)$, with $a \in S$ satisfying $h(a) = 0$. Thus we only need to prove that A_1 is clean in $M_n(S)$.

Clearly, if $\psi(a) \in F[[X]]$ then a is clean in S and there is nothing left to prove. Hence we may assume that $\psi(a) \notin F[[X]]$. Let $k \geq 1$ be the minimal integer such that $\psi(a)X^k \in F[[X]]$.

We shall write matrices in S in block form, with blocks of sizes $k \times k$. Thus, since $h(a) = 0$, we can write

$$a = \begin{pmatrix} T_1 & T_2 & T_3 & \cdots \\ T_0 & T_1 & T_2 & \cdots \\ 0 & T_0 & T_1 & \cdots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

where $T_i \in M_k(F)$, and T_0 is upper triangular and invertible in $M_k(F)$.

Let $u = u_0^k \in U(S)$. Note that by [Proposition 6.1](#) we have $h(u_0) \leq n+1$, which implies $h(u_0^i) \leq n+1 + (i-1)n$ for each $i \geq 0$, hence $h(u) \leq n+1 + (k-1)n = nk+1 \leq (n+1)k$. Therefore we can write the matrix u in the form

$$u = \begin{pmatrix} K_{1,1} & K_{1,2} & K_{1,3} & \cdots \\ \vdots & \vdots & \vdots & \\ K_{n+1,1} & K_{n+1,2} & K_{n+1,3} & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix},$$

with $K_{i,j} \in M_k(R)$. Note that we may replace the blocks $K_{i,j}$, for $j \geq 2$, with arbitrary blocks $X_{i,j} \in M_k(R)$, and the modified matrix will remain invertible in S .

Now, since the ring $M_k(R)$ is clean, we may write $K_{n+1,1} = E + U$ where $E \in \text{Id}(M_k(R))$ and $U \in GL_k(R)$. Define the matrices $V = -T_0U^{-1} \in GL_k(R)$, $X_1 = (K_{n,1} - T_1)U^{-1}E \in M_k(R)$, and $X_i = K_{i-1,1}U^{-1}E \in M_k(R)$ for $i = 2, \dots, n$. Let

$$g = \begin{pmatrix} X_1 & X_1(1 - X_1)V^{-1} & 0 & \cdots \\ VE & VE(1 - X_1)V^{-1} & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \in S$$

and

$$x_i = \begin{pmatrix} X_i & X_i(1 - X_1)V^{-1} & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \in S$$

for $i = 2, \dots, n$. Considering that $X_i \in M_k(R)E$ for each i , we can easily verify that g is an idempotent in S and $x_i = x_i g$ for each $i \geq 2$. Hence

$$E_1 = \begin{pmatrix} g & 0 & \cdots & 0 \\ x_2 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ x_n & 0 & \cdots & 0 \end{pmatrix}$$

is an idempotent in $M_n(S)$. We will verify that $Y = A_1 - E_1$ is invertible in $M_n(S)$, which will conclude the proof.

Denote the shift operators

$$b = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & 0 & \\ & & \ddots & \ddots \end{pmatrix} \in S \quad \text{and} \quad c = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & \ddots \\ & & & \ddots \end{pmatrix} \in S$$

(written in the $k \times k$ -block form as before), so that $cb = 1$. Write $\pi = bc \in \text{Id}(S)$ and $\rho = 1 - \pi$, and define

$$Q = \begin{pmatrix} b^{n-1} & \rho & b\rho & \cdots & b^{n-2}\rho \\ 0 & c & 0 & \cdots & 0 \\ 0 & 0 & c & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & c \end{pmatrix} \in M_n(S).$$

As in the proof of [Lemma 5.3](#), we see that Q is invertible in $M_n(S)$.

Now consider the matrix $Y' = QY$. First we see that Y' is upper triangular since $cx_i = 0$ for every $i \geq 2$. Moreover, the diagonal entries of Y' are exactly x, ca, ca, \dots, ca , where $x = b^{n-1}(a - g) - \sum_{i=0}^{n-2} b^i \rho x_{i+2}$. Clearly, ca is invertible in S . Thus, to prove that Y' is invertible, we only need to prove that x is invertible.

Computing explicitly the matrix x , we see that x is of the form

$$x = \begin{pmatrix} -X_2 & * & \cdots & & \\ \vdots & \vdots & & & \\ -X_n & * & \cdots & & \\ T_1 - X_1 & * & \cdots & & \\ T_0 - VE & * & \cdots & & \\ 0 & T_0 & T_1 & T_2 & \cdots \\ 0 & 0 & T_0 & T_1 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} = \begin{pmatrix} -K_{1,1}U^{-1}E & * & \cdots & & \\ \vdots & \vdots & & & \\ -K_{n-1,1}U^{-1}E & * & \cdots & & \\ T_1 - (K_{n,1} - T_1)U^{-1}E & * & \cdots & & \\ T_0 + T_0U^{-1}E & * & \cdots & & \\ 0 & T_0 & T_1 & T_2 & \cdots \\ 0 & 0 & T_0 & T_1 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Multiplying this matrix from the left by

$$v = \begin{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 & K_{1,1}T_0^{-1} \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & K_{n-1,1}T_0^{-1} \\ 0 & \cdots & 0 & 1 & (K_{n,1} - T_1)T_0^{-1} \\ 0 & \cdots & 0 & 0 & UT_0^{-1} \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} T_0 & T_1 & T_2 & \cdots \\ & T_0 & T_1 & \cdots \\ & & T_0 & \cdots \\ & & & \ddots \end{pmatrix}^{-1} \end{pmatrix} \in U(S)$$

we get

$$vx = \begin{pmatrix} K_{1,1} & * & \cdots & \\ \vdots & \vdots & & \\ K_{n+1,1} & * & \cdots & \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

But we have already seen that this matrix is invertible in S . Thus $x \in U(S)$, as desired. This concludes the proof of the theorem. \square

Remark 6.3. The example presented above also shows that the ‘torsion-free’ assumption in [21, Theorem 3.5] is crucial. Indeed, the ring $M_n(S)$ as given above, together with the ideal $M_n(I)$, fulfills all the assumptions of that theorem, except that the group $K_0(M_n(I))$ is not torsion-free. Note that the connecting homomorphism $K_1(M_n(S)/M_n(I)) \rightarrow K_0(M_n(I))$ is clearly non-zero.

We conclude this paper with the following question:

Question 6.4. Does there exist a ring R such that $M_n(R)$ is clean for every $n \geq 2$, but R is not clean?

References

- [1] P. Ara, Extensions of exchange rings, *J. Algebra* 197 (2) (1997) 409–423.
- [2] P. Ara, K.R. Goodearl, K.C. O’Meara, E. Pardo, Separative cancellation for projective modules over exchange rings, *Israel J. Math.* 105 (1998) 105–137.
- [3] P. Ara, K.R. Goodearl, K.C. O’Meara, R. Raphael, K_1 of separative exchange rings and C^* -algebras with real rank zero, *Pacific J. Math.* 195 (2) (2000) 261–275.
- [4] P. Ara, K.R. Goodearl, E. Pardo, K_0 of purely infinite simple regular rings, *K-Theory* 26 (1) (2002) 69–100.
- [5] V.P. Camillo, D. Khurana, A characterization of unit regular rings, *Comm. Algebra* 29 (5) (2001) 2293–2295.
- [6] V.P. Camillo, D. Khurana, T.Y. Lam, W.K. Nicholson, Y. Zhou, Continuous modules are clean, *J. Algebra* 304 (1) (2006) 94–111.
- [7] V.P. Camillo, H.-P. Yu, Exchange rings, units and idempotents, *Comm. Algebra* 22 (12) (1994) 4737–4749.
- [8] H. Chen, Exchange rings with artinian primitive factors, *Algebr. Represent. Theory* 2 (2) (1999) 201–207.
- [9] G. Cortiñas, Algebraic v. topological K -theory: a friendly match, in: *Topics in Algebraic and Topological K-Theory*, in: *Lecture Notes in Math.*, vol. 2008, Springer, Berlin, 2011, pp. 103–165.
- [10] K.R. Goodearl, *von Neumann Regular Rings*, *Monogr. Stud. Math.*, vol. 4, Pitman (Advanced Publishing Program), Boston, MA, 1979.
- [11] J. Han, W.K. Nicholson, Extensions of clean rings, *Comm. Algebra* 29 (6) (2001) 2589–2595.
- [12] D. Handelman, Perspectivity and cancellation in regular rings, *J. Algebra* 48 (1) (1977) 1–16.
- [13] C. Huh, N.K. Kim, Y. Lee, On exchange rings with primitive factor rings artinian, *Comm. Algebra* 28 (10) (2000) 4989–4993.
- [14] P. Menal, J. Moncasi, Lifting units in self-injective rings and an index theory for Rickart C^* -algebras, *Pacific J. Math.* 126 (2) (1987) 295–329.

- [15] W.K. Nicholson, Lifting idempotents and exchange rings, *Trans. Amer. Math. Soc.* 229 (1977) 269–278.
- [16] W.K. Nicholson, K. Varadarajan, Countable linear transformations are clean, *Proc. Amer. Math. Soc.* 126 (1) (1998) 61–64.
- [17] M. Ó Searcóid, Perturbation of linear operators by idempotents, *Irish Math. Soc. Bull.* (39) (1997) 10–13.
- [18] F. Perera, Lifting units modulo exchange ideals and C^* -algebras with real rank zero, *J. Reine Angew. Math.* 522 (2000) 51–62.
- [19] J. Rosenberg, *Algebraic K-Theory and Its Applications*, Grad. Texts in Math., vol. 147, Springer-Verlag, New York, 1994.
- [20] J. Šter, Corner rings of a clean ring need not be clean, *Comm. Algebra* 40 (5) (2012) 1595–1604.
- [21] J. Šter, Lifting units in clean rings, *J. Algebra* 381 (2013) 200–208.