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The derived superalgebra of skew elements of a semiprime superalgebra with superinvolution

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ABSTRACT

In this paper we investigate the Lie structure of the derived Lie superalgebra $[K, K]$, with K the set of skew elements of a semiprime associative superalgebra A with superinvolution. We show that if U is a Lie ideal of $[K, K]$, then either there exists an ideal J of A such that the Lie ideal $[J \cap K, K]$ is nonzero and contained in U, or A is a subdirect sum of A', A'', where the image of U in A' is central, and A'' is a subdirect product of orders in simple superalgebras, each at most 16-dimensional over its center.

1. Introduction

Let A be an algebra over ϕ, an associative commutative unital ring of scalars with $1/2 \in \phi$. A is said to be a superalgebra if it is a $\mathbf{Z}_2$-graded algebra, that is, $A = A_0 \oplus A_1$,

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with $A_iA_j \subseteq A_{i+j}$, $i, j \in \mathbf{Z}_2$. A_0 is said to be the even part and A_1 is said to be the odd part. Elements in A_0 and A_1 are said to be homogeneous elements.

A Lie superalgebra is a superalgebra with an operation [,] satisfying the following axioms for every a, b, c homogeneous elements in A (where $\bar{a}$ denotes the degree of a, that is $a \in A_{\bar{a}}$)

$$[a,b] = -(-1)^{\bar{a}\bar{b}}[b,a]$$
$$\bigl[a,[b,c]\bigr] = \bigl[[a,b],c\bigr] + (-1)^{\bar{a}\bar{b}}\bigl[b,[a,c]\bigr]$$

Superalgebras have proved to be very useful in mathematics, and, in particular, in algebra. For example in the theory of varieties of algebras, in questions concerning the structure of T-ideals, their nilpotency or solvability [9,21,20]; and also to construct some counterexamples, for instance, of solvable but not nilpotent Jordan, alternative and $(-1,-1)$-algebras, or to construct prime algebras with nonzero absolute zero divisors [18].

In the last two decades, the different kinds of superalgebras have been profusely investigated, and also the relationships among them. In this paper we study some relationships among associative and Lie superalgebras. More specifically we are interested in the description of the Lie structure of the derived superalgebra $[K,K]$, with K the set of skewsymmetric elements of a semiprime superalgebra with superinvolution.

An associative superalgebra is just a superalgebra that is associative as an ordinary algebra.

It is known that, if we take an associative superalgebra, A, and we change the product in A by the superbracket product $[a,b] = ab - (-1)^{\bar{a}\bar{b}}ba$, where $\bar{a}$, $\bar{b}$ denote the degrees of a and b, homogeneous elements in $A = A_0 \oplus A_1$, we obtain a Lie superalgebra, denoted by A^-. Also if A is an associative superalgebra and has a superinvolution, that is, a graded linear map $* : A \longrightarrow A$ such that $a^{**} = a$ and $(ab)^* = (-1)^{\bar{a}\bar{b}}b^*a^*$, for $a, b \in A$ homogeneous elements, the set of skewsymmetric elements, $K = \{x \in A : x^* = -x\}$, is a subalgebra of the Lie superalgebra A^-. In fact, in the classification of the finite dimensional simple Lie superalgebras given by V. Kac in [8], several types are of this kind.

This important fact made that, in [3], C. Gómez-Ambrosi and I. Shestakov investigated the Lie structure of the set of skew elements, K, and also of $[K,K]$, of a simple associative superalgebra with superinvolution over a field of characteristic not 2. More specifically they described the ideals of these Lie superalgebras K and $[K,K]$, also called Lie ideals of K and $[K,K]$. Those results were extended in [4] to prime associative superalgebras with superinvolution for the Lie superalgebras K and $[K,K]$, and later, in [11], to semiprime superalgebras with superinvolution, but only for the Lie superalgebra K.

For the case of associative superalgebras without superinvolution, F. Montaner in [16] and S. Montgomery in [17] studied the Lie ideals of a prime associative superalgebra.

In the nongraded case, there is a parallel situation for associative algebras with involution and Lie algebras. I.N. Herstein [5–7] and W.E. Baxter [1] first studied in associative algebras with involution the Lie structure of K and $[K, K]$, and after several authors contributed to this subject: T.E. Erickson [2], C. Lanski [13], W.S. Martindale III and C.R. Miers [15], etc.

The basis on associative superalgebras can be found in [3] and [16].

In the paper we always deal with nontrivial superalgebras, that is, superalgebras with nonzero odd part. Also, A will denote an associative superalgebra with superinvolution over a ring of scalars ϕ. It is known that the center of A, $Z(A)$, as an associative algebra, is graded, that is, it is a subalgebra of A as graded algebra. We denote by $Z = Z(A)_0$. If $Z \neq 0$ we can construct the superalgebra $Z^{-1}A = \{z^{-1}a : 0 \neq z \in Z, a \in A\}$. When A is prime, $Z^{-1}A$ is a central prime associative superalgebra over the field $Z^{-1}Z$. Throughout the paper we will say that A is a central order in $Z^{-1}A$, although we are conscious that usually this term is used when we consider the extended centroid instead of Z.

As A is a superalgebra with superinvolution, we can consider in A the set of symmetric elements, denoted by H, and the set of skewsymmetric elements, denoted by K. If we define the following product in A: $a \circ b = ab + (-1)^{\bar{a}\bar{b}}ba$, it is easy to check that $H \circ H \subseteq H, H \circ K \subseteq K, K \circ K \subseteq H, [H, H] \subseteq H, [H, K] \subseteq H$ and $[K, K] \subseteq K$. These relations will be used very often during the paper.

If we denote $Z_H = Z \cap H$, we recall that a superinvolution $*$ in A is said to be of the first kind if $Z_H = Z$, and of the second kind if $Z_H \neq Z$. To extend the superinvolution $*$ on A to a localization of A, we consider $V = Z_H$. If A is prime, then $V - \{0\}$ consists of regular symmetric elements and if $Z \neq 0$ then $V = Z_H \neq 0$. Now $V^{-1}A = \{v^{-1}a : v \in V - \{0\}, a \in A\}$ is a superalgebra in which we can define the following superinvolution, extension of the superinvolution on A, $(v^{-1}a)^* = v^{-1}a^*$. Although $Z^{-1}A = V^{-1}A$, because for all $0 \neq z \in Z, a \in A$ we have $z^{-1}a = (zz^*)^{-1}(z^*a)$, considering $V^{?1}A$ is easier to check that $(v^{-1}a)^* = v^{-1}a^*$ define a superinvolution on $V^{?1}A$, and also that $H(V^{-1}A, *) = V^{-1}H$ and $K(V^{-1}A, *) = V^{-1}K$. Moreover we have $Z(V^{-1}A)_0 = V^{-1}Z$ and $V^{-1}Z \cap V^{-1}H = V^{-1}Z_H$.

We notice that in every semiprime superalgebra A, the intersection of all the prime ideals P of A is zero. Consequently A is a subdirect product of its prime images. If each prime image of A is a central order in a simple superalgebra at most n^2 dimensional over its center, we say that A verifies $S(n)$. With this definition we try to describe semiprime algebras whose prime images are in some way of a bounded size. We use this term in our final result for $n = 4$. We recall that finite dimensional simple associative superalgebras were classified in [19]. They are either simple as nongraded algebras, or A_0 is simple and then $A_1 = uA_0$ with $u \in Z(A)_1$ and $u^2 = 1$.

Let V be a vector space over a field F and q be a quadratic form on V. The tensor algebra of V is an associative superalgebra $T(V) = T(V)_0 \oplus T(V)_1$, where $T(V)_0 = F + V \otimes V + \dots$ and $T(V)_1 = V + V \otimes V \otimes V + \dots$. If we denote by $I(q)$ the ideal of $T(V)$ generated in $T(V)$ by the elements $v \otimes v - q(v)$ for $v \in V$, we can define the Clifford superalgebra of the quadratic space (V, q) as $C(V, q) = T(V)/I(q)$. We recall that if $dim_F(V) = n$ then $dim_F(C(V, q)) = 2^n$.

The aim of this paper is to describe Lie ideals of the derived subalgebra $[K, K]$ of A^-, when A is a semiprime associative superalgebra with superinvolution. This paper completes [11], where Lie ideals of K for a superalgebra of the same type were studied. In fact, the scheme of the paper is the same, and the final result also. But the work needed to prove it has been much more complicated. We have had to prove more results, and to use powerful theorems about PI-algebras obtained by V. Kharchenko in [10] and F. Montaner in [16]. Specifically the two sections of paper [11] become in this paper five sections. Only the case in which we study the prime image of a Lie ideal when $P^* = P$ is more or less similar to that of the K. The rest has been more laborious, especially the study of the prime image of the Lie ideal when $P^* \neq P$.

In the end we prove that if U is a Lie ideal of $[K, K]$ then one of the following alternatives must hold: either U must contain a nonzero Lie ideal $[J \cap K, K]$, for J an ideal of A, or A is a subdirect sum of A', A'', where the image of U in A' is central and A'' satisfies $S(4)$.

The study of Lie ideals in the derived superalgebra $[K, K]$ is interesting because it is known that, if A is simple, then $[K, K]/(Z \cap [K, K])$ is simple if the dimension of A is greater than 16 (see [3]). So the description of the Lie ideals when A is prime (made in [4]) or semiprime is a natural question. The corresponding results in the nongraded case (see for instance [14,6,7]) have been very useful for several investigations.

If A is an associative superalgebra and M is a ϕ-submodule of A and we denote by $\bar{M}$ the subalgebra generated by M, we say that M is *dense* in A if $\bar{M}$ contains a nonzero ideal of A.

In the following lemma the bracket product is the usual one in nongraded algebras: $[a, b] = ab - ba$.

Lemma 1.1. *(See [7, Theorem 1].) Let A be a semiprime algebra and let L be a Lie ideal of A. If $[a, [a, L]] = 0$, then $[a, L] = 0$.*

But from now on the bracket product $[\ ,\ ]$ will always denote the superbracket one, that is, $[a, b] = ab - (-1)^{\bar{a}\bar{b}}ba$.

Lemma 1.2. *(See [16, Lemmata 1.2, 1.3].) If $A = A_0 \oplus A_1$ is a semiprime superalgebra, then A_0 is a semiprime algebra. Moreover, if A is prime, then either A is prime or A_0 is prime (as algebras).*

Lemma 1.3. *(See [16, Lemma 1.8].) Let $A = A_0 \oplus A_1$ be a prime superalgebra. Then*

(i) *If $x_1 \in A_1$ centralizes a nonzero ideal I of A_0, then $x_1 \in Z(A)$.*
(ii) *If x_1^2 belongs to the center of a nonzero ideal I of A_0, then $x_1^2 \in Z(A)$.*

Lemma 1.4. *(See [4, Corollary 2].) Let A be a semiprime superalgebra and L a Lie ideal of A. Then either $[L, L] = 0$, or L is dense in A.*

Lemma 1.5. *(See [4, Theorem 2.1].) Let A be a prime nontrivial associative superalgebra. If L is a Lie ideal of A, then either $L \subseteq Z$ or L is dense in A, except if A is a central order in a 4-dimensional Clifford superalgebra.*

During the paper we will use very often the following identities in a superalgebra A, for a, b, c homogeneous elements in A:

$$[a, bc] = [a, b]c + (-1)^{\bar{a}\bar{b}} b[a, c], \tag{1}$$

$$[ab, c] = a[b, c] + (-1)^{\bar{b}\bar{c}} [a, c]b, \tag{2}$$

$$[a, b \circ c] = [a, b] \circ c + (-1)^{\bar{a}\bar{b}} b \circ [a, c], \tag{3}$$

$$[a \circ b, c] = a \circ [b, c] + (-1)^{\bar{b}\bar{c}} [a, c] \circ b. \tag{4}$$

2. Lie structure of $[K, K]$

Let A be an associative superalgebra and M, S be ϕ-submodules of A. Define $(M : S) = \{a \in A : aS \subseteq M\}$.

Let U be a Lie ideal of $[K, K]$. We recall (see Lemma 4.1 in [3]) that K^2 is a Lie ideal of A. So $\overline{K^2}$ is also a Lie ideal of A, because for every k, l homogeneous elements in K^2 and for every a homogeneous element in A we have $[kl, a] = k[l, a] + (-1)^{\bar{l}\bar{a}} [k, a]l \in \overline{K^2}$.

Lemma 2.1. *If A is semiprime, then either U is dense in A or $[u \circ v, w] = 0$ for every homogeneous elements $u, v \in [U, U], w \in U$.*

Proof. We present the proof of this in six steps. Let $u, v \in [U, U], w \in U$.

1. $[u \circ v, w] \in (\overline{U} : A)$. We have

$$[u \circ v, k] = u \circ [v, k] + (-1)^{\bar{k}\bar{v}} [u, k] \circ v \in \overline{U}$$

for every homogeneous elements $u, v \in [U, U]$ and $k \in K$, because

$$\big[[U, U], K\big] \subseteq \big[U, [U, K]\big] \subseteq \big[U, [K, K]\big] \subseteq U.$$

And also for every homogeneous elements $u, v \in [U, U]$ and $h \in H$ we get

$$[u \circ v, h] = [u, v \circ h] + (-1)^{\bar{u}\bar{v}} [v, u \circ h] \in U,$$

because $K \circ H \subseteq K$. Since $A = H \oplus K$ it follows that $[u \circ v, A] \subseteq \overline{U}$ for every homogeneous elements $u, v \in [U, U]$. But for every homogeneous elements $a \in A$, $w \in U$

$$[u \circ v, wa] = [u \circ v, w]a + (-1)^{(\bar{u}+\bar{v})\bar{w}} w[u \circ v, a]$$

and so $[u \circ v, w]A \subseteq \bar{U}$, that is, $[u \circ v, w] \in (\bar{U} : A)$.

2. $[u \circ v, A] \subseteq \overline{K^2}, \overline{[K,K]}$ and $[u \circ v, w] \in (\overline{K^2} : A), (\overline{[K,K]} : A)$. We notice that from the above equations we can also deduce that $[u \circ v, A] \subseteq \overline{[K,K]}$ and that $[u \circ v, w] \in (\overline{[K,K]} : A)$.

3. $A[u \circ v, w]A \subseteq \overline{K^2}$. We claim that $A[u \circ v, w] \subseteq (\overline{K^2} : A)$. Let $a, b \in A$ be homogeneous elements, then

$$a[u \circ v, w]b = \big[a, [u \circ v, w]b\big] + (-1)^{(\bar{u}+\bar{v}+\bar{w})\bar{a}+\bar{b}\bar{a}}[u \circ v, w]ba \subseteq \overline{K^2},$$

because of step 2 and because $\overline{K^2}$ is a Lie ideal of A.

4. $\overline{K}.(\overline{[K,K]} : A) \subseteq (\overline{[K,K]} : A)$. Let $k \in K$, $x \in (\overline{[K,K]} : A)$, $a \in A$ be homogeneous elements, then

$$(kx)a = [k, xa] + (-1)^{(\bar{x}+\bar{a})\bar{k}}(xa)k \in \overline{[K,K]},$$

because $x \in (\overline{[K,K]} : A)$ and because if $l, m \in [K,K]$ are homogeneous elements then from (1)

$$[k, lm] = [k, l]m + (-1)^{\bar{k}\bar{l}}l[k, m] \in \overline{[K,K]}.$$

5. $\overline{[K,K]}.(\bar{U} : A) \subseteq (\bar{U} : A)$. It is the same proof as in step 4. Let $k \in [K,K]$, $x \in (\bar{U} : A)$, $a \in A$ be homogeneous elements, then

$$(kx)a = [k, xa] + (-1)^{(\bar{x}+\bar{a})\bar{k}}(xa)k \in \bar{U},$$

because $x \in (\bar{U} : A)$ and because if $l, m \in U$ are homogeneous elements then

$$[k, lm] = [k, l]m + (-1)^{\bar{k}\bar{l}}l[k, m] \in \overline{U},$$

since U is a Lie ideal of $[K,K]$.

6. $A[u \circ v, w]A[u \circ v, w]A[u \circ v, w]A \subseteq \bar{U}$. From steps 1–5 we deduce that

$$A[u \circ v, w]A[u \circ v, w]A[u \circ v, w]A \subseteq \overline{K^2}\big(\overline{[K,K]} : A\big)A(\bar{U} : A)A \subseteq \overline{[K,K]}(\bar{U} : A)A \subseteq \bar{U}.$$

So, if $[u \circ v, w] \neq 0$, since A is semiprime, $0 \neq J = A[u \circ v, w]A[u \circ v, w]A[u \circ v, w]A \subseteq \bar{U}$, and then U is dense in A. $\square$

We note that the ideal contained in $\bar{U}$ in the above lemma, $J = A[u \circ v, w]A[u \circ v, w]A[u \circ v, w]A$, is also a $*$-ideal, that is, $J^* \subseteq J$.

Theorem 2.2. *Let A be a semiprime superalgebra with superinvolution, then either K is dense or A satisfies $S(2)$.*

Proof. Consider the Lie ideal of A, K^2. From Lemma 1.4, either K^2 is dense in A, or $[K^2, K^2] = 0$. In the first case, K is dense in A, clearly. In the second case, by Theorem 1.1 in [12], A satisfies $S(2)$. $\square$

Lemma 2.3. *Let A be semiprime, and let U be a Lie ideal of $[K, K]$ such that $[u \circ v, w] = 0$ for every $u, v \in [U, U], w \in U$. Then*

(i) $u \circ v \in Z$ *for every* $u, v \in [U, U]_i$.
(ii) $(u \circ v)^2 = 0$ *for every* $u \in [U, U]_0$, $v \in [U, U]_1$.
(iii) $u \circ v = 0$ *for every* $u, v \in [U, U]_1$.

Proof. From step 1 and its proof in Lemma 2.1, we know that $[u \circ v, h] \in U$ and $[u \circ v, k] \in \bar{U}$ for every homogeneous elements $u, v \in [U, U], h \in H, k \in K$. Therefore $[u \circ v, a] \in \overline{U}$ for every $a \in A$. So, from (1), $[u \circ v, [u \circ v, a]] = 0$. Now, if $u \circ v$ is even, we obtain from Lemma 1.1 that $u \circ v \in Z$ and we have (i). And if $u \circ v$ is odd, then, from (4),

$$[u \circ v, u \circ v] = (-1)^{\bar{u}\bar{u}+\bar{u}\bar{v}} u \circ [u \circ v, v] + [u \circ v, u] \circ v = 0,$$

that is, $(u \circ v)^2 = 0$, and we have (ii).

Now, suppose that $\gamma = u \circ v$ with $u, v \in [U, U]_1$. Then

$$\begin{aligned} \gamma\big(u^2 \circ v\big) &= u^2 \circ \gamma v = \frac{1}{2}\big(u^2 \circ \big((u \circ v) \circ v\big)\big) = -\frac{1}{2}\big(u^2 \circ \big[v^2, u\big]\big) \\ &= -\frac{1}{2}\big(\big[u^2 \circ v^2, u\big]\big) = -\frac{1}{2}\big(\big[[u, u] \circ [v, v], u\big]\big) = 0, \end{aligned}$$

because $\gamma \in Z$, because of the hypothesis and from (3). A similar argument shows that $\gamma(v^2 \circ u) = 0$. Notice that $0 = [u \circ v, u] = [uv - vu, u] = uvu - vu^2 - u^2v + uvu$, and so $2uvu = u^2 \circ v$. Therefore $\gamma(uvu) = 0$. And since we can also prove that $2vuv = v^2 \circ u$, it is deduced that $\gamma(vuv) = 0$. Now we observe that

$$2\gamma u^3 = \gamma u \circ u^2 = \frac{1}{2}\big((u \circ v) \circ u\big) \circ u^2 = \frac{1}{2}\big[u^2, v\big] \circ u^2 = \frac{1}{2}\big[u^2, v \circ u^2\big] = 0$$

because of the hypothesis and from (4). And the same $\gamma v^3 = 0$. Notice that

$$\begin{aligned} \gamma^2 &= (u \circ v)(u \circ v) = (\gamma u \circ v) = 1/2(\gamma \circ u) \circ v = 1/2\big((u \circ v) \circ u\big) \circ v \\ &= 1/2\big(\big[u^2, v\big] \circ v\big) = 1/2\big(-\big[u^2 \circ v, v\big] + u^2 \circ [v, v]\big) = 1/2\big(u^2 \circ v^2\big), \end{aligned}$$

and so finally

$$\begin{aligned} \gamma^4 = \gamma\gamma\gamma^2 &= \frac{1}{2}\gamma(u \circ v)\big(u^2 \circ v^2\big) = \frac{1}{2}\gamma(uv - vu)\big(u^2v^2 + v^2u^2\big) \\ &= \frac{1}{2}\gamma\big(uvu^2v^2 + uv^3u^2 - vu^3v^2 - vuv^2u^2\big) = 0, \end{aligned}$$

because $\gamma uvu = \gamma vuv = \gamma u^3 = \gamma v^3 = 0$. So, since A is semiprime, we obtain that $\gamma = 0$ and we get (iii). □

In the following two sections we deal with the second case of Lemma 2.1, that is, when $[u \circ v, w] = 0$ for every $u, v \in [U, U]$, $w \in U$, and we will study the prime images of A. If P is a prime ideal of A we have two possible situations: either $P^* \neq P$ or $P^* = P$.

3. Prime images of Lie ideals when $P^* \neq P$

Let P be a prime ideal of A. We will suppose first that $P^* \neq P$. In this case $(P^*+P)/P$ is a nonzero proper ideal of A/P and we claim that $(P^* + P)/P \subseteq (K + P)/P$. Indeed, if $y \in P^*$ then $y + P = (y - y^*) + y^* + P \in (K + P)/P$. Also if U is a Lie ideal of $[K, K]$ we have that $(U + P)/P$ is a ϕ-submodule of A/P and satisfies

$$\bigl[(U + P)/P, \bigl[\bigl(P^* + P\bigr)/P, \bigl(P^* + P\bigr)/P\bigr]\bigr] \subseteq \bigl(\bigl[U, [K, K]\bigr] + P\bigr)/P \subseteq (U + P)/P.$$

Of course if $u \circ v \in Z$ for every $u, v \in [U, U]_0$, $u \circ v = 0$ for every $u, v \in [U, U]_1$, and $(u \circ v)^2 = 0$ for every $u \in [U, U]_0, v \in [U, U]_1$, then the same property is satisfied in A/P, that is, $(u + P) \circ (v + P) \in Z_0(A/P)$ for every $u + P, v + P \in ([U, U]_0 + P)/P$, $(u+P)\circ(v+P) = 0$ for every $u+P, v+P \in ([U, U]_1+P)/P$, and $((u+P)\circ(v+P))^2 = 0$ for every $u+P \in ([U, U]_0+P)/P, v+P \in ([U, U]_1+P)/P$. Let us analyze this situation. We notice that the assumption that A/P has a superinvolution is not required. We state first some useful lemmata.

Lemma 3.1. *Let A be a prime superalgebra, I a nonzero ideal of A, then either $[I, I]$ is dense in A, or A is a central order in a 4-dimensional Clifford superalgebra, or A is commutative.*

Proof. We notice that $[I, I]$ is a Lie ideal of A, and from Lemma 1.5 it follows that either $[I, I]$ is dense in A, or A is a central order in a 4-dimensional Clifford superalgebra, or $[I, I] \subseteq Z$. Suppose that $[I, I] \subseteq Z$, then $[I_0, I_1] = 0$. But then, from Lemma 1.3(i) we deduce that $I_1 \subseteq Z_1(A)$. We observe that $I_1 \neq 0$ because if $I = I_0$ then $I.(A_1 + A_1^2) = 0$, a contradiction with the primeness of A. Therefore $I = I_0 + I_1$ with $I_1 \neq 0$, and this is satisfied for every nonzero ideal of A. Let $J = I_0 I_1 + I_1^2$. Since $I_1 \subseteq Z_1(A)$, J is an ideal of A. Also $J \neq 0$, because if $J = 0$, then $0 \neq I^2 = I_0^2$ by primeness, and $(I^2)_1 = 0$, a contradiction. Since $I_1 \subseteq Z_1(A)$, we get $[x, J] = 0$ for every $x \in I_0$, because of (1). Therefore for every $a \in A$ and $y \in J$ we have $(xa)y = (ay)x = (ax)y$, that is, $(xa - ax)J = 0$, and since A is prime we deduce that $xa = ax$ for every $a \in A$, so $I_0 \subseteq Z$, and then $I \subseteq Z(A)$. Now is easy to prove that A is commutative. For every homogeneous elements $a, b \in A$ and $y \in I$ it follows that

$$(ab)y = (by)a = (yb)a = (ba)y,$$

and by the primeness of A, $ab = ba$ for every homogeneous elements $a, b \in A$. □

Lemma 3.2. *Let A be a prime superalgebra, L a Lie ideal of A such that L is dense in A, and $v \in A_i$ such that $vLv = 0$, then $v = 0$.*

Proof. Let $u \in L_i$ and $a \in A$. Then $v[u, a]v = 0$. Considering now $v[u, u'va]v$ with $u' \in L$, homogeneous, we have $vuu'vav = 0$. Therefore $vuu'vA$ is a right ideal with square zero, that is a contradiction with the primeness of A, so $vuu'v = 0$. In the same way considering $v[u, u'u''va]v$ we obtain that $vuu'u''v = 0$. So, if J is a nonzero ideal such that $J \subseteq \bar{L}$, we deduce that $vJv = 0$ and, because of A is prime, $v = 0$. □

Lemma 3.3. *Let A be a prime superalgebra, L a Lie ideal of A such that L is dense in A, and V a Lie subalgebra of A such that $[V, L] \subseteq V$. If $v^2 = 0$ for every $v \in V_i$, then $V_i = 0$.*

Proof. Consider $l_0 \in L_0$ and $a_0 \in A_0$, then $[l_0, a_0] \in L$ and $[v, [l_0, a_0]]^2 = 0$ for every $v \in V_i$, that is

$$(vl_0a_0 - va_0l_0 - l_0a_0v + a_0l_0v)^2v = 0.$$

Expanding yields

$$vl_0a_0vl_0a_0v - vl_0a_0va_0l_0v - va_0l_0vl_0a_0v + va_0l_0va_0l_0v = 0.$$

Replacing a_0 by a_0v gives $va_0vl_0va_0vl_0v = 0$, and so, $(vl_0va_0)^3 = 0$. Since A_0 is semiprime by Lemma 1.2, it follows from Lemma 1.1 in [5] that $vl_0v = 0$. Now let $l_1 \in L_1$, we can prove in a similar way that $vl_1v = 0$. Indeed, let $a_1 \in A_1$ and notice that $[v, [l_1, a_1]]^2 = 0$ and so

$$(vl_1a_1 + va_1l_1 - l_1a_1v - a_1l_1v)^2v = 0.$$

Expanding and replacing a_1 by a_1v give $va_1vl_1va_1vl_1v = 0$, and so $(vl_1va_1)^3 = 0$. Therefore vl_1vA_1 is a right ideal of A_0. But A_0 is semiprime, by Lemma 1.2. So from Lemma 1.1 in [5] $vl_1vA_1 = 0$ and also $(vl_1v)(A_1 + A_1^2) = 0$. Since A is prime we have $vl_1v = 0$, and so $vLv = 0$. Now, by Lemma 3.9, $V_i = 0$. □

Then, from now on, and until the end of this section, we will suppose that A is a prime superalgebra, I is a nonzero ideal of A and U is a subalgebra of A^- (that is, U is a ϕ-submodule of A and $[U, U] \subseteq U$) such that $[U, [I, I]] \subseteq U$. Moreover U satisfies the following conditions: $u \circ v \in Z$ for every $u, v \in [U, U]_0$, $u \circ v = 0$ for every $u, v \in [U, U]_1$ and $(u \circ v)^2 = 0$ for every $u \in [U, U]_0, v \in [U, U]_1$.

Our aim in this section is to prove the following theorem:

Theorem 3.4. *Let A be a prime superalgebra and let I be a nonzero proper ideal of A. Suppose that U is a subalgebra of A^- such that $[U, [I, I]] \subseteq U$, $u \circ v \in Z$ for every $u, v \in [U, U]_0$, $u \circ v = 0$ for every $u, v \in [U, U]_1$ and $(u \circ v)^2 = 0$ for every $u \in [U, U]_0$,*

$v \in [U,U]_1$. Then either A is commutative, or A is a central order in a 4-dimensional simple superalgebra, or A is a central order in an 8-dimensional simple superalgebra, or $U \subseteq Z$.

Let us outline the plan to prove it, because the proof is obtained after several results. First we introduce the tool to get our purpose. We will consider the following set:

$$T = \{x \in A : [x,A] \subseteq U\}.$$

We will prove that T is a subring, and we will define the subring of T, denoted T' generated by $[[U,[I,I]],[U,[I,I]]]$. We will distinguish to cases: when $[T',[I,I]] = 0$ and when $[T',[I,I]] \neq 0$. In the first case we will prove that $U \subseteq Z$, and in the second case that either A is commutative, or A is a central order in a 4-dimensional simple superalgebra, or A is a central order in an 8-dimensional simple superalgebra. So, in proving this, we will have the theorem.

Let us start from the very beginning, showing that $[[U,[I,I]],[U,[I,I]]] \subseteq T$, and that T is a subring.

Since

$$\big[\big[[U,[I,I]],[U,[I,I]]\big],A\big] \subseteq \big[[U,[I,I]],\big[[U,[I,I]],A\big]\big] \subseteq \big[[U,[I,I]],[I,I]\big] \subseteq U,$$

we have $[[U,[I,I]],[U,[I,I]]] \subseteq T$. We notice that T is a subring of A because for every homogeneous elements $t,s \in T$, from (2)

$$[ts,a] = [t,sa] + (-1)^{\bar{t}\bar{s}+\bar{a}\bar{t}}[s,at] \in U.$$

Let T' be the subring generated by $[[U,[I,I]],[U,[I,I]]]$. Since

$$\big[\big[[U,[I,I]],[U,[I,I]]\big],[I,I]\big] \subseteq \big[[U,[I,I]],\big[[U,[I,I]],[I,I]\big]\big] \subseteq \big[[U,[I,I]],[U,[I,I]]\big]$$

it follows that $[T',[I,I]] \subseteq T'$. Now we consider two cases:

(a) $[T',[I,I]] = 0$,
(b) $[T',[I,I]] \neq 0$.

We will suppose until the end of the section that A is neither commutative, nor a central order in a 4-dimensional simple superalgebra, nor a central order in an 8-dimensional simple superalgebra. Hence, from Lemma 3.1, we can also suppose that there exists a nonzero ideal of A, J, such that $J \subseteq \overline{[I,I]}$.

Let's go to consider the first case.

Case (a): $[T',[I,I]] = 0$.

Lemma 3.5. *In the above situation, if $[T',[I,I]] = 0$, then $U \subseteq Z$.*

Proof. If $[T', [I, I]] = 0$, it follows from (1) that $[T', J] = 0$, and from Lemma 2.3 in [11], we deduce that $T' \subseteq Z$. Therefore

$$\big[[U,[I,I]]_i, [U,[I,I]]_i\big] \subseteq Z, \tag{5}$$

$$\big[[U,[I,I]]_0, [U,[I,I]]_1\big] = 0 \tag{6}$$

We will prove that, with the above supposition of being A neither commutative, nor a central order in a 4-dimensional simple superalgebra, nor a central order in an 8-dimensional simple superalgebra, then $U \subseteq Z$. We present the proof in 7 steps.

1. $[[U,U],[I,I]]_0 \subseteq Z(A_0)$. From Lemma 1.2, A_0 is semiprime, and $[I,I]_0$ is a Lie ideal of A_0, $[[U,U],[I,I]]_0 \subseteq [I,I]_0$, $[[[U,U],[I,I]]_0,[I,I]_0] \subseteq [[U,U],[I,I]]_0$ and $[[U,U],[I,I]_0,[[U,U],[I,I]]_0] \subseteq Z(A_0)$ because of (5). So we have the conditions of Lemma 1.1 in [14] and therefore we can conclude by this lemma that $[[U,U],[I,I]]_0 \subseteq Z(A_0)$.

2. $[[U,U]_0,[I,I]_0] = 0$. We notice that A_0 is semiprime by Lemma 1.2, $[I,I]_0$ is a Lie ideal of A_0 and $[[U,U]_0,[[U,U]_0,[I,I]_0]] = 0$ by step 1. So we can apply Lemma 1.1 and we obtain that $[[U,U]_0,[I,I]_0] = 0$.

3. Either $[[U,U],[I,I]]_0 = 0$ or $[[U,U],[I,I]]_1 = 0$. Let $u \in [[U,U],[I,I]]_0$ and $v \in [[U,U],[I,I]]_1$. By (6) we have $uv = vu$. But, from the hypothesis, $(u \circ v)^2 = 0$, hence $4u^2v^2 = 0$. So, since $u \in [[U,U],[I,I]]_0 \subseteq Z$ and A is prime, either $[[U,U],[I,I]]_0 = 0$ or $v^2 = 0$ for every $v \in [[U,U],[I,I]]_1$. If $v^2 = 0$ for every $v \in [[U,U],[I,I]]_1$, from Lemma 3.3, taking $L = [I,I]$, $V = [[U,U],[I,I]]$, we deduce that $[[U,U],[I,I]]_1 = 0$.

4. If $[[U,U],[I,I]]_0 = 0$ we claim that $[U,U] \subseteq Z$. We notice that

$$[U,U]_1[I,I]_0 \subseteq \big[[U,U]_1,[I,I]_0\big] + [I,I]_0[U,U]_1 \subseteq [U,U]_1 + [I,I]_0[U,U]_1,$$

$$[U,U]_1[I,I]_1 \subseteq \big[[U,U]_1,[I,I]_1\big] + [I,I]_1[U,U]_1 \subseteq [I,I]_1[U,U]_1,$$

because of the hypothesis of this step. Therefore $[U,U]_1[I,I] \subseteq [U,U]_1 + [I,I][U,U]_1$. In general, we can prove by induction on m that

$$[U,U]_1[I,I]^m \subseteq [U,U]_1 + \sum_i [I,I]^i[U,U]_1,$$

and so $[U,U]_1 J \subseteq [U,U]_1 + \sum_i [I,I]^i[U,U]_1$. Now since

$$\big[[U,U]_1, \big[[U,U],[I,I]\big]_1\big] \subseteq \big[[U,U]_1,[I,I]_1\big] \subseteq \big[[U,U],[I,I]\big]_0 = 0$$

because of the hypothesis of this step, and $[U,U]_1 \circ [[U,U],[I,I]]_1 = 0$ because of our hypothesis, it follows that $[U,U]_1[[U,U],[I,I]]_1 = 0$, and therefore

$$[U,U]_1 J \big[[U,U],[I,I]\big]_1 = 0.$$

But A is prime, so either $[U,U]_1 = 0$ or $[[U,U],[I,I]]_1 = 0$. If $[U,U]_1 = 0$, then $[[U,U]_0,[I,I]_1] = 0$, and from the hypothesis of this step $[[U,U]_0,[I,I]] = 0$. But then, from (2), $[[U,U]_0,J] = 0$ and so, by Lemma 2.3 in [11], $[U,U]_0 \subseteq Z$ and $[U,U] \subseteq Z$. If $[[U,U],[I,I]]_1 = 0$, since $[[U,U]_0,[I,I]_0] = [[U,U]_1,[I,I]_1] = 0$ by the hypothesis of this step, we get $[[U,U],[I,I]] = 0$ and so $[[U,U],J] = 0$, from (2), and as above $[U,U] \subseteq Z$.

5. If $[[U,U],[I,I]]_0 \neq 0$, then by step 3 $[[U,U],[I,I]]_1 = 0$ and also we claim that $[U,U] \subseteq Z$. We have $[[U,U]_0,[I,I]_1] = 0$, and by step 2 $[[U,U]_0,[I,I]_0] = 0$. Therefore, from (2), $[[U,U]_0,J] = 0$, and, by Lemma 2.3 in [11], $[U,U]_0 \subseteq Z$. From the hypothesis about U, for every $u \in [U,U]_0, v \in [U,U]_1$ we have $(u \circ v)^2 = 0$. So, since $(u \circ v)^2 = 4u^2v^2 = 0$, we obtain from the primeness of A that either $[U,U]_0 = 0$ or $v^2 = 0$ for every $v \in [U,U]_1$. If $[U,U]_0 = 0$, then $[[U,U],[I,I]]_0 \subseteq [U,U]_0 = 0$, a contradiction with our assumption. If $v^2 = 0$ for every $v \in [U,U]_1$, from Lemma 3.3 applied to $[I,I]$ and $[U,U]$ we obtain that $[U,U]_1 = 0$ and so $[U,U] \subseteq Z$.

6. If $I \cap Z \neq 0$, then $U \subseteq Z$. Indeed, if $I \cap Z \neq 0$, we can consider $Z^{-1}A$, which is a prime superalgebra over the field $Z^{-1}Z$. Since $Z^{-1}I \cap Z^{-1}Z \neq 0$ it holds that $Z^{-1}I = Z^{-1}A$ and so $Z^{-1}ZU$ is a Lie ideal of $[Z^{-1}A, Z^{-1}A]$. From Theorem 3.3 in [16], and since A is a central order neither in a commutative algebra, nor a 4-dimensional simple superalgebra, nor an 8-dimensional superalgebra, we deduce that either $Z^{-1}ZU \subseteq Z^{-1}Z$ or there exists a nonzero ideal of $Z^{-1}A$, $Z^{-1}N$, such that $[Z^{-1}N, Z^{-1}A] \subseteq Z^{-1}ZU$. In the last case, since, by steps 4 and 5, $[U,U] \subseteq Z$, we have that $[[Z^{-1}N, Z^{-1}A],[Z^{-1}N, Z^{-1}A]] \subseteq Z^{-1}Z$. We notice that if $[[Z^{-1}N, Z^{-1}A],[Z^{-1}N, Z^{-1}A]] = 0$, from Lemma 1.5 and its proof and our hypothesis, $[Z^{-1}N, Z^{-1}A] \subseteq Z^{-1}Z$, and then $Z^{-1}N = Z^{-1}A$ if $[Z^{-1}N, Z^{-1}A] \neq 0$. And if $[Z^{-1}N, Z^{-1}A] = 0$, by Lemma 2.3 in [11], $Z^{-1}N \subseteq Z^{-1}Z$ and $(Z^{-1}N)_1 = 0$, a contradiction with the primeness of $Z^{-1}A$. If $[[Z^{-1}N, Z^{-1}A],[Z^{-1}N, Z^{-1}A]] \neq 0$, then also $Z^{-1}N = Z^{-1}A$. So $[[Z^{-1}A, Z^{-1}A],[Z^{-1}A, Z^{-1}A]] \subseteq Z^{-1}Z$ and then the superalgebra $Z^{-1}A$ verifies the identity $[Z^{-1}A,[[Z^{-1}A, Z^{-1}A],[Z^{-1}A, Z^{-1}A]]] = 0$. Now by Lemma 2.6 in [16] we have a contradiction with our supposition of A not being a central order neither in a commutative algebra, nor a 4-dimensional simple superalgebra, nor an 8-dimensional superalgebra (notice that the product $\circ$ in [16] is our product $[\ ,\ ]$ in the odd part). So $Z^{-1}ZU \subseteq Z^{-1}Z$ and then $U \subseteq Z$.

7. If $I \cap Z = 0$, then $U \subseteq Z$. We consider $[U,[I,I]]$ and we notice that $[[U,[I,I]],[U,[I,I]]] \subseteq [U,U] \cap I \subseteq Z \cap I = 0$. Therefore for every $v \in [U,[I,I]]_1$, $v^2 = [v,v] \in [[U,[I,I]],[U,[I,I]]] = 0$. From Lemma 3.1 applied to $[U,[I,I]]$ and $[I,I]$ it follows that $[U,[I,I]]_1 = 0$. Now let $u \in U_0$, then $[u,[u,[I,I]_0]] \subseteq [U,U] \cap I \subseteq Z \cap I = 0$. By Lemmata 1.1 and 1.2 it is deduced that $[u,[I,I]_0] = 0$, that is, $[U_0,[I,I]_0] = 0$. Now we have $[U_0,[I,I]] = 0$, and therefore, from (2), $[U_0,J] = 0$. So $U_0 \subseteq Z$ because of Lemma 2.3 in [11]. But we have proved that $[U_1,[I,I]_0] \subseteq [U,[I,I]]_1 = 0$, and now we have $[U_1,[I,I]_1] \subseteq U_0 \cap I \subseteq Z \cap I = 0$, therefore $[U_1,[I,I]] = 0$, and then, from (2), $[U_1,J] = 0$. Again, by Lemma 2.3 in [11], $U_1 \subseteq Z$ and $U \subseteq Z$. □

Now we consider the second case.

Case (b): $[T', [I, I]] \neq 0$.

We recall that $[T', [I, I]] \subseteq T'$.

Lemma 3.6. *If $[T', [I, I]] \neq 0$, then either T' is dense or $[t, u][u, s] = 0$ for every $u \in [T', [I, I]]_i$, such that $[u, u] = 0$, and $t, s \in T'$, homogeneous.*

Proof. We notice that if $u \in [T', [I, I]]_0$, then $[u, u] = 0$, and if $u \in [T', [I, I]]_1$, then $[u, u] = 0$ is equivalent to $u^2 = 0$. We will prove that if there exist $t, s \in T'$, homogeneous, and $u \in [T', [I, I]]_i$, with $[u, u] = 0$, such that $[t, u][u, s] \neq 0$, then T' is dense. First we see that $[t, u][u, s]A \subseteq T'$. We have $[u, s]a = [u, sa] - (-1)^{\bar{s}\bar{u}} s[u, a]$, from (2), for every homogeneous element $a \in A$, therefore $[t, u][u, s]a = [t, u][u, sa] - (-1)^{\bar{u}\bar{s}}[t, u]s[u, a]$. But

$$
\begin{aligned}
[t, u][u, sa] &= \big[t, u[u, sa]\big] - (-1)^{\bar{t}\bar{u}} u\big[t, [u, sa]\big] \\
&= (-1)^{\bar{u}}\big[t, [u, usa]\big] - (-1)^{\bar{t}\bar{u}} u\big[t, [u, sa]\big] \in T',
\end{aligned}
$$

because T' is a subring and $[I, I]$ is a Lie ideal of A. And also

$$
[t, u]s[u, a] = [t, u]\big[s, [u, a]\big] + (-1)^{\bar{s}(\bar{u}+\bar{a})}[t, u][u, a]s \in T',
$$

because

$$
[t, u][u, a] = \big[t, u[u, a]\big] - u\big[t, [u, a]\big] = \big[t, [u, ua]\big] - u\big[t, [u, a]\big] \in T'.
$$

Therefore $[t, u]s[u, a] \in T'$, and also $[t, u][u, s]a \in T'$ for every $a \in A$, $u \in [T', [I, I]]_i$, t, s homogeneous elements in T'.

Next we will show that $\overline{[I, I]}[t, u][u, s]A \subseteq T'$. Since $[T', [I, I]] \subseteq T'$ it follows that $[I, I][t, u][u, s]A \subseteq [[I, I], [t, u][u, s]A] + [t, u][u, s]A[I, I] \subseteq T'$. Notice that also $[I, I]^2[t, u][u, s]A \subseteq [[I, I], T'] + [t, u],\ [u, s]A \subseteq T'$. Using induction over n it is easy to prove that $[I, I]^n[t, u][u, s]A \subseteq T'$, and so that $J[t, u][u, s]A \subseteq T'$. Therefore either T' is dense in A or, because of the primeness of A, $[t, u][u, s] = 0$ for every $t, s \in T'$, $u \in [T', [I, I]]_i$. □

Lemma 3.7. *If T' is dense, then A is a central order in a superalgebra B satisfying the condition $[[B, B], [B, B]]_1 \circ [[B, B], [B, B]]_1 = 0$.*

Proof. Let $N = J[t, u][u, s]A \neq 0$, since $[T', A] \subseteq U$ and $N \subseteq T'$, we have $[N, A] \subseteq U$. From the hypothesis about U $u \circ v \in Z$ for every $u, v \in [U, U]_0$, therefore $u \circ v \in Z$ for every $u, v \in [[N, A], [N, A]]_0$.

We suppose first that $u \circ v = 0$ for every $u, v \in [[N, A], [N, A]]_0$. Then $1/2(u \circ u) = u^2 = 0$ for every $u \in [[N, A], [N, A]]_0$, and since A_0 is semiprime because of Lemma 1.2, then it follows from Lemma 1 in [14] that $[[N, A], [N, A]]_0 = 0$. Therefore $L = [[N, A], [N, A]]$ is a Lie ideal of A such that $[L, L] = 0$ and from Theorem 3.2 in [16] we get that $[[N,A],[N,A]] \subseteq Z$ (because A in neither commutative, nor a central order in a 4-dimensional simple superalgebra, nor a central order in an 8-dimensional simple superalgebra). But then $[[N,A],[N,A]] = 0$, and again from Theorem 3.2 in [16], $[N,A]$ is a Lie ideal of A such that $[N,A] \subseteq Z$, and so $[N,A]_1 = 0$. Hence $[N_1, A_1^2] = 0$ and since $A_1^2 \neq 0$ because A is prime we obtain that $N_1 \subseteq Z(A)$ because of Lemma 1.3. Besides also $[N_0, A_1] = 0$ and so $[N_0, A_1^2] = 0$, from (2), what means that $[N, A_1 + A_1^2] = 0$ and from Lemma 2.3 in [11] it follows that $N_0 \subseteq Z$. So we have a nonzero ideal, N, of A such that $N \subseteq Z(A)$, with A prime, and we can deduce that A is commutative like in the proof of Lemma 3.1, that is a contradiction with our hypothesis.

Therefore there exist $u, v \in [[N,A],[N,A]]_0$ such that $0 \neq u \circ v \in Z$. Then we may form the localization $Z^{-1}A$. Since $[N,A] \subseteq U$ we have

$$\big[\big[Z^{-1}N, Z^{-1}A\big], \big[Z^{-1}N, Z^{-1}A\big]\big] \subseteq \big[Z^{-1}ZU, Z^{-1}ZU\big],$$

and so from the hypothesis about U, for every $u, v \in [[Z^{-1}N, Z^{-1}A],[Z^{-1}N, Z^{-1}A]]_0$ we get $u \circ v \in Z^{-1}Z \cap Z^{-1}N$. But $Z^{-1}Z$ is a field and so $Z^{-1}N$ has some invertible element forcing $Z^{-1}N = Z^{-1}A$. Therefore $[Z^{-1}N, Z^{-1}A] = [Z^{-1}A, Z^{-1}A] \subseteq Z^{-1}(ZU)$ and again, by the hypothesis about U, it follows that $[[Z^{-1}A, Z^{-1}A],[Z^{-1}A, Z^{-1}A]]_1 \circ [[Z^{-1}A, Z^{-1}A],[Z^{-1}A, Z^{-1}A]]_1 = 0$. $\square$

Now we will study superalgebras B satisfying the condition $[[B,B],[B,B]]_1 \circ [[B,B],[B,B]]_1 = 0$. We notice that the $\mathbf{Z}_2$-grading is given by the automorphism σ of the algebra defined by $x_i^\sigma = (-1)^i x_i$, on homogeneous elements x_i. Then, we have the group of automorphisms $G = \{1, \sigma\}$ acting on A. Superidentities in B are then special types of G-identities, as defined in [10], that is identities involving elements and images of elements under the action of G, a group of automorphisms. Therefore we can apply results about G-identities, in particular the following one due to V.K. Kharchenko. We denote by R^G the set of elements fixed under every automorphism of G.

Proposition 3.8. *(See [10, Theorem 1].) Suppose G is a finite commutative group of automorphisms of a semiprime algebra R over a commutative domain K containing a primitive root of degree $n = |G|$, and suppose that R^G is prime. If R satisfies a nontrivial G-identity, then R is a PI-algebra.*

In our case, the group G has two elements, and here the conditions on K always hold because we can consider our algebras as $\mathbf{Z}$-algebras, and every automorphism is $\mathbf{Z}$-automorphism. Then, an algebra satisfying $[[B,B],[B,B]]_1 \circ [[B,B],[B,B]]_1 = 0$ is in fact a PI-algebra, if $B^G = B_0$ is a prime algebra. We consider next the case when B_0 is not prime.

Lemma 3.9. *If B is a prime superalgebra and satisfies the condition $[[B,B],[B,B]]_1 \circ [[B,B],[B,B]]_1 = 0$, and B_0 is not prime, then B_0 is commutative and B is a PI-algebra.*

Proof. By Lemma 1.5 in [16], B has an ideal I which is a Morita superalgebra. This means that we have a Morita context (R, S, M, N, μ, τ), where R and S are associative ϕ-algebras, M is an R–S-bimodule, N is an S–R-bimodule and $\mu: M \otimes_R N \to R$, $\tau: N \otimes_R M \to S$ are bimodule homomorphisms, such that I is the set of matrices

$$I = \begin{pmatrix} R & M \\ N & S \end{pmatrix},$$

with the known algebra structure given by the Morita context, and the following grading as superalgebra

$$I_0 = \begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix}, \qquad I_1 = \begin{pmatrix} 0 & M \\ N & 0 \end{pmatrix}.$$

Moreover, Lemma 1.5 in [16] and its proof say that R and S are prime algebras and orthogonal ideals of B_0, and also that I_0 intersects nontrivially every nonzero ideal of B_0.

We have that I satisfies $[[I,I],[I,I]]_1 \circ [[I,I],[I,I]]_1 = 0$, hence

$$\left[\left[\begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix}\right], \left[\begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}\right]\right] \\ \circ \left[\left[\begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix}\right], \left[\begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ N & 0 \end{pmatrix}\right]\right] = 0.$$

So $[R,R]RMNR[R,R] = 0$, but since R is prime either $[R,R] = 0$ or $RMNR = 0$. If $RMNR = 0$, then $MN = 0$ because R is prime, and so NM is a trivial ideal (that is, with $(NM)^2 = 0$) of S, which is also prime, therefore $NM = 0$ and I_1 is a trivial ideal of I. But then II_1I is also a trivial ideal of B, and because B is prime and $I \neq 0$ we have $I_1 = 0$, and as a consequence R and S are orthogonal ideals of A, a contradiction with the primeness of B. Thus $[R,R] = 0$. Similarly we can prove that $[S,S] = 0$, and so I_0 is commutative. But then for every $y, z \in I_0$ and $a, b \in B_0$ it follows, from (2) and (1), that

$$y[a,b]z = y[a,bz] - yb[a,z] = [ya,bz] - [y,bz]a - [yba,z] + [yb,z]a = 0,$$

and so $[([B_0,B_0],B_0 + [B_0,B_0])I_0]^2 \subseteq I_0[B_0,B_0]I_0 = 0$. But B_0 is semiprime because of Lemma 1.2, and $[B_0,B_0]I_0$ is an ideal of B_0, therefore $([B_0,B_0]B_0 + [B_0,B_0])I_0 = 0$. Since I_0 intersects nontrivially every nonzero ideal of B_0 we have $([B_0,B_0]B_0 + [B_0,B_0]) \cap I_0 = N \neq 0$ satisfies that $N^2 = 0$. So $[B_0,B_0]B_0 + [B_0,B_0] = 0$, and B_0 is commutative. □

Lemma 3.10. *If B is a prime superalgebra satisfying the condition $[[B,B],[B,B]]_1 \circ [[B,B],[B,B]]_1 = 0$, then B is a central order in $\Omega \oplus \Omega.v$ with $v^2 \in \Omega$ (where Ω is the field of fractions of Z).*

Proof. By Proposition 3.8 and Lemma 3.9, B is a PI-algebra. Then by Lemma 1.7 in [16] B is a central order in a simple superalgebra which is finite dimensional over Ω, that is, $C = Z^{-1}B$ is simple and finite dimensional over $\Omega = Z^{-1}Z$. Take $\bar{\Omega}$ an algebraic closure of Ω. Then $\bar{C} = Z^{-1}B \otimes \bar{\Omega}$ is a simple superalgebra, finite dimensional over $\bar{\Omega}$ and satisfies $[[\bar{C},\bar{C}],[\bar{C},\bar{C}]]_1 \circ [[\bar{C},\bar{C}],[\bar{C},\bar{C}]]_1 = 0$. But finite dimensional simple associative superalgebras were classified in [19] and over an algebraically closed field we obtain that $\bar{C} = \bar{\Omega} \oplus \bar{\Omega}.u$ with $u^2 = 1$, so $C = \Omega \oplus \Omega.v$ with $v^2 \in \Omega$. $\square$

So, from Lemmata 3.7 and 3.10 we can deduce:

Corollary 3.11. *If T' is dense in A, then A is either commutative, or a central order in a 4-dimensional simple superalgebra, or a central order in an 8-dimensional simple superalgebra.*

So, because of our assumption of $[T',[I,I]] \neq 0$ and of A being neither commutative, nor a central order in a 4-dimensional simple superalgebra, nor a central order in an 8-dimensional simple superalgebra, T' is not dense in A, and then by Lemma 3.6 $[t,u][u,s] = 0$ for every $u \in [T',[I,I]]_i$ such that $[u,u] = 0$, and for every $t,s \in T'$, homogeneous. We will prove that this cannot occur either, because we arrive to a contradiction.

Lemma 3.12. *If $[t,u][u,s] = 0$ for every $u \in [T',[I,I]]_i$ such that $[u,u] = 0$, and for every $t,s \in T'$, then $[T',[I,I]] = 0$.*

Proof. We prove the result in 4 steps.

1. $[X,X] = [X_1,X_1]$ with $X = [T',[I,I]]$. Indeed, we have $[t,u]^2 = 0$ for every $t \in T', u \in X_0$. Let $x,y \in X$, homogeneous, and $u \in X_i$ such that $u^2 = 0$. From our assumption $[u,x][u,y] = 0$, and expanding this gives $uxuy - (-1)^{\bar{y}\bar{u}}uxyu + (-1)^{\bar{x}\bar{u}+\bar{y}\bar{u}}xuyu = 0$. Right multiplication by u gives $uxuyu = 0$. Since $[y,l] \in X$ for every $l \in [I,I]_i$, we obtain that $[y,[l,u]] \in X$. So $uxu[y,[l,u]]u = 0$. Expanding this expression yields $uxuluyu = 0$. From Lemma 3.2 we deduce that $uXu = 0$. If $u,u' \in X$ are homogeneous elements with $u^2 = (u')^2 = 0$, we conclude that

$$\left(uu'\right)^2 = uu'uu' \in uXuu' = 0.$$

If $l \in [I,I]_i$ we have

$$0 = u\left[u',l\right]uu' = uu'luu',$$

so $uu'[I,I]uu' = 0$ and from Lemma 3.2,

$$uu' = 0 \quad \text{for every } u,u' \in X, \text{ homogeneous, with } u^2 = \left(u'\right)^2 = 0. \tag{$*$}$$

Now consider $x, y \in X_1$, $u, v \in X_0$. We have $[x, u]^2 = 0 = [y, v]^2$, and so $[x, u][y, v] = 0$, because of $(*)$. Since $[X_0, X_1]$ is additively generated by the elements $[x, u]$ with $x \in X_1$, $u \in X_0$, we have $v^2 = 0$ for every $v \in [X_0, X_1]$. From Lemma 3.3, $[X_0, X_1] = 0$, and $[X, X] = [X_0, X_0] + [X_1, X_1]$. Now consider $K = [X_0, X_0]$. We notice that K is a subalgebra of A^- and $[K, [I, I]] \subseteq K$. From our assumption for every $x, y, u, v \in X_0$ we have $[x, u]^2 = [y, v]^2 = 0$, and so, by $(*)$ we obtain that $[x, u][y, v] = 0$. Again, since $[X_0, X_0]$ is additively generated by the elements $[x, u]$ with $x, u \in X_0$, we deduce that for every $v \in K = K_0$, $v^2 = 0$. From Lemma 3.3, $K = 0$. Therefore $[X, X] = [X_1, X_1]$.

2. $[W_0, [I, I]_0] = 0$, and $B_1 = 0$ with $B = [W, [I, I]]$, $W = [S, S]$ and $S = [U, [I, I]]$. We have $B \subseteq [U, U]$. Since $W \subseteq T'$, $B = [W, [I, I]] \subseteq [T', [I, I]] = X$. So, from step 1, for every $b_0, b_0' \in B_0, [b_0, b_0'] = 0$. But $B \subseteq [U, U]$, and then $b_0 \circ b_0' \in Z$ for every $b_0 \in B_0$. Therefore $B_0^2 \subseteq Z$. Now Lemma 4 in [7] yields $[B_0, [I, I]_0] = 0$. Hence $[[W_0, [I, I]_0], [I, I]_0] = 0$, and by Theorem 1 in [7] $[W_0, [I, I]_0] = 0$. Also, since $B \subseteq [U, U]$, we have $(b_0 \circ b_1)^2 = 0$ for every $b_0 \in B_0, b_1 \in B_1$. But $B \subseteq X$, and so $[b_0, b_1] = 0$. Now applying that $B_0^2 \subseteq Z$, we obtain that $b_0^2 b_1^2 = 0$ for every $b_0 \in B_0, b_1 \in B_1$. Now we consider the ideal of A, $b_0^2 A$ and then $b_1^2(b_0^2 A) = 0$. Hence, from the primeness of A, either $b_0^2 = 0$ or $b_1^2 = 0$. From Lemma 3.3, if $b_0^2 = 0$, $B_0 = 0$ and then $[B_1, B_1] = 0$ and therefore $[b_1, b_1] = b_1^2 = 0$. So in any case $b_1^2 = 0$ for every $b_1 \in B_1$, and again from Lemma 3.3 $B_1 = 0$.

3. $W_1 = 0$ with $W = [S, S]$, $S = [U, [I, I]]$. Since $W \subseteq [U, U]$, $w_0 \circ w_0' \in Z$ for every $w_0, w_0' \in W_0$. But $W_0 \subseteq [I, I]_0$, and then by step 2 $[w_0, w_0'] = 0$. Therefore $W_0^2 \subseteq Z$. Moreover, since $W \subseteq [U, U]$, we have also $(w_0 \circ w_1)^2 = 0$, and because of step 2 also $[w_0, w_1] \in B_1 = 0$, for every $w_0 \in W_0$, $w_1 \in W_1$. So $w_0^2 w_1^2 = 0$. Hence $w_0^2 A$ is an ideal of A such that $w_0^2 A w_1^2 = 0$, and then either $w_0^2 = 0$ or $w_1^2 = 0$. Now, as in the proof of step 2, we can deduce that $W_1 = 0$.

4. $B = 0$ and then $[T', [I, I]] = 0$, a contradiction. Indeed, from step 2, $B_1 = 0$. And since $B = [W, [I, I]]$, from steps 2 and 3, $B = B_0 = [W_0, [I, I]_0] + [W_1, [I, I]_1] = 0$. But T' is the subring of T generated by $W = [S, S] = [[U, [I, I]], [U, [I, I]]]$. So if $[w, y] = 0$ for every $w \in W, y \in [I, I]$, homogeneous, then, since $[w'w, y] = w'[w, y] + (-1)^{\bar{w}\bar{y}}[w', y]w = 0$, we deduce that $[T', [I, I]] = 0$. □

So, in the last lemma we have arrived to a contradiction with our assumption in case (b): $[T', [I, I]] = 0$. This case is only possible if A is either commutative, or a central order in a 4-dimensional simple superalgebra, or a central order in an 8-dimensional simple superalgebra, as we have seen in Corollary 3.11. Hence, now we can deduce Theorem 3.4.

So the prime images of Lie ideals U of $[K, K]$ satisfying $[u \circ v, w] = 0$ for every $u, v \in [U, U], w \in U$ when the prime ideal P satisfies $P^* \neq P$ are like this.

Corollary 3.13. *Let A be semiprime, and let U be a Lie ideal of $[K, K]$ such that $[u \circ v, w] = 0$ for every $u, v \in [U, U]$, $w \in U$. If P is a prime ideal of A such that $P^* \neq P$ then either the projection of U in A/P is central, or A/P is commutative, or A/P is* *a central order in a 4-dimensional simple superalgebra or in an 8-dimensional simple superalgebra.*

4. Prime images of Lie ideals when $P^* = P$

Next we consider the cases when $P^* = P$, for P a prime ideal of A. So we have a superinvolution on A/P induced by the superinvolution on A. Recall that a superinvolution on A is said to be of the first kind if $Z_H = Z$, and it is said to be of the second kind if $Z_H \neq Z$.

Lemma 4.1. *Let A be a prime superalgebra with a superinvolution $*$ of the second kind. Let U be a Lie ideal of $[K, K]$ such that $u \circ v \in Z$ for every $u, v \in [U, U]_0$, $u \circ v = 0$ for every $u, v \in [U, U]_1$ and $(u \circ v)^2 = 0$ for every $u \in [U, U]_0, v \in [U, U]_1$. Then either $U \subseteq Z$ or A satisfies $S(3)$.*

Proof. If $*$ is of the second kind we know that $Z_H = \{x \in Z : x^* = x\} \neq Z$. We may localize A by $V = Z_H - \{0\}$ and replace U by $V^{-1}(Z_H U)$ and A by $V^{-1}A$. The hypothesis remains unchanged, so we keep for this superalgebra the same notation A, and now Z is a field. Let $0 \neq t \in Z_K$. Then $H = tK$ and $A = tK + K$. It follows that $[ZU, [A, A]] \subseteq ZU$, $u \circ v \in Z$ for every $u, v \in Z[U, U]_0$, $u \circ v = 0$ for every $u, v \in Z[U, U]_1$ and $(u \circ v)^2 = 0$ for every $u \in [U, U]_0, v \in [U, U]_1$. By Corollary 3.11, either $ZU \subseteq Z$, which implies that $U \subseteq Z$, or A satisfies $S(3)$. □

Lemma 4.2. *Let A be a prime superalgebra with a superinvolution $*$ of the first kind. Let U be a Lie ideal of $[K, K]$ such that $u \circ v \in Z$ for every $u, v \in [U, U]_0$, $u \circ v = 0$ for every $u, v \in [U, U]_1$ and $(u \circ v)^2 = 0$ for every $u \in [U, U]_0, v \in [U, U]_1$. Then either $U = 0$ or A satisfies $S(4)$.*

Proof. Since $*$ is of the first kind, $Z_K = K \cap Z = 0$. So, from Theorem 4.1 and Lemma 4.1 in [4], either $[K, K]$ is dense in A or A satisfies $S(2)$. If $u^2 = 0$ for every $u \in [U, U]_0$, applying Theorem 3.3 in [4] we obtain that $[U, U] = 0$. But then by Lemma 4.5 in [4] we obtain that either $U = 0$ or A satisfies $S(2)$. Suppose then that $u^2 \neq 0$ for some $u \in [U, U]_0$. By Theorem 3.4 in [4] we get that either $[U, U] \subseteq Z$ or A satisfies $S(4)$. But if $[U, U] \subseteq Z$ then $[[U, U], [U, U]] = 0$ and applying twice Lemma 4.5 in [4] yields $U = 0$. □

5. The proof of the main result

Combining the above results of Sections 3 and 4 we obtain

Theorem 5.1. *Let A be a semiprime superalgebra and U a Lie ideal of $[K, K]$ with $u \circ v \in Z$ for every $u, v \in [U, U]_0$, $u \circ v = 0$ for every $u, v \in [U, U]_1$ and $(u \circ v)^2 = 0$ for every*

$u \in [U,U]_0, v \in [U,U]_1$. Then A is the subdirect sum of two semiprime homomorphic images A', A'', such that the image of U in A' is central and A'' satisfies $S(4)$.

Proof. Let $T' = \{P : P$ is a prime ideal of A such that the image of U in A/P is central$\}$ and let $T'' = \{P : P$ is a prime ideal of A such that A/P satisfies $S(4)\}$.

If we consider P, a prime ideal of A such that $P^* \neq P$, we know from Corollary 3.13 that either A/P is a central order in a simple superalgebra at most 8-dimensional over its center, or $(U + P)/P$ is central. If we consider P a prime ideal of A such that $P^* = P$, it follows from Lemmata 4.1, 4.2 that either A/P is a central order in a simple superalgebra at most 16-dimensional over its center, or the image of U in A/P is central.

So every prime ideal of A belongs either T' or T''. Then A' is obtained by taking the quotient of A by the intersection of all the prime ideals in T', and A'' is obtained by taking the quotient of A by the intersection of all the prime ideals in T''. This proves the theorem. □

We finally arrive at the main theorem on the Lie structure of $[K,K]$.

Theorem 5.2. *Let A be a semiprime superalgebra with superinvolution $*$, and let U be a Lie ideal of $[K,K]$. Then either A is a subdirect sum of two semiprime homomorphic images A', A'', with the image of U in A' being central and A'' satisfying $S(4)$, or $U \supseteq [J \cap K, K] \neq 0$ for some ideal J of A.*

Proof. We consider $V = [U,U]$, which is also a Lie ideal of $[K,K]$. From Lemmata 2.1 and 2.3 we know that either V is dense in A, and so there exists a nonzero ideal J such that $J \subseteq \bar{V}$, or the conditions (i), (ii) and (iii) in Lemma 2.3 are satisfied by V.

In the second case we obtain by Theorem 5.1 the first part of the theorem for V. We claim that we obtain also this for U. Indeed, if $(V + P)/P$ is central in A/P for some P prime ideal of A, we notice that then the conditions (i), (ii) and (iii) in Lemma 2.3 are also satisfied by $(U + P)/P$ in A/P. So from Corollary 3.13 and Lemmata 4.1 and 4.2 we have that either $(U + P)/P$ is central or A/P verifies $S(4)$. So, like in Theorem 5.1, we have the first part of the theorem for U.

In the first case, that is, if $J \subseteq \bar{V}$, we will show that $[J \cap K, K] \subseteq U$.

The identity

$$[xy, z] = [x, yz] + (-1)^{\bar{x}\bar{y}+\bar{x}\bar{z}}[y, zx]$$

can be used to show that $[\bar{V}, A] = [V, A]$. Hence $[J \cap K, K] \subseteq [\bar{V}, A] = [V, A] = [V, H] + [V, K]$. But $[V, H] \subseteq H$, and $[V, K] \subseteq K$, so $[J \cap K, K] \subseteq [[U, U], K] \subseteq U$.

Finally, we prove that if $[J \cap K, K] = 0$ we would be in the situation of the first part of the theorem, that is, that either $(U + P)/P$ is central or A/P satisfies $S(4)$ for every prime ideal P of A. Suppose that $[J \cap K, K] = 0$, then $[u \circ v, w] = 0$ for every $u, v \in [J \cap K, J \cap K]$, $w \in J \cap K$ because $[uv, w] = u[v, w] + (-1)^{\bar{v}\bar{w}}[u, w]v = 0$. So by Lemmata 4.1, 4.2 and Corollary 3.13 it follows that for each prime image, A/P, of

A either its center contains $((J \cap K) + P)/P$, or A/P is a central order in a simple superalgebra at most 16-dimensional over its center.

We claim that if the image of $J \cap K$ in A/P for some prime ideal P of A is central, then either $(U + P)/P$ is central or A/P satisfies $S(4)$.

Let P be a prime ideal such that $P^* \neq P$. If $(J + P)/P \neq 0$, then since A/P is a prime superalgebra we get $((J \cap P^*) + P)/P \neq 0$, and so we have $((J \cap P^*) + P)/P \subseteq ((J \cap K) + P)/P \subseteq Z_0(A/P)$, that is, A/P is commutative. So A/P is commutative unless $J \subseteq P$. And if $J \subseteq P$, then by the proof of Lemma 2.1 we know that $A[u \circ v, w]A[u \circ v, w]A[u \circ v, w]A \subseteq P$ for every $u, v \in [V, V], w \in V$. Because P is a prime ideal we deduce that $[u \circ v, w] \in P$ for every $u, v \in [V, V], w \in V$. But now, by Lemma 2.3, $(V + P)/P$ satisfies the conditions (i), (ii) and (iii) in this lemma. So by Corollary 3.13 we obtain that either $(V + P)/P \subseteq Z_0(A/P)$, or A/P satisfies $S(4)$. Applying the same to $(U + P)/P$, which satisfies also that $[u \circ v, w] = 0$ for every u, $v \in ([U, U] + P)/P$, $w \in (U + P)/P$, we obtain that either $(U + P)/P \subseteq Z_0(A/P)$ or A/P satisfies $S(4)$.

And, if P is a prime ideal such that $P^* = P$, then A/P has a superinvolution induced by $*$ and $K(A/P) = (K+P)/P$. In this case, if $((J \cap K)+P)/P = 0$, we get $(J+P)/P \subseteq (H+P)/P = H(A/P)$, and therefore $(J+P)/P$ is supercommutative. But then for every $a, b \in A/P$ and $y, z \in (J + P)/P$ it follows that

$$\begin{aligned} yabz &= (-1)^{(\bar{b}+\bar{z})(\bar{y}+\bar{a})}(bz)(ya) = (-1)^{\bar{b}(\bar{y}+\bar{a})}b(ya)z \\ &= (-1)^{\bar{b}\bar{y}+\bar{b}\bar{a}+(\bar{a}+\bar{z})\bar{y}}b(az)y = (-1)^{\bar{b}\bar{a}}ybaz. \end{aligned}$$

Since A/P is prime, $ab = (-1)^{\bar{a}\bar{b}}ba$, that is, A/P is supercommutative. Now, from Lemma 1.9 in [16], A/P is a central order in a simple superalgebra at most 4-dimensional over its center. And, if $((J \cap K) + P)/P \neq 0$, then $Z_0(A/P) \neq 0$. So by localizing at $V = (Z_0(A/P) \cap H(A/P)) - \{0\}$ we can suppose that $Z_0(A/P)$ is a field, which we denote by Z. We will replace $V^{-1}(A/P)$ by A/P and $V^{-1}((J + P)/P)$ by $(J + P)/P$. Then, if $0 \neq t \in ((J \cap K) + P)/P$, we have $tH = K$ with $H = H(A/P)$, $K = K(A/P)$. So $K = tH \subseteq ((K \cap J) + P)/P \subseteq Z(A/P)$, and $H \subseteq t^{-1}Z(A/P) \subseteq Z(A/P)$. Therefore A/P is a field. □

Let $AnnT = \{x \in A : xT = Tx = 0\}$. Finally we have

Corollary 5.3. *Let A be a semiprime superalgebra with superinvolution $*$, and let U be a Lie ideal of $[K, K]$. Then either $[J \cap K, K] \subseteq U$ where J is a nonzero ideal of A or there exists a semiprime ideal T of A such that $A/AnnT$ satisfies $S(4)$ and $(U+T)/T \subseteq Z_0(A/T)$.*

Proof. By Theorem 4.4 we have that either the first conclusion holds, or, for each prime ideal P of A, either A/P satisfies $S(4)$ or $(U+P)/P \subseteq Z_0(A/P)$. Let T be the intersection of the prime ideals P of A such that $(U + P)/P \subseteq Z_0(A/P)$. Then $AnnT$ contains the intersection of those prime ideals P such that A/P satisfies $S(4)$. So we get that $A/AnnT$ satisfies $S(4)$, and this proves the result. $\square$

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