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Walks on graphs and their connections with tensor invariants and centralizer algebras

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ABSTRACT

The number of walks of k steps from the node 0 to the node λ on the McKay quiver determined by a finite group G and a G -module V is the multiplicity of the irreducible G -module G_λ in the tensor power $V^{\otimes k}$, and it is also the dimension of the irreducible module labeled by λ for the centralizer algebra $Z_k(G) = \text{End}_G(V^{\otimes k})$. This paper explores ways to effectively calculate that number using the character theory of G . We determine the corresponding Poincaré series. The special case $\lambda = 0$ gives the Poincaré series for the tensor invariants $T(V)^G = \bigoplus_{k=0}^{\infty} (V^{\otimes k})^G$ and a tensor analog of Molien's formula for polynomial invariants. When G is abelian, we show that the exponential generating function for the number of walks is a product of generalized hyperbolic functions. Many graphs (such as circulant graphs) can be viewed as McKay quivers, and the methods presented here provide efficient ways to compute the number of walks on them.

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1. Introduction

Let G be a finite group, and assume that the elements λ of $\Lambda(G)$ index the irreducible complex representations of G , hence also the conjugacy classes of G . Let G_λ denote the irreducible G -module indexed by λ , and let χ_λ be its character. The module G_0 denotes the trivial one-dimensional G -module with $\chi_0(g) = 1$ for all $g \in G$.

The *McKay quiver* $Q_V(G)$ (also known as the *representation graph*) associated to a finite-dimensional G -module V over the complex field $\mathbb{C}$ has nodes corresponding to the irreducible G -modules $\{G_\lambda \mid \lambda \in \Lambda(G)\}$. For $\nu \in \Lambda(G)$, there are $a_{\nu,\lambda}$ arrows from ν to λ in $Q_V(G)$ if

$$G_\nu \otimes V = \bigoplus_{\lambda \in \Lambda(G)} a_{\nu,\lambda} G_\lambda. \quad (1)$$

If $a_{\nu,\lambda} = a_{\lambda,\nu}$, then we draw $a_{\nu,\lambda}$ edges without arrows between ν and λ . The number of arrows $a_{\nu,\lambda}$ from ν to λ in $Q_V(G)$ is the multiplicity of G_λ as a summand of $G_\nu \otimes V$. Since each step on the graph is achieved by tensoring with V ,

$$\begin{aligned} m_k^\lambda &:= \text{number of walks of } k \text{ steps from } 0 \text{ to } \lambda \\ &= \text{multiplicity of } G_\lambda \text{ in } G_0 \otimes V^{\otimes k} \cong V^{\otimes k}. \end{aligned}$$

For a faithful G -module V , any irreducible G -module G_λ occurs in $V^{\otimes \ell}$ for some ℓ by Burnside's theorem (in fact, for some ℓ such that $0 \leq \ell \leq |G|$ by Brauer's strengthening of that result [8, Thm. 9.34]). This implies that there is a directed path with ℓ steps from G_0 to G_λ in $Q_V(G)$.

The *centralizer algebra*,

$$Z_k(G) = \{z \in \text{End}(V^{\otimes k}) \mid z(g.w) = g.z(w) \quad \forall g \in G, w \in V^{\otimes k}\},$$

plays a critical role in studying $V^{\otimes k}$, as it contains the projection maps onto the irreducible summands of $V^{\otimes k}$.

Let $\Lambda_k(G)$ denote the subset of $\Lambda(G)$ corresponding to the irreducible G -modules that occur in $V^{\otimes k}$ with multiplicity at least one. *Schur–Weyl duality* establishes essential connections between the representation theories of G and $Z_k(G)$:

- $Z_k(G)$ is a semisimple associative $\mathbb{C}$ -algebra whose irreducible modules $Z_k^\lambda(G)$ are in bijection with the elements λ of $\Lambda_k(G)$.
- $\dim Z_k^\lambda(G) = m_k^\lambda$, the number of walks of k steps from the trivial G -module G_0 to G_λ on $Q_V(G)$.
- If $d_\lambda = \dim G_\lambda$, then the tensor space $V^{\otimes k}$ has the following decompositions:

$$V^{\otimes k} \cong \bigoplus_{\lambda \in \Lambda_k(G)} m_k^\lambda G_\lambda \quad \text{as a } G\text{-module,}$$

$$\begin{aligned} &\cong \bigoplus_{\lambda \in \Lambda_k(\mathbf{G})} d_\lambda Z_k^\lambda(\mathbf{G}) && \text{as a } Z_k(\mathbf{G})\text{-module,} \\ &\cong \bigoplus_{\lambda \in \Lambda_k(\mathbf{G})} (\mathbf{G}_\lambda \otimes Z_k^\lambda(\mathbf{G})) && \text{as a } (\mathbf{G}, Z_k(\mathbf{G}))\text{-bimodule.} \end{aligned}$$

In Corollary 2.2 below, we contribute to this list of Schur–Weyl duality results three more important relations that are expressed in terms of the characters χ_V of V and χ_λ of $\mathbf{G}_\lambda$:

- $\dim Z_k^\lambda(\mathbf{G}) = |\mathbf{G}|^{-1} \sum_{g \in \mathbf{G}} \chi_V(g)^k \overline{\chi_\lambda(g)}$,
- $\dim (V^{\otimes k})^{\mathbf{G}} = |\mathbf{G}|^{-1} \sum_{g \in \mathbf{G}} \chi_V(g)^k$, where $(V^{\otimes k})^{\mathbf{G}} = \{w \in V^{\otimes k} \mid g.w = w \ \forall g \in \mathbf{G}\}$ (the space of $\mathbf{G}$ -invariants in $V^{\otimes k}$),
- $\dim Z_k(\mathbf{G}) = |\mathbf{G}|^{-1} \sum_{g \in \mathbf{G}} \chi_V(g)^{2k}$, when V is a self-dual $\mathbf{G}$ -module.

Hence, the dimension of the space of $\mathbf{G}$ -invariants in $V^{\otimes k}$ is the average of the character values $\chi_{V^{\otimes k}}(g) = \chi_V(g)^k$ as g ranges over the elements of $\mathbf{G}$ (compare [12, Sec. 2.2]), and when V and its dual V^* are isomorphic as $\mathbf{G}$ -modules, the dimension of the centralizer algebra $Z_k(\mathbf{G})$ is the average of the values $\chi_V(g)^{2k}$.

Therefore, Schur–Weyl duality tells us that the following numbers are the same, and our aim in this paper is to demonstrate various ways to compute these values effectively:

- (1) the number of walks of k steps from 0 to $\lambda \in \Lambda(\mathbf{G})$ on $Q_V(\mathbf{G})$,
 - (2) the $(0, \lambda)$ -entry $(A^k)_{0, \lambda}$ of A^k , where $A = (a_{\nu, \lambda})$ is the adjacency matrix of $Q_V(\mathbf{G})$,
 - (3) the multiplicity m_k^λ of the irreducible $\mathbf{G}$ -module $\mathbf{G}_\lambda$ in $V^{\otimes k}$,
 - (4) the dimension of the irreducible module $Z_k^\lambda(\mathbf{G})$ labeled by $\lambda \in \Lambda_k(\mathbf{G})$ for the centralizer algebra $Z_k(\mathbf{G}) = \text{End}_{\mathbf{G}}(V^{\otimes k})$,
 - (5) the number of paths from 0 at level 0 to λ at level k on the Bratteli diagram $\mathcal{B}_V(\mathbf{G})$ (see Section 3.2 for the definition).
- (*) Moreover, when $\lambda = 0$, these values are all equal to the dimension $\dim (V^{\otimes k})^{\mathbf{G}}$ of the space of $\mathbf{G}$ -invariants.

Many graphs can be viewed as McKay quivers $Q_V(\mathbf{G})$ for some choice of $\mathbf{G}$ and V , and the methods described here provide an efficient approach to computing walks on them. This is true, for example, of circulant graphs, as illustrated in Section 3.3. In [6], it is shown that the adjacency matrix A for a McKay quiver $Q_V(\mathbf{G})$ can always be used to construct an *avalanche-finite matrix*, which has many interesting properties related to chip-firing dynamics. Recent work of Grinberg, Huang, and Reiner [13] extends the results of [6] to a Hopf algebra setting that includes modular representations of groups.

We fix a set $\{c_\mu\}_{\mu \in \Lambda(G)}$ of conjugacy class representatives of G , and let $\mathcal{C}_\mu$ denote the conjugacy class of c_μ . Then c_0 is the identity element, and $|\mathcal{C}_0| = 1$. From (1), we note that

$$\sum_{\lambda \in \Lambda(G)} a_{\nu, \lambda} \chi_\lambda(c_\mu) = \chi_V(c_\mu) \chi_\nu(c_\mu),$$

which implies that the eigenvalues of the adjacency matrix A of $Q_V(G)$ are the character values $\chi_V(c_\mu)$ as μ ranges over the elements of $\Lambda(G)$, and the eigenvector corresponding to $\chi_V(c_\mu)$ is the column vector with entries $\chi_\nu(c_\mu)$ for $\nu \in \Lambda(G)$. The matrix of these eigenvectors is exactly the character table of G . (Compare [27, Sec. 1] which considers the matrix $dI - A$, where $d = \chi_V(c_0) = \dim V$.)

In this paper, we prove the following result giving a formula for the number of walks:

Theorem 1.1. (Theorem 2.1) *Assume V is a finite-dimensional module over $\mathbb{C}$ for the finite group G . The number of walks of k -steps from node ν to node λ on the McKay quiver $Q_V(G)$ is*

$$\begin{aligned} (A^k)_{\nu, \lambda} &= |G|^{-1} \sum_{\mu \in \Lambda(G)} |\mathcal{C}_\mu| \chi_\nu(c_\mu) \chi_V(c_\mu)^k \overline{\chi_\lambda(c_\mu)} = |G|^{-1} \sum_{g \in G} \chi_\nu(g) \chi_V(g)^k \overline{\chi_\lambda(g)} \\ &= |G|^{-1} \sum_{\mu \in \Lambda(G)} |\mathcal{C}_\mu| \chi_\nu(c_\mu) \chi_V(c_\mu)^k \chi_\lambda(c_\mu^{-1}) = |G|^{-1} \sum_{g \in G} \chi_\nu(g) \chi_V(g)^k \chi_\lambda(g^{-1}). \end{aligned}$$

As a consequence of Theorem 2.1, we determine that the Poincaré series for the number of walks from 0 on λ on $Q_V(G)$ (hence also for the multiplicities of the G -module G_λ in the tensor powers $V^{\otimes k}$ and for the dimensions of the centralizer algebra modules $\dim Z_k^\lambda(G)$) is given by

$$P^\lambda(t) = \sum_{k=0}^{\infty} (A^k)_{0, \lambda} t^k = |G|^{-1} \sum_{\mu \in \Lambda(G)} |\mathcal{C}_\mu| \frac{\overline{\chi_\lambda(c_\mu)}}{1 - \chi_V(c_\mu)t} = |G|^{-1} \sum_{g \in G} \frac{\overline{\chi_\lambda(g)}}{1 - \chi_V(g)t}. \quad (2)$$

Since the space $T(V)^G = \bigoplus_{k=0}^{\infty} (V^{\otimes k})^G$ of G -invariants in $T(V) = \bigoplus_{k=0}^{\infty} V^{\otimes k}$ is the sum of the trivial G -summands G_0 in $T(V)$, it follows that the Poincaré series for the tensor invariants is given by

$$P^0(t) = |G|^{-1} \sum_{\mu \in \Lambda(G)} |\mathcal{C}_\mu| \frac{1}{1 - \chi_V(c_\mu)t} = |G|^{-1} \sum_{g \in G} \frac{1}{1 - \chi_V(g)t}. \quad (3)$$

(An alternate derivation of (3) can be found in [9].) The results in (2) and (3) are tensor analogues of Molien's 1897 formulas for polynomials that have played a prominent role in combinatorics, coding theory, commutative algebra, and physics (see, for example, Stanley [24], Sloane [23], Murai [19], and Forger [11]). To see this comparison, let

$\{z_1, z_2, \dots, z_n\}$ be a basis for V , and let $S(V) = \mathbb{C}[z_1, z_2, \dots, z_n]$ be the symmetric algebra of polynomials in the z_i . Assume $S_k(V)$ is the space of polynomials in $S(V)$ of total degree k , and let $S_k^\lambda(V)$ be the sum of all the copies of G_λ in $S_k(V)$ (the λ -isotypic component). According to [18], the Poincaré series are then given by the following expressions, which are similar to the ones for tensors:

$$P_S^\lambda(t) = \sum_{k=0}^{\infty} \dim S_k^\lambda(V) t^k = |G|^{-1} \sum_{\mu \in \Lambda(G)} |\mathcal{C}_\mu| \frac{\overline{\chi_\lambda(\mathbf{c}_\mu)}}{\det_V(I - t\mathbf{c}_\mu)} = |G|^{-1} \sum_{g \in G} \frac{\overline{\chi_\lambda(g)}}{\det_V(I - tg)},$$

$$P_S^0(t) = |G|^{-1} \sum_{\mu \in \Lambda(G)} |\mathcal{C}_\mu| \frac{1}{\det_V(I - t\mathbf{c}_\mu)} = |G|^{-1} \sum_{g \in G} \frac{1}{\det_V(I - tg)}.$$

Theorem 2.1 of [4] shows that the Poincaré series $P^\lambda(t)$ can be expressed as a quotient of two determinants under the assumption that the module V is isomorphic to its dual G -module. But that assumption is unnecessary if the matrix A is replaced by its transpose in computing the determinant in the numerator, as in the statement of Theorem 1.3 below. As we illustrate next, a proof of Theorem 1.3 can be deduced from the following proposition, which we frame in the context of walks on arbitrary finite directed graphs, since it holds in that generality.

More specifically, let $\mathfrak{G}$ be a directed graph with finite vertex set Γ and adjacency matrix $\mathfrak{A} = (\mathfrak{a}_{\alpha,\gamma})_{\alpha,\gamma \in \Gamma}$. Then $\mathfrak{a}_{\alpha,\gamma}$ is the number of arrows from α to γ in $\mathfrak{G}$, and $(\mathfrak{A}^k)_{\alpha,\gamma}$ is the number of walks of k steps from α to γ on $\mathfrak{G}$. We consider the corresponding generating function for the number of walks from α to γ ,

$$w_{\alpha,\gamma}(t) = \sum_{k=0}^{\infty} (\mathfrak{A}^k)_{\alpha,\gamma} t^k,$$

where $\mathfrak{A}^0 = I$, the identity matrix.

Proposition 1.2. *Let δ_α be the $|\Gamma| \times 1$ matrix with 1 in row α and zeros elsewhere so that entry γ of δ_α is the Kronecker delta $\delta_{\alpha,\gamma}$, and assume M_α^γ is the matrix $I - t\mathfrak{A}^T$ with column γ replaced by δ_α (here T denotes the transpose). Then*

$$w_{\alpha,\gamma}(t) = \frac{\det(M_\alpha^\gamma)}{\det(I - t\mathfrak{A})}.$$

Proof. First a simple observation: $(\mathfrak{A}^{k+1})_{\alpha,\gamma} = \sum_{\beta \in \Gamma} (\mathfrak{A}^k)_{\alpha,\beta} \mathfrak{a}_{\beta,\gamma}$, for all $k \geq 0$. Then

$$\begin{aligned} w_{\alpha,\gamma}(t) &= \sum_{k=0}^{\infty} (\mathfrak{A}^k)_{\alpha,\gamma} t^k = \delta_{\alpha,\gamma} + t \sum_{k \geq 1} (\mathfrak{A}^k)_{\alpha,\gamma} t^{k-1} \\ &= \delta_{\alpha,\gamma} + t \sum_{k \geq 0} (\mathfrak{A}^{k+1})_{\alpha,\gamma} t^k = \delta_{\alpha,\gamma} + t \sum_{k \geq 0} \left(\sum_{\beta \in \Gamma} (\mathfrak{A}^k)_{\alpha,\beta} \mathfrak{a}_{\beta,\gamma} \right) t^k \end{aligned}$$

$$= \delta_{\alpha, \gamma} + t \sum_{\beta \in \Gamma} \mathbf{a}_{\beta, \gamma} \left(\sum_{k \geq 0} (\mathfrak{A}^k)_{\alpha, \beta} t^k \right) = \delta_{\alpha, \gamma} + t \sum_{\beta \in \Gamma} \mathbf{a}_{\beta, \gamma} \mathbf{w}_{\alpha, \beta}(t).$$

Letting $\mathbf{w}_\alpha$ be the $|\Gamma| \times 1$ matrix with $\mathbf{w}_{\alpha, \gamma}(t)$ in row γ , we see from the above calculation that the matrix equation $\mathbf{w}_\alpha^T (\mathbf{I} - t\mathfrak{A}) = \delta_\alpha^T$, or equivalently $(\mathbf{I} - t\mathfrak{A}^T) \mathbf{w}_\alpha = \delta_\alpha$, holds. It follows then from Cramer's rule that

$$\mathbf{w}_{\alpha, \gamma}(t) = \frac{\det(\mathbf{M}_\alpha^\gamma)}{\det(\mathbf{I} - t\mathfrak{A}^T)} = \frac{\det(\mathbf{M}_\alpha^\gamma)}{\det(\mathbf{I} - t\mathfrak{A})}. \quad \square$$

In applying this proposition to our group-walking setup for $\mathbf{G}$ and $\mathbf{V}$, we assume that the elements of $\Lambda(\mathbf{G})$ have been ordered in some fashion so that 0 is always the first element relative to the ordering. This gives the ordering on rows and columns of the adjacency matrix $\mathbf{A}$ used in the next result, which is an immediate consequence of Proposition 1.2.

Theorem 1.3. *Let $\mathbf{G}$ be a finite group with irreducible modules $\mathbf{G}_\lambda$, $\lambda \in \Lambda(\mathbf{G})$, over $\mathbb{C}$, and let $\mathbf{V}$ be a finite-dimensional $\mathbf{G}$ -module. Let $\mathbf{A} = (a_{\nu, \lambda})$ be the adjacency matrix of the McKay quiver $\mathbf{Q}_\mathbf{V}(\mathbf{G})$, and let $\mathbf{M}^\lambda$ be the matrix $\mathbf{I} - t\mathbf{A}^T$ with the column indexed by λ replaced by $\delta_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. Then the Poincaré series $\mathbf{P}^\lambda(t) = \sum_{k=0}^\infty (\mathbf{A}^k)_{0, \lambda} t^k$ has the following expression,*

$$\mathbf{P}^\lambda(t) = \frac{\det(\mathbf{M}^\lambda)}{\det(\mathbf{I} - t\mathbf{A})} = \frac{\det(\mathbf{M}^\lambda)}{\prod_{\mu \in \Lambda(\mathbf{G})} (1 - \chi_\mathbf{V}(\mathbf{c}_\mu)t)}.$$

In [17], John McKay described a remarkable correspondence between the finite subgroups $\mathbf{G}$ of the special unitary group SU_2 and the simply laced affine Dynkin diagrams. Almost a century earlier, Felix Klein had determined that a finite subgroup of SU_2 must be isomorphic to one of the following: (a) a cyclic group $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ of order n , (b) a binary dihedral group $\mathbf{D}_n$ of order $4n$, or (c) one of the 3 exceptional groups: the binary tetrahedral group $\mathbf{T}$ of order 24, the binary octahedral group $\mathbf{O}$ of order 48, or the binary icosahedral group $\mathbf{I}$ of order 120. McKay's observation was that the graph $\mathbf{Q}_\mathbf{V}(\mathbf{G})$ for $\mathbf{G} = \mathbb{Z}_n, \mathbf{D}_n, \mathbf{T}, \mathbf{O}, \mathbf{I}$ relative to its defining representation $\mathbf{V} = \mathbb{C}^2$ corresponds exactly to the affine Dynkin diagram $\hat{\mathbf{A}}_{n-1}, \hat{\mathbf{D}}_{n+2}, \hat{\mathbf{E}}_6, \hat{\mathbf{E}}_7, \hat{\mathbf{E}}_8$, respectively, where the node labeled by 0 corresponding to the trivial $\mathbf{G}$ -module is the affine node. The matrix $\mathbf{C} = 2\mathbf{I} - \mathbf{A}$, where $\mathbf{A}$ is adjacency matrix of $\mathbf{Q}_\mathbf{V}(\mathbf{G})$, is the associated affine Cartan matrix. In this case, the Poincaré series for the tensor invariants in Theorem 1.3 specializes to the following:

Theorem 1.4. [4, Thm. 3.1] *Let $\mathbf{G}$ be a finite subgroup of SU_2 and $\mathbf{V} = \mathbb{C}^2$. Then the Poincaré series for the $\mathbf{G}$ -invariants $\mathbf{T}(\mathbf{V})^\mathbf{G}$ in $\mathbf{T}(\mathbf{V}) = \bigoplus_{k=0}^\infty \mathbf{V}^{\otimes k}$ is*

$$P^0(t) = \frac{\det(I - t\mathring{A})}{\det(I - tA)} = \frac{\det(I - t\mathring{A})}{\prod_{\mu \in \Lambda(G)} (1 - \chi_V(c_\mu)t)},$$

where A is the adjacency matrix of $Q_V(G)$ (i.e. the affine Dynkin diagram corresponding to G), and $\mathring{A}$ is the adjacency matrix of the finite Dynkin diagram obtained by removing the affine node.

As shown in [4, Sec. 3], the eigenvalues of $\mathring{A}$ and A are related to the exponents of the finite and affine root systems respectively, and the determinants in this formula can be expressed as Chebyshev polynomials of the second kind. Results in a similar vein for the doubly laced root systems can be found in [3].

We illustrate the usefulness of the results in our paper by computing several examples, as described below for various choices of G and V . We obtain new expressions for the dimensions of the tensor invariants, the multiplicities of irreducible summands, and the dimensions of centralizer algebras and their irreducible modules and the related exponential generating functions and Poincaré series using the methods presented here.

When G is abelian, the conjugacy classes consist of a single element of G , so we will always identify $\Lambda(G)$ with G when G is abelian. Here is a brief summary of the examples studied in this work and the results shown for them.

1. $G = \mathbb{Z}_r$ (a cyclic group of order r) and $V = G_1 \oplus G_{r-1}$:

In Section 3.1, we obtain a formula for the number of walks of k steps on a circular graph with r nodes.

2. $G = \mathbb{Z}_{13}$ and $V = \bigoplus_j G_j$, where $j = 1, 3, 4, 9, 10, 12$:

As shown in Section 3.3, this example leads to an expression for the number of walks on the Paley graph $\mathcal{P}_{13}$ of order 13. Paley graphs arise in studying quadratic residues in finite fields, and the key fact germane to the results here is that Paley graphs are circulant graphs (their adjacency matrices are circulant matrices). The same method used for $\mathcal{P}_{13}$ can be applied to compute the number of walks on any circulant graph. In Section 3.4, we adopt a different approach and determine closed-form formulas for the number of walks of k steps from 0 to any node on a Paley (di)graph $\mathcal{P}_p$ of order p for an arbitrary odd prime p using Theorem 2.1 and number-theoretic properties of Gauss sums. When $p \equiv 1 \pmod{4}$, $\mathcal{P}_p$ is an undirected graph, and when $p \equiv 3 \pmod{4}$, $\mathcal{P}_p$ is a directed graph (digraph).

3. $G = S_n$, the symmetric group on n letters, and V is its n -dimensional permutation module:

Our results here lead to a proof of the relation

$$\dim Z_k(S_n) = (n!)^{-1} \sum_{\sigma \in S_n} F(\sigma)^{2k} = \sum_{\ell=0}^n \left\{ \begin{matrix} 2k \\ \ell \end{matrix} \right\} \quad (4)$$

between the number of fixed points $F(\sigma)$ of permutations σ , and the Stirling numbers $\left\{ \begin{smallmatrix} 2k \\ \ell \end{smallmatrix} \right\}$ of the second kind, which count the number of ways to partition a set of $2k$ objects into ℓ nonempty disjoint parts. (Note that $\left\{ \begin{smallmatrix} 0 \\ \ell \end{smallmatrix} \right\} = 0$ unless $\ell = 0$, in which case it is 1.) The first relation in (4) was shown by Farina and Halverson in [10] under the additional assumption that $n \geq 2k$ using the orthogonality of the characters of the partition algebra $P_k(n)$, which is isomorphic to the centralizer algebra $Z_k(S_n) = \text{End}_{S_n}(V^{\otimes k})$ when $n \geq 2k$. By instead using the character theory of the symmetric group S_n , we are able to deduce the first equality in (4) readily from our results without imposing restrictions on n and k . The second equality follows from our expressions in (19) as explained in the next paragraph. It can also be deduced from Polya's Theorem [16, I, 2, Ex. 9] and is related to moments of random permutations.

The partitions λ of n index the irreducible S_n -modules. Applying Corollary 2.2(i) and [5, Thm. 5.5(a)], we determine in (19) that

$$\begin{aligned} \dim Z_k^\lambda(S_n) &= (n!)^{-1} \sum_{\sigma \in S_n} F(\sigma)^k \overline{\chi_\lambda(\sigma)} = (n!)^{-1} \sum_{\sigma \in S_n} F(\sigma)^k \chi_\lambda(\sigma) \\ &= \sum_{\ell=0}^n \left\{ \begin{smallmatrix} k \\ \ell \end{smallmatrix} \right\} K_{\lambda, (n-\ell, 1^\ell)}, \end{aligned} \quad (5)$$

where $K_{\lambda, (n-\ell, 1^\ell)}$ is the *Kostka number*, and $(n-\ell, 1^\ell)$ is the partition of n with one part of size $n-\ell$ and ℓ parts of size 1. Equation (4) is a special case of (5), since $\dim Z_k(S_n) = \dim Z_{2k}^0(S_n)$, and the relevant Kostka numbers are all 1 in this case. It follows from (5) with $\lambda = 0$ that the dimension of the S_n -invariants in $V^{\otimes k}$ is given by

$$\dim (V^{\otimes k})^{S_n} = (n!)^{-1} \sum_{\sigma \in S_n} F(\sigma)^k = \sum_{\ell=0}^n \left\{ \begin{smallmatrix} k \\ \ell \end{smallmatrix} \right\},$$

and the Poincaré series for the tensor invariants is given by

$$P^0(t) = \sum_{k=0}^{\infty} \dim (V^{\otimes k})^{S_n} t^k = (n!)^{-1} \sum_{\sigma \in S_n} \frac{1}{1 - F(\sigma)t}.$$

It would be nice to have a bijective combinatorial proof of the identity in (5).

4. $G = \mathbb{Z}_r \wr S_n$ (the wreath product) and V is its n -dimensional module over $\mathbb{C}$ on which G acts by $n \times n$ monomial matrices with entries of the form ω^j for $j = 0, 1, \dots, r-1$, where ω is a primitive r th root of unity for $r \geq 2$:

In Theorem 4.1(a) below, we prove that

$$\dim (V^{\otimes k})^G = \frac{1}{r^n n!} \sum_{m=1}^n r^m F_n(m)^k \left(\sum_{\ell_1, \ell_2, \dots, \ell_m} \left(\begin{smallmatrix} k \\ \ell_1, \ell_2, \dots, \ell_m \end{smallmatrix} \right) \right),$$

where the inner sum of multinomial coefficients is over all $0 \leq \ell_1, \ell_2, \dots, \ell_m \leq k$ such that $\ell_1 + \ell_2 + \dots + \ell_m = k$ and $\ell_1 \equiv \ell_2 \equiv \dots \equiv \ell_m \equiv 0 \pmod{r}$, and

$F_n(m) = \frac{n!}{m!} \sum_{j=0}^{n-m} \frac{(-1)^j}{j!}$ is the number of permutations in S_n with exactly m fixed points. Part (b) of Theorem 4.1 gives an expression for the exponential generating function $g^0(t) = \sum_{k=0}^{\infty} \dim(V^{\otimes k})^G \frac{t^k}{k!}$ in terms of a generalized hyperbolic function.

Equation (26) gives a second expression for the dimension of the invariants using the fact that the irreducible modules for $G = \mathbb{Z}_r \wr S_n$ are indexed by r -tuples $\underline{\alpha} = (\alpha^{(1)}, \alpha^{(2)}, \dots, \alpha^{(r)})$ of partitions $\alpha^{(i)}$ with $\sum_{i=1}^r |\alpha^{(i)}| = n$:

$$\dim(V^{\otimes k})^G = \sum_{\underline{\alpha} \in \Lambda(G)} \frac{\left(\sum_{i=1}^r F(\alpha^{(i)}) \omega^{i-1} \right)^k}{r^{p(\underline{\alpha})} \prod_{j=1}^n j^{p_j(\underline{\alpha})} \left(\prod_{i=1}^r p_j(\alpha^{(i)})! \right)} \quad \text{for } G = \mathbb{Z}_r \wr S_n.$$

In this formula $p_j(\alpha^{(i)})$ is the number of parts of $\alpha^{(i)}$ of size j ; $p_j(\underline{\alpha}) = \sum_{i=1}^r p_j(\alpha^{(i)})$; $p(\underline{\alpha}) = \sum_{j=1}^n p_j(\underline{\alpha})$; and $F(\alpha^{(i)}) = p_1(\alpha^{(i)})$, the number of parts of $\alpha^{(i)}$ of size 1, (the number of fixed points of a permutation with cycle type $\alpha^{(i)}$). It is desirable to have a direct proof of the equivalence of these two formulas for $\dim(V^{\otimes k})^G$.

When $r = 2$, the group $G = \mathbb{Z}_2 \wr S_n$ is the Weyl group corresponding to the root systems B_n and C_n . In this case, the exponential generating function $g^0(t)$ for the G -invariants in Theorem 4.1 is a linear combination of powers of hyperbolic cosines. This result can be found in part (b) of Theorem 4.2. Part (a) of that result presents formulas for the dimensions of the tensor invariants from the perspective of equation (26).

5. G is the general linear group $GL_2(\mathbb{F}_q)$ of invertible 2×2 matrices over a finite field $\mathbb{F}_q$ of q elements, where q is odd, or G is the special linear subgroup $SL_2(\mathbb{F}_q)$ of matrices of determinant 1. The G -module V is the $(q+1)$ -dimensional module over $\mathbb{C}$ obtained by inducing the trivial module for the Borel subgroup B of upper-triangular matrices in G :

The module V decomposes as a G -module, $V = G_0 \oplus V_q$, where G_0 is the trivial G -module and V_q is the q -dimensional irreducible Steinberg module. In Theorems 5.1 and 5.2, we derive formulas for the dimension of the spaces $(V^{\otimes k})^G$ and $(V_q^{\otimes k})^G$ of G -invariants and determine the Poincaré series for the tensor invariants $T(V)^G$ and $T(V_q)^G$.

6. G is an arbitrary finite abelian group, say $G = \mathbb{Z}_{r_1} \times \dots \times \mathbb{Z}_{r_n}$, and $V = G_{\varepsilon_1} \oplus G_{\varepsilon_2} \oplus \dots \oplus G_{\varepsilon_n}$, where ε_j is the element of G with 1 as its j th component and 0 as its other components:

In Section 6, we show that the exponential generating function for the number of walks on the McKay quiver (equivalently, for the multiplicities of the irreducible G -modules in $V^{\otimes k}$; also, for the dimensions of the irreducible modules $Z_k^\lambda(G)$ for the centralizer algebra $Z_k(G)$), is a product of generalized hyperbolic functions. We de-

duce that the number of walks can be expressed as a sum of multinomial coefficients. When $r_1 = r_2 = \cdots = r_n = 2$, we obtain a formula for the number of walks on a hypercube of dimension n and the expression for the exponential generating function for the number of walks as a product of hyperbolic sines and cosines that was given in [7, Cor. 4.29]. In Sections 6.2 and 6.3, we exhibit a basis for $Z_k(G)$ and view $Z_k(G)$ as a diagram algebra by giving a diagrammatic realization of the basis elements.

2. Walks and Poincaré series

2.1. Expressions for counting walks, multiplicities, and centralizer algebra dimensions

There is a Hermitian inner product on the class functions of a finite group G defined by

$$\langle \phi, \psi \rangle = |G|^{-1} \sum_{g \in G} \phi(g) \overline{\psi(g)} = |G|^{-1} \sum_{\mu \in \Lambda(G)} |\mathcal{C}_\mu| \phi(\mathbf{c}_\mu) \overline{\psi(\mathbf{c}_\mu)},$$

where “ $-$ ” denotes the complex conjugate. The irreducible characters χ_λ for $\lambda \in \Lambda(G)$ satisfy the well-known orthogonality relation relative to this inner product (see for example, [12, (2.10)]):

$$\langle \chi_\nu, \chi_\lambda \rangle = |G|^{-1} \sum_{g \in G} \chi_\nu(g) \overline{\chi_\lambda(g)} = \delta_{\nu, \lambda}. \quad (6)$$

Therefore, if U is a G -module over $\mathbb{C}$ with character χ_U , then (6) implies that

$$\langle \chi_U, \chi_\lambda \rangle = |G|^{-1} \sum_{g \in G} \chi_U(g) \overline{\chi_\lambda(g)} = |G|^{-1} \sum_{\mu \in \Lambda(G)} |\mathcal{C}_\mu| \chi_U(\mathbf{c}_\mu) \overline{\chi_\lambda(\mathbf{c}_\mu)}$$

is the multiplicity of G_λ as a summand of U . Applying this to the G -module $U = G_\nu \otimes V^{\otimes k}$, which has character $\chi_\nu \chi_V^k$, gives the following result.

Theorem 2.1. *Assume V is finite-dimensional module for the finite group G . The number of walks of k -steps from node ν to node λ on the McKay quiver $Q_V(G)$ (equivalently, the multiplicity of G_λ in $G_\nu \otimes V^{\otimes k}$) is equal to*

$$(A^k)_{\nu, \lambda} = |G|^{-1} \sum_{\mu \in \Lambda(G)} |\mathcal{C}_\mu| \chi_\nu(\mathbf{c}_\mu) \chi_V(\mathbf{c}_\mu)^k \overline{\chi_\lambda(\mathbf{c}_\mu)}. \quad (7)$$

Corollary 2.2. *Under the hypotheses of Theorem 2.1, the following hold:*

- (i) *The dimension of the irreducible module $Z_k^\lambda(G)$ for the algebra $Z_k(G) = \text{End}_G(V^{\otimes k})$ is*

$$\dim Z_k^\lambda(\mathbf{G}) = (A^k)_{0,\lambda} = |\mathbf{G}|^{-1} \sum_{\mu \in \Lambda(\mathbf{G})} |\mathcal{C}_\mu| \chi_V(\mathbf{c}_\mu)^k \overline{\chi_\lambda(\mathbf{c}_\mu)} = |\mathbf{G}|^{-1} \sum_{g \in \mathbf{G}} \chi_V(g)^k \overline{\chi_\lambda(g)}. \quad (8)$$

(ii) The dimension of the space of $\mathbf{G}$ -invariants in $V^{\otimes k}$ is

$$\dim (V^{\otimes k})^{\mathbf{G}} = (A^k)_{0,0} = |\mathbf{G}|^{-1} \sum_{\mu \in \Lambda(\mathbf{G})} |\mathcal{C}_\mu| \chi_V(\mathbf{c}_\mu)^k = |\mathbf{G}|^{-1} \sum_{g \in \mathbf{G}} \chi_V(g)^k. \quad (9)$$

(iii) When V is a self-dual $\mathbf{G}$ -module,

$$\dim Z_k(\mathbf{G}) = \dim Z_{2k}^0(\mathbf{G}) = (A^{2k})_{0,0} = |\mathbf{G}|^{-1} \sum_{\mu \in \Lambda(\mathbf{G})} |\mathcal{C}_\mu| \chi_V(\mathbf{c}_\mu)^{2k} = |\mathbf{G}|^{-1} \sum_{g \in \mathbf{G}} \chi_V(g)^{2k}. \quad (10)$$

2.2. Poincaré series

It is a consequence of the results in (8) and (10) that the Poincaré series

$$P^\lambda(t) := \sum_{k=0}^{\infty} (A^k)_{0,\lambda} t^k = \sum_{k=0}^{\infty} m_k^\lambda t^k = \sum_{k=0}^{\infty} \dim Z_k^\lambda(\mathbf{G}) t^k$$

has the following expression

$$\begin{aligned} P^\lambda(t) &= |\mathbf{G}|^{-1} \sum_{\mu \in \Lambda(\mathbf{G})} |\mathcal{C}_\mu| \frac{\overline{\chi_\lambda(\mathbf{c}_\mu)}}{1 - \chi_V(\mathbf{c}_\mu)t} = |\mathbf{G}|^{-1} \sum_{g \in \mathbf{G}} \frac{\overline{\chi_\lambda(g)}}{1 - \chi_V(g)t} \\ &= \frac{\det(M^\lambda)}{\det(I - tA)} = \frac{\det(M^\lambda)}{\prod_{\mu \in \Lambda(\mathbf{G})} (1 - \chi_V(\mathbf{c}_\mu)t)}, \end{aligned} \quad (11)$$

where M^λ is the matrix $I - tA^T$ with the column indexed by λ replaced by $\delta_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ as

in Theorem 1.3. Then a special case of this formula is the Poincaré series for the tensor invariants $T(V)^{\mathbf{G}}$ in $T(V) = \bigoplus_{k=0}^{\infty} V^{\otimes k}$:

$$\begin{aligned} P^0(t) &= |\mathbf{G}|^{-1} \sum_{\mu \in \Lambda(\mathbf{G})} |\mathcal{C}_\mu| \frac{1}{1 - \chi_V(\mathbf{c}_\mu)t} = |\mathbf{G}|^{-1} \sum_{g \in \mathbf{G}} \frac{1}{1 - \chi_V(g)t} \\ &= \frac{\det(M^0)}{\det(I - tA)} = \frac{\det(M^0)}{\prod_{\mu \in \Lambda(\mathbf{G})} (1 - \chi_V(\mathbf{c}_\mu)t)}. \end{aligned} \quad (12)$$

The expressions in (11) and (12) are analogs of Molien's formulas

$$P_S^\lambda(t) = \sum_{k=0}^{\infty} \dim S_k^\lambda(V) t^k = |G|^{-1} \sum_{\mu \in \Lambda(G)} |\mathcal{C}_\mu| \frac{\overline{\chi_\lambda(c_\mu)}}{\det_V(I - tc_\mu)} = |G|^{-1} \sum_{g \in G} \frac{\overline{\chi_\lambda(g)}}{\det_V(I - tg)},$$

$$P_S^0(t) = |G|^{-1} \sum_{\mu \in \Lambda(G)} |\mathcal{C}_\mu| \frac{1}{\det_V(I - tc_\mu)} = |G|^{-1} \sum_{g \in G} \frac{1}{\det_V(I - tg)},$$

for multiplicities of G -modules and invariants in polynomials, as described in the Introduction.

3. Cyclic examples

3.1. $G = \mathbb{Z}_r$

When $G = \mathbb{Z}_r = \mathbb{Z}/r\mathbb{Z}$, we identify the elements of $\Lambda(G)$ with the elements $\{0, 1, \dots, r-1\}$ of $\mathbb{Z}_r$. Then for $a \in G$, the character χ_a of G_a is given by $\chi_a(b) = \omega^{ab}$ for $a, b \in G$, where $\omega = e^{2\pi i/r}$. We assume $V = G_1 \oplus G_{r-1}$. The McKay quiver $Q_V(\mathbb{Z}_r)$ is a circular graph with r nodes, and a step from a node on the graph amounts to moving one step to the left or to the right. Then for $b \in G$, we have $\chi_V(b) = \chi_1(b) + \chi_{r-1}(b) = \omega^b + \omega^{-b} = 2\cos(2\pi ib/r)$. Therefore

$$\chi_{V^{\otimes k}}(b) = \chi_V(b)^k = (\omega^b + \omega^{-b})^k = \sum_{\ell=0}^k \binom{k}{\ell} \omega^{(k-\ell)b} \omega^{-\ell b} = \sum_{\ell=0}^k \binom{k}{\ell} \omega^{(k-2\ell)b}.$$

Now using the fact that

$$\sum_{b=0}^{r-1} \omega^{mb} = \begin{cases} r & \text{if } m \equiv 0 \pmod{r}, \\ 0 & \text{otherwise,} \end{cases} \quad (13)$$

and Theorem 2.1, we have the following expression for the number of walks of k steps from a to c on $Q_V(\mathbb{Z}_r)$:

$$\begin{aligned} (A^k)_{a,c} &= r^{-1} \sum_{b \in \mathbb{Z}_r} \chi_a(b) \chi_V(b)^k \overline{\chi_c(b)} = r^{-1} \sum_{b=0}^{r-1} \omega^{(a-c)b} \sum_{\ell=0}^k \binom{k}{\ell} \omega^{(k-2\ell)b} \\ &= r^{-1} \sum_{\ell=0}^k \binom{k}{\ell} \sum_{b=0}^{r-1} \omega^{(k-2\ell+a-c)b} = \sum_{\substack{0 \leq \ell \leq k \\ k-2\ell \equiv c-a \pmod{r}}} \binom{k}{\ell}. \end{aligned} \quad (14)$$

Therefore, the dimension of the irreducible module $Z_k^c(\mathbb{Z}_r)$ for the centralizer algebra $Z_k(\mathbb{Z}_r) = \text{End}_{\mathbb{Z}_r}(V^{\otimes k})$ is

$$\dim Z_k^c(\mathbb{Z}_r) = (A^k)_{0,c} = \sum_{\substack{0 \leq \ell \leq k \\ k-2\ell \equiv c \pmod{r}}} \binom{k}{\ell}.$$

In particular, in order for the irreducible $\mathbb{Z}_r$ -module labeled by c to occur in $V^{\otimes k}$ with multiplicity at least one, equivalently, in order for $\dim Z_k^c(\mathbb{Z}_r)$ to be nonzero, it must be that $k - c \equiv 2\ell \pmod r$ for some ℓ . Let ℓ_c be the least nonnegative integer with that property. Then

$$\dim Z_k^c(\mathbb{Z}_r) = \sum_{\substack{0 \leq \ell \leq k \\ \ell \equiv \ell_c \pmod{\tilde{r}}}} \binom{k}{\ell},$$

where $\tilde{r} = r$ if r is odd, and $\tilde{r} = r/2$ if r is even. Since the module V is self dual,

$$\dim Z_k(\mathbb{Z}_r) = \dim Z_{2k}^0(\mathbb{Z}_r) = \sum_{\substack{0 \leq \ell \leq 2k \\ k - \ell \equiv 0 \pmod{\tilde{r}}}} \binom{2k}{\ell}.$$

(Compare [2, Thm. 2.17(i) and Thm. 2.8(d)].) These formulas can be interpreted as computing Pascal's triangle on a cylinder of diameter $\tilde{r}$. (See [2, Sec. 4.2] for more details.)

Here is a specific example to demonstrate the above results.

Example 3.1. When $k = 6$ and $r = 10$,

$$\begin{aligned} \dim Z_6(\mathbb{Z}_{10}) &= \sum_{\substack{0 \leq \ell \leq 12 \\ 6 - \ell \equiv 0 \pmod 5}} \binom{12}{\ell} \\ &= \binom{12}{1} + \binom{12}{6} + \binom{12}{11} = 12 + 924 + 12 = 948. \end{aligned}$$

3.2. Bratteli diagram

The *Bratteli diagram* $\mathcal{B}_V(\mathbf{G})$ is an infinite graph with vertices labeled by the elements of $\Lambda_k(\mathbf{G})$ on level k . A walk of k steps on the quiver $\mathbf{Q}_V(\mathbf{G})$ from 0 to λ is a sequence

$$(\lambda^{(0)} = 0, \lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(k)} = \lambda)$$

starting at $\lambda^{(0)} = 0$ and ending at $\lambda^{(k)} = \lambda$, such that $\lambda^{(j)} \in \Lambda_j(\mathbf{G})$ for each $1 \leq j \leq k$, and $\lambda^{(j-1)}$ is connected to $\lambda^{(j)}$ by an edge in $\mathbf{Q}_V(\mathbf{G})$. Such a walk is equivalent to a unique path of length k on the Bratteli diagram $\mathcal{B}_V(\mathbf{G})$ from 0 at the top to $\lambda \in \Lambda_k(\mathbf{G})$ on level k . The subscript on vertex $\lambda \in \Lambda_k(\mathbf{G})$ in $\mathcal{B}_V(\mathbf{G})$ indicates the number m_k^λ of paths from 0 on the top to λ at level k . This can be easily computed by summing, in a Pascal triangle fashion, the subscripts of the vertices at level $k - 1$ that are connected to λ . This is dimension of the irreducible $Z_k(\mathbf{G})$ -module $Z_k^\lambda(\mathbf{G})$, which is also the multiplicity of $\mathbf{G}_\lambda$ in $V^{\otimes k}$. The sum of the squares of those dimensions at level k is the number on the right, which is the dimension of the centralizer algebra $Z_k(\mathbf{G})$ by Wedderburn theory.

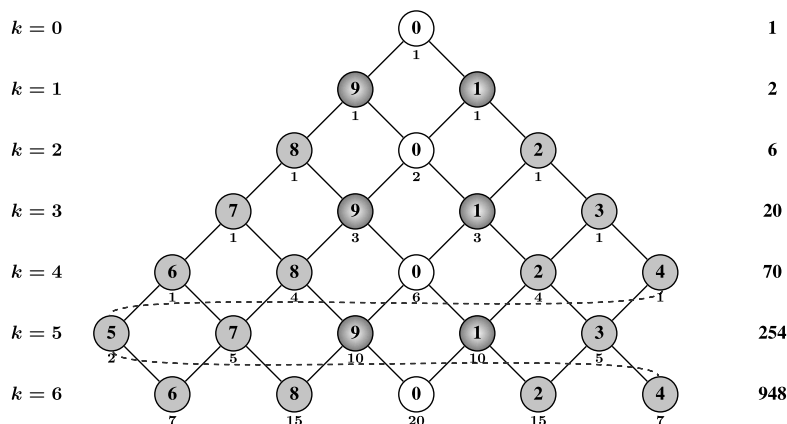


Fig. 1. Bratteli diagram $\mathcal{B}_V(G)$ for $G = \mathbb{Z}_{10}$ and $V = G_1 \oplus G_9$.

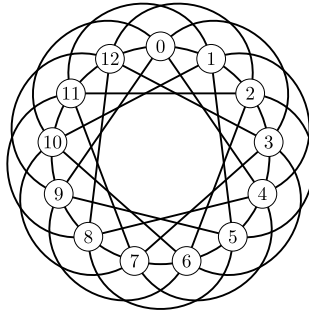
The Bratteli diagram for the cyclic group $G = \mathbb{Z}_{10}$ in Example 3.1 is pictured below and also in [2, Sec. 4.2]. The label inside the node is the index of the irreducible G -module. The trivial module is indicated in white and the module $V = G_1 \oplus G_9$ in black. The subscript on node λ on level k indicates the number of paths from 0 at the top to λ at level k (equivalently, the number of walks from 0 to λ of k steps on the McKay quiver $Q_V(G)$; also the multiplicity of G_λ in $V^{\otimes k}$; also the dimension of the irreducible module $Z_k^\lambda(G)$ for the centralizer algebra $Z_k(G) = \text{End}_G(V^{\otimes k})$). The right-hand column displays the dimension of the centralizer algebra, which for $Z_6(\mathbb{Z}_{10})$ is 948, as computed above. Since the dimension of the irreducible module $Z_6^8(\mathbb{Z}_{10})$ is the number of walks of 6 steps from 0 to 8 on the McKay quiver for $G = \mathbb{Z}_{10}$ and $V = G_1 \oplus G_9$, we have from (14),

$$\dim Z_6^8(\mathbb{Z}_{10}) = \sum_{\substack{0 \leq \ell \leq 6 \\ 6-2\ell \equiv 8 \pmod{5}}} \binom{6}{\ell} = \binom{6}{4} = 15.$$

This is the subscript on the node labeled 8 on level $k = 6$ of the Bratteli diagram in Fig. 1.

3.3. Circulant graphs

The Paley graphs are a family of graphs constructed from quadratic residues in finite fields. The Paley graph $\mathcal{P}_{13}$ of order 13 is pictured in Fig. 2. Every Paley graph is a circulant graph, which is equivalent to saying its adjacency matrix is a circulant matrix. There are many different characterizations of circulant graphs and circulant matrices. (The article by Kra and Simanca [15] nicely summarizes many of them.) Most relevant here is the fact that a graph is circulant if and only if its automorphism group contains a cyclic group acting transitively on its nodes. For $\mathcal{P}_{13}$ this group is $\mathbb{Z}_{13}$. In the notation of the previous example, we can take the module V so that $\chi_V = \sum_j \chi_j$, where the sum is over $j = 1, 3, 4, 9, 10, 12$ (the quadratic residues mod 13). Then a step on $\mathcal{P}_{13}$

Fig. 2. Paley graph $\mathcal{P}_{13}$.

corresponds to tensoring with this particular choice of $\mathbb{Z}_{13}$ -module V . Using that fact and Theorem 2.1, we have the following (where ω is a primitive 13th root of 1):

Corollary 3.2. *The number of walks of k steps from 0 to $c \in \{0, 1, \dots, 12\}$ on the Paley graph $\mathcal{P}_{13}$ is*

$$\begin{aligned} (A^k)_{0,c} &= (13)^{-1} \sum_{\substack{0 \leq \ell_1, \ell_2, \dots, \ell_6 \leq k \\ \ell_1 + \ell_2 + \dots + \ell_6 = k}} \binom{k}{\ell_1, \ell_2, \dots, \ell_6} \left(\sum_{b=0}^{12} \omega^{(\ell_1 + 3\ell_2 + 4\ell_3 + 9\ell_4 + 10\ell_5 + 12\ell_6 - c)b} \right) \\ &= \sum_{\substack{0 \leq \ell_1, \ell_2, \dots, \ell_6 \leq k, \ell_1 + \ell_2 + \dots + \ell_6 = k \\ \ell_1 + 3\ell_2 + \dots + 12\ell_6 \equiv c \pmod{13}}} \binom{k}{\ell_1, \ell_2, \dots, \ell_6}. \end{aligned}$$

Walks on any circulant graph can be enumerated by exactly the same type of argument.

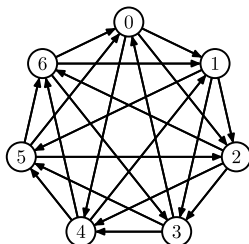
3.4. Paley (di)graphs $\mathcal{P}_p$ of order p an odd prime

Suppose p is an odd prime and $\omega = e^{2\pi i/p}$. The nodes in the Paley (di)graph $\mathcal{P}_p$ are labeled by the elements in $\{0, 1, \dots, p-1\}$, and the ones connected to 0 are labeled by the distinct square values x^2 in $\mathbb{Z}_p^\times = \{1, 2, \dots, p-1\}$ (the quadratic residues modulo p). As noted earlier, for $p = 13$ these are the values $x^2 = 1, 3, 4, 9, 10, 12$. When $p \equiv 1 \pmod{4}$, $\mathcal{P}_p$ is an undirected graph, and for $p \equiv 3 \pmod{4}$ it is a digraph, as illustrated in Fig. 3 below for $p = 7$.

We take V so that $Q_V(\mathbb{Z}_p)$ is $\mathcal{P}_p$. Then

$$\chi_V(b) = f(b) := \sum_{x^2 \in \mathbb{Z}_p^\times} \omega^{bx^2}$$

for $b \in \mathbb{Z}_p$, and we know from (8) that the number of walks of k steps from 0 to c on the graph $\mathcal{P}_p$ is given by

Fig. 3. Paley digraph $\mathcal{P}_7$.

$$(\mathbf{A}^k)_{0,c} = \frac{1}{p} \sum_{b \in \mathbb{Z}_p} \chi_V(b)^k \overline{\chi_c(b)} = \frac{1}{p} \sum_{b=0}^{p-1} f(b)^k \omega^{-cb} \quad (15)$$

We evaluate this expression using well-known facts about Gauss sums, which can be found for example in [14, Chap. 8]. Suppose

$$\xi = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{4}, \\ i = \sqrt{-1} & \text{if } p \equiv 3 \pmod{4}. \end{cases} \quad (16)$$

The Gauss sum $g(b) = \sum_{x=0}^{p-1} \omega^{bx^2}$ equals p when $b = 0$, and for $b \in \mathbb{Z}_p^\times$

$$g(b) = \left(\frac{b}{p}\right) g(1) = \begin{cases} \xi \sqrt{p} & \text{if } b \text{ is a quadratic residue modulo } p, \\ -\xi \sqrt{p} & \text{if } b \text{ is a quadratic nonresidue modulo } p, \end{cases}$$

where $\left(\frac{b}{p}\right)$ is the Legendre symbol, which is 1 if b is a quadratic residue and -1 otherwise. Since the number of quadratic residues equals the number of quadratic nonresidues, it follows that

$$f(b) = \frac{1}{2} (g(b) - 1) = \begin{cases} \frac{1}{2} (\xi \sqrt{p} - 1) & \text{if } b \text{ is a nonzero quadratic residue modulo } p, \\ -\frac{1}{2} (\xi \sqrt{p} + 1) & \text{if } b \text{ is a quadratic nonresidue modulo } p, \\ \frac{1}{2} (p - 1) & \text{if } b = 0. \end{cases}$$

Our aim in this section is to prove

Theorem 3.3. Assume $\mathcal{P}_p$ is the Paley (di)graph of order p a prime and ξ is as in (16). Then the number of walks of k steps from 0 to c on $\mathcal{P}_p$ is given by one of the following:

(i) If c is a nonzero quadratic residue, then

$$(A^k)_{0,c} = \begin{cases} \frac{1}{2^{k+1}p} \left(2(p-1)^k + (\sqrt{p}-1)^{k+1} + (-1)^{k+1} (\sqrt{p}+1)^{k+1} \right) \\ \quad \text{if } p \equiv 1 \pmod{4}, \\ \frac{1}{2^{k+1}p} \left(2(p-1)^k + (p+1) \left((i\sqrt{p}-1)^{k-1} + (-1)^k (i\sqrt{p}+1)^{k-1} \right) \right) \\ \quad \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

(ii) If c is a quadratic nonresidue, then

$$(A^k)_{0,c} = \begin{cases} \frac{p-1}{2^{k+1}p} \left(2(p-1)^{k-1} + (\sqrt{p}-1)^{k-1} + (-1)^k (\sqrt{p}+1)^{k-1} \right) \\ \quad \text{if } p \equiv 1 \pmod{4}, \\ \frac{1}{2^{k+1}p} \left(2(p-1)^k - (i\sqrt{p}+1)^{k+1} + (-1)^{k+1} (i\sqrt{p}+1)^{k+1} \right) \\ \quad \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

(iii) If $c = 0$, then

$$(A^k)_{0,0} = \frac{p-1}{2^{k+1}p} \left(2(p-1)^{k-1} + (\xi\sqrt{p}-1)^k + (-1)^k (\xi\sqrt{p}+1)^k \right).$$

Proof. Since the quadratic nonresidues modulo p are all of the form ax^2 for some fixed quadratic nonresidue a , we have from (15)

$$\begin{aligned} (A^k)_{0,c} &= \frac{1}{p} \left(\left(\frac{p-1}{2} \right)^k + \sum_{x^2 \in \mathbb{Z}_p^\times} \left(\frac{\xi\sqrt{p}-1}{2} \right)^k \omega^{-x^2c} + \sum_{x^2 \in \mathbb{Z}_p^\times} (-1)^k \left(\frac{\xi\sqrt{p}+1}{2} \right)^k \omega^{-ax^2c} \right) \\ &= \frac{1}{p} \left(\left(\frac{p-1}{2} \right)^k + \left(\frac{\xi\sqrt{p}-1}{2} \right)^k \sum_{x^2 \in \mathbb{Z}_p^\times} \omega^{-x^2c} + (-1)^k \left(\frac{\xi\sqrt{p}+1}{2} \right)^k \sum_{x^2 \in \mathbb{Z}_p^\times} \omega^{-ax^2c} \right). \end{aligned} \quad (17)$$

Now if $c \neq 0$, then

$$g(-c) = \left(\frac{-c}{p} \right) g(1) = \left(\frac{c}{p} \right) \left(\frac{-1}{p} \right) g(1) = \begin{cases} g(c) & \text{if } p \equiv 1 \pmod{4}, \\ -g(c) & \text{if } p \equiv 3 \pmod{4}, \end{cases}$$

so that

$$f(-c) = \begin{cases} f(c) & \text{if } p \equiv 1 \pmod{4}, \\ -(f(c) + 1) & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Therefore when $c \neq 0$,

$$(A^k)_{0,c} = \begin{cases} \frac{1}{p} \left(\left(\frac{p-1}{2} \right)^k + \left(\frac{\sqrt{p}-1}{2} \right)^k f(c) + (-1)^k \left(\frac{\sqrt{p}+1}{2} \right)^k f(ac) \right) & \text{if } p \equiv 1 \pmod{4}, \\ \frac{1}{p} \left(\left(\frac{p-1}{2} \right)^k - \left(\frac{i\sqrt{p}-1}{2} \right)^k (f(c) + 1) + (-1)^{k+1} \left(\frac{i\sqrt{p}+1}{2} \right)^k (f(ac) + 1) \right) & \text{if } p \equiv 3 \pmod{4}. \end{cases} \quad (18)$$

We examine the expression in (18) for the scenarios in (i) and (ii) of Theorem 3.3.

(i) When $c \in \mathbb{Z}_p^\times$ is a quadratic residue modulo p , then

$$(A^k)_{0,c} = \begin{cases} \frac{1}{2^{k+1}p} \left(2(p-1)^k + (\sqrt{p}-1)^{k+1} + (-1)^{k+1} (\sqrt{p}+1)^{k+1} \right) & \text{if } p \equiv 1 \pmod{4}, \\ \frac{1}{2^{k+1}p} \left(2(p-1)^k - (i\sqrt{p}-1)^k (i\sqrt{p}+1) + (-1)^{k+1} (i\sqrt{p}+1)^k (i\sqrt{p}-1) \right) & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

$$= \frac{1}{2^{k+1}p} \left(2(p-1)^k + (p+1) \left((i\sqrt{p}-1)^{k-1} + (-1)^k (i\sqrt{p}+1)^{k-1} \right) \right)$$

(ii) When c is a quadratic nonresidue modulo p ,

$$(A^k)_{0,c} = \begin{cases} \frac{p-1}{2^{k+1}p} \left(2(p-1)^{k-1} + (\sqrt{p}-1)^{k-1} + (-1)^k (\sqrt{p}+1)^{k-1} \right) & \text{if } p \equiv 1 \pmod{4}, \\ \frac{1}{2^{k+1}p} \left(2(p-1)^k - (i\sqrt{p}+1)^{k+1} + (-1)^{k+1} (i\sqrt{p}+1)^{k+1} \right) & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

(iii) Finally, when $c = 0$, then (17) implies

$$(A^k)_{0,0} = \frac{1}{2^{k+1}p} \left(2(p-1)^k + (\xi\sqrt{p}-1)^k (p-1) + (-1)^k (\xi\sqrt{p}+1)^k (p-1) \right)$$

$$= \frac{p-1}{2^{k+1}p} \left(2(p-1)^{k-1} + (\xi\sqrt{p}-1)^k + (-1)^k (\xi\sqrt{p}+1)^k \right),$$

to give the assertion in part (iii). $\square$

4. The groups S_n and $\mathbb{Z}_r \wr S_n$

4.1. The symmetric group S_n

The irreducible modules for the symmetric group S_n are in one-to-one correspondence with the partitions $\lambda \vdash n$, and the conjugacy classes are determined by the cycle decom-

position of the permutations, hence they also are indexed by the partitions of n . If V is taken to be the n -dimensional permutation module on which S_n acts by permuting the basis elements, then for all $\sigma \in S_n$,

$$\chi_V(\sigma) = \text{tr}_V(\sigma) = F(\sigma),$$

where $F(\sigma)$ is the number of fixed points of σ . As a result, we know from (12) that the Poincaré series for the tensor invariants $T(V)^{S_n}$ is given by

$$\begin{aligned} P^0(t) &= (n!)^{-1} \sum_{\mu \vdash n} |\mathcal{C}_\mu| \frac{1}{1 - F(c_\mu)t} = (n!)^{-1} \sum_{\sigma \in S_n} \frac{1}{1 - F(\sigma)t} \\ &= \frac{\det(M^0)}{\det(I - tA)} = \frac{\det(M^0)}{\prod_{\mu \vdash n} (1 - F(c_\mu)t)}, \end{aligned}$$

where M^0 and A are as in Theorem 1.3. For the centralizer algebra $Z_k(S_n) = \text{End}_{S_n}(V^{\otimes k})$ and its irreducible module $Z_k^\lambda(S_n)$,

$$\begin{aligned} \dim Z_k^\lambda(S_n) &= (n!)^{-1} \sum_{\sigma \in S_n} F(\sigma)^k \overline{\chi_\lambda(\sigma)} = (n!)^{-1} \sum_{\sigma \in S_n} F(\sigma)^k \chi_\lambda(\sigma^{-1}) \\ &= (n!)^{-1} \sum_{\sigma \in S_n} F(\sigma)^k \chi_\lambda(\sigma), \\ \dim Z_k(S_n) &= (n!)^{-1} \sum_{\mu \vdash n} |\mathcal{C}_\mu| F(c_\mu)^{2k} = (n!)^{-1} \sum_{\sigma \in S_n} F(\sigma)^{2k}. \end{aligned} \tag{19}$$

The last equality in the first line of (19) comes from the fact that σ and σ^{-1} have the same cycle structures, hence belong to the same conjugacy class of S_n . The centralizer algebra $Z_k(S_n)$ for the S_n -action on the k -fold tensor power of its permutation module V is a homomorphic image of the partition algebra $P_k(n) \rightarrow Z_k(S_n) = \text{End}_{S_n}(V^{\otimes k})$, and $Z_k(S_n)$ is isomorphic to $P_k(n)$ when $n \geq 2k$. Parts (a) and (c) of [5, Thm. 5.5] give expressions for the dimension of $Z_k^\lambda(S_n)$ and $Z_k(S_n)$ respectively in terms of Stirling numbers of the second kind, and these expressions combine with the ones above to show that

$$\begin{aligned} (n!)^{-1} \sum_{\sigma \in S_n} F(\sigma)^k \chi_\lambda(\sigma) &= (n!)^{-1} \sum_{\mu \vdash n} |\mathcal{C}_\mu| F(c_\mu)^k \chi_\lambda(c_\mu) \\ &= \dim Z_k^\lambda(S_n) = \sum_{\ell=0}^n K_{\lambda, (n-\ell, 1^\ell)} \begin{Bmatrix} k \\ \ell \end{Bmatrix}, \\ (n!)^{-1} \sum_{\sigma \in S_n} F(\sigma)^{2k} &= \dim Z_k(S_n) = \sum_{\ell=0}^n \begin{Bmatrix} 2k \\ \ell \end{Bmatrix}. \end{aligned} \tag{20}$$

The *Kostka number* $K_{\lambda, (n-\ell, 1^\ell)}$ counts the number of semistandard tableaux of shape λ with $n - \ell$ entries equal to 0 and one entry equal to each of the numbers $1, 2, \dots, \ell$

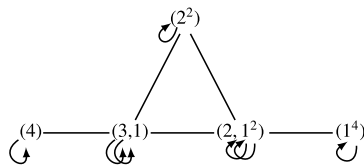


Fig. 4. McKay quiver $Q_V(S_4)$ for $V = (S_4)_{(4)} \oplus (S_4)_{(3,1)}$.

such that the entries weakly increase across the rows and strictly increase down the columns of the Young diagram of λ (more details on Kostka numbers can be found in [22, Sec. 2.11] or [26, Sec. 7.10]). The first equality in the second relation in (20) was proven by Farina and Halverson in [10] under the additional assumption that $n \geq 2k$.

In that case, $Z_k(S_n) \cong P_k(n)$, and the right-hand side $\sum_{\ell=0}^n \left\{ \begin{smallmatrix} 2k \\ \ell \end{smallmatrix} \right\} = \sum_{\ell=0}^{2k} \left\{ \begin{smallmatrix} 2k \\ \ell \end{smallmatrix} \right\}$ equals the Bell number $B(2k)$. The relations in (20) hold for all $n, k \in \mathbb{Z}_{\geq 1}$.

Next we examine the particular case of the symmetric group S_4 to illustrate the above results.

4.2. The special case of the symmetric group S_4

The irreducible modules and conjugacy classes for the symmetric group S_4 are indexed by the partitions $\lambda \vdash 4$, where $\lambda \in \{(4), (3,1), (2^2), (2,1^2), (1^4)\}$. The trivial module corresponds to the partition (4) with just one part, and the 4-dimensional permutation module for S_4 is given by $V = (S_4)_{(4)} \oplus (S_4)_{(3,1)}$. The corresponding McKay quiver $Q_V(S_4)$ is pictured in Fig. 4.

Hence, by (7), the dimensions of the irreducible modules $Z_k^\lambda(S_4)$ for the centralizer algebra $Z_k(S_4) = \text{End}_{S_4}(V^{\otimes k})$ are given by

$$\dim Z_k^\lambda(S_4) = (A^k)_{(4),\lambda} = (24)^{-1} \sum_{\mu \vdash 4} |\mathcal{C}_\mu| \chi_V(c_\mu)^k \chi_\lambda(c_\mu).$$

The necessary information to evaluate this expression is displayed in the table below and can be gotten from the character table for S_4 (see for example [12, Sec. 2.3]).

$\lambda \setminus \mu$	(1^4)	$(2, 1^2)$	(2^2)	$(3, 1)$	(4)
$ \mathcal{C}_\mu $	1	6	3	8	6
$\chi_{(4)}(c_\mu)$	1	1	1	1	1
$\chi_{(3,1)}(c_\mu)$	3	1	-1	0	-1
$\chi_{(2^2)}(c_\mu)$	2	0	2	-1	0
$\chi_{(2,1^2)}(c_\mu)$	3	-1	-1	0	1
$\chi_{(1^4)}(c_\mu)$	1	-1	1	1	-1
$\chi_V^k(c_\mu)$	4^k	2^k	0	1	0

From this we determine that for $k \geq 1$,

$$\begin{aligned}
 \dim Z_k^{(4)}(S_4) &= \frac{1}{24} (4^k + 6 \cdot 2^k + 8) \quad \left(= \sum_{\ell=1}^4 \left\{ \begin{matrix} k \\ \ell \end{matrix} \right\} \right) \\
 \dim Z_k^{(3,1)}(S_4) &= \frac{1}{24} (3 \cdot 4^k + 6 \cdot 2^k) \quad \left(= \left\{ \begin{matrix} k \\ 1 \end{matrix} \right\} + 2 \left\{ \begin{matrix} k \\ 2 \end{matrix} \right\} + 3 \left\{ \begin{matrix} k \\ 3 \end{matrix} \right\} + 3 \left\{ \begin{matrix} k \\ 4 \end{matrix} \right\} \right) \\
 \dim Z_k^{(2^2)}(S_4) &= \frac{1}{24} (2 \cdot 4^k - 8) \quad \left(= \left\{ \begin{matrix} k \\ 2 \end{matrix} \right\} + 2 \left\{ \begin{matrix} k \\ 3 \end{matrix} \right\} + 2 \left\{ \begin{matrix} k \\ 4 \end{matrix} \right\} \right) \\
 \dim Z_k^{(2,1^2)}(S_4) &= \frac{1}{24} (3 \cdot 4^k - 6 \cdot 2^k) \quad \left(= \left\{ \begin{matrix} k \\ 2 \end{matrix} \right\} + 3 \left\{ \begin{matrix} k \\ 3 \end{matrix} \right\} + 3 \left\{ \begin{matrix} k \\ 4 \end{matrix} \right\} \right) \\
 \dim Z_k^{(1^4)}(S_4) &= \frac{1}{24} (4^k - 6 \cdot 2^k + 8) \quad \left(= \left\{ \begin{matrix} k \\ 3 \end{matrix} \right\} + \left\{ \begin{matrix} k \\ 4 \end{matrix} \right\} \right) \\
 \dim Z_k(S_4) &= \dim Z_{2k}^{(4)}(S_4) = \frac{1}{24} (4^{2k} + 6 \cdot 2^{2k} + 8) \quad \left(= \sum_{\ell=1}^4 \left\{ \begin{matrix} 2k \\ \ell \end{matrix} \right\} \right).
 \end{aligned} \tag{21}$$

On the right-hand side above, we have given expressions for the dimensions in terms of Stirling numbers of the second kind, which were derived using the following closed-form formula:

$$\left\{ \begin{matrix} k \\ \ell \end{matrix} \right\} = \frac{1}{\ell!} \sum_{j=0}^{\ell} (-1)^{\ell-j} \binom{\ell}{j} j^k.$$

The coefficients of the Stirling numbers $\left\{ \begin{matrix} k \\ \ell \end{matrix} \right\}$ are the Kostka numbers $K_{\lambda, (n-\ell, 1^{\ell})}$ for $n = 4$, and they enumerate the semistandard tableaux of shape λ and type $(4-\ell, 1^{\ell})$ as pictured below for $\lambda = (2^2)$:

$$\begin{array}{ccc}
 \begin{array}{|c|c|} \hline 0 & 0 \\ \hline 1 & 2 \\ \hline \end{array} &
 \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 2 & 3 \\ \hline \end{array} &
 \begin{array}{|c|c|} \hline 0 & 2 \\ \hline 1 & 3 \\ \hline \end{array} &
 \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} &
 \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \\
 \ell = 2 & & \ell = 3 & & \ell = 4
 \end{array}$$

The top levels of the Bratteli diagram for the group $G = S_4$ and its 4-dimensional permutation module V are exhibited in Fig. 5. The subscripts in the columns of $\mathcal{B}_V(S_4)$ are given by the formulas in (21).

4.3. The group $\mathbb{Z}_r \wr S_n$

In this section, G is the wreath product $\mathbb{Z}_r \wr S_n$ viewed as $n \times n$ monomial matrices with entries of the form ω^j for $j = 0, 1, \dots, r-1$, where $\omega = e^{2\pi i/r}$, a primitive r th root of unity for $r \geq 2$. The module V is the space of $n \times 1$ column vectors with complex entries on

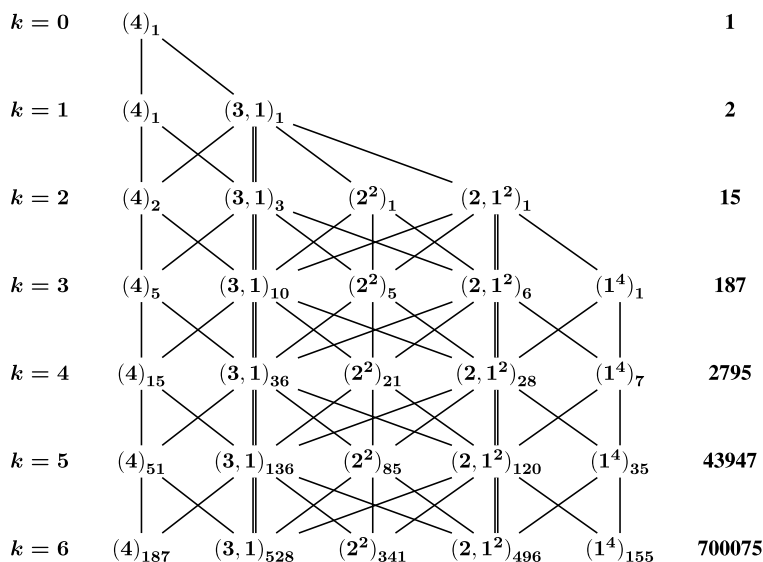


Fig. 5. Levels $k = 0, 1, \dots, 6$ of the Bratteli diagram $\mathcal{B}_V(S_4)$ for S_4 and its permutation module V .

which G acts by matrix multiplication. Our main result (Theorem 4.1 below) is a formula for the dimension of the G -invariants $(V^{\otimes k})^G$ in $V^{\otimes k}$, equivalently, for the dimension $\dim Z_k^0(G) = |G|^{-1} \sum_{g \in G} \chi_V(g)^k$ of the irreducible module labeled by 0 for the centralizer algebra $Z_k(G) = \text{End}_G(V^{\otimes k})$. Our formula will depend on the number of entries on the main diagonal of a monomial matrix in G (the number of fixed points of the underlying permutation in S_n), and so for $m = 1, 2, \dots, n$, we set $F_n(m) := |\{\sigma \in S_n \mid F(\sigma) = m\}|$. This number, which is sometimes referred to as a *rencontres number*, counts the number of “partial derangements” of n with m fixed points. It equals $\binom{n}{m} D_{n-m}$, where D_{n-m} is the number of *derangements* of $n-m$ (permutations in S_{n-m} with no fixed points). From known expressions for the derangement numbers (see [25, Example 2.2.1]), we have

$$F_n(m) = \binom{n}{m} D_{n-m} = \binom{n}{m} (n-m)! \sum_{j=0}^{n-m} \frac{(-1)^j}{j!} = \frac{n!}{m!} \sum_{j=0}^{n-m} \frac{(-1)^j}{j!}. \quad (22)$$

Theorem 4.1. Assume $G = \mathbb{Z}_r \wr S_n$ and V is the n -dimensional G -module on which G acts by monomial matrices. Then the following hold:

- (a) The dimension of the space of G -invariants in $V^{\otimes k}$ (equivalently, $\dim Z_k^0(G)$) is given by

$$\dim (V^{\otimes k})^G = \frac{1}{r^n n!} \sum_{m=1}^n r^m F_n(m)^k \left(\sum_{\ell_1, \ell_2, \dots, \ell_m} \binom{k}{\ell_1, \ell_2, \dots, \ell_m} \right), \quad (23)$$

where the sum is over all $0 \leq \ell_1, \ell_2, \dots, \ell_m \leq k$ such that $\ell_1 + \ell_2 + \dots + \ell_m = k$ and $\ell_1 \equiv \ell_2 \equiv \dots \equiv \ell_m \equiv 0 \pmod{r}$, and $F_n(m) = \frac{n!}{m!} \sum_{j=0}^{n-m} \frac{(-1)^j}{j!}$. In particular, the space

$(V^{\otimes k})^G$ of invariants is 0 unless $k \equiv 0 \pmod{r}$.

(b) The exponential generating function for the G -invariants is

$$g^0(t) = \sum_{k=0}^{\infty} \dim(V^{\otimes k})^G \frac{t^k}{k!} = \frac{1}{r^n n!} \sum_{m=1}^n r^m h_1(F_n(m)t, r)^m, \quad (24)$$

where $h_1(t, r) = \sum_{q=0}^{\infty} \frac{t^{qr}}{(qr)!}$ is a generalized hyperbolic function.

Proof. (a) We know from Theorem 2.1 that $\dim(V^{\otimes k})^G = (A^k)_{0,0} = |G|^{-1} \sum_{g \in G} \chi_V(g)^k$, from which we have

$$\begin{aligned} \dim(V^{\otimes k})^G &= \frac{1}{r^n n!} \sum_{m=1}^n F_n(m)^k \sum_{b_1, b_2, \dots, b_m \in \{0, 1, \dots, r-1\}} (\omega^{b_1} + \omega^{b_2} + \dots + \omega^{b_m})^k \\ &= \frac{1}{r^n n!} \sum_{m=1}^n F_n(m)^k \\ &\quad \times \left(\sum_{\ell_1 + \ell_2 + \dots + \ell_m = k} \binom{k}{\ell_1, \ell_2, \dots, \ell_m} \left(\sum_{b_1=0}^{r-1} \omega^{\ell_1 b_1} \right) \left(\sum_{b_2=0}^{r-1} \omega^{\ell_2 b_2} \right) \dots \left(\sum_{b_m=0}^{r-1} \omega^{\ell_m b_m} \right) \right) \\ &= \frac{1}{r^n n!} \sum_{m=1}^n F_n(m)^k r^m \left(\sum_{\substack{\ell_1 + \ell_2 + \dots + \ell_m = k \\ \ell_1 \equiv \ell_2 \equiv \dots \equiv \ell_m \equiv 0 \pmod{r}}} \binom{k}{\ell_1, \ell_2, \dots, \ell_m} \right) \quad \text{by (13).} \end{aligned}$$

(b) It is a consequence of (23) that for $G = \mathbb{Z}_r \wr S_n$,

$$\dim(V^{\otimes k})^G = \frac{1}{r^n n!} \sum_{m=1}^n r^m F_n(m)^k \left(\sum_{(q_1 + q_2 + \dots + q_m)r = k} \binom{k}{q_1 r, q_2 r, \dots, q_m r} \right).$$

Therefore, the exponential generating function for the invariants is given by

$$\begin{aligned} g^0(t) &= \sum_{k=0}^{\infty} \dim(V^{\otimes k})^G \frac{t^k}{k!} \\ &= \frac{1}{r^n n!} \sum_{m=1}^n r^m \sum_{k=0}^{\infty} F_n(m)^k \left(\sum_{(q_1 + q_2 + \dots + q_m)r = k} \binom{k}{q_1 r, q_2 r, \dots, q_m r} \frac{t^k}{k!} \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{r^n n!} \sum_{m=1}^n r^m \left(\sum_{q_1=0}^{\infty} \frac{(F_n(m)t)^{q_1 r}}{(q_1 r)!} \right) \left(\sum_{q_2=0}^{\infty} \frac{(F_n(m)t)^{q_2 r}}{(q_2 r)!} \right) \cdots \left(\sum_{q_m=0}^{\infty} \frac{(F_n(m)t)^{q_m r}}{(q_m r)!} \right) \\
&= \frac{1}{r^n n!} \sum_{m=1}^n r^m \left(h_1(F_n(m)t, r) \right)^m,
\end{aligned}$$

where $h_1(t, r)$ is the generalized hyperbolic function $h_1(t, r) = \sum_{q=0}^{\infty} \frac{t^{qr}}{(qr)!}$ (see (32) and (33) below for more details). $\square$

4.4. $G = \mathbb{Z}_r \wr S_n$ for some special choices of r and n

Assume $G = \mathbb{Z}_r \wr S_2$ and $V = \mathbb{C}^2$. Then since $F_2(1) = \binom{2}{1} D_1 = 0$, and $F_2(2) = \binom{2}{2} D_0 = 1$, we have

$$\dim(V^{\otimes k})^G = \dim Z_k^0(G) = \frac{1}{2} \sum_{\substack{\ell_1 + \ell_2 = k \\ \ell_1 \equiv \ell_2 \equiv 0 \pmod{r}}} \binom{k}{\ell_1, \ell_2}.$$

So, for example, when $r = 2$,

$$\begin{aligned}
\dim(V^{\otimes k})^G &= \begin{cases} \frac{1}{2} \sum_{\ell=0}^{\frac{1}{2}k} \binom{k}{2\ell} = \frac{1}{2} 2^{k-1} = 2^{k-2} & \text{if } k \text{ is even and } k \geq 2, \\ 0 & \text{if } k \text{ is odd and } k \geq 1. \end{cases} \\
P^0(t) &= \sum_{k=0}^{\infty} \dim(V^{\otimes k})^G t^k = 1 + t^2 \sum_{j=0}^{\infty} (4t^2)^j = 1 + \frac{t^2}{1-4t^2} = \frac{1-3t^2}{1-4t^2}.
\end{aligned}$$

4.5. The group $G = \mathbb{Z}_r \wr S_n$ – a different approach

The irreducible modules $G_{\underline{\alpha}}$ for $G = \mathbb{Z}_r \wr S_n$, hence also the G -conjugacy classes $C_{\underline{\alpha}}$, are labeled by r -tuples of partitions $\underline{\alpha} = (\alpha^{(1)}, \alpha^{(2)}, \dots, \alpha^{(r)})$ such that $n = \sum_{i=1}^r |\alpha^{(i)}|$ (see for example [1, Sec. 2]).

For $x \in \mathbb{C}$, let $J_{\ell}(x)$ be the $\ell \times \ell$ Jordan block matrix given by

$$J_{\ell}(x) = \begin{pmatrix} 0 & 1 & 0 & & \cdots & & 0 \\ 0 & 0 & 1 & 0 & & \cdots & 0 \\ & & & \ddots & & & \\ \vdots & & & & \ddots & & \vdots \\ & & & & & \ddots & 0 \\ 0 & 0 & & & & & 0 & 1 \\ x & 0 & & \cdots & & & 0 & 0 \end{pmatrix}$$

Then a conjugacy class representative of G corresponding to $\underline{\alpha}$ is

$$c_{\underline{\alpha}} = \bigoplus_{i=1}^r \bigoplus_p J_{\alpha_p^{(i)}}(\omega^{i-1}),$$

where $\omega = e^{2\pi i/r}$, the parts $\alpha_p^{(i)}$ of the i th partition $\alpha^{(i)}$ are $\alpha_1^{(i)} \geq \alpha_2^{(i)} \geq \dots$, and this sum represents the $n \times n$ matrix with blocks down the main diagonal starting with $J_{\alpha_1^{(1)}}(\omega^0)$, then $J_{\alpha_2^{(1)}}(\omega^0), \dots$, and continuing down to $J_{\alpha_\ell^{(r)}}(\omega^{r-1})$ corresponding to the last part $\alpha_\ell^{(r)}$ of the last partition $\alpha^{(r)}$.

For a partition λ , assume $p_j(\lambda)$ is the number of parts of λ equal to j . Set

$$z_\lambda = \prod_{j=1}^n j^{p_j(\lambda)} p_j(\lambda)!.$$

This is the order of the centralizer of an element of $S_{|\lambda|}$ with cycle structure given by the partition λ . Now for $\underline{\alpha} = (\alpha^{(1)}, \alpha^{(2)}, \dots, \alpha^{(r)})$, we define

$$p_j(\underline{\alpha}) = \sum_{i=1}^r p_j(\alpha^{(i)}) \quad \text{and} \quad p(\underline{\alpha}) = \sum_{j=1}^n p_j(\underline{\alpha}). \quad (25)$$

Thus, $p_j(\underline{\alpha})$ is the total number of parts equal to j in the partitions comprising $\underline{\alpha}$, and $p(\underline{\alpha})$ is the total number of nonzero parts in the partitions of $\underline{\alpha}$. Then according to [1, Sec. 2], the size of the centralizer of $c_{\underline{\alpha}}$ in G is given by

$$z_{\underline{\alpha}} = \prod_{i,j} (rj)^{p_j(\alpha^{(i)})} p_j(\alpha^{(i)})! = r^{p(\underline{\alpha})} \prod_{j=1}^n j^{p_j(\underline{\alpha})} \left(\prod_{i=1}^r p_j(\alpha^{(i)})! \right) = r^{p(\underline{\alpha})} \prod_{i=1}^r z_{\alpha^{(i)}}.$$

Hence, the size of the conjugacy class $\mathcal{C}_{\underline{\alpha}}$ corresponding to the element $c_{\underline{\alpha}}$ is given by

$$|\mathcal{C}_{\underline{\alpha}}| = \frac{|G|}{z_{\underline{\alpha}}} = \frac{|G|}{r^{p(\underline{\alpha})} \prod_{j=1}^n j^{p_j(\underline{\alpha})} (\prod_{i=1}^r p_j(\alpha^{(i)})!)}.$$

Thus, we know that

$$\begin{aligned} \dim(V^{\otimes k})^G &= \dim Z_k^0(G) = |G|^{-1} \sum_{\underline{\alpha} \in \Lambda(G)} |\mathcal{C}_{\underline{\alpha}}| \chi_V(c_{\underline{\alpha}})^k \\ &= \sum_{\underline{\alpha} \in \Lambda(G)} \frac{\chi_V(c_{\underline{\alpha}})^k}{r^{p(\underline{\alpha})} \prod_{j=1}^n j^{p_j(\underline{\alpha})} (\prod_{i=1}^r p_j(\alpha^{(i)})!)}. \end{aligned}$$

Observe that

$$\chi_V(\mathbf{c}_{\underline{\alpha}}) = \text{tr}_V(\mathbf{c}_{\underline{\alpha}}) = \sum_{i=1}^r \mathbf{p}_1(\alpha^{(i)}) \omega^{i-1} = \sum_{i=1}^r F(\alpha^{(i)}) \omega^{i-1}$$

where $\mathbf{p}_1(\alpha^{(i)})$ is the number of parts equal to 1 in $\alpha^{(i)}$, as the only contributions to the trace come from the matrix blocks of size one in $\mathbf{c}_{\underline{\alpha}}$. Since that is the number of fixed points of a permutation of cycle type $\alpha^{(i)}$, we write $F(\alpha^{(i)})$ by a slight abuse of notation. Therefore, we obtain a second expression for the dimension of the $\mathbf{G}$ -invariants in $V^{\otimes k}$ using the definitions in (25):

$$\dim(V^{\otimes k})^{\mathbf{G}} = \dim Z_k^0(\mathbf{G}) = \sum_{\underline{\alpha} \in \Lambda(\mathbf{G})} \frac{\left(\sum_{i=1}^r F(\alpha^{(i)}) \omega^{i-1} \right)^k}{r^{\mathbf{p}(\underline{\alpha})} \prod_{j=1}^n j^{\mathbf{p}_j(\underline{\alpha})} (\prod_{i=1}^r \mathbf{p}_j(\alpha^{(i)})!)} \quad \text{for } \mathbf{G} = \mathbb{Z}_r \wr S_n. \quad (26)$$

4.6. The group $\mathbf{G} = \mathbb{Z}_2 \wr S_n$

The group $\mathbf{G} = \mathbb{Z}_2 \wr S_n$ is the Weyl group for a root system of type B_n or C_n . The irreducible $\mathbf{G}$ -modules are labeled by pairs $\underline{\alpha} = (\alpha^{(1)}, \alpha^{(2)})$ of partitions such that $|\alpha^{(1)}| + |\alpha^{(2)}| = n$. Since $\omega = -1$ in this case, we have the following formula for the dimension of the space of $\mathbf{G}$ -invariants in $V^{\otimes k}$ in part (a) of the next Theorem. Part (b) is an immediate consequence of Theorem 4.1 (b).

Theorem 4.2. Assume $\mathbf{G} = \mathbb{Z}_2 \wr S_n$ is the Weyl group for a root system of type B_n or C_n and V is the n -dimensional $\mathbf{G}$ -module on which $\mathbf{G}$ acts by monomial matrices with entries ± 1 . Then

(a)

$$\dim(V^{\otimes k})^{\mathbf{G}} = \dim Z_k^0(\mathbf{G}) = \sum_{\underline{\alpha} \in \Lambda(\mathbf{G})} \frac{\left(F(\alpha^{(1)}) - F(\alpha^{(2)}) \right)^k}{2^{\mathbf{p}(\underline{\alpha})} \prod_{j=1}^n j^{\mathbf{p}_j(\underline{\alpha})} (\mathbf{p}_j(\alpha^{(1)})! \cdot \mathbf{p}_j(\alpha^{(2)})!)} \quad (27)$$

where $\underline{\alpha}$ ranges over pairs $\underline{\alpha} = (\alpha^{(1)}, \alpha^{(2)})$ of partitions such that $|\alpha^{(1)}| + |\alpha^{(2)}| = n$; $\mathbf{p}_j(\alpha^{(i)})$ and $\mathbf{p}(\underline{\alpha})$ are as in (25); and $F(\alpha^{(i)})$ is the number of fixed points of a permutation of cycle type $\alpha^{(i)}$.

(b) The exponential generating function for the $\mathbf{G}$ -invariants is

$$g^0(t) = \sum_{k=0}^{\infty} \dim(V^{\otimes k})^{\mathbf{G}} \frac{t^k}{k!} = \frac{1}{2^n n!} \sum_{m=1}^n 2^m \left(\cosh(F_n(m)t) \right)^m.$$

Remark 4.3. In [28], Tanabe investigated the centralizer algebra $Z_k(\mathbf{G})$, where $\mathbf{G}$ is a complex reflection group $\mathbf{G}(m, p, n)$ viewed as $n \times n$ matrices acting on $V = \mathbb{C}^n$. The group $\mathbf{G}(r, 1, n)$ is the wreath product $\mathbb{Z}_r \wr S_n$. Using results from [28], we showed in [7] for $\mathbf{G} = \mathbb{Z}_2 \wr S_n$ that

$$\dim Z_k(\mathbf{G}) = \sum_{s=1}^n T(k, s),$$

where $T(k, s)$ is the number of set partitions of a set of size $2k$ into s nonempty disjoint parts of *even* size. The numbers $T(k, s)$ correspond to sequence A156289 in the Online Encyclopedia of Integer Sequences [OEIS] and have many different interpretations. They are known to satisfy

$$T(k, s) = \frac{1}{s! 2^{s-1}} \sum_{j=1}^s (-1)^{s-j} \binom{2s}{s-j} j^{2k} = \sum_{\lambda} \frac{1}{\prod_{j \geq 1} p_j(\lambda)} \binom{2k}{2\lambda_1, 2\lambda_2, \dots, 2\lambda_s},$$

where the last sum is over all partitions $\lambda = \{\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_s > 0\}$ of k into s nonzero parts λ_i (see [7, Sec. 4.2] for details). In particular, since $\mathbf{V}$ is self-dual, we see that

$$\dim (\mathbf{V}^{\otimes 2k})^{\mathbf{G}} = \dim Z_k(\mathbf{G}) = \sum_{s=1}^n T(k, s), \quad \text{for } \mathbf{G} = \mathbb{Z}_2 \wr S_n. \quad (28)$$

It would be interesting to show the equivalence of the formulas in Theorem 4.1 and (27) and then relate them (with $2k$ in place of k) to (28).

5. $\mathbf{G} = \mathbf{GL}_2(\mathbb{F}_q)$ and $\mathbf{G} = \mathbf{SL}_2(\mathbb{F}_q)$

Let $\mathbb{F}_q$ be a finite field of q elements. Then $q = p^\ell$ for some prime p and some $\ell \geq 1$, and we assume p is odd to simplify considerations. In this section, $\mathbf{G}$ is the general linear group $\mathbf{GL}_2(\mathbb{F}_q)$ of 2×2 invertible matrices over $\mathbb{F}_q$ or the special linear subgroup $\mathbf{SL}_2(\mathbb{F}_q)$ of matrices with determinant equal to 1. We assume $\mathbf{V} = \text{Ind}_{\mathbf{B}}^{\mathbf{G}} \mathbf{B}_0$, the $\mathbf{G}$ -module induced from the trivial module $\mathbf{B}_0$ for the subgroup $\mathbf{B}$ of upper triangular matrices in $\mathbf{G}$, and $\mathbf{V}_q$ is its q -dimensional irreducible summand, which is the Steinberg module. (Here we write $\mathbf{V}_q$ rather than the customary St , to emphasize its analogy to $\mathbf{V}$ in previous sections.) Our aim in this section is to develop a formula for $\dim (\mathbf{V}^{\otimes k})^{\mathbf{G}}$ and for $\dim (\mathbf{V}_q^{\otimes k})^{\mathbf{G}}$ and to determine the corresponding Poincaré series for the tensor invariants.

5.1. $\mathbf{G} = \mathbf{GL}_2(\mathbb{F}_q)$

Let $\mathbf{B} = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \mid x, z \in \mathbb{F}_q^\times, y \in \mathbb{F}_q \right\}$ be the Borel subgroup of upper-triangular matrices in $\mathbf{G} = \mathbf{GL}_2(\mathbb{F}_q)$ and $\mathbf{V}$ be the induced $\mathbf{G}$ -module $\mathbf{V} = \text{Ind}_{\mathbf{B}}^{\mathbf{G}} \mathbf{B}_0 = \mathbb{C}[\mathbf{G}] \otimes_{\mathbb{C}[\mathbf{B}]} \mathbf{B}_0$. Since the order of $\mathbf{G}$ is $q(q+1)(q-1)^2$ and the order of $\mathbf{B}$ is $q(q-1)^2$, we have $\dim \mathbf{V} = q+1$. The module $\mathbf{V}$ decomposes into a sum $\mathbf{V} = \mathbf{G}_0 \oplus \mathbf{V}_q$ of a copy of the trivial $\mathbf{G}$ -module $\mathbf{G}_0$ and a copy of a q -dimensional irreducible $\mathbf{G}$ -module $\mathbf{V}_q$ (the Steinberg module).

Let ε be a non-square in $\mathbb{F}_q^\times$, and define the following elements of G ,

$$\mathbf{a}_x = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}, \quad \mathbf{b}_x = \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}, \quad \mathbf{c}_{x,y} = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, \quad \mathbf{d}_{x,y} = \begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix}$$

$$(x \in \mathbb{F}_q^\times) \quad (x \in \mathbb{F}_q^\times) \quad (x, y \in \mathbb{F}_q^\times, x \neq y) \quad (y \in \mathbb{F}_q^\times)$$

We will use the information in the table below, which can be derived from [12, Sec. 5.2]. As before, $\mathbf{c}_\mu$, $\mu \in \Lambda(G)$, is a representative of the conjugacy class $\mathcal{C}_\mu$ of G .

$\mathbf{c}_\mu$	$\mathbf{a}_x$	$\mathbf{b}_x$	$\mathbf{c}_{x,y}$	$\mathbf{d}_{x,y}$
No. of such classes	$q-1$	$q-1$	$\frac{1}{2}(q-1)(q-2)$	$\frac{1}{2}q(q-1)$
$ \mathcal{C}_\mu $	1	q^2-1	q^2+q	q^2-q
$\chi_V(\mathbf{c}_\mu)$	$q+1$	1	2	0
$\chi_{V_q}(\mathbf{c}_\mu)$	q	0	1	-1

(29)

Therefore, we have the following consequence of Theorem 2.1.

Theorem 5.1. Assume $G = \mathrm{GL}_2(\mathbb{F}_q)$ where q is odd.

- (a) For the G -module $V = \mathrm{Ind}_B^G B_0 = G_0 \oplus V_q$ induced from the trivial module B_0 for the Borel subgroup B of upper-triangular matrices in G ,

$$\dim (V^{\otimes k})^G = \begin{cases} 1 & \text{when } k = 0, \\ \frac{1}{q(q-1)} ((q+1)^{k-1} + q(q-2) \cdot 2^{k-1} + q-1) & \text{when } k \geq 1. \end{cases}$$

The Poincaré series for the G -invariants $T(V)^G$ in $T(V) = \bigoplus_{k=0}^{\infty} V^{\otimes k}$ is

$$P^0(t) = \sum_{k=0}^{\infty} \dim (V^{\otimes k})^G t^k = \frac{1 - (q+3)t + (2q+3)t^2 - qt^3}{(1-t)(1-2t)(1-(1+q)t)}.$$

- (b) For the Steinberg module V_q , $\dim (V_q^{\otimes k})^G = 1$ when $k = 0$, and

$$\dim (V_q^{\otimes k})^G = \frac{1}{2(q^2-1)} (2q^{k-1} - q(q-1)(-1)^{k-1} + (q+1)(q-2)) \quad \text{when } k \geq 1,$$

$$= \begin{cases} \frac{q^{2\ell}-1}{q^2-1} = \sum_{j=0}^{\ell-1} q^{2j} & \text{if } k = 2\ell+1 \geq 1, \\ 1 + q \frac{q^{2\ell-2}-1}{q^2-1} = 1 + \sum_{j=0}^{\ell-2} q^{2j+1} & \text{if } k = 2\ell \geq 2. \end{cases}$$

The Poincaré series $P_q^0(t)$ for the G -invariants $T(V_q)^G$ in $T(V_q) = \bigoplus_{k=0}^{\infty} V_q^{\otimes k}$ is

$$P_q^0(t) = \sum_{k=0}^{\infty} \dim(V_q^{\otimes k})^G t^k = \frac{1 - qt + t^3}{(1-t)(1+t)(1-qt)}.$$

Proof. (a) From Theorem 2.1 and Table 29 we know that for $k \geq 1$,

$$\begin{aligned} \dim(V_q^{\otimes k})^G &= \dim Z_k^0(G) = \frac{1}{|G|} \sum_{\mu \in \Lambda(G)} |\mathcal{C}_\mu| \chi_V(c_\mu)^k \\ &= \frac{1}{(q-1)^2 q(q+1)} \left((q-1)(q+1)^k + (q-1)(q^2-1)1^k + \right. \\ &\quad \left. \frac{1}{2}q(q+1)(q-1)(q-2)2^k + \frac{1}{2}q^2(q-1)^2 0^k \right) \\ &= \frac{1}{q(q-1)} ((q+1)^{k-1} + q(q-2) \cdot 2^{k-1} + q-1). \end{aligned}$$

Therefore,

$$\begin{aligned} P^0(t) &= \sum_{k=0}^{\infty} \dim(V_q^{\otimes k})^G t^k = 1 + \frac{1}{q(q-1)} \left(\sum_{k=1}^{\infty} (q+1)^{k-1} + q(q-2) \cdot 2^{k-1} + (q-1) \right) t^k \\ &= 1 + \frac{1}{q(q-1)} \left(t \sum_{k=1}^{\infty} (q+1)^{k-1} t^{k-1} + q(q-2)t \sum_{k=1}^{\infty} 2^{k-1} t^{k-1} + (q-1)t \sum_{k=1}^{\infty} t^{k-1} \right) \\ &= 1 + \frac{1}{q(q-1)} \left(\frac{t}{1-(q+1)t} + \frac{q(q-2)t}{1-2t} + \frac{(q-1)t}{1-t} \right) \\ &= \frac{1 - (q+3)t + (2q+3)t^2 - qt^3}{(1-t)(1-2t)(1-(q+1)t)}. \end{aligned}$$

(b) Now for V_q and $k \geq 1$, we have

$$\begin{aligned} \dim(V_q^{\otimes k})^G &= \frac{1}{|G|} \sum_{\mu \in \Lambda(G)} |\mathcal{C}_\mu| \chi_{V_q}(c_\mu)^k \\ &= \frac{1}{(q-1)^2 q(q+1)} \left((q-1)q^k + (q-1)(q^2-1)0^k + \right. \\ &\quad \left. \frac{1}{2}q(q+1)(q-1)(q-2)1^k + \frac{1}{2}q^2(q-1)^2(-1)^k \right) \\ &= \frac{1}{2(q^2-1)} (2q^{k-1} + q(q-1)(-1)^k + (q+1)(q-2)) \end{aligned}$$

$$= \begin{cases} \frac{q^{2\ell} - 1}{q^2 - 1} = \sum_{j=0}^{\ell-1} q^{2j} & \text{if } k = 2\ell + 1 \geq 1, \\ 1 + q \frac{q^{2\ell-2} - 1}{q^2 - 1} = 1 + q \sum_{j=0}^{\ell-2} q^{2j} & \text{if } k = 2\ell \geq 2, \end{cases}$$

and hence,

$$\begin{aligned} P^0(t) &= 1 + \frac{1}{2(q^2 - 1)} \sum_{k=1}^{\infty} (2q^{k-1} + (q+1)(q-2) + q(q-1)(-1)^k) t^k \\ &= \frac{1 - qt + t^3}{(1-t)(1+t)(1-qt)}. \quad \square \end{aligned}$$

5.2. $G = \mathrm{SL}_2(\mathbb{F}_q)$

For the group $G = \mathrm{SL}_2(\mathbb{F}_q)$ (q odd), we introduce the following elements of G :

$$u_x = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \quad (x \neq 0), \quad v_y = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}, \quad w_{x,y} = \begin{pmatrix} x & y \\ y\varepsilon & x \end{pmatrix} \quad (x^2 - \varepsilon y^2 = 1).$$

We will use the information in the following table, which can be derived from [20, Chap. 3] or [12, Sec. 5.2]. As before, c_μ , $\mu \in \Lambda(G)$, is a representative of the conjugacy class $\mathcal{C}_\mu$ of G .

c_μ	± 1	$u_x, x \neq \pm 1$	$v_y, y = 1, \varepsilon$	$-v_y, y = -1, -\varepsilon$	$w_{x,y}, x \neq \pm 1$
No. of such classes	2	$\frac{1}{2}(q-3)$	2	2	$\frac{1}{2}(q-1)$
$ \mathcal{C}_\mu $	1	$q(q+1)$	$\frac{1}{2}(q^2-1)$	$\frac{1}{2}(q^2-1)$	$q(q-1)$
$\chi_V(c_\mu)$	$q+1$	2	1	1	0
$\chi_{V_q}(c_\mu)$	q	1	0	0	-1

(30)

The order of $G = \mathrm{SL}_2(\mathbb{F}_q)$ is $q(q-1)(q+1)$ and the order of its Borel subgroup B of upper triangular matrices is $q(q-1)$. Therefore, the induced G -module $V = \mathrm{Ind}_B^G B_0$ has dimension $q+1$, and $V = G_0 \oplus V_q$, where V_q is the q -dimensional irreducible Steinberg module for G . Using Table 30 and Theorem 2.1, we have the next result.

Theorem 5.2. Assume $G = \mathrm{SL}_2(\mathbb{F}_q)$, where q is odd.

- (a) For $V = \mathrm{Ind}_B^G B_0 = G_0 \oplus V_q$, the G -module over $\mathbb{C}$ induced from the trivial module B_0 for the Borel subgroup B of upper-triangular matrices in G , we have

$$\dim (V^{\otimes k})^G = \begin{cases} 1 & \text{when } k = 0 \\ \frac{1}{q(q-1)} (2(q+1)^{k-1} + q(q-3) \cdot 2^{k-1} + 2(q-1)) & \text{when } k \geq 1. \end{cases}$$

The Poincaré series for the G -invariants $T(V)^G$ in $T(V) = \bigoplus_{k=0}^{\infty} V^{\otimes k}$ is

$$P^0(t) = \sum_{k=0}^{\infty} \dim(V^{\otimes k})^G t^k = \frac{1 - (q+3)t + (2q+3)t^2 - (q-1)t^3}{(1-t)(1-2t)(1-(q+1)t)}.$$

(b) For the Steinberg module V_q , $\dim(V_q^{\otimes k})^G = 1$ when $k = 0$, and

$$\begin{aligned} \dim(V_q^{\otimes k})^G &= \frac{1}{2(q^2-1)} (4q^{k-1} + (q-1)^2(-1)^k + (q-3)(q+1)) \quad \text{when } k \geq 1, \\ &= \begin{cases} \frac{2(q^{2\ell}-1)}{q^2-1} = 2 \sum_{j=0}^{\ell-1} q^{2j} & \text{if } k = 2\ell + 1 \geq 1, \\ 1 + 2q \frac{(q^{2\ell-2}-1)}{q^2-1} = 1 + 2 \sum_{j=0}^{\ell-2} q^{2j+1} & \text{if } k = 2\ell \geq 2. \end{cases} \end{aligned}$$

(b) The Poincaré series $P_q^0(t)$ for the G -invariants $T(V_q)^G$ in $T(V_q) = \bigoplus_{k=0}^{\infty} V_q^{\otimes k}$ is

$$P_q^0(t) = \sum_{k=0}^{\infty} \dim(V_q^{\otimes k})^G t^k = \frac{1 - qt + 2t^3}{(1+t)(1-t)(1-qt)}.$$

Proof. The proofs are analogous to those for Theorem 5.1 and are left to the reader. $\square$

6. The case G is abelian and exponential generating functions

It is convenient to regard an arbitrary finite abelian group $(G, +)$ as a multiplicative group and write e^a for $a \in G$, so that the group operation is given by $e^a e^b = e^{a+b}$, $a, b \in G$, where the sum $a + b$ is addition in G . The identity element is e^0 . Since G is abelian, the irreducible G -modules are all one-dimensional, and we label them and the conjugacy classes with the elements of G . Thus, for $a \in G$, let $G_a = \mathbb{C}x_a$, where $e^b x_a = \chi_a(b)x_a$, and let χ_a denote the corresponding character. The characters satisfy

$$\begin{aligned} \chi_a(b + b') &= \chi_a(b)\chi_a(b') \quad \text{for all } a, b, b' \in G, \text{ and} \\ \chi_{a+a'}(b) &= \chi_a(b)\chi_{a'}(b) \quad \text{for all } a, a', b \in G, \end{aligned}$$

as $G_a \otimes G_{a'} \cong G_{a+a'}$, for all $a, a' \in G$. Since $\chi_a(b)\chi_{-a}(b) = \chi_{a-a'}(b) = \chi_0(b) = 1$ and $\chi_a(0) = 1$ for all $a, b \in G$, the following hold:

$$\begin{aligned} \chi_{-a}(b) &= \chi_a(b)^{-1} = \overline{\chi_a(b)} \\ \chi_a(-b) &= \chi_a(b)^{-1} = \overline{\chi_a(b)}. \end{aligned}$$

By the fundamental theorem of finite abelian groups, we may suppose that $G = \mathbb{Z}_{r_1} \times \mathbb{Z}_{r_2} \times \cdots \times \mathbb{Z}_{r_n}$ where the r_j are powers of not necessarily distinct primes. The

elements of $\mathbf{G}$ have the form $\mathbf{e}^{\mathbf{b}}$, where $\mathbf{b} = (b_1, b_2, \dots, b_n)$ and $b_j \in \mathbb{Z}_{r_j}$ for each j . Set $\omega_j = e^{2\pi i/r_j}$. Then $\mathbf{G}_{\mathbf{a}} = \mathbb{C}x_{\mathbf{a}}$, where

$$\mathbf{e}^{\mathbf{b}}x_{\mathbf{a}} = \chi_{\mathbf{a}}(\mathbf{b})x_{\mathbf{a}} \quad \text{and} \quad \chi_{\mathbf{a}}(\mathbf{b}) = \omega_1^{a_1b_1}\omega_2^{a_2b_2}\dots\omega_n^{a_nb_n}.$$

Let ε_j be the n -tuple with 1 in position j and 0 for all its other components. Here we suppose that $\mathbf{V} = \mathbf{G}_{\varepsilon_1} \oplus \mathbf{G}_{\varepsilon_2} \oplus \dots \oplus \mathbf{G}_{\varepsilon_n}$, so for $\mathbf{b} = (b_1, b_2, \dots, b_n) \in \mathbf{G}$, the character values are given by

$$\chi_{\mathbf{V}}(\mathbf{b}) = \sum_{j=1}^n \chi_{\varepsilon_j}(\mathbf{b}) = \sum_{j=1}^n \omega_j^{b_j} \quad \chi_{\mathbf{V}^{\otimes k}}(\mathbf{b}) = \chi_{\mathbf{V}}(\mathbf{b})^k = \left(\sum_{j=1}^n \omega_j^{b_j} \right)^k. \quad (31)$$

We have the following corollary to Theorem 2.1:

Corollary 6.1. *The number of walks of k -steps from node $\mathbf{a}$ to node $\mathbf{c}$ on the McKay quiver $\mathbf{Q}_{\mathbf{V}}(\mathbf{G})$ for $\mathbf{G} = \mathbb{Z}_{r_1} \times \mathbb{Z}_{r_2} \times \dots \times \mathbb{Z}_{r_n}$ and $\mathbf{V} = \mathbf{G}_{\varepsilon_1} \oplus \mathbf{G}_{\varepsilon_2} \oplus \dots \oplus \mathbf{G}_{\varepsilon_n}$ is*

$$(\mathbf{A}^k)_{\mathbf{a}, \mathbf{c}} = \sum_{0 \leq \ell_1, \ell_2, \dots, \ell_n \leq k} \binom{k}{\ell_1, \ell_2, \dots, \ell_n}$$

where the sum is over all $\ell_1, \ell_2, \dots, \ell_n$ such that $\ell_1 + \ell_2 + \dots + \ell_n = k$ and $c_i - a_i \equiv \ell_i \pmod{r_i}$ for all $i \in [1, n] = \{1, 2, \dots, n\}$.

Proof. Now

$$\begin{aligned} (\mathbf{A}^k)_{\mathbf{a}, \mathbf{c}} &= \sum_{0 \leq \ell_1, \dots, \ell_n \leq k} |\mathbf{G}|^{-1} \sum_{\mathbf{b} \in \mathbf{G}} \chi_{\mathbf{a}}(\mathbf{b}) \chi_{\mathbf{V}}^k(\mathbf{b}) \overline{\chi_{\mathbf{c}}(\mathbf{b})} = |\mathbf{G}|^{-1} \sum_{\mathbf{b} \in \mathbf{G}} \chi_{\mathbf{a}-\mathbf{c}}(\mathbf{b}) \chi_{\mathbf{V}}^k(\mathbf{b}) \\ &= |\mathbf{G}|^{-1} \sum_{\mathbf{b} \in \mathbf{G}} \omega_1^{(a_1-c_1)b_1} \dots \omega_n^{(a_n-c_n)b_n} \left(\sum_{j=1}^n \omega_j^{b_j} \right)^k \\ &= |\mathbf{G}|^{-1} \sum_{\mathbf{b} \in \mathbf{G}} \omega_1^{(a_1-c_1)b_1} \dots \omega_n^{(a_n-c_n)b_n} \left(\sum_{0 \leq \ell_1, \dots, \ell_n \leq k} \binom{k}{\ell_1, \ell_2, \dots, \ell_n} \omega_1^{\ell_1 b_1} \dots \omega_n^{\ell_n b_n} \right) \\ &= |\mathbf{G}|^{-1} \sum_{0 \leq \ell_1, \ell_2, \dots, \ell_n \leq k} \binom{k}{\ell_1, \ell_2, \dots, \ell_n} \left(\left(\sum_{b_1=0}^{r_1-1} \omega_1^{(a_1-c_1+\ell_1)b_1} \right) \dots \right. \\ &\quad \left. \left(\sum_{b_n=0}^{r_n-1} \omega_n^{(a_n-c_n+\ell_n)b_n} \right) \right) \\ &= \sum_{0 \leq \ell_1, \ell_2, \dots, \ell_n \leq k} \binom{k}{\ell_1, \ell_2, \dots, \ell_n} \end{aligned}$$

by applying (13) repeatedly, where the sum is over all $\ell_1, \ell_2, \dots, \ell_n$ such that $\ell_1 + \ell_2 + \dots + \ell_n = k$ and $\ell_i \equiv c_i - a_i \pmod{r_i}$ for all $i \in [1, n]$. $\square$

6.1. Exponential generating functions

For $\mathbf{c} \in \mathbf{G} = \mathbb{Z}_{r_1} \times \mathbb{Z}_{r_2} \times \dots \times \mathbb{Z}_{r_n}$ and $\mathbf{V} = \mathbf{G}_{\varepsilon_1} \oplus \mathbf{G}_{\varepsilon_2} \oplus \dots \oplus \mathbf{G}_{\varepsilon_n}$, let $g^{\mathbf{c}}(t) := \sum_{k=0}^{\infty} (A^k)_{0,\mathbf{c}} \frac{t^k}{k!}$ denote the exponential generating function for walks of k steps from 0 to $\mathbf{c}$ on the McKay quiver $\mathbf{Q}_{\mathbf{V}}(\mathbf{G})$ (and also for the multiplicity of $\mathbf{G}_{\mathbf{c}}$ in $\mathbf{V}^{\otimes k}$ and for dimension of the irreducible module $Z_k^{\mathbf{c}}(\mathbf{G})$ for the centralizer algebra). We determine an expression for $g^{\mathbf{c}}(t)$ in terms of generalized hyperbolic functions.

The *generalized hyperbolic function* $h_j(t, r)$ for $j \in \mathbb{Z}$ is defined by

$$h_j(t, r) := r^{-1} \sum_{m=0}^{r-1} \omega^{(1-j)m} e^{\omega^m t},$$

where $\omega = e^{2\pi i/r}$. In particular,

$$h_1(t, r) = r^{-1} \sum_{m=0}^{r-1} e^{\omega^m t}, \quad (32)$$

so that $h_1(t, 1) = e^t$ and $h_1(t, 2) = \cosh t$. Because

$$h_{j+r}(t, r) = h_j(t, r), \quad \text{for } j \in \mathbb{Z},$$

there are r distinct generalized hyperbolic functions $h_j(t, r)$ for a fixed value of r .

Theorem 6.2. For $\mathbf{G} = \mathbb{Z}_{r_1} \times \mathbb{Z}_{r_2} \times \dots \times \mathbb{Z}_{r_n}$ and $\mathbf{c} = (c_1, c_2, \dots, c_n) \in \mathbf{G}$, the exponential generating function for the number of walks of k steps from 0 to $\mathbf{c}$ on $\mathbf{Q}_{\mathbf{V}}(\mathbf{G})$ is

$$g^{\mathbf{c}}(t) = \sum_{k=0}^{\infty} (A^k)_{0,\mathbf{c}} \frac{t^k}{k!} = h_{1+c_1}(t, r_1) h_{1+c_2}(t, r_2) \cdots h_{1+c_n}(t, r_n).$$

Before giving the proof, we note the following immediate consequences.

Corollary 6.3. For $\mathbf{G} = \mathbb{Z}_{r_1} \times \mathbb{Z}_{r_2} \times \dots \times \mathbb{Z}_{r_n}$ and $\mathbf{V} = \mathbf{G}_{\varepsilon_1} \oplus \mathbf{G}_{\varepsilon_2} \oplus \dots \oplus \mathbf{G}_{\varepsilon_n}$,

- (a) $g^0(t) = \sum_{k=0}^{\infty} (A^k)_{0,0} \frac{t^k}{k!} = h_1(t, r_1) h_1(t, r_2) \cdots h_1(t, r_n).$
- (b) When $\mathbf{G} = \mathbb{Z}_r^n$, then $g^0(t) = h_1(t, r)^n.$

Remark 6.4. Part (b) of this corollary generalizes [7, Cor. 4.29], which says that the exponential generating function for the number of walks on a hypercube of order n is given by $g^0(t) = (\cosh t)^n = h_1(t, 2)^n$. Theorem 4.25 of [7] shows that for $\mathbb{Z}_2^n$,

$$g^c(t) = (\cosh t)^{n-h(c)} (\sinh t)^{h(c)},$$

where $h(c)$ is the Hamming weight of c (the number of ones in c). This follows directly from Theorem 6.2, since each component of c equal to 1 contributes a factor $h_2(t, 2) = \sinh t$, and each component of c equal to 0 gives a factor $h_1(t, 2) = \cosh t$.

Proof of Theorem 6.2. Observe that by (31) and Corollary 6.1,

$$\begin{aligned} g^c(t) &= \sum_{k=0}^{\infty} (A)_{0,c}^k \frac{t^k}{k!} \\ &= |G|^{-1} \sum_{k=0}^{\infty} \sum_{b=(b_1, \dots, b_n) \in G} \omega_1^{-b_1 c_1} \dots \omega_n^{-b_n c_n} \left(\sum_{j=1}^n \omega_j^{b_j} \right)^k \frac{t^k}{k!} \\ &= r_1^{-1} \dots r_n^{-1} \sum_{k=0}^{\infty} \sum_{b \in G} \omega_1^{-b_1 c_1} \dots \omega_n^{-b_n c_n} \left(\sum_{\ell_1 + \dots + \ell_n = k} \frac{\omega_1^{b_1 \ell_1} t^{\ell_1}}{\ell_1!} \dots \frac{\omega_n^{b_n \ell_n} t^{\ell_n}}{\ell_n!} \right) \\ &= \left(r_1^{-1} \sum_{b_1=0}^{r_1-1} \sum_{\ell_1=0}^{\infty} \omega_1^{-b_1 c_1} \frac{\omega_1^{b_1 \ell_1} t^{\ell_1}}{\ell_1!} \right) \times \dots \times \left(r_n^{-1} \sum_{b_n=0}^{r_n-1} \sum_{\ell_n=0}^{\infty} \omega_n^{-b_n c_n} \frac{\omega_n^{b_n \ell_n} t^{\ell_n}}{\ell_n!} \right) \\ &= \left(r_1^{-1} \sum_{b_1=0}^{r_1-1} \omega_1^{-b_1 c_1} e^{\omega_1^{b_1} t} \right) \times \dots \times \left(r_n^{-1} \sum_{b_n=0}^{r_n-1} \omega_n^{-b_n c_n} e^{\omega_n^{b_n} t} \right) \\ &= h_{1+c_1}(t, r_1) h_{1+c_2}(t, r_2) \dots h_{1+c_n}(t, r_n). \quad \square \end{aligned}$$

Using (13) and the definition of the generalized hyperbolic function $h_j(t, r)$, one sees that the Taylor series expansion of $h_j(t, r)$ is given by

$$h_j(t, r) = \sum_{q=0}^{\infty} \frac{t^{qr+j-1}}{(qr+j-1)!} \quad (33)$$

Suppose $c = (c_1, c_2, \dots, c_n) \in G = \mathbb{Z}_{r_1} \times \mathbb{Z}_{r_2} \times \dots \times \mathbb{Z}_{r_n}$, where $0 \leq c_j < r_j$ for all j , and let $|c| = \sum_{j=1}^n c_j$. We have shown in Theorem 6.2 that the exponential generating function $g^c(t)$ is given by

$$g^c(t) = \sum_{k=0}^{\infty} (A^k)_{0,c} \frac{t^k}{k!} = h_{1+c_1}(t, r_1) h_{1+c_2}(t, r_2) \dots h_{1+c_n}(t, r_n).$$

Combining that with the expressions coming from (33), we have

$$\begin{aligned}
g^c(t) &= h_{1+c_1}(t, r_1) h_{1+c_2}(t, r_2) \cdots h_{1+c_n}(t, r_n) \\
&= \left(\sum_{q_1=0}^{\infty} \frac{t^{q_1 r_1 + c_1}}{(q_1 r_1 + c_1)!} \right) \left(\sum_{q_2=0}^{\infty} \frac{t^{q_2 r_2 + c_2}}{(q_2 r_2 + c_2)!} \right) \cdots \left(\sum_{\ell_n=0}^{\infty} \frac{t^{q_n r_n + c_n}}{(q_n r_n + c_n)!} \right) \\
&= \sum_{k=0}^{\infty} \sum_{q_1 r_1 + \cdots + q_n r_n + |c| = k} \frac{k!}{(q_1 r_1 + c_1)! (q_2 r_2 + c_2)! \cdots (q_n r_n + c_n)!} \frac{t^k}{k!}.
\end{aligned}$$

Setting $q_i r_i + c_i = \ell_i$ for $i = 1, 2, \dots, n$ gives the result in Corollary 6.1 with $\mathbf{a} = \mathbf{0}$, which provides a formula for the dimension of the irreducible module $Z_k^c(\mathbf{G})$ for the centralizer algebra $Z_k(\mathbf{G})$:

$$\dim Z_k^c(\mathbf{G}) = (A^k)_{0,c} = \sum_{0 \leq \ell_1, \ell_2, \dots, \ell_n \leq k} \binom{k}{\ell_1, \ell_2, \dots, \ell_n}. \quad (34)$$

The sum is over all $0 \leq \ell_1, \ell_2, \dots, \ell_n \leq k$ such that $\ell_1 + \ell_2 + \cdots + \ell_n = k$ and $\ell_i \equiv c_i \pmod{r_i}$ for all $i \in [1, n]$. In particular, when $\mathbf{G} = \mathbb{Z}_{r_1} \times \mathbb{Z}_{r_2} \times \cdots \times \mathbb{Z}_{r_n}$ and $\mathbf{c} = \mathbf{0}$, then

$$\dim (V^{\otimes k})^{\mathbf{G}} = \dim Z_k^0(\mathbf{G}) = \sum_{0 \leq \ell_1, \ell_2, \dots, \ell_n \leq k} \binom{k}{\ell_1, \ell_2, \dots, \ell_n},$$

where $\ell_1 + \ell_2 + \cdots + \ell_n = k$ and $\ell_i \equiv 0 \pmod{r_i}$ for all $i \in [1, n]$.

An alternate approach to the result in (34) is via characters. For $\mathbf{G} = \mathbb{Z}_{r_1} \times \cdots \times \mathbb{Z}_{r_n}$ and $\mathbf{V} = \mathbf{G}_{\varepsilon_1} \oplus \mathbf{G}_{\varepsilon_2} \oplus \cdots \oplus \mathbf{G}_{\varepsilon_n}$, where $\mathbf{G}_{\varepsilon_j} = \mathbb{C}x_{\varepsilon_j}$ for all j , the character of the k th tensor power of $\mathbf{V}$ is given by

$$\begin{aligned}
\chi_{V^{\otimes k}} &= \chi_{\mathbf{V}}^k = (\chi_{\varepsilon_1} + \cdots + \chi_{\varepsilon_n})^k \\
&= \sum_{\substack{0 \leq \ell_1, \ell_2, \dots, \ell_n \leq k \\ \ell_1 + \ell_2 + \cdots + \ell_n = k}} \binom{k}{\ell_1, \ell_2, \dots, \ell_n} \chi_{\varepsilon_1}^{\ell_1} \cdots \chi_{\varepsilon_n}^{\ell_n} \\
&= \sum_{\substack{0 \leq \ell_1, \ell_2, \dots, \ell_n \leq k \\ \ell_1 + \ell_2 + \cdots + \ell_n = k}} \binom{k}{\ell_1, \ell_2, \dots, \ell_n} \chi_{\ell_1 \varepsilon_1 + \ell_2 \varepsilon_2 + \cdots + \ell_n \varepsilon_n}.
\end{aligned}$$

Now for $\mathbf{c} = (c_1, c_2, \dots, c_n)$ with $0 \leq c_i < r_i$ for all $i \in [1, n]$, the multiplicity of the character $\chi_{\mathbf{c}}$ in this expression is exactly the number of n -tuples $(\ell_1, \ell_2, \dots, \ell_n)$ such that $\ell_i \equiv c_i \pmod{r_i}$ for all $i \in [1, n]$, as in (34).

Example 6.5. Consider $\mathbf{G} = \mathbb{Z}_4 \times \mathbb{Z}_2$ and the tensor power $V^{\otimes 6}$ for $\mathbf{V} = \mathbf{G}_{\varepsilon_1} \oplus \mathbf{G}_{\varepsilon_2}$. Then

$$\begin{aligned}
(\chi_{\varepsilon_1} + \chi_{\varepsilon_2})^6 &= \chi_{6\varepsilon_1} + 6\chi_{5\varepsilon_1 + \varepsilon_2} + 15\chi_{4\varepsilon_1 + 2\varepsilon_2} + 20\chi_{3\varepsilon_1 + 3\varepsilon_2} \\
&\quad + 15\chi_{2\varepsilon_1 + 4\varepsilon_2} + 6\chi_{\varepsilon_1 + 5\varepsilon_2} + \chi_{6\varepsilon_2} \\
&= 16\chi_{2\varepsilon_1} + 12\chi_{\varepsilon_1 + \varepsilon_2} + 16\chi_0 + 20\chi_{3\varepsilon_1 + \varepsilon_2}.
\end{aligned}$$

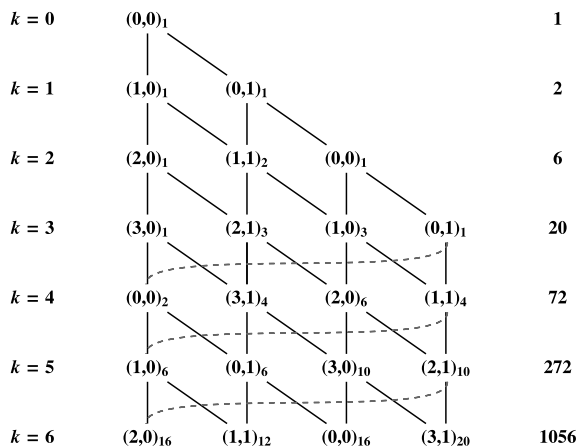


Fig. 6. Levels $k = 0, 1, \dots, 6$ of the Bratteli diagram for $\mathbb{Z}_4 \times \mathbb{Z}_2$.

Thus, $\dim Z_6^{(2,0)}(\mathbb{G}) = 16$, $\dim Z_6^{(1,1)}(\mathbb{G}) = 12$, $\dim Z_6^{(0,0)}(\mathbb{G}) = 16$, and $\dim Z_6^{(3,1)}(\mathbb{G}) = 20$.

6.2. The Bratteli diagram and a basis for $Z_k(\mathbb{G})$ when $\mathbb{G} = \mathbb{Z}_{r_1} \times \mathbb{Z}_{r_2} \times \dots \times \mathbb{Z}_{r_n}$ and $V = G_{\varepsilon_1} \oplus G_{\varepsilon_2} \oplus \dots \oplus G_{\varepsilon_n}$

A walk of k steps on the McKay quiver $Q_V(\mathbb{G})$ from 0 to c corresponds to a path

$$(c^{(0)}, c^{(1)}, \dots, c^{(k)})$$

on the Bratteli diagram $\mathcal{B}_V(\mathbb{G})$ starting at $c^{(0)} = 0 = (0, 0, \dots, 0)$ at level 0 and ending at $c = c^{(k)}$ at level k such that $c^{(i)} \in \mathbb{G}$ for each $1 \leq i \leq k$, and $c^{(i)} = c^{(i-1)} + \varepsilon_{\gamma_i}$ for some $\gamma_i \in [1, n]$, where $c^{(i)}$ is connected to $c^{(i-1)}$ by the edge corresponding to γ_i in $Q_V(\mathbb{G})$. The subscript on node c at level k in $\mathcal{B}_V(\mathbb{G})$ indicates the number of such paths, which is the multiplicity of the irreducible $\mathbb{G}$ -module G_c in $V^{\otimes k}$ and also equal to the dimension of the irreducible $Z_k(\mathbb{G})$ -module $Z_k^c(\mathbb{G})$. The sum of the squares of those dimensions at level k is the number on the right, which is the dimension of the centralizer algebra $Z_k(\mathbb{G})$. Levels $0, 1, \dots, 6$ of the Bratteli diagram for $\mathbb{Z}_4 \times \mathbb{Z}_2$ are displayed in Fig. 6. The nodes of the diagram correspond to elements $c = (c_1, c_2) \in \mathbb{Z}_4 \times \mathbb{Z}_2$ and have $c_1 \in \{0, 1, 2, 3\}$ and $c_2 \in \{0, 1\}$.

Remark 6.6. The subscripts in the last row of the Bratteli diagram in Fig. 6, exactly match with the dimensions determined in Example 6.5. The sequence of numbers in the right-hand column of Fig. 6 (i.e. the dimension $d(k)$ of the centralizer algebra $Z_k(\mathbb{Z}_4 \times \mathbb{Z}_2)$) satisfies $d(k) = a(k-1)$ in sequence [21, A063376], where $a(-1) = 1$ and $a(k-1) = 2^{k-1} + 4^{k-1}$ for $k \geq 1$. Among the objects that $a(k-1)$ counts is the number of closed walks of length $2k$ at a vertex of the circular graph on 8 nodes, which is the same as $\dim Z_k(\mathbb{G})$ for $\mathbb{G} = \mathbb{Z}_8$ and $V = G_1 \oplus G_7$ (see Section 3.1).

Much of the next result is evident from the above considerations.

Theorem 6.7. Assume $G = \mathbb{Z}_{r_1} \times \mathbb{Z}_{r_2} \times \cdots \times \mathbb{Z}_{r_n}$ and $V = G_{\varepsilon_1} \oplus G_{\varepsilon_2} \oplus \cdots \oplus G_{\varepsilon_n}$. Then the following hold:

(i) For $c = (c_1, \dots, c_n) \in G$, a basis for the irreducible $Z_k(G)$ -module $Z_k^c(G) \subseteq V^{\otimes k}$ is

$$\left\{ x(\gamma) := x_{\varepsilon_{\gamma_1}} \otimes \cdots \otimes x_{\varepsilon_{\gamma_k}} \mid \gamma_i \in [1, n] \text{ for all } i \in [1, k], \text{ and } \sum_{i=1}^k \varepsilon_{\gamma_i} = c \right\}.$$

(ii) $e^a x(\gamma) = \chi_c(a) x(\gamma)$ for all $a \in G$ and all $x(\gamma)$ in (i), where $\chi_c(a) = \prod_{j=1}^n \omega_j^{a_j c_j}$ and $\omega_j = e^{2\pi i/r_j}$ for all $j \in [1, n]$, so that $Z_k^c(G)$ is also a G -submodule of $V^{\otimes k}$; it is the sum of all the copies of the irreducible G -module G_c in $V^{\otimes k}$.

(iii) For $\gamma = (\gamma_1, \dots, \gamma_k), \beta = (\beta_1, \dots, \beta_k) \in [1, n]^k$ with $\sum_{i=1}^k \varepsilon_{\gamma_i} = \sum_{i=1}^k \varepsilon_{\beta_i}$, let $E_\gamma^\beta \in \text{End}(V^{\otimes k})$ be defined by $E_\gamma^\beta x(\alpha) = \delta_{\alpha, \gamma} x(\beta)$ for $\alpha \in [1, n]^k$. Then $E_\vartheta^\eta E_\gamma^\beta = \delta_{\beta, \vartheta} E_\eta^\beta$ for all such ϑ, η , and the E_γ^β determine a basis for $Z_k(G) = \text{End}_G(V^{\otimes k})$.

Proof. From the calculation below it is easy to see that the transformations E_γ^β for $\gamma, \beta \in [1, n]^k$ as in (iii) of Theorem 6.7 commute with the action of G on $V^{\otimes k}$, hence belong to $Z_k(G)$. Indeed, suppose $\alpha \in [1, n]^k$ with $\sum_{i=1}^k \varepsilon_{\alpha_i} = c' \in G$, and assume $a \in G$. Then

$$\begin{aligned} e^a E_\gamma^\beta(x(\alpha)) &= \delta_{\alpha, \gamma} e^a x(\beta) = \delta_{\alpha, \gamma} \chi_c(a) x(\beta) \\ E_\gamma^\beta e^a(x(\alpha)) &= \chi_{c'}(a) \delta_{\alpha, \gamma} x(\beta). \end{aligned}$$

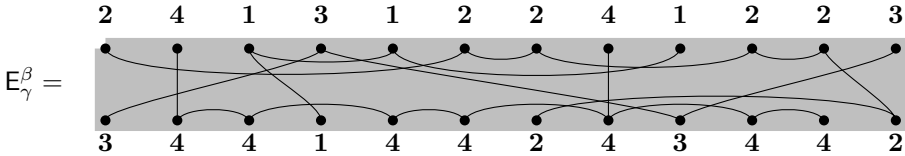
Both expressions are 0 when $\alpha \neq \gamma$, and when $\alpha = \gamma$, then $c' = c$, and the two expressions are identical. The transformations E_γ^β are clearly linearly independent. The number of tuples $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_k)$ such that $\sum_{i=1}^k \varepsilon_{\gamma_i} = c$ is the number of paths from 0 at level 0 to c at level k of the Bratteli diagram $\mathcal{B}_V(G)$, which is $\dim Z_k^c(G)$. Therefore, the number of E_γ^β in (iii) equals $(\dim Z_k^c(G))^2$, and since $\dim Z_k(G) = \sum_{c \in G} (\dim Z_k^c(G))^2$, taking the union of the sets of transformations E_γ^β as c ranges over all the elements of G will give a basis for $Z_k(G)$. $\square$

Remark 6.8. The condition $\sum_{i=1}^k \varepsilon_{\gamma_i} = \sum_{i=1}^k \varepsilon_{\beta_i}$ in Theorem 6.7 is equivalent to saying $(\#\gamma_i = j) \equiv (\#\beta_i = j) \pmod{r_j}$ for all $j = 1, \dots, n$. That interpretation leads to the diagrammatic point of view that we describe next.

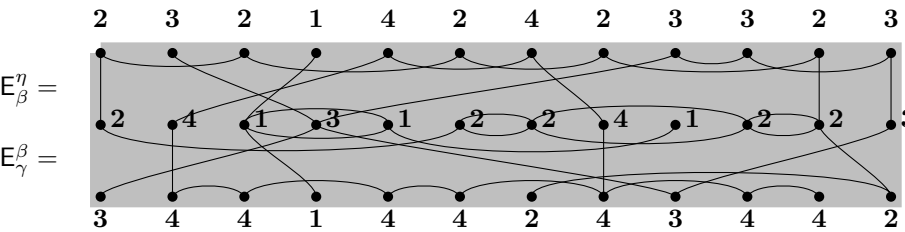
6.3. A diagram basis for $Z_k(G)$ for $G = \mathbb{Z}_{r_1} \times \mathbb{Z}_{r_2} \times \cdots \times \mathbb{Z}_{r_n}$

In this section, we present a realization $Z_k(G)$ as a diagram algebra. We identify the basis element E_γ^β with a diagram having two rows of k nodes. The components of $\gamma = (\gamma_1, \dots, \gamma_k)$, which lie in $[1, n]$, label the nodes on the bottom row, and those of $\beta = (\beta_1, \dots, \beta_k)$ the top row. Nodes having the same labels are connected, but

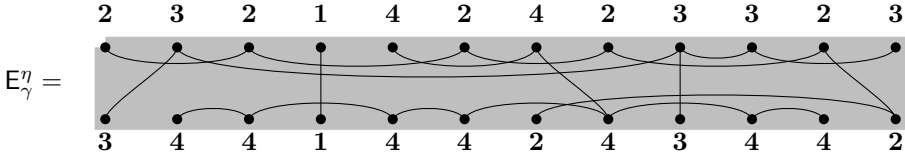
the way the edges are drawn is immaterial. What matters is that nodes with identical labels are all connected somehow, and those with different labels are not. Thus, for $\gamma = (3, 4, 4, 1, 4, 4, 2, 4, 3, 4, 4, 2)$ and $\beta = (2, 4, 1, 3, 1, 2, 2, 4, 1, 2, 2, 3)$ in $[1, 4]^{12}$, the basis element E_γ^β is identified with the diagram



Observe that in this example $(\#\gamma_i = j) \equiv (\#\beta_i = j) \pmod{r_j}$ for $r_1 = 2, r_2 = 3, r_3 = 2, r_4 = 5$. Thus, E_γ^β is a legitimate basis element for $Z_{12}(G)$, where $G = \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_5$. Since $E_\vartheta^\eta E_\gamma^\beta = \delta_{\beta, \vartheta} E_\gamma^\eta$, the top row of E_γ^β must exactly match the bottom row of E_ϑ^η to achieve a nonzero product. Thus for E_β^η with $\eta = (2, 3, 2, 1, 4, 2, 4, 2, 3, 3, 2, 3)$, we place the diagram for E_β^η on top of the diagram for E_γ^β and concatenate the two diagrams, as pictured below.



The result is $E_\beta^\eta E_\gamma^\beta = E_\gamma^\eta$ where



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