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"Strong" Euler class of a stably free module of odd rank

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ABSTRACT

Let R be a commutative Noetherian ring of dimension $n \geq 3$. Following a suggestion of Fasel, we establish a group homomorphism ϕ from van der Kallen's group $\mathrm{Um}_{n+1}(R)/E_{n+1}(R)$ to the n-th Euler class group $E^n(R)$ so that: (1) when n is even, ϕ coincides with the homomorphism given by Bhatwadekar and Sridharan through Euler classes; (2) when n is odd, ϕ is non-trivial in general for an important class of rings; (3) the sequence $Um_{n+1}(R)/E_{n+1}(R) \xrightarrow{\phi} E^n(R) \longrightarrow E_0^n(R) \to 0$ is exact, where $E_0^n(R)$ is the n-th weak Euler class group. (If $X = \mathrm{Spec}(R)$ is a smooth affine variety of dimension n over $\mathbb{R}$ so that the complex points of X are complete intersections and the canonical module K_R is trivial, then the sequence is proved to be exact on the left as well.) More generally, let R be a commutative Noetherian ring of dimension d and n be an integer such that $n \leq d \leq 2n-3$. We also indicate how to extend our arguments to this setup to obtain a group homomorphism from $\mathrm{Um}_{n+1}(R)/E_{n+1}(R)$ to $E^n(R)$.

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1. Introduction

This paper may be regarded as a supplementary note to the articles [4,5,11,15]. Let R be a Noetherian ring of (Krull) dimension n. Our purpose is to understand the relation between the following groups:

(1) $\underline{Um_{n+1}(R)/E_{n+1}(R)}$: the orbit space of unimodular rows of length $n+1$ under the natural action of elementary $(n+1)\times(n+1)$ matrices. This space is equipped with a group structure, introduced by van der Kallen [14].
(2) $\underline{E^n(R)}$: the n-th Euler class group of R defined by Bhatwadekar and Sridharan [2,4] which detects the obstruction for a projective R-module of rank n (with trivial determinant) to split off a free summand of rank one.
(3) $\underline{E_0^n(R)}$: the n-th weak Euler class group of R defined by Bhatwadekar and Sridharan [3,4] which is a certain quotient of $E^n(R)$ and is an analogue of the Chow group $CH_0(R)$ for regular R.

When n is *even* and $\mathbb{Q}\subset R$, Bhatwadekar and Sridharan established a wonderful relation between these groups in [4, 7.6] by showing that there is an exact sequence:

$$(*)\quad \mathrm{Um}_{n+1}(R)/E_{n+1}(R)\longrightarrow E^n(R)\longrightarrow E_0^n(R)\longrightarrow 0$$

Let $[a_1,\cdots,a_{n+1}]\in Um_{n+1}(R)/E_{n+1}(R)$. The first map in the above exact sequence is given by the Euler class of the stably free R-module P associated to the unimodular row $[a_1,\cdots,a_{n+1}]$ (we shall freely use the same notation for a unimodular row and the elementary orbit represented by it). We may loosely call it the Euler class of $[a_1,\cdots,a_{n+1}]$ and denote it by $e[a_1,\cdots,a_{n+1}]$. Thus, $e[a_1,\cdots,a_{n+1}]=0$ if and only if $P\simeq Q\oplus R$ for some R-module Q, and equivalently, $[a_1,\cdots,a_{n+1}]$ is the first row of a right-invertible $2\times(n+1)$ matrix. If $X=\mathrm{Spec}(R)$ is a smooth affine variety of dimension n (n even) over $\mathbb{R}$ such that $X(\mathbb{R})$, the smooth real manifold consisting of all real points of X, is orientable and every complex maximal ideal of R is a complete intersection, a remarkable result of Fasel [11, 5.9] essentially asserts that $e[a_1,\cdots,a_{n+1}]=0$ if and only if $[a_1,\cdots,a_{n+1}]$ is the first row of an $(n+1)\times(n+1)$ elementary matrix.

All the results mentioned above are heavily dependent on the fact that n is even. Whereas, if n is odd, the Euler class $e[a_1,\cdots,a_{n+1}]$ is always trivial (note that if n is odd, $[a_1,\cdots,a_{n+1}]$ is the first row of a right-invertible $2\times(n+1)$ matrix). It is therefore natural to ask whether for odd n one can define a morphism $\phi: Um_{n+1}(R)/E_{n+1}(R)\longrightarrow E^n(R)$ which is non-trivial for some important class of rings and further, we have an exact sequence as above. In this article, we answer this question affirmatively in 3.4, 3.6, 3.8, when R is a commutative Noetherian ring (unlike [4] we do not assume that $\mathbb{Q}\subset R$). We call the element $\phi[a_1,\cdots,a_{n+1}]\in E^n(R)$ the *"strong" Euler class* of $[a_1,\cdots,a_{n+1}]$. The definition of ϕ is for general n and it coincides with $e[a_1,\cdots,a_{n+1}]$ when n is even and $\mathbb{Q}\subset R$ (see 3.10).

If $X = \mathrm{Spec}(R)$ is a smooth affine variety of dimension n over $\mathbb{R}$ such that $X(\mathbb{R})$ is orientable and every complex maximal ideal of R is a complete intersection, we crucially use a structure theorem of Fasel [11, 4.9, 5.7] to show that in this setup the following sequence of groups is exact (Fasel did it for even n in [11, 5.9]):

$$0 \longrightarrow Um_{n+1}(R)/E_{n+1}(R) \xrightarrow{\phi} E^n(R) \longrightarrow E_0^n(R) \longrightarrow 0$$

As a consequence, it follows that if a unimodular row $[a_1, \cdots, a_{n+1}]$ has trivial strong Euler class (i.e., $\phi[a_1, \cdots, a_{n+1}] = 0$ in $E^n(R)$), then it is the first row of an $(n+1)\times(n+1)$ elementary matrix. This justifies the adjective "strong" as there are interesting cases to show that when n is odd, $\phi[a_1, \cdots, a_{n+1}]$ could be non-zero while $e[a_1, \cdots, a_{n+1}]$ is always zero. For example, the unimodular row associated to the tangent bundle of the real 3-sphere has trivial Euler class but its strong Euler class is non-zero.

This note grew out of a remark made by Jean Fasel to us (through personal communication) on our article [10]. In [10] we proved that if $R \hookrightarrow S$ is a subintegral extension of Noetherian rings of dimension n, then: (1) $E^n(R) \simeq E^n(S)$; (2) $Um_{n+1}(R)/E_{n+1}(R) \simeq Um_{n+1}(S)/E_{n+1}(S)$ (see also [12]); and (3) $E_0^n(R) \simeq E_0^n(S)$ (*provided n is even and* $\mathbb{Q} \subset R$). To prove (3) for general n, Fasel suggested that we should try to show that there is a similar exact sequence (as $(*)$ above) when n is odd (so that we can use the isomorphisms of (1) and (2) to conclude our result). The suggestion of the map ϕ is also due to Fasel. Thus, Question 3.29 (1) of [10] is now completely settled (see 3.12 below). Further, we do not need the assumption that $\mathbb{Q} \subset R$.

In an earlier version of this article we could only prove the main results (see 3.4, 3.6, 3.8) in the case when R is a smooth affine domain over an infinite field (see Section 4). The referee showed us how to generalize our results to Noetherian domains. In this current version these results are finally proved for an arbitrary Noetherian ring.

2. Some preliminaries

All the rings considered in this paper are commutative Noetherian. The modules are assumed to be finitely generated.

Unless specified otherwise, R will stand for a ring of dimension $n \geq 3$.

2.1. The Euler class group

We recall the definition of the n-th Euler class group $E^n(R)$.

Definition 2.1. Consider all ideals $\mathfrak{n}$ of R such that $\mathfrak{n}$ is an $\mathfrak{m}$-primary ideal for some maximal ideal $\mathfrak{m}$ of height n and $\mu(\mathfrak{n}/\mathfrak{n}^2) = n$ (where $\mu(-)$ stands for minimal number of generators). Let G be the free abelian group on the set of pairs $(\mathfrak{n}, \omega_{\mathfrak{n}})$, where $\mathfrak{n}$ is as above and $\omega_{\mathfrak{n}} : (R/\mathfrak{n})^n \twoheadrightarrow \mathfrak{n}/\mathfrak{n}^2$ is a surjection. Let J be an ideal of height n such that $\mu(J/J^2) = n$ and let $\omega_J : (R/J)^n \twoheadrightarrow J/J^2$ be a surjection. Let $J = \mathfrak{n}_1 \cap \cdots \cap \mathfrak{n}_r$ be the (irredundant) primary decomposition of J. Then $\mathfrak{n}_i$ is $\mathfrak{m}_i$-primary for some maximal ideal $\mathfrak{m}_i$ of height n for $1 \le i \le r$. Note that ω_J naturally induces surjections $\omega_{\mathfrak{n}_i} : (R/\mathfrak{n}_i)^n \twoheadrightarrow \mathfrak{n}_i/\mathfrak{n}_i^2$ for $1 \le i \le r$. The pair (J, ω_J) is associated to the element $(\mathfrak{n}_1, \omega_{\mathfrak{n}_1}) + \cdots + (\mathfrak{n}_r, \omega_{\mathfrak{n}_r})$ of G. By a slight abuse of notations, $(\mathfrak{n}_1, \omega_{\mathfrak{n}_1}) + \cdots + (\mathfrak{n}_r, \omega_{\mathfrak{n}_r})$ is denoted by (J, ω_J). Let H be the subgroup of G generated by the set of pairs (J, ω_J) where J is an ideal of height n and the surjection $\omega_J : (R/J)^n \twoheadrightarrow J/J^2$ has a surjective lift $\theta : R^n \twoheadrightarrow J$ (in other words, ω_J is induced by a set of n generators of J). The n-th Euler class group of R is defined as $E^n(R) := G/H$.

The above definition is, *a priori*, slightly different from the one given in [4]. We give a quick proof to show their equivalence.

Proposition 2.2. *The Euler class group defined above is equivalent to the Euler class group defined in* [4].

Proof. In [4] two pairs (J, ω_J) and $(J, \omega_J \sigma)$ are identified, where $\sigma \in SL_n(R/J)$. Therefore, we are required to prove that $(J, \omega_J) - (J, \omega_J \sigma) \in H$, where H is as in the above definition. Using the moving lemma [4, 2.14] it follows that there is an ideal J' and a surjection $\beta : R^n \twoheadrightarrow J \cap J'$ such that: (1) $J + J' = R$, (2) $\mathrm{ht}(J') \ge n$, and (3) $\beta \otimes R/J = \omega_J$.

If $J' = R$, then β is a surjective lift of ω_J and therefore $(J, \omega_J) \in H$. It is then enough to prove that $\omega_J \sigma$ also has a surjective lift.

If $\mathrm{ht}(J') = n$, then β induces a surjection, say, $\omega_{J'} : (R/J')^n \twoheadrightarrow J'/J'^2$ and we have

$$(J \cap J', \overline{\beta}) = (J, \omega_J) + (J', \omega_{J'})$$

in G, where $\overline{\beta} = \beta \otimes R/(J \cap J') : (R/J \cap J')^n \twoheadrightarrow (J \cap J')/(J \cap J')^2$. Using the Chinese Remainder Theorem we can find $\widetilde{\sigma} \in SL_n(R/J \cap J')$ such that $\widetilde{\sigma} = \sigma$ modulo J and $\widetilde{\sigma}$ is the identity matrix modulo J'. Therefore, we have

$$(J \cap J', \overline{\beta}\widetilde{\sigma}) = (J, \omega_J \sigma) + (J', \omega_{J'})$$

in G. Note that, as $(J \cap J', \overline{\beta}) \in H$, proving $(J \cap J', \overline{\beta}\widetilde{\sigma}) \in H$ will suffice.

Therefore, we are finally reduced to prove that if I is an ideal of height n and $\alpha : R^n \twoheadrightarrow I$ is a surjection, then the surjection $\widetilde{\alpha}\tau : (R/I)^n \twoheadrightarrow I/I^2$ has a surjective lift $\theta : R^n \twoheadrightarrow I$, where τ is any matrix in $SL_n(R/I)$ and $\widetilde{\alpha} = \alpha \otimes R/I$. Now note that $\dim(R/I) = 0$ and therefore $SL_n(R/I) = E_n(R/I)$. Since elementary matrices can be lifted off a surjection, it is easy to see that $\widetilde{\alpha}\tau$ has the desired lift. $\square$

Remark 2.3. For the convenience of the reader we remind that by applying the *moving lemma* [4, 2.14] and *addition principle* [4, 3.2], an element of $E^n(R)$ can be represented by a pair (J, ω_J), where $J \subset R$ is an ideal of height n and $\omega_J : (R/J)^n \twoheadrightarrow J/J^2$ is a surjection. We shall call these representatives of elements of $E^n(R)$ as *Euler cycles*.

We now quote one crucial result from [4].

Theorem 2.4. *(See [4, 4.2].) Let R be a ring of dimension n and J be an ideal of height n such that $\mu(J/J^2) = n$. Let $\omega_J : (R/J)^n \twoheadrightarrow J/J^2$ be a surjection. Suppose that the image of (J, ω_J) is zero in $E^n(R)$. Then ω_J has a surjective lift $\theta : R^n \twoheadrightarrow J$.*

Remark 2.5. Given a projective R-module P of rank n with trivial determinant and an isomorphism $\chi : R \xrightarrow{\sim} \wedge^n(P)$, Bhatwadekar and Sridharan associate the *Euler class* of (P, χ), denoted $e(P, \chi)$, in the group $E^n(R)$ and prove [4, 4.4] that $e(P, \chi) = 0$ in $E^n(R)$ if and only if $P \simeq Q \oplus R$ for some R-module Q. To define the Euler class of (P, χ) they need the additional assumption that $\mathbb{Q} \subset R$.

2.2. Segre class of ideals

We also need to recall the *Segre class* of an ideal as defined in [9]. Let $J \subset R$ be an ideal of height at least $n - 1$ with $\mu(J/J^2) = n$, and $\omega_J : (R/J)^n \twoheadrightarrow J/J^2$ be a surjection. With this data, the Segre class $\langle J, \omega_J \rangle$ has been defined in [9], which takes values in $E^n(R)$. Note that we are using a notation different from the one in [9].

Definition 2.6. Let R be a ring of dimension n. Let $J \subset R$ be an ideal of height $\geq n - 1$ such that J/J^2 is generated by n elements. Let $\omega_J : (R/J)^n \twoheadrightarrow J/J^2$ be a surjection induced by $J = (a_1, \cdots, a_n) + J^2$. Applying a "moving lemma" [9, 2.7] we can find $c_1, \cdots, c_n \in J$ such that $(c_1, \cdots, c_n) = J \cap J_1$ where $\mathrm{ht}(J_1) \geq n$, $J_1 + J = R$ and $c_i = a_i$ modulo J^2. If J_1 is a proper ideal then $J_1 = (c_1, \cdots, c_n) + J_1^2$ and it induces a surjection $\omega_{J_1} : (R/J_1)^n \twoheadrightarrow J_1/J_1^2$. The *Segre class* of the pair $\{J, \omega_J\}$ is defined as

$$\langle J, \omega_J \rangle := -(J_1, \omega_{J_1})$$

in $E^n(R)$. If $J_1 = R$ then $J = (c_1, \cdots, c_n)$, and in that case the Segre class $\langle J, \omega_J \rangle$ is defined to be trivial in $E^n(R)$.

It has been proved in [9, 3.2] that $\langle J, \omega_J \rangle$ is well-defined. The Segre classes behave very much like the Euler cycles. First of all, if in the above definition we have $\mathrm{ht}(J) = n$, then $\langle J, \omega_J \rangle$ is precisely (J, ω_J) in $E^n(R)$. Further, the following result has been proved in [9].

Theorem 2.7. *(See [9, 3.3].) Let R be a ring of dimension n. Let $J \subset R$ be an ideal of height $\geq n - 1$ and $\omega_J : (R/J)^n \twoheadrightarrow J/J^2$ be a surjection. Then, $\langle J, \omega_J \rangle = 0$ in $E^n(R)$ if and only if ω_J can be lifted to a surjection $\theta : R^n \twoheadrightarrow J$.*

The Segre class is additive in the following sense.

Theorem 2.8. *(See [9, 3.4].) Let R be a ring of dimension n. Let $J_1, J_2 \subset R$ be two comaximal ideals, each of height $\geq n-1$. Suppose that there are surjections $\omega_{J_1} : (R/J_1)^n \twoheadrightarrow J_1/J_1^2$ and $\omega_{J_2} : (R/J_2)^n \twoheadrightarrow J_2/J_2^2$. Then*

$$\langle J_1 \cap J_2, \omega_{J_1 \cap J_2} \rangle = \langle J_1, \omega_{J_1} \rangle + \langle J_2, \omega_{J_2} \rangle$$

in $E^n(R)$, where $\omega_{J_1 \cap J_2} : (R/(J_1 \cap J_2))^n \twoheadrightarrow J_1 \cap J_2/(J_1 \cap J_2)^2$ is the surjection naturally induced by ω_{J_1} and ω_{J_2}.

The following proposition will be crucially used in the next section. We prove this one and 2.10 from first principle.

Proposition 2.9. *Let R be a ring of dimension n. Let $J \subset R$ be an ideal of height $\geq n-1$ and $\omega_J : (R/J)^n \twoheadrightarrow J/J^2$ be a surjection. Let $\bar{\sigma} \in E_n(R/J)$. Then $\langle J, \omega_J \rangle = \langle J, \omega_J \bar{\sigma} \rangle$ in $E^n(R)$.*

Proof. Let $\sigma \in E_n(R)$ be a lift of $\bar{\sigma}$.

Let us first assume that $\langle J, \omega_J \rangle = 0$. Then by 2.7, there is a surjection $\theta : R^n \twoheadrightarrow J$ such that θ lifts ω_J. It is easy to see that $\theta\sigma : R^n \twoheadrightarrow J$ is a surjective lift of $\omega_J \bar{\sigma}$ and therefore $\langle J, \omega_J \bar{\sigma} \rangle = 0$ in $E^n(R)$.

Next we assume that $\langle J, \omega_J \rangle \neq 0$. Write $\overline{J} = J/J^2$. Now ω_J is given by a set of generators $(\bar{a}_1, \cdots, \bar{a}_n)$ of $\overline{J}$. By [9, 2.7] we can find $c_1, \cdots, c_n \in J$ such that $(c_1, \cdots, c_n) = J \cap J_1$ where $\mathrm{ht}(J_1) \geq n$, $J_1 + J = R$ and $c_i = a_i$ modulo J^2. As $\langle J, \omega_J \rangle \neq 0$, J_1 is a proper ideal of height n. Let us write $\widetilde{J}_1 = J_1/J_1^2$. Then $\widetilde{J}_1$ is generated by $(\widetilde{c}_1, \cdots, \widetilde{c}_n)$ and if $\omega_{J_1} : (R/J_1)^n \twoheadrightarrow J_1/J_1^2$ denotes the corresponding surjection, from the definition of the Segre class above we have $\langle J, \omega_J \rangle = -(J_1, \omega_{J_1})$ in $E^n(R)$. We need to prove that $\langle J, \omega_J \bar{\sigma} \rangle = -(J_1, \omega_{J_1})$, where $\bar{\sigma} \in E_n(R/J)$.

Let $(\bar{a}_1, \cdots, \bar{a}_n)\bar{\sigma} = (\bar{b}_1, \cdots, \bar{b}_n)$. Then $\overline{J} = (\bar{b}_1, \cdots, \bar{b}_n)$. Let us write $\widetilde{\sigma}$ for the image of σ in $E_n(R/J_1)$.

Let $(c_1, \cdots, c_n)\sigma = (d_1, \cdots, d_n)$. Then $J \cap J_1 = (d_1, \cdots, d_n)$. As $c_i - a_i \in J^2$, it follows that $d_i - b_i \in J^2$. Now note that $J_1 = (d_1, \cdots, d_n) + J_1^2$ and the corresponding Euler cycle is $(J_1, \omega_{J_1} \widetilde{\sigma})$. From the definition of the Segre class it follows that the Segre class of the pair $\{J, \omega_J \bar{\sigma}\}$ is $\langle J, \omega_J \bar{\sigma} \rangle = -(J_1, \omega_{J_1} \widetilde{\sigma})$.

As $\widetilde{\sigma} \in E_n(R/J_1)$, from the proof of 2.2 above it is easy to see that $(J_1, \omega_{J_1}) = (J_1, \omega_{J_1} \widetilde{\sigma})$. Therefore, $\langle J, \omega_J \rangle = \langle J, \omega_J \bar{\sigma} \rangle$ in $E^n(R)$. □

Notation. Let $J \subset R$ be an ideal of height $\geq n-1$ and $\omega_J : (R/J)^n \twoheadrightarrow J/J^2$ be a surjection. Let $\bar{u} \in (R/J)^*$ and σ be any diagonal matrix in $GL_n(R/J)$ with determinant $\bar{u}$. We shall denote the composite surjection (to be consistent with such notations for Euler cycles)

$$(R/J)^n \overset{\sigma}{\underset{\sim}{\to}} (R/J)^n \overset{\omega_J}{\twoheadrightarrow} J/J^2$$

by $\bar{u}\omega_J$. It is easy to check that the element $\langle J, \bar{u}\omega_J \rangle \in E^n(R)$ is independent of σ (one has to use the fact that a diagonal matrix with determinant 1 is elementary).

Proposition 2.10. *Let $J \subset R$ be an ideal of height $\geq n-1$ and $\omega_J : (R/J)^n \twoheadrightarrow J/J^2$ be a surjection. Let $\bar{u} \in (R/J)^*$. Then $\langle J, \bar{u}^2\omega_J \rangle = \langle J, \omega_J \rangle$ in $E^n(R)$.*

Proof. We first assume that $\langle J, \omega_J \rangle = 0$ in $E^n(R)$. We then have $J = (a_1, \cdots, a_n)$ and ω_J is induced by $(a_1, \cdots, a_n)$. From the above discussion it is clear that without loss of generality we may assume that $\bar{u}^2\omega_J$ is induced by $J = (a_1, \cdots, u^2 a_n) + J^2$. By [1, 3.4], $\bar{u}^2\omega_J$ can be lifted to a surjection $\theta : R^n \twoheadrightarrow J$ and therefore $\langle J, \bar{u}^2\omega_J \rangle = 0$.

Next let $\langle J, \omega_J \rangle \neq 0$. Let ω_J be induced by $J = (a_1, \cdots, a_n) + J^2$. Applying [9, 2.7] we can find $c_1, \cdots, c_n \in J$ such that $(c_1, \cdots, c_n) = J \cap J_1$ where $\mathrm{ht}(J_1) \geq n$, $J_1 + J = R$ and $c_i = a_i$ modulo J^2. In this case J_1 is a proper ideal. Then $J_1 = (c_1, \cdots, c_n) + J_1^2$ and it induces a surjection $\omega_{J_1} : (R/J_1)^n \twoheadrightarrow J_1/J_1^2$. By definition of the Segre class, $\langle J, \omega_J \rangle = -(J_1, \omega_{J_1})$ in $E^n(R)$.

Using the Chinese Remainder Theorem choose $v \in R$ such that $v \equiv u$ modulo J and $v \equiv 1$ modulo J_1. Then v is a unit modulo $J \cap J_1$ and we have, $J \cap J_1 = (c_1, \cdots, v^2 c_n) + (J \cap J_1)^2$. Again applying [1, 3.4] it follows that $J \cap J_1 = (d_1, \cdots, d_n)$ with $d_i - c_i \in (J \cap J_1)^2$ for $i = 1, \cdots, n-1$ and $d_n - v^2 c_n \in (J \cap J_1)^2$. Then, clearly $d_i - a_i \in J^2$ for $i = 1, \cdots, n-1$ and $d_n - u^2 a_n \in J^2$. Further note that $d_i - c_i \in J_1^2$ for $i = 1, \cdots, n$ and therefore $d_1, \cdots, d_n$ will still induce $\omega_{J_1} : (R/J_1)^n \twoheadrightarrow J_1/J_1^2$. From the definition of the Segre class we have $\langle J, \bar{u}^2\omega_J \rangle = -(J_1, \omega_{J_1})$ in $E^n(R)$ and we are done. $\square$

Remark 2.11. Let $\mathrm{ht}(J) = n$ and ω_1, ω_2 be two surjections from $(R/J)^n$ to J/J^2. It then follows from [4, 2.2, 5.0] that $(J, \omega_2) = (J, \bar{u}\omega_1)$ in $E^n(R)$ for some $\bar{u} \in (R/J)^*$.

2.3. Weak Euler class group

We recall the definition of the n-th weak Euler class group of R from [4].

Definition 2.12. Let R be a ring of dimension n. Let S be the set of ideals $\mathfrak{n} \subset R$ such that $\mu(\mathfrak{n}/\mathfrak{n}^2) = n$ and $\mathfrak{n}$ is $\mathfrak{m}$-primary for some maximal ideal $\mathfrak{m}$ of height n. Let G_0 be the free abelian group on the set S. Let J be an ideal of height n such that $\mu(J/J^2) = n$. Let $J = \mathfrak{n}_1 \cdots \mathfrak{n}_r$ be the (irredundant) primary decomposition of J, where $\mathfrak{n}_i$ is $\mathfrak{m}_i$-primary for some maximal ideal $\mathfrak{m}_i$ of height n $(1 \leq i \leq r)$. The element $\sum_{i=1}^{r} \mathfrak{n}_i$ in G is associated to J and is denoted as (J). Let H_0 be the subgroup of G_0 generated by all elements (J) where J is an ideal of height n such that $\mu(J) = n$. The n-th weak Euler class group of R is defined as $E_0^n(R) := G_0/H_0$.

Remark 2.13. There is an obvious canonical surjective group homomorphism ψ : $E^n(R) \twoheadrightarrow E^n_0(R)$ (under which an element (J, ω_J) is mapped to (J)).

The following proposition will be useful in the next section. It can be proved following the proof of [3, 3.3].

Proposition 2.14. *Let H' be the subgroup of $E^n(R)$ generated by all (J, ω_J) where J is an ideal of height n which is generated by n elements and $\omega_J : (R/J)^n \twoheadrightarrow J/J^2$ is a surjection. Then $Ker(\psi) = H'$.*

2.4. The map of Bhatwadekar and Sridharan through Euler class

Let R be a commutative Noetherian ring of dimension $n \geq 3$ and let $\mathbb{Q} \subset R$. Let us now briefly indicate how the group homomorphism from van der Kallen's group $Um_{n+1}(R)/E_{n+1}(R)$ to the Euler class group $E^n(R)$ is defined in [4].

Let $[a_0, a_1, \cdots, a_n] \in Um_{n+1}(R)$. Let $b_0, \cdots, b_n \in R^{n+1}$ be such that $\sum_{i=0}^n a_i bi = 1$. Let P be the kernel of the surjection $\theta : R^{n+1} \twoheadrightarrow R$ which sends e_i to a_i, $i = 0, \cdots, n$, where $\{e_0, \cdots, e_n\}$ is the standard basis of R^n. Write $p_i = a_i \sum_{i=0}^n b_i e_i - e_i$. Then $p_0, \cdots, p_n$ generate P and $\sum_{i=0}^n b_i p_i = 0$. It can be checked that

$$\chi := \sum_{i=0}^{n} (-1)^i a_i p_0 \wedge \cdots \wedge p_{i-1} \wedge p_{i+1} \wedge \cdots \wedge p_n$$

defines an isomorphism $\chi : R \simeq \wedge^n(P)$, and χ does not depend on the choice of b_i. In this setup, Bhatwadekar and Sridharan define a map from $Um_{n+1}(R)/E_{n+1}(R)$ to $E^n(R)$ which sends the orbit of $[a_0, a_1, \cdots, a_n]$ to $e(P, \chi)$. As their definition is through the Euler class of a projective module, they need the assumption that $\mathbb{Q} \subset R$.

We shall loosely call this map as $e : Um_{n+1}(R)/E_{n+1}(R) \to E^n(R)$ and denote the image of $[a_0, a_1, \cdots, a_n]$ as $e[a_0, a_1, \cdots, a_n]$. It is proved in [4] that e turns out to be a group homomorphism.

In [4], they also give an explicit description of this map. Let $[a_0, a_1, \cdots, a_n] \in Um_{n+1}(R)/E_{n+1}(R)$. Performing elementary transformations we may assume that the ideal $J = (a_1, \cdots, a_n)$ has height n. Note that there is a surjective map $\beta : P \twoheadrightarrow J$ defined by $\beta(p_0) = b_0 a_0 - 1$ and $\beta(p_i) = b_0 a_i$ for $i \geq 1$. Let $\omega_J : R^n \twoheadrightarrow J$ be the surjection induced by $a_1, \cdots, a_n$. They show that the pair (β, χ) induces the Euler class $e(P, \chi) = (J, \bar{b}_0^{n-1} \omega_J)$, where bar denotes reduction modulo J. Note that $\bar{a}_0 \bar{b}_0 \equiv 1$ modulo J.

If n is odd then we know that P has a unimodular element. We also observe that in this case, $e(P, \chi) = (J, \bar{b}_0^{n-1} \omega_J) = (J, \omega_J) = 0$ (using [4, 5.4] or 2.10 above).

If n is even, then using [4, 5.4] or 2.10 and $\bar{a}_0 \bar{b}_0 \equiv 1$ modulo J we have

$$e(P, \chi) = (J, \bar{b}_0^{n-1} \omega_J) = (J, \bar{b}_0 \omega_J) = (J, \bar{a}_0^2 \bar{b}_0 \omega_J) = (J, \bar{a}_0 \omega_J)$$

3. A group homomorphism and an exact sequence

Let R be a commutative Noetherian ring of dimension $n \geq 3$. Our aim in this section is to define a group homomorphism $\phi : \mathrm{Um}_{n+1}(R)/E_{n+1}(R) \to E^n(R)$ so that we have an exact sequence

$$Um_{n+1}(R)/E_{n+1}(R) \xrightarrow{\phi} E^n(R) \xrightarrow{\psi} E_0^n(R) \longrightarrow 0$$

We first state a couple of lemmas. The following lemma is standard. For a proof, see [13, 7.1.3].

Lemma 3.1. *Let B be a ring and $[a_1, \cdots, a_r, s] \in Um_{r+1}(B)$. Then there exists $\lambda_1, \cdots, \lambda_r \in B$ such that $ht(a_1 + \lambda_1 s, \cdots, a_r + \lambda_r s) \geq r$.*

In this section we shall frequently use the following easy derivative of the preceding lemma.

Lemma 3.2. *Let B be a ring and $[a_1, \cdots, a_r, s] \in Um_{r+1}(B)$. Then there exists $\lambda_1, \cdots, \lambda_{r-1} \in B$ such that $ht(a_1 + \lambda_1 s, \cdots, a_{r-1} + \lambda_{r-1} s, a_r) \geq r - 1$.*

Proof. Let 'bar' denote reduction modulo (a_r). Apply the above lemma to the unimodular row $[\bar{a}_1, \cdots, \bar{a}_{r-1}, \bar{s}] \in Um_r(\overline{B})$. $\square$

Let us now fix some notations. Let $J \subset R$ be an ideal of height $\geq n - 1$ such that $\mu(J/J^2) = n$. Let $J = (a_1, \cdots, a_n) + J^2$ and $\omega_J : (R/J)^n \twoheadrightarrow J/J^2$ be the corresponding surjection. We shall denote the Segre class of the pair $\{J, \omega_J\}$ in $E^n(R)$ by $\big\langle J, (a_1, \cdots, a_n)\big\rangle$. If J is n-generated, say, $J = (b_1, \cdots, b_n)$, then in our notation, $\big\langle (b_1, \cdots, b_n), (b_1, \cdots, b_n)\big\rangle = 0$. Also, we shall put $\big\langle R, (a_1, \cdots, a_n)\big\rangle = 0$ for any $a_1, \cdots, a_n$.

We now prove a proposition which will be very useful for subsequent discussions.

Proposition 3.3. *Let $J \subset R$ be an ideal of height $\geq n - 1$. Assume that $J = (a_1, \cdots, a_n)$ and u be a unit modulo J. Let $\lambda_1, \cdots, \lambda_{n-1} \in R$. Then in $E^n(R)$, we have*

$$\big\langle J, (a_1 + \lambda_1 a_n, \cdots, a_{n-1} + \lambda_{n-1} a_n, ua_n)\big\rangle = \big\langle J, (a_1, \cdots, ua_n)\big\rangle$$

Proof. Note that we have $J = (a_1, \cdots, a_n) = (a_1 + \lambda_1 a_n, \cdots, a_{n-1} + \lambda_{n-1} a_n, a_n)$ and

(1) $J = (a_1, \cdots, ua_n) + J^2$
(2) $J = (a_1 + \lambda_1 a_n, \cdots, a_{n-1} + \lambda_{n-1} a_n, ua_n) + J^2$

Let $\omega_1 : (R/J)^n \twoheadrightarrow J/J^2$ be the surjection corresponding to (1) above and ω_2 be the same for (2). We are required to prove that $\big\langle J, \omega_1\big\rangle = \big\langle J, \omega_2\big\rangle$ in $E^n(R)$.

Let $v \in R$ be such that $uv = 1$ modulo J. Writing $\overline{J} = J/J^2$, from (2) we have

$$\overline{J} = (\bar{a}_1 + \overline{\lambda}_1\bar{a}_n, \cdots, \bar{a}_{n-1} + \overline{\lambda}_{n-1}\bar{a}_n, \bar{u}\bar{a}_n)$$

Let $\bar{\sigma} \in E_n(R/J)$ correspond to the elementary transformation which adds $-(\bar{v}\overline{\lambda}_i)(\bar{u}\bar{a}_n)$ to the i-th generator of $\overline{J}$ above for $i = 1, \cdots, n-1$. As $\bar{u}\bar{v} = 1$, this transformation results in $\overline{J} = (\bar{a}_1, \cdots, \bar{a}_{n-1,}\bar{u}\bar{a}_n)$. In other words, $\langle J, \omega_1 \rangle = \langle J, \omega_2\bar{\sigma} \rangle$. As $\bar{\sigma} \in E_n(R/J)$, applying 2.9 we have the result. □

Definition of a map: We do not assume n to be odd or even. Further, unlike [4] we do not need the assumption that $\mathbb{Q} \subset R$. We first define a map $\varphi : Um_{n+1}(R) \to E^n(R)$. Let $[a_1, \cdots, a_{n+1}] \in Um_{n+1}(R)$. If either a_n or a_{n+1} is zero, we define $\varphi[a_1, \cdots, a_{n+1}] = 0$. So let us assume that $a_n \neq 0$, $a_{n+1} \neq 0$. Using 3.2 we can choose $\lambda_1, \cdots, \lambda_{n-1} \in R$ such that the ideal $J_1 = (a_1 + \lambda_1 a_n, \cdots, a_{n-1} + \lambda_{n-1}a_n, a_{n+1})$ has height $\geq n-1$. Similarly, choose $\mu_1, \cdots, \mu_{n-1} \in R$ so that the ideal $J_2 = (a_1 + \mu_1 a_{n+1}, \cdots, a_{n-1} + \mu_{n-1}a_{n+1}, a_n)$ has height $\geq n-1$. Note that $J_1 + (a_n) = R = J_2 + (a_{n+1})$ and we have

(1) $J_1 = (a_1 + \lambda_1 a_n, \cdots, a_{n-1} + \lambda_{n-1}a_n, a_n a_{n+1}) + J_1^2$
(2) $J_2 = (a_1 + \mu_1 a_{n+1}, \cdots, a_{n-1} + \mu_{n-1}a_{n+1}, a_n a_{n+1}) + J_2^2$

For $i = 1, \cdots, n-1$ write $b_i = a_i + \lambda_i a_n + \mu_i a_{n+1}$. With our notations fixed above, applying 3.3 we have,

(3) $\langle J_1, (a_1 + \lambda_1 a_n, \cdots, a_{n-1} + \lambda_{n-1}a_n, a_n a_{n+1}) \rangle = \langle J_1, (b_1, \cdots, b_{n-1}, a_n a_{n+1}) \rangle$
(4) $\langle J_2, (a_1 + \mu_1 a_{n+1}, \cdots, a_{n-1} + \mu_{n-1}a_{n+1}, a_n a_{n+1}) \rangle = \langle J_2, (b_1, \cdots, b_{n-1}, a_n a_{n+1}) \rangle$

Let 'bar' denote reduction modulo $(b_1, \cdots, b_{n-1})$. We then have

$$\overline{J}_1 = (\bar{a}_n\bar{a}_{n+1}) + \overline{J}_1^2, \;\; \overline{J}_2 = (\bar{a}_n\bar{a}_{n+1}) + \overline{J}_2^2 \text{ and } \overline{J}_1 \cap \overline{J}_2 = (\bar{a}_n\bar{a}_{n+1})$$

Therefore, it follows that in $E^n(R)$ we have

(5) $\langle J_1, (b_1, \cdots, b_{n-1}, a_n a_{n+1}) \rangle + \langle J_2, (b_1, \cdots, b_{n-1}, a_n a_{n+1}) \rangle = 0$

We now define $\varphi : Um_{n+1}(R) \to E^n(R)$, as follows:

$$\varphi[a_1, \cdots, a_{n+1}] = \langle J_2, (a_1 + \mu_1 a_{n+1}, \cdots, a_{n-1} + \mu_{n-1}a_{n+1}, a_n a_{n+1}) \rangle$$

Note that from (3), (4) and (5) we have

$$\begin{aligned} &\langle J_2, (a_1 + \mu_1 a_{n+1}, \cdots, a_{n-1} + \mu_{n-1}a_{n+1}, a_n a_{n+1}) \rangle \\ &\quad = -\langle J_1, (a_1 + \lambda_1 a_n, \cdots, a_{n-1} + \lambda_{n-1}a_n, a_n a_{n+1}) \rangle \end{aligned}$$

in $E^n(R)$. As one expression is independent of $\lambda_1, \cdots, \lambda_{n-1}$ and the other is of $\mu_1, \cdots, \mu_{n-1}$, it follows that φ is well-defined. We thus get a set-theoretic map $\varphi : Um_{n+1}(R) \to E^n(R)$.

Proposition 3.4. *The map $\varphi : Um_{n+1}(R) \to E^n(R)$, defined above, induces a set-theoretic map $\phi : Um_{n+1}(R)/E_{n+1}(R) \to E^n(R)$.*

Proof. It is enough to check that φ is invariant under the action of $E_n(R)$. We first record two immediate observations from the definition of φ which are going to be useful.

(a) If $[a_1, \cdots, a_n]$ is unimodular, then $\varphi[a_1, \cdots, a_{n+1}] = 0$.
(b) $\varphi[a_1, \cdots, a_{n+1}] = -\varphi[a_1, \cdots, a_{n-1}, a_{n+1}, a_n]$.

Our aim is to show that φ is invariant under the action of elementary generators of $E_n(R)$. This is indeed sufficient.

Claim 1. $\varphi[a_1, \cdots, a_{n+1}] = \varphi[a_1, \cdots, a_n, a_{n+1} + \tau a_n]$ *for any* $\tau \in R$.

Proof. If $a_n = 0$, the result is obvious. On the other hand if either a_{n+1} or $a_{n+1} + \tau a_n$ is zero, then $[a_1, \cdots, a_n]$ is unimodular. Therefore, we may assume that $a_n \neq 0$, $a_{n+1} \neq 0$ and $a_{n+1} + \tau a_n \neq 0$.

As before applying 3.2 we can choose $\mu_1, \cdots, \mu_{n-1} \in R$ so that the ideal

$$J = (a_1 + \mu_1(a_{n+1} + \tau a_n), \cdots, a_{n-1} + \mu_{n-1}(a_{n+1} + \tau a_n), a_n)$$

has height at least $n - 1$. Applying 3.3 we then have the following equalities

$$\begin{aligned}
&\varphi[a_1, \cdots, a_n, a_{n+1} + \tau a_n] \\
&\quad = \big\langle J, (a_1 + \mu_1(a_{n+1} + \tau a_n), \cdots, a_{n-1} + \mu_{n-1}(a_{n+1} + \tau a_n), a_n(a_{n+1} + \tau a_n))\big\rangle \\
&\quad = \big\langle J, (a_1 + \mu_1 a_{n+1}, \cdots, a_{n-1} + \mu_{n-1} a_{n+1}, a_n(a_{n+1} + \tau a_n))\big\rangle \\
&\quad = \big\langle J, (a_1 + \mu_1 a_{n+1}, \cdots, a_{n-1} + \mu_{n-1} a_{n+1}, a_n a_{n+1})\big\rangle \text{ (as } \tau a_n^2 \in J^2\text{)},
\end{aligned}$$

and this is independent of τ. This proves the claim. $\square$

From observation (b) above and Claim 1 it then follows that

(c) $\varphi[a_1, \cdots, a_{n+1}] = \varphi[a_1, \cdots, a_n + \tau a_{n+1}, a_{n+1}]$ for any $\tau \in R$.

It is easy to see that φ is also invariant under elementary operations on $(a_1, \cdots, a_{n-1})$. Further, φ does not change if we add a multiple of a_n to a_{n-1}. Therefore it does not change if we add multiples of a_n, a_{n+1} to $a_1, \cdots, a_{n-1}$. Finally, we will be done if we prove the following claim.

Claim 2. $\varphi[a_1, \cdots, a_{n+1}] = \varphi[a_1, \cdots, a_n, a_{n+1} + \tau a_1]$.

Proof. Arguing as before, we can assume that each of $a_1, a_n, a_{n+1}, a_{n+1} + \tau a_1$ is not zero. We choose $\mu_2, \cdots, \mu_n \in R$ so that the ideal

$$K = (a_1, a_2 + \mu_2(a_{n+1} + \tau a_1), \cdots, a_n + \mu_n(a_{n+1} + \tau a_1))$$

has height at least $n - 1$. We can also assume that $a_n + \mu_n(a_{n+1} + \tau a_1) \neq 0$ and $a_n + \mu_n a_{n+1} \neq 0$. We have the following equations in $E^n(R)$:

$$\begin{aligned}
&\varphi[a_1, \cdots, a_n, a_{n+1} + \tau a_1] \\
&\quad = \varphi[a_1, \cdots, a_{n-1}, a_n + \mu_n(a_{n+1} + \tau a_1), a_{n+1} + \tau a_1] \text{ (by (c) above)} \\
&\quad = \big\langle K, (a_1, a_2 + \mu_2(a_{n+1} + \tau a_1), \cdots, a_{n-1} + \mu_{n-1}(a_{n+1} + \tau a_1), \\
&\qquad (a_n + \mu_n(a_{n+1} + \tau a_1))(a_{n+1} + \tau a_1))\big\rangle \text{ (from the definition of } \varphi) \\
&\quad = \big\langle K, (a_1, a_2 + \mu_2 a_{n+1}, \cdots, a_{n-1} + \mu_{n-1} a_{n+1}, (a_n + \mu_n a_{n+1}) a_{n+1})\big\rangle \text{ (by 2.9)} \\
&\quad = \varphi[a_1, a_2, \cdots, a_n + \mu_n a_{n+1}, a_{n+1}] \text{ (from definition of } \varphi) \\
&\quad = \varphi[a_1, \cdots, a_{n+1}] \text{ (by (c) above)} \qquad \square
\end{aligned}$$

Remark 3.5. From now on we shall freely use the same symbol for a unimodular row $[a_1, \cdots, a_{n+1}]$ and the elementary orbit it represents.

Theorem 3.6. *The map $\phi : Um_{n+1}(R)/E_{n+1}(R) \to E^n(R)$ is a group homomorphism.*

Proof. By [15, 3.3], it is enough to prove that if $[x, a_1, \cdots, a_n]$ and $[y, a_1, \cdots, a_n]$ are unimodular with $x + y = 1$, then

$$\phi[x, a_1, \cdots, a_n] + \phi[y, a_1, \cdots, a_n] = \phi[xy, a_1, \cdots, a_n].$$

If any of x, y or xy is zero, then $[a_1, \cdots, a_n]$ is unimodular and therefore the above equation is trivial. So, assume them to be non-zero. Let bar denote reduction modulo xy. Adding suitable multiples of $\overline{a_n}$ to $\overline{a_1}, \cdots, \overline{a_{n-1}}$ we may assume that $\mathrm{ht}(\overline{a_1}, \cdots, \overline{a_{n-1}}) \geq n - 1$. It then follows that $\mathrm{ht}(xy, a_1, \cdots, a_{n-1}) \geq n - 1$, $\mathrm{ht}(x, a_1, \cdots, a_{n-1}) \geq n - 1$ and $\mathrm{ht}(y, a_1, \cdots, a_{n-1}) \geq n - 1$.

Let $J_1 = (x, a_1, \cdots, a_{n-1})$ and $J_2 = (y, a_1, \cdots, a_{n-1})$. Then $J_1 + J_2 = R$ and $J_1 \cap J_2 = (xy, a_1, \cdots, a_{n-1})$. By the definition of ϕ we have

(1) $\phi[x, a_1, \cdots, a_n] = \big\langle J_1, (x, a_1, \cdots, a_{n-2}, a_{n-1} a_n)\big\rangle$
(2) $\phi[y, a_1, \cdots, a_n] = \big\langle J_2, (y, a_1, \cdots, a_{n-2}, a_{n-1} a_n)\big\rangle$
(3) $\phi[xy, a_1, \cdots, a_n] = \big\langle J_1 \cap J_2, (xy, a_1, \cdots, a_{n-2}, a_{n-1} a_n)\big\rangle$

It is now easy to see that

$$\begin{aligned}\phi[xy, a_1, \cdots, a_n] &= \big\langle J_1 \cap J_2, (xy, a_1, \cdots, a_{n-2}, a_{n-1}a_n)\big\rangle \\ &= \big\langle J_1, (x, a_1, \cdots, a_{n-2}, a_{n-1}a_n)\big\rangle + \big\langle J_2, (y, a_1, \cdots, a_{n-2}, a_{n-1}a_n)\big\rangle \\ &= \phi[x, a_1, \cdots, a_n] + \phi[y, a_1, \cdots, a_n] \qquad \square\end{aligned}$$

Remark 3.7. The upshot of the above theorem and the proposition is that now we have a map from the orbit space $Um_{n+1}(R)/E_{n+1}(R)$ and therefore if we pick a representative $[a_1, \cdots, a_{n+1}]$ (a unimodular row), we are free to apply elementary transformations on $[a_1, \cdots, a_{n+1}]$. Therefore, we may use 3.1 and perform elementary transformations if necessary, to ensure that $\mathrm{ht}(a_1, \cdots, a_n) \geq n$. If $[a_1, \cdots, a_n]$ is unimodular, then $\phi[a_1, \cdots, a_{n+1}] = 0$. Otherwise, $J = (a_1, \cdots, a_n)$ is a proper ideal of height n. Let $\omega_J : R^n \twoheadrightarrow J$ be the surjection induced by $a_1, \cdots, a_n$. We then have $\phi[a_1, \cdots, a_{n+1}] = (J, \bar{a}_{n+1}\omega_J)$ in $E^n(R)$. In what follows, we shall use this description of ϕ.

Theorem 3.8. *The following sequence of groups is exact*

$$Um_{n+1}(R)/E_{n+1}(R) \xrightarrow{\phi} E^n(R) \xrightarrow{\psi} E_0^n(R) \longrightarrow 0$$

Proof. Clearly, the sequence is a complex and we only need to prove exactness at the middle. Note that if $(J, \omega_J) \in E^n(R)$ is such that $(J) = 0$ in $E_0^n(R)$, then by 2.14,

$$\sum_{i=1}^{p}(J_i, \omega_i) + (J, \omega_J) = \sum_{k=p+1}^{q}(J_k, \omega_k),$$

where $J_1, \cdots, J_q$ are ideals of height n such that each of them is generated by n elements. Take one of them, say, J_1. Let $J_1 = (\alpha_1, \cdots, \alpha_n)$. If $\omega_1' : R^n \twoheadrightarrow J_1$ denotes the surjection induced by $\alpha_1, \cdots, \alpha_n$, then by 2.11, $(J_1, \omega_1) = (J_1, \bar{u}\omega_1')$ for some u which is unit modulo J_1. Then (J_1, ω_1) is the image of the unimodular row $[\alpha_1, \cdots, \alpha_n, u]$ under the map ϕ. Since this is true for each of $1, \cdots, q$ and since ϕ is a morphism, it follows that (J, ω_J) is image of a unimodular row under ϕ. $\square$

Remark 3.9. Here are some easy consequences of 3.8. Let (I, ω_I) be an element of $E^n(R)$ such that its image in $E_0^n(R)$ is zero. From the exact sequence it is almost immediate that then $(I, \omega_I) = (J, \omega_J)$ in $E^n(R)$ for some ideal J of height n and $\omega_J : (R/J)^n \twoheadrightarrow J/J^2$ such that J is generated by n elements.

Next, let P be a projective R-module of rank n with trivial determinant, such that its weak Euler class $e(P)$ is trivial in $E_0^n(R)$. Then P maps onto an ideal J of height n which is generated by n elements. To see this, fix $\chi : R \xrightarrow{\sim} \wedge^n(P)$ and choose a surjection $\alpha : P \twoheadrightarrow I$ where I is an ideal of height n. Then (α, χ) will induce the Euler class $e(P, \chi) = (I, \omega_I)$ (see [4] for definitions of the Euler class and the weak Euler class of a projective module). As $e(P) = 0$, it follows that $(I) = 0$ in $E_0^n(R)$. By the above paragraph, $(I, \omega_I) = (J, \omega_J)$ in $E^n(R)$ for some ideal J of height n and $\omega_J : (R/J)^n \twoheadrightarrow J/J^2$ such that J is generated by n elements. Then $e(P, \chi) = (J, \omega_J)$ in $E^n(R)$ and by [4, 4.3] there is a surjection $\beta : P \twoheadrightarrow J$. These results are new for odd n. For even n there are stronger results (see [4, Section 6]).

In Section 2, we have recalled the definition of the map $e : Um_{n+1}(R)/E_{n+1}(R) \longrightarrow E^n(R)$ given by Bhatwadekar and Sridharan in [4] (with the assumption $\mathbb{Q} \subset R$) and remarked that for odd n this is the zero map. We prove below that if n is even and $\mathbb{Q} \subset R$, the maps ϕ and e are the same.

Proposition 3.10. *When n is even, the map ϕ coincides with the map given by e.*

Proof. Let n be even and let $[u, a_1, \cdots, a_n] \in Um_{n+1}(R)$. We may as before assume that $\mathrm{ht}(a_1, \cdots, a_n) = n$. Write $J = (a_1, \cdots, a_n)$ and let $\omega_J : R^n \twoheadrightarrow J$ be induced by $a_1, \cdots, a_n$. By [4, Page 214] (see also Section 2 above), $e[u, a_1, \cdots, a_n] = (J, \bar{u}\omega_J)$.

By elementary transformations $[u, a_1, \cdots, a_n]$ can be changed to $[a_1, -u, a_2, \cdots, a_n]$. Doing it successively we observe that as n is even, $[u, a_1, \cdots, a_n]$ and $[a_1, \cdots, a_n, u]$ are in the same elementary orbit. Therefore, $\phi[u, a_1, \cdots, a_n] = \phi[a_1, \cdots, a_n, u] := (J, \bar{u}\omega_J) = e[u, a_1, \cdots, a_n]$. This completes the proof. $\square$

Remark 3.11. Let R be a ring of dimension d and n be an integer such that $n \leq d \leq 2n-3$. Suitably modifying the definitions of the Euler class group and the weak Euler class group from Section 2, one can also define the n-th Euler class group $E^n(R)$ (see [5]) and the n-th weak Euler class group $E_0^n(R)$. Also note that, in this range, van der Kallen [15] showed that $Um_{n+1}(R)/E_{n+1}(R)$ carries a group structure, extending his results from [14]. Emulating the process of defining the morphism from van der Kallen's group to the Euler class group as done in this section, one can also define a group morphism $\phi : Um_{n+1}(R)/E_{n+1}(R) \longrightarrow E^n(R)$ (it has been done in [5] when R is a regular domain containing an infinite field and n is even). In order to carry this out one has to extend the definition of the Segre class and establish analogues of 2.7, 2.9, 2.10, 2.14, which is not difficult to do. The resulting sequence $Um_{n+1}(R)/E_{n+1}(R) \to E^n(R) \to E_0^n(R) \to 0$ becomes a complex. However, we are not sure of its exactness at the middle if $n < d$. On a related note, we would like to mention that if R contains an infinite field then van der Kallen [16] has recently proved that $\phi : Um_{n+1}(R)/E_{n+1}(R) \longrightarrow E^n(R)$ is a group homomorphism using a different method.

Applying the main theorem (3.8 above) we now settle Question 3.29 (1) of [10] affirmatively and improve [10, 3.25]. Let $R \hookrightarrow S$ be a subintegral ring extension with $\dim(R) = n = \dim(S)$. If $\mathbb{Q} \subset R$ and n is even, we proved [10, 3.25] that the weak Euler class groups $E_0(R)$ and $E_0(S)$ are isomorphic. We then asked [10, 3.29 (1)] whether the result is true if n is odd. We now remove the condition $\mathbb{Q} \subset R$ and prove the desired isomorphism for all n.

Theorem 3.12. *Let $R \hookrightarrow S$ be a subintegral ring extension with dim$(R) = n =$ dim(S). Then the weak Euler class groups $E_0^n(R)$ and $E_0^n(S)$ are isomorphic.*

Proof. By 3.8 we have the following commutative diagram with exact rows:

$$\begin{tikzcd} \frac{Um_{n+1}(R)}{E_{n+1}(R)} \arrow[r] \arrow[d, "\varphi"] & E^n(R) \arrow[r] \arrow[d, "\Phi"] & E_0^n(R) \arrow[r] \arrow[d, "\Phi_0"] & 0 \\ \frac{Um_{n+1}(S)}{E_{n+1}(S)} \arrow[r] & E^n(S) \arrow[r] & E_0^n(S) \arrow[r] & 0 \end{tikzcd}$$

By [10, 6.1], the map φ is an isomorphism. By [10, 3.11], the map Φ is an isomorphism. As a consequence, Φ_0 is an isomorphism. $\square$

4. The smooth case: homotopy invariance

Let R be a smooth affine domain of dimension $n \geq 3$ over an infinite field. The purpose of this short section is to show that for such a ring it is easy to define a map $\phi : \mathrm{Um}_{n+1}(R)/E_{n+1}(R) \to E^n(R)$ which coincides with the one defined in the previous section. Before getting into the explicit construction, a remark is in order.

Let $u \in Um_{n+1}(R)$ and let $u\sigma = v$, where $\sigma \in E_{n+1}(R)$. As σ is elementary, there is a $\tau \in E_{n+1}(R[T])$ such that $\tau(0) = \mathrm{id}$ and $\tau(1) = \sigma$. If we write $u\tau = w(T)$, then $w(T) \in Um_{n+1}(R[T])$ and one has $w(0) = u$, $w(1) = v$. Following Fasel's terminology, u and v are 'naively homotopic'. On the other hand, as R is a smooth affine domain over an infinite field, the Euler class group is (naive) homotopy invariant in the following sense (see [2, 4.6]. The main ingredient to prove [2, 4.6] is [2, 3.8] where the field k is assumed to be perfect. In [7, 4.12] it has been shown that we do not need to assume that k is perfect).

Theorem 4.1. *Let R be a smooth affine domain of dimension $n \geq 3$ over an infinite field k. Let $I \subset R[T]$ be an ideal of height n such that I/I^2 is generated by n elements and let $\omega : (R[T]/I)^n \twoheadrightarrow I/I^2$ be a surjection. Assume further that both $I(0)$ and $I(1)$ are proper ideals of R of height n. Let $\omega(0) : (R/I(0))^n \twoheadrightarrow I(0)/I(0)^2$ and $\omega(1) : (R/I(1))^n \twoheadrightarrow I(1)/I(1)^2$ be the surjections induced by ω. Then $(I(0), \omega(0)) = (I(1), \omega(1))$ in $E^n(R)$.*

We exploit the above homotopy invariance to define the desired map, as follows.

Let $[a_1, \cdots, a_n, a_{n+1}] \in Um_{n+1}(R)/E_{n+1}(R)$. We can perform elementary operations and assume that $\mathrm{ht}(a_1, \cdots, a_n) = n$. Let J_0 be the ideal $(a_1, \cdots, a_n)$ and $\omega_0 : R^n \twoheadrightarrow J_0$ be surjection induced by $(a_1, \cdots, a_n)$. As a_{n+1} is a unit modulo J_0, we have an Euler cycle $(J_0, \overline{a_{n+1}}\omega_0) \in E^n(R)$ (where "bar" means reduction modulo J_0). We associate to $[a_1, \cdots, a_n, a_{n+1}]$ the element $(J_0, \overline{a_{n+1}}\omega_0) \in E^n(R)$. We show below that this association is invariant under elementary action and thus defines a map $\phi : Um_{n+1}(R)/E_{n+1}(R) \to E^n(R)$.

Let $[a_1, \cdots, a_n, a_{n+1}], [b_1, \cdots, b_n, b_{n+1}] \in Um_{n+1}(R)$ be such that $\mathrm{ht}(a_1, \cdots, a_n) = n = \mathrm{ht}(b_1, \cdots, b_n)$ and there is $\sigma \in E_{n+1}(R)$ such that

$$[a_1, \cdots, a_n, a_{n+1}]\sigma = [b_1, \cdots, b_n, b_{n+1}]$$

Let $J_1 = (b_1, \cdots, b_n)$ and $\omega_1 : R^n \twoheadrightarrow J_1$ be the surjection induced by $(b_1, \cdots, b_n)$. It is enough to prove that $(J_0, \overline{a_{n+1}}\omega_0) = (J_1, \widetilde{b_{n+1}}\omega_1)$ in $E^n(R)$, where "tilde" means reduction modulo J_1.

As σ is elementary, there is a $\tau \in E_{n+1}(R[T])$ such that $\tau(0) = \mathrm{id}$ and $\tau(1) = \sigma$. Let

$$[a_1, \cdots, a_n, a_{n+1}]\tau = [f_1(T), \cdots, f_n(T), f_{n+1}(T)].$$

Then $[f_1(T), \cdots, f_n(T), f_{n+1}(T)] \in Um_{n+1}(R[T])$. As J_0 and J_1 are both of height n, it is easy to see that the ideal $(f_1(T), \cdots, f_n(T), T^2 - T) \subset R[T]$ has height $n+1$. Since $[f_1(T), \cdots, f_n(T), f_{n+1}(T)]$ is a unimodular row, we have

$$\mathrm{ht}(f_1(T), \cdots, f_n(T), (T^2 - T)f_{n+1}(T)) = n + 1.$$

Adding suitable multiples of $(T^2 - T)f_{n+1}(T)$ to $f_1(T), \cdots, f_n(T)$ if necessary, we can assume that $\mathrm{ht}(f_1(T), \cdots, f_n(T)) = n$. Write $I = (f_1(T), \cdots, f_n(T))$. Then we have $I(0) = J_0$ and $I(1) = J_1$. Let $\omega' : R[T]^n \twoheadrightarrow I$ denote the surjection induced by $f_1(T), \cdots, f_n(T)$. If we denote the surjection $\widehat{f_{n+1}(T)}\omega'$ by ω ("hat" means reduction modulo I), then we have an element $(I, \omega) \in E^n(R[T])$ such that $(I(0), \omega(0)) = (J_0, \overline{a_{n+1}}\omega_0)$ and $(I(1), \omega(1)) = (J_1, \widetilde{b_{n+1}}\omega_1)$ in $E^n(R)$. As R is a smooth affine domain over an infinite field, by 4.1 we obtain that $(J_0, \overline{a_{n+1}}\omega_0) = (J_1, \widetilde{b_{n+1}}\omega_1)$.

Remark 4.2. If R is not smooth, then 4.1 is no longer true. An example based on [2, 6.4] has been given in [8, 5.21]. Therefore, the above line of argument does not work if R is not smooth.

5. On real varieties

We shall be working with the following setup: $X = \mathrm{Spec}(R)$ is a smooth affine variety of dimension $n \geq 3$ over $\mathbb{R}$ such that

(1) The complex points of X are complete intersections.
(2) The set $X(\mathbb{R})$ of real points is non-empty and therefore it is a (smooth) real manifold equipped with Euclidean topology.
(3) $X(\mathbb{R})$ is orientable (in other words, the canonical module K_R is trivial).

We first recall some structure theorems. The reader may note that due to the assumption (1) above, the Euler class groups $E^n(R)$ and $E^n(\mathbb{R}(X))$ are isomorphic, where

$\mathbb{R}(X)$ is the ring obtained from R by inverting all the functions which do not have any real zeros. For the same reason, $E_0^n(R) \simeq E_0^n(\mathbb{R}(X))$.

Theorem 5.1. *Let $X = Spec(R)$ be as above. Let $C_1, \cdots, C_t$ be the compact connected components of $X(\mathbb{R})$ in the Euclidean topology. Then,*

(1) $E^n(R) \simeq \bigoplus_{i=1}^t \mathbb{Z}$ *(see [3]);*
(2) $E_0^n(R) \simeq \bigoplus_{i=1}^t \mathbb{Z}/2\mathbb{Z}$ *(see [3]);*
(3) $Um_{n+1}(R)/E_{n+1}(R) \simeq \bigoplus_{i=1}^t \mathbb{Z}$ *(see [11, 4.9, 5.7]).*

We now give a precise description of the map $\phi : Um_{n+1}(R)/E_{n+1}(R) \to E^n(R)$. For simplicity, we assume that $X(\mathbb{R})$ is compact and connected. The reader can easily figure out the general case from the structure theorem given above.

Proposition 5.2. *Let $X = Spec(R)$ be as above and assume that $X(\mathbb{R})$ is compact and connected. Then $Im(\phi) = 2\mathbb{Z} \subset E^n(R)$.*

Proof. Let $[a_1, \cdots, a_{n+1}] \in Um_{n+1}(R)/E_{n+1}(R)$. Applying Swan's Bertini theorem as stated in [3, 2.11] we may actually assume that the ideal $J = (a_1, \cdots, a_n)$ is a reduced ideal of height n. Then it follows from [3, 4.2] that J is supported on an even number of real maximal ideals, say, $J = \mathfrak{m}_1 \cdots \mathfrak{m}_{2r}$. Then $a_1, \cdots, a_n$ induces surjection $\omega_i : (R/\mathfrak{m}_i)^n \twoheadrightarrow \mathfrak{m}_i/\mathfrak{m}_i^2$ for each i. Let $\omega_J : R^n \twoheadrightarrow J$ denote the surjection induced by $a_1, \cdots, a_n$. Then, we have

$$0 = (J, \omega_J) = (\mathfrak{m}_1, \omega_1) + \cdots + (\mathfrak{m}_{2r}, \omega_{2r})$$

We know that for a real maximal ideal $\mathfrak{m}$ and a surjection $\omega_\mathfrak{m} : (R/\mathfrak{m})^n \twoheadrightarrow \mathfrak{m}/\mathfrak{m}^2$, the Euler cycle $(\mathfrak{m}, \omega_\mathfrak{m})$ is 1 or -1 in $E^n(R)$ (see [3]). It is also known that $(\mathfrak{m}, \omega_\mathfrak{m}) + (\mathfrak{m}, -\omega_\mathfrak{m}) = 0$.

Therefore, it follows that exactly half of the cycles on the right hand side of the above equation are 1 and the rest are -1. Now

$$(J, a_{n+1}\omega_J) = (\mathfrak{m}_1, a_{n+1}\omega_1) + \cdots + (\mathfrak{m}_{2r}, a_{n+1}\omega_{2r})$$

(with a slight abuse of notations, we are no longer putting "bar" on a_{n+1}). If the sign of a_{n+1} modulo $\mathfrak{m}_i$ is positive, then $(\mathfrak{m}_i, a_{n+1}\omega_i) = (\mathfrak{m}_i, \omega_i)$ and if a_{n+1} modulo $\mathfrak{m}_i$ is negative, then $(\mathfrak{m}_i, a_{n+1}\omega_i) = (\mathfrak{m}_i, -\omega_i)$. It is now easy to see that $\phi[a_1, \cdots, a_{n+1}] = (J, a_{n+1}\omega_J)$ is an even integer. Therefore, $Im(\phi) \subseteq 2\mathbb{Z} \subset E^n(R)$.

Conversely, we show that 2 has a preimage in $Um_{n+1}(R)/E_{n+1}(R)$. Take any two real maximal ideals $\mathfrak{m}$ and $\mathfrak{m}'$ and choose surjections $\omega_\mathfrak{m} : (R/\mathfrak{m})^n \twoheadrightarrow \mathfrak{m}/\mathfrak{m}^2$ and $\omega_{\mathfrak{m}'} : (R/\mathfrak{m}')^n \twoheadrightarrow \mathfrak{m}'/\mathfrak{m}'^2$ so that $(\mathfrak{m}, \omega_\mathfrak{m}) = (\mathfrak{m}', \omega_{\mathfrak{m}'}) = 1$. By [3, 4.8], the ideal $I = \mathfrak{m} \cap \mathfrak{m}'$ is a complete intersection. Let $I = (a_1, \cdots, a_n)$ and let $\omega_I : R^n \twoheadrightarrow I$

denote the corresponding surjection. On the other hand, $\omega_{\mathfrak{m}}$ and $\omega_{\mathfrak{m}'}$ together will induce a surjection $\widetilde{\omega_I} : (R/I)^n \twoheadrightarrow I/I^2$. There is a $\lambda \in R$ such that λ is a unit modulo I and $(I, \widetilde{\omega_I}) = (I, \bar{\lambda}\omega_I)$. As $\bar{\lambda} \in (R/I)^*$, it follows that $[a_1, \cdots, a_n, \lambda] \in Um_{n+1}(R)$. It is now easy to see that

$$\phi([a_1, \cdots, a_n, \lambda]) = (I, \widetilde{\omega_I}) = (\mathfrak{m}, \omega_{\mathfrak{m}}) + (\mathfrak{m}', \omega_{\mathfrak{m}'}) = 2. \qquad \square$$

The following corollary is now obvious. Fasel proved essentially the same result in [11, 5.9] for even n.

Corollary 5.3. *With $X = Spec(R)$ as in the above proposition, we have the following exact sequence of groups:*

$$0 \longrightarrow Um_{n+1}(R)/E_{n+1}(R) \longrightarrow E^n(R) \longrightarrow E_0(R) \longrightarrow 0$$

And, as a consequence we finally have:

Corollary 5.4. *Let $X = Spec(R)$ be as in the above proposition. Let $[a_1, \cdots, a_{n+1}] \in Um_{n+1}(R)/E_{n+1}(R)$. The strong Euler class $\phi([a_1, \cdots, a_{n+1}])$ is trivial in $E^n(R)$ if and only if $[a_1, \cdots, a_{n+1}]$ is the first row of an elementary matrix $\sigma \in E_{n+1}(R)$.*

Proof. Follows from the injectivity of ϕ. $\square$

Remark 5.5. Let $R = \dfrac{\mathbb{R}[X_1, \cdots, X_{n+1}]}{(X_1^2 + \cdots + X_{n+1}^2 - 1)}$ be the coordinate ring of the real n-sphere. Let $x_1, \cdots, x_{n+1}$ be the images of $X_1, \cdots, X_{n+1}$ in R. Then $R = \mathbb{R}[x_1, \cdots, x_{n+1}]$ with $x_1^2 + \cdots + x_{n+1}^2 = 1$. The stably free R-module associated to the unimodular row $(x_1, \cdots, x_{n+1}) \in Um_{n+1}(R)$ is the tangent bundle. Fasel [11] proves that the group $Um_{n+1}(R)/E_{n+1}(R)$ is generated by the orbit of $[x_1, \cdots, x_{n+1}]$. We can explicitly compute the strong Euler class of $[x_1, \cdots, x_{n+1}]$ as follows. Let $J = (x_1, \cdots, x_n)$. We observe that $J = \mathfrak{m}_1 \cap \mathfrak{m}_2$ where $\mathfrak{m}_1 = (x_1, \cdots, x_n, 1 + x_{n+1})$ and $\mathfrak{m}_2 = (x_1, \cdots, x_n, 1 - x_{n+1})$ are both real maximal ideals. The elements $x_1, \cdots, x_n$ will induce surjections $\omega_J : R^n \twoheadrightarrow J$, $\omega_1 : (R/\mathfrak{m}_1)^n \twoheadrightarrow \mathfrak{m}_1/\mathfrak{m}_1^2$, and $\omega_2 : (R/\mathfrak{m}_2)^n \twoheadrightarrow \mathfrak{m}_2/\mathfrak{m}_2^2$. Therefore we have $0 = (J, \omega_J) = (\mathfrak{m}_1, \omega_1) + (\mathfrak{m}_2, \omega_2)$. The strong Euler class of the tangent bundle is

$$\phi[x_1, \cdots, x_{n+1}] = (J, x_{n+1}\omega_J) = (\mathfrak{m}_1, x_{n+1}\omega_1) + (\mathfrak{m}_2, x_{n+1}\omega_2) = (\mathfrak{m}_1, -\omega_1) + (\mathfrak{m}_2, \omega_2)$$

(note that x_{n+1} is -1 modulo $\mathfrak{m}_1$ and 1 modulo $\mathfrak{m}_2$). Therefore, up to a sign, the strong Euler class is 2. Consequently, for any n, the unimodular row $[x_1, \cdots, x_{n+1}]$ cannot be completed to an elementarily matrix. Whereas, it is well known that for even n, it cannot be completed to an invertible matrix. For $S^3(\mathbb{R})$ and $S^7(\mathbb{R})$ the row can be completed to an invertible matrix but as we have seen, not to an elementarily matrix.

Remark 5.6. Let $\mathbf{R}$ be an Archimedean real closed field and $X = \mathrm{Spec}(A)$ be a smooth affine variety over $\mathbf{R}$ of dimension $n \geq 3$ such that: (1) the canonical module $\wedge^n(\Omega^*_{A/\mathbf{R}})$ is free, and (2) every maximal ideal m of R such that $R/m \simeq \overline{\mathbf{R}}$ is complete intersection (where $\overline{\mathbf{R}}$ is the algebraic closure of $\mathbf{R}$). Let $B = A \otimes_{\mathbf{R}} \mathbb{R}$ and write $Y = \mathrm{Spec}(B)$. Then, it follows from a (much more general) result of Bhatwadekar and Sane [6] that $E^n(A) \simeq E^n(B)$ and $E^n_0(A) \simeq E^n_0(B)$. It can be easily deduced using a direct limit argument and the exact sequence in 5.3 that $Um_{n+1}(A)/E_{n+1}(A)$ is also isomorphic to $Um_{n+1}(B)/E_{n+1}(B)$ and the sequence for the ring A is also exact at left as in 5.3. From this one can also deduce the structure of isomorphic classes of stably free A-modules of rank n for n even in exactly the same way it has been done in [11]. However, at this point we do not know how to extend these results to an arbitrary real closed field.

Remark 5.7. It will be interesting if one can prove 5.3 without using Fasel's structure theorem for $Um_{n+1}(R)/E_{n+1}(R)$ (which will then yield Fasel's structure theorem). In other words, we are asking whether the structure theorem for $Um_{n+1}(R)/E_{n+1}(R)$ can be deduced from the same for $E^n(R)$ and $E^n_0(R)$.

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