Journal of Algebra 525 (2019) 191–233

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Affine walled Brauer–Clifford superalgebras ☆

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A R T I C L E   I N F O

Article history:
Received 24 October 2017
Available online 2 February 2019
Communicated by Leonard L. Scott, Jr.

MSC:
180
250
270

Keywords:
Affine walled Brauer–Clifford superalgebras
Queer Lie superalgebra
Schur–Weyl duality

A B S T R A C T

In this paper, a notion of affine walled Brauer–Clifford superalgebras $BC^{\mathrm{aff}}_{r,t}$ is introduced over an arbitrary integral domain R containing 2^{-1}. These superalgebras can be considered as affinization of walled Brauer superalgebras in [9]. By constructing infinitely many homomorphisms from $BC^{\mathrm{aff}}_{r,t}$ to a class of level two walled Brauer–Clifford superalgebras over $\mathbb{C}$, we prove that $BC^{\mathrm{aff}}_{r,t}$ is free over R with infinite rank. We explain that any finite dimensional irreducible $BC^{\mathrm{aff}}_{r,t}$-module over an algebraically closed field F of characteristic not 2 factors through a cyclotomic quotient of $BC^{\mathrm{aff}}_{r,t}$, called a cyclotomic (or level k) walled Brauer–Clifford superalgebra $BC_{k,r,t}$. Using a previous method on cyclotomic walled Brauer algebras in [15], we prove that $BC_{k,r,t}$ is free over R with super rank $(k^{r+t}2^{r+t-1}(r+t)!, k^{r+t}2^{r+t-1}(r+t)!)$ if and only if it is admissible in the sense of Definition 6.4.

☆ H. Rui is supported partially by NSFC (grant No. 11571108). L. Song is supported partially by NSFC (grant No. 11501368) and the Fundamental Research Funds for the Central Universities (grant No. 22120180001). Y. Su is supported partially by NSFC (grant No. 11371278 and 11431010).

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https://doi.org/10.1016/j.jalgebra.2019.01.014
0021-8693/

1. Introduction

In his pioneer's work, Schur considered $V^{\otimes r}$, the r-th tensor product of the natural module V of the general linear group $GL_n(\mathbb{C})$. This is a left $GL_n(\mathbb{C})$-module such that $GL_n(\mathbb{C})$ acts on $V^{\otimes r}$ diagonally. There is a right action of the symmetric group Σ_r on $V^{\otimes r}$, and two such actions commute with each other [7]. This enabled Schur to establish a duality between the polynomial representations of $GL_n(\mathbb{C})$ and the representations of symmetric groups over $\mathbb{C}$. Later on, such a result was generalized by Brauer [1], Sergeev [17], Lehrer–Zhang [12] and so on. In these cases, the r-th tensor product $V^{\otimes r}$ is considered where V is the natural module of a symplectic group or an orthogonal group or a queer Lie superalgebra $\mathfrak{q}(n)$ or an orthosymplectic supergroup and so on. The Brauer algebras and the Hecke–Clifford superalgebras etc naturally appear as endomorphism algebras of $V^{\otimes r}$.

Let V^* be the linear dual of the natural $GL_n(\mathbb{C})$-module V. Koike [11] considered the mixed tensor modules $V^{\otimes r}\otimes(V^*)^{\otimes t}$ for various $r, t \in \mathbb{Z}^{\geq 0}$. This led him to introduce the notion of walled Brauer algebras in [11] (see also [19]). Shader and Moon [18] set up super Schur–Weyl dualities between walled Brauer algebras and general linear Lie superalgebras, by studying mixed tensor modules of general linear Lie superalgebras $\mathfrak{gl}_{m|n}$. Brundan and Stroppel [4] established super Schur–Weyl dualities between level two Hecke algebras $H_r^{p,q}$ and $\mathfrak{gl}_{m|n}$, by studying tensor modules $K_\lambda\otimes V^{\otimes r}$ of Kac modules K_λ with the r-th power $V^{\otimes r}$ of the natural module V of $\mathfrak{gl}_{m|n}$. This led them to obtain various results including the celebrated one on Morita equivalences between blocks of categories of finite dimensional $\mathfrak{gl}_{m|n}$-modules and categories of finite dimensional left modules over some generalized Khovanov's diagram algebras [5]. By studying tensor modules $M^{r,t} := V^{\otimes r}\otimes K_\lambda\otimes(V^*)^{\otimes t}$ of Kac modules K_λ with the r-th power of the natural module V and the t-th power of the dual natural module V^* of $\mathfrak{gl}_{m|n}$, two of the authors [14,15] introduced a new class of associative algebras, referred to as affine walled Brauer algebras, over a commutative ring containing 1.[1] They established super Schur–Weyl dualities between level two walled Brauer algebras $B_{2,r,t}$ and general linear Lie superalgebras, which enables them to classify highest weight vectors of $\mathfrak{gl}_{m|n}$-modules $M^{r,t}$, and to determine decomposition numbers of $B_{2,r,t}$ arising from super Schur–Weyl duality. In order to further study representation theory of queer Lie superalgebras and to establish higher level mixed Schur–Weyl duality between queer Lie superalgebras and some associative algebras, a natural question is, what kind of algebras may come into play if one replaces general linear Lie superalgebras $\mathfrak{gl}_{m|n}$ by queer Lie superalgebras $\mathfrak{q}(n)$. This is one of the motivations of the present paper to introduce the notion of affine walled Brauer–Clifford superalgebras. Another motivation comes from two of the authors' work on the Jucys–Murphy elements of walled Brauer algebras in [14].

[1] See [16] (resp., [2]) where the affine walled Brauer algebra is defined over $\mathbb{C}$ (resp., over R) in terms of affine oriented Brauer category.

In 2014, Jung and Kang [9] introduced the notion of walled Brauer superalgebras or walled Brauer–Clifford superalgebras $BC_{r,t}$ so as to establish the mixed Schur–Weyl–Sergeev duality for queer Lie superalgebras $\mathfrak{q}(n)$. The superalgebra $BC_{r,t}$ can be considered as a generalization of a Hecke–Clifford superalgebra and a walled Brauer algebra. In the present paper, we construct Jucys–Murphy elements for $BC_{r,t}$ and study their properties in detail. Through these elements, we can introduce the notion of affine Brauer–Clifford superalgebras $BC_{r,t}^{\mathrm{aff}}$ in a ring theoretical way. Using arguments similar to those in [14], we construct infinitely many homomorphisms between the affine Brauer–Clifford superalgebra $BC_{r,t}^{\mathrm{aff}}$ and walled Brauer–Clifford superalgebras, and thus we are able to prove that the affine walled Brauer–Clifford superalgebra is free over R with infinite rank if the defining parameter ω_1 is zero. However, many affine walled Brauer–Clifford superalgebras which appear in the higher level mixed Schur–Weyl–Sergeev duality have non-zero defining parameter ω_1. In order to overpass this, we consider level two mixed Schur–Weyl–Sergeev duality for $\mathfrak{q}(n)$ and prove that a class of level two walled Brauer–Clifford superalgebras over $\mathbb{C}$ with non-zero ω_1 have required super-dimensions. Using these level two walled Brauer–Clifford superalgebras instead of walled Brauer–Clifford superalgebras used before, we can establish infinitely many superalgebra homomorphisms and hence prove the freeness of the affine walled Brauer–Clifford superalgebra over R no matter whether ω_1 is zero or not. This is one of the points which is different from the work in [14].

It is a natural problem to give a classification of finite dimensional irreducible $BC_{r,t}^{\mathrm{aff}}$-modules over an algebraically closed field of characteristic not 2. By introducing cyclotomic quotients of $BC_{r,t}^{\mathrm{aff}}$, called cyclotomic walled Brauer–Clifford superalgebras, we are able to prove that any finite dimensional irreducible $BC_{r,t}^{\mathrm{aff}}$-module factors through a cyclotomic walled Brauer–Clifford superalgebra. We define this superalgebra over R and prove that it is free over R with required rank if and only if it is admissible in the sense of Definition 6.4. In a sequel, we will classify finite dimensional irreducible modules for affine and cyclotomic walled Brauer–Clifford superalgebras over an arbitrary (algebraically closed) field with characteristic not 2.

We notice that in July, 2017 (at that time we had obtained our affine and cyclotomic walled Brauer–Clifford algebras), Brundan, Comes and Kujawa [6] introduced the affine oriented Brauer–Clifford supercategories $\mathcal{AOBC}$ and their cyclotomic quotients $\mathcal{COBC}$. They proved that any morphism space of $\mathcal{AOBC}$ is free over an integral domain R. In the second version of [6], they showed that our degenerate affine walled Brauer–Clifford algebra $BC_{r,t}^{\mathrm{aff}}$ is isomorphic to certain endomorphism algebra in (specialized) $\mathcal{AOBC}$. In the third version of [6], they proved that any morphism space of $\mathcal{COBC}$ is free over R. As an application, they showed that our admissible cyclotomic walled Brauer–Clifford superalgebra $BC_{k,r,t}$ (see Definition 6.4) is isomorphic to certain endomorphism algebra in (specialized) $\mathcal{COBC}$. For more details, see [6, section 7.2]. As explained in [6, section 1.4], their basis theorems imply our basis theorems and the converse is not true. However, our method used in this paper is quite different from that in [6].

We organize our paper as follows. In section 2, we recall the notion of walled Brauer–Clifford superalgebras $BC_{r,t}$ in [9]. Several properties on the Jucys–Murphy elements of $BC_{r,t}$ are given. This leads us to introduce the notion of affine walled Brauer–Clifford superalgebras in section 3. We give infinitely many homomorphisms between $BC^{\mathrm{aff}}_{r,t}$ and walled Brauer–Clifford superalgebras. We also define cyclotomic walled Brauer–Clifford superalgebras. In section 4, we use higher level mixed Schur–Weyl–Sergeev dualities to prove that a class of level two walled Brauer–Clifford algebras with non-zero parameter ω_1 have required super-dimensions over $\mathbb{C}$. In section 5, we construct infinitely many homomorphisms between $BC^{\mathrm{aff}}_{r,t}$ and level two walled Brauer–Clifford superalgebras which appear in the higher level mixed Schur–Weyl–Sergeev dualities in section 4. This in turn enables us to mimic arguments in [14] to prove the freeness of $BC^{\mathrm{aff}}_{r,t}$ over R. In particular, $BC^{\mathrm{aff}}_{r,t}$ is of infinite super-rank. In section 6, we prove that a cyclotomic walled Brauer–Clifford superalgebra is free over R with required super rank if and only if it is admissible.

2. Walled Brauer–Clifford superalgebras

Throughout, we assume that R is an integral domain containing 2^{-1}. Let Σ_r be the *symmetric group* on r letters. Then Σ_r is generated by $s_1, \dots, s_{r-1}$, subject to the relations (for all admissible i and j):

$$s_i^2 = 1, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad s_i s_j = s_j s_i, \text{ if } |i - j| > 1. \tag{2.1}$$

Each s_i can be identified with the simple reflection $(i, i+1)$, where $(i, j) \in \Sigma_r$, which switches i, j and fixes others. In this paper, we always assume that Σ_r acts on the right of the set $\{1, 2, \dots, r\}$.

The *Hecke–Clifford algebra* HC_r was introduced by Sergeev [17] in order to study $V^{\otimes r}$, where V is the natural module for the queer Lie superalgebra $\mathfrak{q}(n)$. It is the associative R-superalgebra generated by even elements $s_1, \dots, s_{r-1}$ and odd elements $c_1, \dots, c_r$ subject to (2.1) together with the following defining relations (for all admissible i, j):

$$c_i^2 = -1, \quad c_i c_j = -c_j c_i, \quad w^{-1} c_i w = c_{(i)w}, \forall w \in \Sigma_r. \tag{2.2}$$

In this paper, we denote $\mathbb{Z}_i = \{0, 1, \dots, i-1\}$. We always use α_j to denote the j-th coordinate of $\alpha \in \mathbb{Z}_i^r$ for $1 \le j \le r$. Let $|\alpha| = \sum_{j=1}^r \alpha_j$. The Hecke–Clifford algebra HC_r is free over R with basis $\{c^\alpha w \mid w \in \Sigma_r, \alpha \in \mathbb{Z}_2^r\}$, where $c^\alpha = c_1^{\alpha_1} \cdots c_r^{\alpha_r}$ (see [10]). Since $s_1, \dots, s_{r-1}$ (resp., $c_1, \dots, c_r$) are even (resp., odd), the even (resp., odd) subspace of HC_r is spanned by $\{c^\alpha w \mid w \in \Sigma_r, \alpha \in \mathbb{Z}_2^r, |\alpha| \in 2\mathbb{Z}\}$ (resp., $\{c^\alpha w \mid w \in \Sigma_r, \alpha \in \mathbb{Z}_2^r, |\alpha| \notin 2\mathbb{Z}\}$). In particular, the super rank of HC_r is $(2^{r-1} r!, 2^{r-1} r!)$.

We need $\overline{HC}_r$ as follows. As the R-superalgebra, it is generated by the even elements $\overline{s}_1, \dots, \overline{s}_{r-1}$ and odd elements $\overline{c}_1, \dots, \overline{c}_r$ subject to the relations for all admissible i and j:

$$\overline{s}_i^2 = 1, \quad \overline{s}_i\overline{s}_{i+1}\overline{s}_i = \overline{s}_{i+1}\overline{s}_i\overline{s}_{i+1}, \quad \text{and } \overline{s}_i\overline{s}_j = \overline{s}_j\overline{s}_i, \text{ if } |i-j|>1,$$
$$\overline{c}_i^2 = 1, \quad \overline{c}_i\overline{c}_j = -\overline{c}_j\overline{c}_i, \quad \text{and } w^{-1}\overline{c}_i w = \overline{c}_{(i)w}, \forall w \in \Sigma_r. \tag{2.3}$$

In this case, we identify $\overline{s}_i$ with s_i. If $\sqrt{-1} \in R$, then HC_r is $\overline{HC}_r$ by setting $\overline{c}_i = \sqrt{-1}c_i$ and $\overline{s}_i = s_i$. Let

$$L_1 = 0, \text{ and } L_i = \mathfrak{L}_i + c_i\mathfrak{L}_i c_i, \; 2 \le i \le r, \tag{2.4}$$

where $\mathfrak{L}_i = \sum_{j=1}^{i-1}(j,i)$. These elements, which are known as Jucys–Murphy elements of HC_r, satisfy the following relations for all admissible i,j,k:

$$L_iL_j = L_jL_i, \quad s_iL_k = L_ks_i, \quad \text{if } k \ne i, i+1,$$
$$s_iL_is_i = L_{i+1} - (1 - c_ic_{i+1})s_i, \quad c_iL_k = (-1)^{\delta_{i,k}}L_kc_i, \tag{2.5}$$

where $\delta_{i,k} = 1$ if $i = k$, and 0 otherwise. If we denote by $\overline{L}_1, \ldots, \overline{L}_r$ the Jucys–Murphy elements of $\overline{HC}_r$, then $\overline{L}_1 = 0$ and $\overline{L}_i = \overline{\mathfrak{L}}_i - \overline{c}_i\overline{\mathfrak{L}}_i\overline{c}_i$, where $\overline{\mathfrak{L}}_j = \sum_{k=1}^{j-1}(\overline{k}, \overline{j})$. In this case, we identify $\overline{i}$ with i for all $1 \le i \le r$. So, Σ_r can be identified with the symmetric group on the set $\{\overline{1}, \ldots, \overline{r}\}$, and (2.5) turns out to be

$$\overline{L}_i\overline{L}_j = \overline{L}_j\overline{L}_i, \quad \overline{s}_i\overline{L}_k = \overline{L}_k\overline{s}_i, \quad \text{if } k \ne i, i+1,$$
$$\overline{s}_i\overline{L}_i\overline{s}_i = \overline{L}_{i+1} - (1 + \overline{c}_i\overline{c}_{i+1})\overline{s}_i, \quad \overline{c}_i\overline{L}_k = (-1)^{\delta_{i,k}}\overline{L}_k\overline{c}_i. \tag{2.6}$$

Considering $-L_i$ (resp., $-\overline{L}_i$) as abstract generators x_i (resp., $\overline{x}_i$) yields the notion of the affine Hecke–Clifford algebra $HC_r^{\rm aff}$ (resp., $\overline{HC}_r^{\rm aff}$) as follows.

The *affine Hecke–Clifford* algebra $HC_r^{\rm aff}$ is the associative R-superalgebra generated by even elements $s_1, \ldots, s_{r-1}, x_1$ and odd elements $c_1, \ldots, c_r$ subject to (2.1)–(2.2), together with the following defining relations (for all admissible i and j):

$$x_1x_2 = x_2x_1, \quad x_1c_i = (-1)^{\delta_{i,1}}c_ix_1, \quad s_jx_1 = x_1s_j, \text{if } j \ne 1, \tag{2.7}$$

where $x_2 = s_1x_1s_1 - (1 - c_1c_2)s_1$. Later on, we need $\overline{HC}_r^{\rm aff}$ as follows. As the R-superalgebra, it is generated by even elements $\overline{s}_1, \ldots, \overline{s}_{r-1}, \overline{x}_1$ and odd elements $\overline{c}_1, \ldots, \overline{c}_r$ subject to (2.3) together with the following defining relations (for all admissible i and j):

$$\overline{x}_1\overline{x}_2 = \overline{x}_2\overline{x}_1, \quad \overline{x}_1\overline{c}_i = (-1)^{\delta_{i,1}}\overline{c}_i\overline{x}_1, \quad \overline{s}_j\overline{x}_1 = \overline{x}_1\overline{s}_j, \text{if } j \ne 1, \tag{2.8}$$

where $\overline{x}_2 = \overline{s}_1\overline{x}_1\overline{s}_1 - (1 + \overline{c}_1\overline{c}_2)\overline{s}_1$. Certainly, $\overline{HC}_r^{\rm aff}$ is $HC_r^{\rm aff}$ if $\sqrt{-1} \in R$. For $1 \le i \le r$, define

$$x_i = x_i' - L_i, \text{ and } \overline{x}_i = \overline{x}_i' - \overline{L}_i, \tag{2.9}$$

where $x'_i = s_{i-1}\cdots s_1x_1s_1\cdots s_{i-1}$, $x'_1 = x_1$ and $\overline{x}'_i = \overline{s}_{i-1}\cdots\overline{s}_1\overline{x}_1\overline{s}_1\cdots\overline{s}_{i-1}$, $\overline{x}'_1 = \overline{x}_1$. Then we have the following relations for all admissible i and j:

$$\begin{aligned} x_{i+1} &= s_ix_is_i - (1-c_ic_{i+1})s_i, \text{ and } x_ix_j = x_jx_i, \\ \overline{x}_{i+1} &= \overline{s}_i\overline{x}_i\overline{s}_i - (1+\overline{c}_i\overline{c}_{i+1})\overline{s}_i, \text{ and } \overline{x}_i\overline{x}_j = \overline{x}_j\overline{x}_i. \end{aligned} \tag{2.10}$$

For all $\alpha \in \mathbb{N}^r$, define $x^\alpha = x_1^{\alpha_1}\cdots x_r^{\alpha_r}$ and $\overline{x}^\alpha = \overline{x}_1^{\alpha_1}\cdots\overline{x}_r^{\alpha_r}$. It is proven in [10] that HC_r^{aff} has basis $\{x^\alpha c^\beta w \mid w \in \mathfrak{S}_r, \alpha \in \mathbb{N}^r, \beta \in \mathbb{Z}_2^r\}$. The even (resp., odd) subspace of HC_r^{aff} is spanned by all $x^\alpha c^\beta w$ such that $|\beta| \in 2\mathbb{Z}$ (resp., $|\beta| \notin 2\mathbb{Z}$). Similar results hold for $\overline{HC}_r^{\mathrm{aff}}$.

We are going to recall the definition of the *walled Brauer–Clifford superalgebra* $BC_{r,t}$. This superalgebra was introduced by Jung and Kang in [9] so as to study the mixed tensor product of the natural module and its linear dual for the queer Lie superalgebra $\mathfrak{q}(n)$. The original $BC_{r,t}$ is defined via (r,t)-*superdiagrams* in [9]. In this paper, we use its equivalent definition.

Definition 2.1. [9, Theorem 5.1] The walled Brauer–Clifford superalgebra $BC_{r,t}$ is the associative R-superalgebra generated by even generators e_1, $s_1,\dots,s_{r-1}$, $\overline{s}_1,\dots,\overline{s}_{t-1}$, and odd generators $c_1,\dots,c_r,\overline{c}_1,\dots,\overline{c}_t$ subject to (2.1)–(2.3) together with the following defining relations for all admissible i,j:

(1) $e_1c_1 = e_1\overline{c}_1$, $c_1e_1 = \overline{c}_1e_1$,
(2) $\overline{s}_jc_i = c_i\overline{s}_j$, $s_i\overline{c}_j = \overline{c}_js_i$,
(3) $c_i\overline{c}_j = -\overline{c}_jc_i$, $s_i\overline{s}_j = \overline{s}_js_i$,
(4) $e_1^2 = 0$,
(5) $e_1s_1e_1 = e_1 = e_1\overline{s}_1e_1$,
(6) $s_ie_1 = e_1s_i$, $\overline{s}_ie_1 = e_1\overline{s}_i$, if $i \neq 1$,
(7) $e_1s_1\overline{s}_1e_1s_1 = e_1s_1\overline{s}_1e_1\overline{s}_1$,
(8) $s_1e_1s_1\overline{s}_1e_1 = \overline{s}_1e_1s_1\overline{s}_1e_1$,
(9) $c_ie_1 = e_1c_i$ and $\overline{c}_ie_1 = e_1\overline{c}_i$, if $i \neq 1$,
(10) $e_1c_1e_1 = 0 = e_1\overline{c}_1e_1$.

Lemma 2.2. *There is a unique R-linear anti-involution $\tau : BC_{r,t} \to BC_{r,t}$, which fixes all of its generators.*

Proof. It follows from Definition 2.1, immediately. □

It is known that the subalgebra of $BC_{r,t}$ generated by even generators $s_1,\dots,s_{r-1}$, $\overline{s}_1,\dots,\overline{s}_{t-1}$ and e_1 is isomorphic to the walled Brauer algebra $B_{r,t}(0)$ in [11,19]. This enables us to freely use results on $B_{r,t}(0)$ in [14] so as to simplify our presentation. Write $s_{i,j} = s_is_{i+1,j}$ if $i<j$ and $s_{i,i} = 1$ and $s_{i,j} = s_{i,j+1}s_j$ if $i>j$. Similarly, we have $\overline{s}_{i,j}$'s, etc. Following [13], define $D_{r,t}^f = \{1\}$ if $f=0$ and

$$D_{r,t}^f = \Big\{ s_{f,i_f}\overline{s}_{f,j_f}\cdots s_{1,i_1}\overline{s}_{1,j_1} \,\Big|\, k \le j_k \le t, 1 \le k \le f, \\ 1 \le i_1 < i_2 < \ldots < i_f \le r \Big\} \text{ if } 0 < f \le \min\{r,t\}. \tag{2.11}$$

Definition 2.3. Define $e^0 = 1$ and $e^f = e_1e_2\cdots e_f$ if $0 < f \le \min\{r,t\}$, where $e_i = e_{i,i}$ and $e_{i,j} = (\overline{1},\overline{j})(1,i)e_1(1,i)(\overline{1},\overline{j})$ for all admissible i,j.

Theorem 2.4. *[9, Theorem 5.1] The walled Brauer–Clifford superalgebra $BC_{r,t}$ has R-basis*

$$\mathcal{S} = \Big\{c^\alpha d_1^{-1}e^f wd_2\overline{c}^\beta \mid 0 \le f \le \min\{r,t\}, w \in \Sigma_{r-f}\times\Sigma_{\overline{t-f}}, d_1,d_2 \in D^f_{r,t},$$
$$(\alpha,\beta)\in\mathbb{Z}_2^r\times\mathbb{Z}_2^t\Big\}. \tag{2.12}$$

In particular, the super rank of $BC_{r,t}$ is $(2^{r+t-1}(r+t)!, 2^{r+t-1}(r+t)!)$.

Proof. The basis of $BC_{r,t}$ given in (2.12) is a refinement of X given in the proof of [9, Theorem 5.1]. We remark that each $d_1^{-1}e^f wd_2$ corresponds to a unique walled Brauer diagram in [14]. □

Corollary 2.5. *For any positive integer k, the subalgebra of $BC_{k+r,k+t}$ generated by even elements $e_{k+1}, s_{k+1},\dots, s_{k+r-1}$, $\overline{s}_{k+1},\dots,\overline{s}_{k+t-1}$ and odd elements $c_{k+1},\overline{c}_{k+1}$ is isomorphic to $BC_{r,t}$.*

Proof. Easy exercise using Theorem 2.4 and Definition 2.1. □

Lemma 2.6. *Let $BC_{k-1,k-1}$ be the subalgebra of $BC_{k,k}$ generated by $e_1, s_1,\dots,s_{k-2}$, $\overline{s}_1,\dots,\overline{s}_{k-2}$ and $c_1,\overline{c}_1$. Then $e_kBC_{k,k}$ is a left $BC_{k-1,k-1}$-module generated by all $e_kc_k^\sigma s_{k,j}\overline{s}_{k,l}$ such that $\sigma\in\mathbb{Z}_2$ and $1\le j,l\le k$.*

Proof. It is enough to prove that the left $BC_{k-1,k-1}$-module V_k generated by all $e_kc_k^\sigma s_{k,j}\overline{s}_{k,l}$ is a right $BC_{k,k}$-module. If so, then $V_k = e_kBC_{k,k}$ by the fact that $e_k\in V_k$. We have $V_ks_i\subset V_k$ and $V_kc_1\subset V_k$ since

$$e_kc_k^\sigma s_{k,j}\overline{s}_{k,l}s_i = \begin{cases} s_ie_kc_k^\sigma s_{k,j}\overline{s}_{k,l} & \text{if } i<j,\\ e_kc_k^\sigma s_{k,j+1}\overline{s}_{k,l} & \text{if } i=j,\\ s_{i-1}e_kc_k^\sigma s_{k,j}\overline{s}_{k,l} & \text{if } i>j,\end{cases} \quad \text{and}$$

$$e_kc_k^\sigma s_{k,j}\overline{s}_{k,l}c_1 = \begin{cases} \varepsilon c_1e_kc_k^\sigma s_{k,j}\overline{s}_{k,l} & \text{if } j>1,\\ e_kc_k^{\sigma+1}s_{k,j}\overline{s}_{k,l} & \text{if } j=1,\end{cases}$$

where $\varepsilon = 1$ (resp., -1) if $\sigma = 0$ (resp., 1). Similarly, $V_k\overline{s}_i\subset V_k$ and $V_k\overline{c}_1\subset V_k$. Finally, $V_ke_1\subset V_k$ since

$$e_k c_k^\sigma s_{k,j}\overline{s}_{k,l} e_k = \begin{cases} 0 & \text{if } j=k=l, \\ c_{k-1}^\sigma s_{k-1,j} e_k & \text{if } l=k>j, \\ \overline{c}_{k-1}^\sigma \overline{s}_{k-1,l} e_k & \text{if } j=k>l, \\ e_{k-1} c_{k-1}^\sigma s_{k-1,j}\overline{s}_{k-1,l} e_k & \text{if } j,l<k. \qquad \square \end{cases} \tag{2.13}$$

Proposition 2.7. *We have $e_k BC_{k,k} e_k = e_k BC_{k-1,k-1}$ for all $k \geq 2$ and $e_1 BC_{1,1} e_1 = 0$.*

Proof. We have $e_k BC_{k,k} e_k \subseteq e_k BC_{k-1,k-1}$ by Lemma 2.6 and (2.13). When $k \geq 2$, the inverse inclusion follows from the equations $e_k = e_k s_{k-1} e_k$ and $e_k x = x e_k$ for any $x \in BC_{k-1,k-1}$. $\square$

Definition 2.8. For all admissible i, j, let $y_i = \sum_{j=1}^{i-1}(e_{i,j} + \overline{e}_{i,j}) - L_i$, and $\overline{y}_i = \sum_{j=1}^{i-1}(e_{j,i} - \overline{e}_{j,i}) - \overline{L}_i$, where $\overline{e}_{i,j} = c_i e_{i,j} c_i$. Then $y_i = \mathfrak{y}_i + c_i \mathfrak{y}_i c_i$ and $\overline{y}_j = \overline{\mathfrak{y}}_j - \overline{c}_j \overline{\mathfrak{y}}_j \overline{c}_j$, where $\mathfrak{y}_i$ (resp., $\overline{\mathfrak{y}}_j$) is y_i (resp., $\overline{y}_j$) in [14, (3.5)] in the case $\delta_1 = 0$. So,

$$\mathfrak{y}_i = \sum_{j=1}^{i-1}(e_{i,j} - (j,i)), \text{ and } \overline{\mathfrak{y}}_i = \sum_{j=1}^{i-1}(e_{j,i} - (\overline{j},\overline{i})). \tag{2.14}$$

Lemma 2.9. *With the notations above, the following results hold in $BC_{r,t}$ for all admissible i, j:*

(1) $s_j y_i = y_i s_j$, $\overline{s}_j \overline{y}_i = \overline{y}_i \overline{s}_j$ *if* $j \neq i-1, i$,

(2) $s_j \overline{y}_i = \overline{y}_i s_j$, $\overline{s}_j y_i = y_i \overline{s}_j$ *if* $j \neq i-1$,

(3) $y_i c_i = -c_i y_i$, $\overline{y}_i \overline{c}_i = -\overline{c}_i \overline{y}_i$,

(4) $y_i c_j = c_j y_i$, $\overline{y}_i \overline{c}_j = \overline{c}_j \overline{y}_i$ *if* $i \neq j$,

(5) $y_i \overline{c}_j = \overline{c}_j y_i$, $\overline{y}_i c_j = c_j \overline{y}_i$ *if* $j \geq i$,

(6) $y_i y_{i+1} = y_{i+1} y_i$, $\overline{y}_i \overline{y}_{i+1} = \overline{y}_{i+1} \overline{y}_i$,

(7) $y_i(e_i + \overline{y}_i - \overline{e}_i) = (e_i + \overline{y}_i - \overline{e}_i) y_i$,

(8) $e_i \overline{y}_i = e_i(L_i - \overline{L}_i)$, $e_i y_i = e_i(\overline{L}_i - L_i)$,

(9) $e_i s_i y_i s_i = s_i y_i s_i e_i$, $e_j \overline{s}_j \overline{y}_j \overline{s}_j = \overline{s}_j \overline{y}_j \overline{s}_j e_j$,

(10) $y_i \tilde{y}_i = \tilde{y}_i y_i$, $\overline{y}_i \tilde{\overline{y}}_i = \tilde{\overline{y}}_i \overline{y}_i$,

(11) $e_i y_i^k c_i e_i = 0$, $\forall k \in \mathbb{N}$,

(12) $e_i y_i^{2n} e_i = 0$, $e_i \overline{y}_i^{2n} e_i = 0$, $\forall n \in \mathbb{N}$,

(13) $e_i y_i e_i = e_i \overline{y}_i e_i = 0$,

where $\tilde{y}_i = s_i y_i s_i - (1 - c_i c_{i+1}) s_i$ *and* $\tilde{\overline{y}}_i = \overline{s}_i \overline{y}_i \overline{s}_i - (1 + \overline{c}_i \overline{c}_{i+1}) \overline{s}_i$.

Proof. We assume $\sqrt{-1} \in R$. Then $BC_{r,t} \cong BC_{t,r}$. It is reasonable since we can embed R into a larger integral domain containing $\sqrt{-1}$. The required isomorphism sends (a): $\sqrt{-1}c_i$ (resp., $\sqrt{-1}\overline{c}_j$) in $BC_{r,t}$ to $\overline{c}_i$ (resp., c_j) in $BC_{t,r}$; (b): e_1 to e_1; (c): s_i (resp., $\overline{s}_j$) in $BC_{r,t}$ to $\overline{s}_i$ (resp., s_j) in $BC_{t,r}$. So, it suffices to verify one of equations in (1)–(6), (8)–(13) except (11).

(1) If $j \neq i, i-1$, then $s_j c_i = c_i s_j$ and $s_j \mathfrak{y}_i = \mathfrak{y}_i s_j$ by (2.2) and [14, Lemma 3.3(6)]. So,

$$s_j y_i = s_j(\mathfrak{y}_i + c_i \mathfrak{y}_i c_i) = (\mathfrak{y}_i + c_i \mathfrak{y}_i c_i) s_j = y_i s_j.$$

(2) If $j \neq i-1$, then $s_j\overline{\mathfrak{y}}_i = \overline{\mathfrak{y}}_i s_j$ by [14, Lemma 3.3(7)]. By Definition 2.1(2),

$$s_j\overline{y}_i = s_j(\overline{\mathfrak{y}}_i - \overline{c}_i\overline{\mathfrak{y}}_i\overline{c}_i) = s_j\overline{y}_i.$$

(3) Since $c_i^2 = -1$, we have $c_iy_i = c_i(\mathfrak{y}_i + c_i\mathfrak{y}_ic_i) = -\mathfrak{y}_ic_i + c_i\mathfrak{y}_i = -y_ic_i$.

(4) By (2.2), (2.5) and Definition 2.1(9), $c_ic_j = -c_jc_i$, $c_jL_i = L_ic_j$ and $e_{i,k}c_j = c_je_{i,k}$ if $i \neq j$. By Definition 2.8, we have $y_ic_j = c_jy_i$.

(5) If $j > k$, then $e_{i,k}\overline{c}_j = \overline{c}_je_{i,k}$. By Definition 2.1(2)–(3) and Definition 2.8, we have $y_i\overline{c}_j = \overline{c}_jy_i$.

(6) Since $\mathfrak{y}_i = \sum_{j=1}^{i-1}(e_{i,j} - (j,i))$ (see (2.14)), we have $\mathfrak{y}_ic_{i+1} = c_{i+1}\mathfrak{y}_i$. By [14, Lemma 3.3(9)], $\mathfrak{y}_i\mathfrak{y}_{i+1} = \mathfrak{y}_{i+1}\mathfrak{y}_i$. Thus,

$$\begin{aligned}y_iy_{i+1} &= (\mathfrak{y}_i + c_i\mathfrak{y}_ic_i)y_{i+1} \overset{(4)}{=} \mathfrak{y}_iy_{i+1} + c_i\mathfrak{y}_iy_{i+1}c_i\\ &= \mathfrak{y}_i\mathfrak{y}_{i+1} + c_{i+1}\mathfrak{y}_i\mathfrak{y}_{i+1}c_{i+1} + c_i\mathfrak{y}_i\mathfrak{y}_{i+1}c_i + c_ic_{i+1}\mathfrak{y}_i\mathfrak{y}_{i+1}c_{i+1}c_i.\end{aligned}$$

Applying the anti-involution τ on the above equation yields $y_iy_{i+1} = y_{i+1}y_i$.

(7) We have

$$\begin{aligned}y_i(e_i - \overline{e}_i + \overline{y}_i) &= (\mathfrak{y}_i + c_i\mathfrak{y}_ic_i)(e_i + \overline{\mathfrak{y}}_i - \overline{c}_i\overline{\mathfrak{y}}_i\overline{c}_i - c_ie_ic_i)\\ &= \mathfrak{y}_i(e_i + \overline{\mathfrak{y}}_i) - \overline{c}_i\mathfrak{y}_i(e_i + \overline{\mathfrak{y}}_i)\overline{c}_i + c_i\mathfrak{y}_i(e_i + \overline{\mathfrak{y}}_i)c_i - \overline{c}_ic_i\mathfrak{y}_i(e_i + \overline{\mathfrak{y}}_i)c_i\overline{c}_i.\end{aligned}$$

Applying the anti-involution τ on the above equation and using $(e_i + \overline{\mathfrak{y}}_i)\mathfrak{y}_i = \mathfrak{y}_i(e_i + \overline{\mathfrak{y}}_i)$ (see [14, Lemma 3.3(4)]) yields (7).

(8) By [14, Lemma 3.3(1)], $e_i\mathfrak{y}_i = e_i(-\mathfrak{L}_i + \overline{\mathfrak{L}}_i)$. So,

$$\begin{aligned}e_iy_i &= e_i\mathfrak{y}_i + e_ic_i\mathfrak{y}_ic_i = e_i\mathfrak{y}_i(1 + \overline{c}_ic_i) = -e_i(\mathfrak{L}_i - \overline{\mathfrak{L}}_i)(1 + \overline{c}_ic_i)\\ &= -e_i\mathfrak{L}_i + e_i\overline{\mathfrak{L}}_i - e_i(\mathfrak{L}_i - \overline{\mathfrak{L}}_i)\overline{c}_ic_i\\ &= -e_i(\mathfrak{L}_i + c_i\mathfrak{L}_ic_i) + e_i(\overline{\mathfrak{L}}_i - \overline{c}_i\overline{\mathfrak{L}}_i\overline{c}_i) = -e_i(L_i - \overline{L}_i).\end{aligned}$$

(9) By [14, Lemma 3.3(5)], $e_is_i\mathfrak{y}_is_i = s_i\mathfrak{y}_is_ie_i$. So,

$$e_is_iy_is_i = e_is_ic_i\mathfrak{y}_ic_is_i + e_is_i\mathfrak{y}_is_i = c_{i+1}e_is_i\mathfrak{y}_is_ic_{i+1} + s_i\mathfrak{y}_is_ie_i = s_iy_is_ie_i.$$

(10) We define $\mathbf{m}_i = s_i\mathfrak{y}_is_i - s_i$. By [14, Lemma 3.3(5)]), $\mathbf{m}_i\mathfrak{y}_i = \mathfrak{y}_i\mathbf{m}_i$. So,

$$\begin{aligned}y_i\tilde{y}_i &= (\mathfrak{y}_i + c_i\mathfrak{y}_ic_i)(s_i(\mathfrak{y}_i + c_i\mathfrak{y}_ic_i)s_i - (1 - c_ic_{i+1})s_i)\\ &= \mathfrak{y}_i\mathbf{m}_i + \mathfrak{y}_i(s_ic_i\mathfrak{y}_ic_is_i + c_ic_{i+1}s_i) + c_i\mathfrak{y}_ic_i(s_i(\mathfrak{y}_i + c_i\mathfrak{y}_ic_i)s_i - (1 - c_ic_{i+1})s_i)\\ &= \mathfrak{y}_i\mathbf{m}_i + c_{i+1}\mathfrak{y}_i\mathbf{m}_ic_{i+1} + c_i\mathfrak{y}_ic_i\mathbf{m}_i + c_i\mathfrak{y}_ic_i(s_ic_i\mathfrak{y}_ic_is_i + c_ic_{i+1}s_i)\\ &= \mathfrak{y}_i\mathbf{m}_i + c_{i+1}\mathfrak{y}_i\mathbf{m}_ic_{i+1} + c_i\mathfrak{y}_ic_is_i\mathfrak{y}_is_i - c_i\mathfrak{y}_ic_is_i - c_i\mathfrak{y}_ic_{i+1}s_i + c_i\mathfrak{y}_ic_is_ic_i\mathfrak{y}_ic_is_i\\ &= \mathfrak{y}_i\mathbf{m}_i + c_{i+1}\mathfrak{y}_i\mathbf{m}_ic_{i+1} + c_i\mathfrak{y}_i\mathbf{m}_ic_i - c_i\mathfrak{y}_ic_is_i - c_ic_{i+1}\mathfrak{y}_is_i\mathfrak{y}_is_ic_ic_{i+1}\\ &= \mathfrak{y}_i\mathbf{m}_i + c_{i+1}\mathfrak{y}_i\mathbf{m}_ic_{i+1} + c_i\mathfrak{y}_i\mathbf{m}_ic_i + c_{i+1}c_i\mathfrak{y}_i\mathbf{m}_ic_ic_{i+1}.\end{aligned} \tag{2.15}$$

Applying the anti-involution τ on (2.15) yields $\tilde{y}_iy_i = y_i\tilde{y}_i$.

(11) If k is odd, then

$$e_i y_i^k c_i e_i = e_i y_i^k \overline{c}_i e_i \overset{(5)}{=} e_i \overline{c}_i y_i^k e_i \overset{(3)}{=} -e_i y_i^k c_i e_i,$$

forcing $e_i y_i^k c_i e_i = 0$. If k is even, then

$$\begin{aligned} e_i y_i^{k-1} \overline{y}_i &= e_i y_i^{k-1}(\overline{y}_i + e_i - \overline{e}_i) - e_i y_i^{k-1} e_i + e_i y_i^{k-1} \overline{e}_i \\ &\overset{(7)}{=} e_i(\overline{y}_i + e_i - \overline{e}_i) y_i^{k-1} - e_i y_i^{k-1} e_i + e_i y_i^{k-1} \overline{e}_i. \end{aligned} \tag{2.16}$$

So, $e_i y_i^{k-1} \overline{y}_i c_i e_i = e_i \overline{y}_i y_i^{k-1} c_i e_i$. On the other hand, $e_i y_i^{k-1} \overline{y}_i c_i e_i = -e_i c_i y_i^{k-1} \overline{y}_i e_i = e_i c_i y_i^k e_i$ and $e_i \overline{y}_i y_i^{k-1} c_i e_i = -e_i y_i^k c_i e_i = -e_i c_i y_i^k e_i$. So, $e_i y_i^k c_i e_i = 0$ for even k.

(12) We have $e_k y_k^{2n} e_k = 0$ since

$$e_k y_k^{2n} e_k = e_k y_k^{2n} \overline{c}_k^2 e_k \overset{(5)}{=} e_k \overline{c}_k y_k^{2n} c_k e_k = e_k c_k y_k^{2n} c_k e_k \overset{(3)}{=} e_k y_k^{2n} c_k^2 e_k = -e_k y_k^{2n} e_k.$$

(13) If $j < i$, then $e_i c_i e_{i,j} c_i e_i = e_i \overline{c}_i e_{i,j} \overline{c}_i e_i = e_i e_{i,j} e_i = e_i$ and $e_i c_i(j,i) c_i e_i = e_i \overline{c}_i(j,i) \overline{c}_i e_i = e_i$. So,

$$e_i y_i e_i = 2 e_i \mathfrak{y}_i e_i = 2 \sum_{j=1}^{i-1} e_i(e_{i,j} - (i,j)) e_i = 0. \quad \square$$

Corollary 2.10. *There is a unique element $\omega_{a,k} \in BC_{k-1,k-1}$ such that $e_k y_k^a e_k = \omega_{a,k} e_k$. Similarly, there is a unique element $\overline{\omega}_{a,k} \in BC_{k-1,k-1}$ such that $e_k \overline{y}_k{}^a e_k = \overline{\omega}_{a,k} e_k$. Moreover, $\omega_{2n,k} = \overline{\omega}_{2n,k} = 0$.*

Proof. The existence of an $\omega_{a,k}$ follows from Proposition 2.7 and the uniqueness of such an element follows from Theorem 2.4. The second statement can be verified similarly. Finally, we have $\omega_{2n,k} = \overline{\omega}_{2n,k} = 0$ by Lemma 2.9(12). $\square$

Lemma 2.11. *For $n \in \mathbb{N}$, $e_i \overline{y}_i^{2n+1} = \sum_{j=0}^n a_{2n+1,j}^{(i)} e_i y_i^{2j+1}$ for some $a_{2n+1,j}^{(i)} \in R[\omega_{3,i}, \ldots, \omega_{2n-1,i}]$ such that*

(1) $a_{2n+1,n}^{(i)} = -1$,
(2) $a_{2n+1,j}^{(i)} = a_{2n-1,j-1}^{(i)}$, $1 \le j \le n$,
(3) $a_{2n+1,0}^{(i)} = \sum_{j=1}^{n-1} a_{2n-1,j}^{(i)} \omega_{2j+1,i}$.

Proof. When $n = 0$, we have $e_i(y_i + \overline{y}_i) = 0$ by Lemma 2.9(8). So, $a_{1,0}^{(i)} = -1$. In general, we have

$$\begin{aligned} e_i y_i^{2j-1} \overline{y}_i^2 &= e_i y_i^{2j-1}(\overline{y}_i + e_i - \overline{e}_i)\overline{y}_i - e_i y_i^{2j-1} e_i \overline{y}_i \quad \text{(by Lemma 2.9(11))} \\ &= e_i(\overline{y}_i + e_i - \overline{e}_i) y_i^{2j-1} \overline{y}_i + \omega_{2j-1,i} e_i y_i \quad \text{(by Lemma 2.9(7))} \\ &= -e_i y_i^{2j} \overline{y}_i + \omega_{2j-1,i} e_i y_i \quad \text{(by Lemma 2.9(8))}. \end{aligned}$$

Similarly, using $\omega_{2j,i} = 0$ yields $e_i y_i^{2j}\overline{y}_i = -e_i y_i^{2j+1}$. So, $e_i y_i^{2j-1}\overline{y}_i^2 = e_i y_i^{2j+1} + w_{2j-1,i} e_i y_i$. By inductive assumption on n and $\omega_{1,i} = 0$, we have the result, immediately. □

Lemma 2.12. *For positive integers n, $e_i\overline{y}_i^{2n} = \sum_{j=0}^{n} a_{2n,j}^{(i)} e_i y_i^{2j}$ for some $a_{2n,j}^{(i)} \in R[\omega_{3,i}, \ldots, \omega_{2n-1,i}]$ such that*

(1) $a_{2n,n}^{(i)} = 1$,
(2) $a_{2n,j}^{(i)} = a_{2n-2,j-1}^{(i)}$, $1 \le j \le n$,
(3) $a_{2n,0}^{(i)} = \sum_{j=1}^{n-1} a_{2n-2,j}^{(i)} \omega_{2j+1,i}$.

Proof. We have $e_i y_i^{2j}\overline{y}_i^2 = -e_i y_i^{2j+1}\overline{y}_i = e_i y_i^{2j+2} + \omega_{2j+1,i} e_i$. By inductive assumption on n and $\omega_{1,i} = 0$, we immediately have the result. □

We can assume $k \ge 2$ (resp., $n \ge 2$) in Lemma 2.13 since $y_1 = \overline{y}_1 = 0$ (resp., $\omega_{1,k} = \overline{\omega}_{1,k} = 0$ by Lemma 2.9(13)).

Lemma 2.13. *We have $\overline{\omega}_{2n-1,k} \in R[\omega_{3,k}, \ldots, \omega_{2n-1,k}]$ if $k, n \in \mathbb{Z}^{\ge 2}$. Furthermore, both $\omega_{2n-1,k}$ and $\overline{\omega}_{2n-1,k}$ are central in $BC_{k-1,k-1}$.*

Proof. By Lemma 2.11 and inductive assumption on k, we have the first statement. To prove the second, note that any $h \in \{e_1, s_1, \ldots, s_{k-2}, c_1\}$ commutes with e_k, y_k. So, $e_k(h\omega_{2n-1,k}) = e_k(\omega_{2n-1,k}h)$. By Theorem 2.4, $h\omega_{2n-1,k} = \omega_{2n-1,k}h$. Finally, we need to check that $e_k(h\omega_{2n-1,k}) = e_k(\omega_{2n-1,k}h)$ for any $h \in \{\overline{s}_1, \ldots, \overline{s}_{k-2}, \overline{c}_1\}$. In this case, we use Lemma 2.11. More explicitly, we can use $\overline{y}_k$ instead of y_k in $e_k y_k^{2n-1} e_k$. Thus $h\omega_{2n-1,k} = \omega_{2n-1,k}h$, as required. □

In the following, we define

$$h_k = y_k + e_k + \overline{e}_k, \text{ and } \overline{h}_k = \overline{y}_k + e_k - \overline{e}_k, \text{ for all admissible } k. \tag{2.17}$$

Lemma 2.14. *For $k, a \in \mathbb{Z}^{\ge 1}$, we have*

$$s_k y_{k+1}^a = h_k^a s_k - \sum_{b=0}^{a-1} h_k^{a-1-b} y_{k+1}^b + \sum_{b=0}^{a-1} (-1)^{a-b} c_k c_{k+1} h_k^{a-b-1} y_{k+1}^b,$$

where h_k is given in (2.17).

Proof. It is easy to verify the result by induction on a. □

Lemma 2.15. *Suppose $1 \le j \le k-1$. Define $z_{j,k} = s_{j,k-1} h_{k-1} s_{k-1,j}$, and $\overline{z}_{j,k} = \overline{s}_{j,k-1} \overline{h}_{k-1} \overline{s}_{k-1,j}$, where h_{k-1} and $\overline{h}_{k-1}$ are given in (2.17). Then*

(1) $z_{j,k} = \sum_{\ell=1}^{k-1} e_{j,\ell} - \sum_{1\le s\le k-1, s\ne j}(s,j) + c_j \left(\sum_{\ell=1}^{k-1} e_{j,\ell} - \sum_{1\le s\le k-1, s\ne j}(s,j)\right) c_j$,
(2) $\overline{z}_{j,k} = \sum_{\ell=1}^{k-1} e_{\ell,j} - \sum_{1\le s\le k-1, s\ne j}(\overline{s},\overline{j}) - \overline{c}_j \left(\sum_{\ell=1}^{k-1} e_{\ell,j} - \sum_{1\le s\le k-1, s\ne j}(\overline{s},\overline{j})\right) \overline{c}_j$.

Proof. Easy exercise. $\square$

Note that $\omega_{0,k} = 0$ and $\omega_{1,k} = \overline{\omega}_{1,k} = 0$ (see Lemma 2.9(13)), and $e_k h = 0$ for $h \in BC_{k-1,k-1}$ if and only if $h = 0$. We will use these facts frequently in the proof of Lemma 2.16, where we use the terminology that a monomial in $z_{j,k+1}$'s and $\overline{z}_{j,k+1}$'s is a *leading term* in an expression if it has the highest degree by defining $\deg z_{i,j} = \deg \overline{z}_{i,j} = 1$.

Lemma 2.16. *For any positive integer n, $\omega_{2n+1,k+1}$ can be written as an R-linear combination of monomials in $z_{j,k+1}$'s and $\overline{z}_{j,k+1}$'s for $1 \le j \le k$ such that the summation of the leading terms of $\omega_{2n+1,k+1}$ is $2\sum_{j=1}^{k}(-z_{j,k+1}^{2n} + \overline{z}_{j,k+1}^{2n})$.*

Proof. By Corollary 2.10 and Lemma 2.9(8), we have

$$\omega_{2n+1,k+1}e_{k+1} = e_{k+1}y_{k+1}^{2n+1}e_{k+1} = e_{k+1}(\overline{L}_{k+1} - L_{k+1})y_{k+1}^{2n}e_{k+1}. \tag{2.18}$$

Note that $(j, k+1) = s_{j,k}s_k s_{k,j}$ and $s_{j,k}, s_{k,j}$ commute with $y_{k+1}, e_{k+1}, c_{k+1}$ (see Lemma 2.9(1) and (2.2)). Considering the right-hand side of (2.18) and expressing L_{k+1} by (2.4), we see that a term of $-e_{k+1}L_{k+1}y_{k+1}^{2n}e_{k+1}$ becomes

$$\begin{aligned}
&- s_{j,k}e_{k+1}(s_k y_{k+1}^{2n} + c_{k+1}s_k c_{k+1}y_{k+1}^{2n})e_{k+1}s_{k,j} = -2s_{j,k}e_{k+1}(s_k y_{k+1}^{2n})e_{k+1}s_{k,j} \\
&= -2s_{j,k}e_{k+1}\left\{h_k^{2n}s_k - \sum_{b=0}^{2n-1} h_k^{2n-b-1}y_{k+1}^{b} + \sum_{b=0}^{2n-1}(-1)^{2n-b}c_k c_{k+1}h_k^{2n-b-1}y_{k+1}^{b}\right\}e_{k+1}s_{k,j} \\
&= -2s_{j,k}h_k^{2n}e_{k+1}s_k e_{k+1}s_{k,j} + 2s_{j,k}\sum_{b=0}^{2n-1} h_k^{2n-b-1}e_{k+1}y_{k+1}^{b}e_{k+1}s_{k,j} \quad \text{(by Lemma 2.9(11))} \\
&= -2s_{j,k}e_{k+1}\left(h_k^{2n} - \sum_{b=0}^{2n-1} h_k^{2n-b-1}\omega_{b,k+1}\right)s_{k,j}.
\end{aligned}$$

By inductive assumption, the right-hand side of the above equation can be written as an R-linear combination of monomials with the required form such that the leading term is $-2z_{j,k+1}^{2n}$. Finally, we consider terms in (2.18) concerning $\overline{L}_{k+1}$, namely we need to deal with

$$e_{k+1}(\overline{j}, \overline{k+1})y_{k+1}^{2n}e_{k+1} - e_{k+1}\overline{c}_{k+1}(\overline{j}, \overline{k+1})\overline{c}_{k+1}y_{k+1}^{2n}e_{k+1} = 2e_{k+1}(\overline{j}, \overline{k+1})y_{k+1}^{2n}e_{k+1}.$$

Applying τ on $e_{k+1}\overline{y}_{k+1}^{2n}$ and using Lemma 2.12 and inductive assumption on n, we can use $\overline{y}_{k+1}^{2n}e_{k+1}$ to replace $y_{k+1}^{2n}e_{k+1}$ in $e_{k+1}(\overline{j}, \overline{k+1})y_{k+1}^{2n}e_{k+1}$ (by forgetting lower terms).

This enables us to consider $e_{k+1}(\overline{j},\overline{k+1})\overline{y}_{k+1}^{2n}e_{k+1}$ instead. As above, this term can be written as the required form with leading term $2\overline{z}_{j,k+1}^{2n}$. The proof is completed. $\square$

Lemma 2.17. *For $(a,k) \in \mathbb{Z}^{\geq 0} \times \mathbb{Z}^{\geq 1}$, both $\omega_{a,k+1}$ and $\overline{\omega}_{a,k+1}$ commute with y_{k+1}, $\overline{y}_{k+1}, c_l, \overline{c}_l$, $l \geq k+1$.*

Proof. Since $\omega_{a,1} = \overline{\omega}_{a,1} = \omega_{1,k} = \overline{\omega}_{1,k} = \omega_{2a,k} = \overline{\omega}_{2a,k} = 0$ for all admissible a, k, we can assume $a, k \in \mathbb{Z}^{\geq 2}$ and $2 \nmid a$. In order to verify that $\omega_{a,k+1}$ and $\overline{\omega}_{a,k+1}$ commute with y_{k+1}, by Lemmas 2.13, 2.15–2.16, it suffices to prove that y_{k+1} commutes with both $z_{j,k+1}$ and $\overline{z}_{j,k+1}$ for $1 \leq j \leq k$. By Lemma 2.15, $z_{j,k+1} = \mathfrak{z}_{j,k+1} + c_j\mathfrak{z}_{j,k+1}c_j$, where

$$\mathfrak{z}_{j,k+1} = \sum_{\ell=1}^{k} e_{j,\ell} - \sum_{1\leq s\leq k, s\neq j} (s,j),$$

which is $z_{j,k+1}$ in [14, Lemma 3.9]. Obviously, $c_{k+1}\mathfrak{z}_{j,k+1} = \mathfrak{z}_{j,k+1}c_{k+1}$. By Lemma 2.9(4), we have $y_{k+1}c_j = c_jy_{k+1}$. So,

$$\begin{aligned} z_{j,k+1}y_{k+1} &= \mathfrak{z}_{j,k+1}(c_{k+1}\mathfrak{y}_{k+1}c_{k+1} + \mathfrak{y}_{k+1}) + c_j\mathfrak{z}_{j,k+1}(c_{k+1}\mathfrak{y}_{k+1}c_{k+1} + \mathfrak{y}_{k+1})c_j \\ &= c_{k+1}\mathfrak{z}_{j,k+1}\mathfrak{y}_{k+1}c_{k+1} + \mathfrak{z}_{j,k+1}\mathfrak{y}_{k+1} + c_j\mathfrak{z}_{j,k+1}\mathfrak{y}_{k+1}c_j \\ &\quad + c_jc_{k+1}\mathfrak{z}_{j,k+1}\mathfrak{y}_{k+1}c_{k+1}c_j. \end{aligned} \tag{2.19}$$

Recall that τ is the R-linear anti-involution in Lemma 2.2. By Definition 2.8 and Lemma 2.15, τ fixes both $z_{j,k+1}$ and y_{k+1}. So, $y_{k+1}z_{j,k+1} = \tau(z_{j,k+1}y_{k+1})$. Since $\mathfrak{y}_{k+1}\mathfrak{z}_{j,k+1} = \mathfrak{z}_{j,k+1}\mathfrak{y}_{k+1}$ (see [14, Lemma 3.11]), we have $z_{j,k+1}y_{k+1} = y_{k+1}z_{j,k+1}$ by (2.19). One can check $\overline{z}_{j,k+1}y_{k+1} = y_{k+1}\overline{z}_{j,k+1}$ similarly via Definition 2.8 and the equation $\overline{z}_{j,k+1} = \overline{\mathfrak{z}}_{j,k+1} - \overline{c}_j\overline{\mathfrak{z}}_{j,k+1}\overline{c}_j$. This proves that y_{k+1} commutes with $\omega_{a,k+1}$ and $\overline{\omega}_{a,k+1}$. We remark that one can check both $\omega_{a,k+1}$ and $\overline{\omega}_{a,k+1}$ commute with $\overline{y}_{k+1}$, similarly. By Lemma 2.15, one can easily check that $z_{j,k+1}$ and $\overline{z}_{j,k+1}$ commute with c_l and $\overline{c}_l$ for all $l \geq k+1$. $\square$

3. Affine walled Brauer–Clifford superalgebras

In this section, we assume that R is an integral domain containing ω_1 and 2^{-1}. Motivated by Definition 2.1 and Lemma 2.9, we introduce the notion of affine walled Brauer–Clifford superalgebra over R as follows.

Definition 3.1. The *affine walled Brauer–Clifford superalgebra* $BC_{r,t}^{\mathrm{aff}}$ is the associative R-superalgebra generated by odd elements $c_1, \ldots, c_r$, $\overline{c}_1, \ldots, \overline{c}_t$ and even elements $e_1, x_1, \overline{x}_1$, $s_1, \ldots, s_{r-1}$, $\overline{s}_1, \ldots, \overline{s}_{t-1}$, and two families of even central elements $\omega_{2k+1}, \overline{\omega}_k$, $k \in \mathbb{Z}^{\geq 1}$ subject to (2.1)–(2.3), (2.7), (2.8) and Definition 2.1(1)–(10) together with the following defining relations for all admissible i:

(1) $e_1(x_1+\overline{x}_1)=(x_1+\overline{x}_1)e_1=0$,
(2) $e_1s_1x_1s_1=s_1x_1s_1e_1$,
(3) $x_1(e_1+\overline{x}_1-c_1e_1c_1)=(e_1-c_1e_1c_1+\overline{x}_1)x_1$,
(4) $e_1\overline{s}_1\overline{x}_1\overline{s}_1=\overline{s}_1\overline{x}_1\overline{s}_1e_1$,
(5) $e_1x_1^{2k+1}e_1=\omega_{2k+1}e_1$, $\forall k\in\mathbb{N}$,
(6) $e_1x_1^{2k}e_1=0$, $\forall k\in\mathbb{N}$,
(7) $e_1\overline{x}_1^ke_1=\overline{\omega}_ke_1$, $\forall k\in\mathbb{Z}^{>0}$,
(8) $x_1\overline{c}_i=\overline{c}_ix_1$,
(9) $\overline{x}_1c_i=c_i\overline{x}_1$,
(10) $x_1\overline{s}_i=\overline{s}_ix_1$,
(11) $\overline{x}_1s_i=s_i\overline{x}_1$.

Recall we have the notations $e_{i,j}$ in Definition 2.3 and $\overline{e}_{i,j}$ in Definition 2.8. In particular, $\overline{e}_1=c_1e_1c_1$. For the simplification of presentation, we set $\omega_{2k}=0$, $\forall k\in\mathbb{N}$. The following result follows from Definition 3.1, immediately.

Lemma 3.2. *There is an R-linear anti-involution $\sigma : BC^{\mathrm{aff}}_{r,t}\to BC^{\mathrm{aff}}_{r,t}$, which fixes all generators of $BC^{\mathrm{aff}}_{r,t}$ in Definition 3.1.*

Lemmas 3.3 and 3.4 can be proven by arguments similar to those for Lemmas 2.11 and 2.12.

Lemma 3.3. *For any $n\in\mathbb{N}$, $e_1\overline{x}_1^{2n+1}=\sum_{j=0}^n a_{2n+1,j}e_1x_1^{2j+1}$ for some $a_{2n+1,j}\in BC^{\mathrm{aff}}_{r,t}$ such that*

(1) $a_{2n+1,n}=-1$,
(2) $a_{2n+1,j}=a_{2n-1,j-1}$ *for all* $1\le j\le n-1$,
(3) $a_{2n+1,0}=\sum_{j=0}^{n-1}a_{2n-1,j}\omega_{2j+1}$.

In particular, $a_{2n+1,j}\in R[\omega_3,\dots,\omega_{2n-1}]$, for all $0\le j\le n$.

Lemma 3.4. *For any positive integer n, $e_1\overline{x}_1^{2n}=\sum_{j=0}^n a_{2n,j}e_1x_1^{2j}$ for some $a_{2n,j}\in BC^{\mathrm{aff}}_{r,t}$ such that*

(1) $a_{2n,n}=1$,
(2) $a_{2n,j}=a_{2n-1,j-1}$ *for all* $1\le j\le n-1$,
(3) $a_{2n,0}=\sum_{j=0}^{n-1}a_{2n-2,j}\omega_{2j+1}$.

In particular, $a_{2n,j}\in R[\omega_3,\dots,\omega_{2n-1}]$, for all $0\le j\le n$.

Corollary 3.5. *If e_1 is $R[\omega_3,\omega_5,\dots,\overline{\omega}_1,\overline{\omega}_2,\dots]$-torsion-free, then $\overline{\omega}_{2n+1}=\sum_{i=0}^n a_{2n+1,i}\omega_{2i+1}$ and $\overline{\omega}_{2n}=0$ for all $n\in\mathbb{N}$. In particular, $\overline{\omega}_1=-\omega_1$.*

Proof. By Definition 3.1(1), $(\omega_1+\overline{\omega}_1)e_1=0$. If e_1 is $R[\omega_3,\omega_5,\dots,\overline{\omega}_1,\overline{\omega}_2,\dots]$-torsion-free, $\overline{\omega}_1=-\omega_1$. In general, by Lemma 3.3, $e_1\overline{x}_1^{2n+1}e_1=\sum_{j=0}^n a_{2n+1,j}e_1x_1^{2j+1}e_1$. So, $\overline{\omega}_{2n+1}=\sum_{j=0}^n a_{2n+1,j}\omega_{2j+1}$. Similarly, by Lemma 3.4 and Definition 3.1(6), $\overline{\omega}_{2n}=\sum_{j=0}^n a_{2n,j}\omega_{2j}=0$. □

Assumption 3.6. From here onwards, we always assume that $\overline{\omega}_{2n} = 0$ and $\overline{\omega}_{2n+1}$'s are given in Corollary 3.5. Otherwise, we would have $e_1 = 0$ provided that R is a field, in which case, $BC_{r,t}^{\mathrm{aff}}$ turns out to be $HC_r^{\mathrm{aff}} \boxtimes \overline{HC}_t^{\mathrm{aff}}$, the outer tensor product of two affine Hecke–Clifford superalgebras! We remark that

$$(x \boxtimes y)(x_1 \boxtimes y_1) = (-1)^{[y][x_1]} x x_1 \boxtimes y y_1,$$

for any homogenous elements $x, x_1 \in HC_r^{\mathrm{aff}}$ and $y, y_1 \in \overline{HC}_t^{\mathrm{aff}}$, where $[x]$, called the parity of x, is 1 (resp., 0) if x is odd (resp., even).

Theorem 3.7. *For any $k \in \mathbb{Z}^{>0}$, there is a superalgebra homomorphism $\Phi_k : BC_{r,t}^{\mathrm{aff}} \to BC_{r+k,t+k}$ sending $s_i, \overline{s}_j, e_1, x_1, \overline{x}_1, c_l, \overline{c}_m, \omega_a, \overline{\omega}_a$ to $s_{k+i}, \overline{s}_{k+j}, e_{k+1}, y_{k+1}, \overline{y}_{k+1}, c_{k+l}, \overline{c}_{k+m}, \omega_{a,k+1}, \overline{\omega}_{a,k+1}$ for all admissible a, i, j, l, m's, respectively.*

Proof. It is enough to verify the images of generators of $BC_{r,t}^{\mathrm{aff}}$ satisfy the defining relations for $BC_{r,t}^{\mathrm{aff}}$ in Definition 3.1. We say Φ_k satisfies the relation if the images of generators satisfy this relation.

By Lemmas 2.15–2.17, the images of ω_a and $\overline{\omega}_a$ commute with the images of other generators. By Corollary 2.5, Φ_k satisfies (2.1)–(2.3) and Definition 2.1(1)–(10). Φ_k satisfies (2.7) and (2.8) by Lemma 2.9(1), (3), (4), (10). Further, Φ_k satisfies Definition 3.1(1)–(4) by Lemma 2.9(7)–(9). In this case, we need $(y_{k+1} + \overline{y}_{k+1}) e_{k+1} = 0$, which can be obtained by applying the anti-involution τ on Lemma 2.9(8). Φ_k satisfies Definition 3.1(5)–(7) by Corollary 2.10 and Lemma 2.9(11)–(13). Finally, Φ_k satisfies Definition 3.1(8)–(11) by Lemma 2.9(2), (5). $\square$

In [14], two of the authors proved the freeness of the affine walled Brauer algebra via bases of infinitely many walled Brauer algebras. The key point is the existence of infinitely many homomorphisms between the affine walled Brauer algebra and walled Brauer algebras [14, Theorem 3.12]. In the current case, Theorem 3.7 is the counterpart of [14, Theorem 3.12]. Since $\omega_{1,k} = 0$ for all k, what we can do is to use Theorem 3.7 to prove the freeness of affine walled Brauer–Clifford superalgebras with parameter $\omega_1 = 0$. However, many affine walled Brauer–Clifford algebras which appear in the higher mixed Schur–Weyl–Sergeev dualities have non-zero parameter ω_1. For details, see section 4. For this reason, we use level two walled Brauer–Clifford superalgebras (with special parameters) instead of walled Brauer–Clifford superalgebras later on. This is one of the points which is different from the work in [14].

In $BC_{r,t}^{\mathrm{aff}}$, we define $x_i, x_i', \overline{x}_j$ and $\overline{x}_j'$ as in (2.9) for all admissible i and j.

Lemma 3.8. *We have the following results for all admissible i and j:*

(1) $x_i c_i = -c_i x_i$ *and* $\overline{x}_i \overline{c}_i = -\overline{c}_i \overline{x}_i$,

(2) $x_i c_j = c_j x_i$, $\overline{x}_i \overline{c}_j = \overline{c}_j \overline{x}_i$ *if* $i \neq j$,

(3) $x_i \overline{c}_j = \overline{c}_j x_i$ *and* $\overline{x}_i c_j = c_j \overline{x}_i$.

Proof. (1) and (2) follow from (2.2), (2.3), (2.7) and (2.8). (3) follows from Definition 3.1(8)–(11). □

Lemma 3.9. *We have the following results for all admissible* i, j*:*

(1) $x'_i(\overline{x}'_j+e_{i,j}-\overline{e}_{i,j}) = (\overline{x}'_j+e_{i,j}-\overline{e}_{i,j})x'_i$,

(2) $\overline{x}'_j(x'_i+e_{i,j}+\overline{e}_{i,j}) = (x'_i+e_{i,j}+\overline{e}_{i,j})\overline{x}'_j$,

(3) $e_{i,j}(x'_i + \overline{x}'_j) = 0$,

(4) $(x'_i + \overline{x}'_j)e_{i,j} = 0$,

(5) $e_i {x'_i}^k c_i e_i = 0$, $\forall k \in \mathbb{N}$,

(6) $e_i {\overline{x}'_i}^k c_i e_i = 0$, $\forall k \in \mathbb{N}$.

Proof. Multiplying $(1,i)(\overline{1},\overline{j})$ on both sides of Definition 3.1(3) yields (1). By Definition 3.1(1), (3), we know that $\overline{x}_1(x_1 + e_1 + \overline{e}_1) = (x_1 + e_1 + \overline{e}_1)\overline{x}_1$. So (2) can be proven similarly. Multiplying $(1,i)(\overline{1},\overline{j})$ on both side of Definition 3.1(1) yields (3) and (4). We have $c_i x'_i = -x'_i c_i$ (resp., $\overline{c}_i x'_i = x'_i \overline{c}_i$) by (2.5) and Lemma 3.8(1) (resp., Definition 3.1(8)–(11)). Also, (1) is the counterpart of Lemma 2.9(7). So, (5) and (6) can be verified by arguments similar to those in the proof of Lemma 2.9(11). We leave the details to the reader. □

Lemma 3.10. *We have the following results for all admissible* i, j, k, l*:*

(1) $e_{i,k}x'_j = x'_j e_{i,k}$, *if* $i \neq j$,

(2) $e_{i,k}\overline{x}'_l = \overline{x}'_l e_{i,k}$, *if* $k \neq l$,

(3) $e_{i,j}(x'_i)^a e_{i,j} = \omega_a e_{i,j}$, $\forall a \in \mathbb{N}$,

(4) $e_{i,j}(\overline{x}'_j)^a e_{i,j} = \overline{\omega}_a e_{i,j}, \forall a \in \mathbb{N}$.

Proof. We have $e_1 x'_2 = x'_2 e_1$ by Definition 3.1(2). Multiplying $(2,j)$ on both sides of the equation yields $e_1 x'_j = x'_j e_1$. Since $i \neq j$, multiplying $(1,i)(\overline{1},\overline{k})$ on both sides of $e_1 x'_j = x'_j e_1$ yields (1). (2) can be verified similarly. (3) and (4) follow from Definition 3.1(5)–(7). □

We consider $BC^{\mathrm{aff}}_{r,t}$ as a filtrated superalgebra by setting

$$\deg s_i = \deg \overline{s}_j = \deg e_1 = \deg c_n = \deg \overline{c}_m = \deg \omega_a = \deg \overline{\omega}_a = 0 \text{ and } \deg x_k = \deg \overline{x}_\ell = 1,$$

for all admissible a, i, j, k, ℓ, m, n. Let $(BC^{\mathrm{aff}}_{r,t})^{(k)}$ be the super R-submodule spanned by monomials with degrees less than or equal to k for $k \in \mathbb{Z}^{\geq 0}$. Then we have the following filtration

$$BC^{\mathrm{aff}}_{r,t} \supset \ldots \supset (BC^{\mathrm{aff}}_{r,t})^{(1)} \supset (BC^{\mathrm{aff}}_{r,t})^{(0)} \supset (BC^{\mathrm{aff}}_{r,t})^{(-1)} = 0. \tag{3.1}$$

Let $\mathrm{gr}(BC^{\mathrm{aff}}_{r,t}) = \oplus_{i\geq 0}(BC^{\mathrm{aff}}_{r,t})^{[i]}$, where $(BC^{\mathrm{aff}}_{r,t})^{[i]} = (BC^{\mathrm{aff}}_{r,t})^{(i)}/(BC^{\mathrm{aff}}_{r,t})^{(i-1)}$. Then $\mathrm{gr}(BC^{\mathrm{aff}}_{r,t})$ is a $\mathbb{Z}$-graded superalgebra associated to $BC^{\mathrm{aff}}_{r,t}$. We use the same symbols to denote elements in $\mathrm{gr}(BC^{\mathrm{aff}}_{r,t})$. Moreover, $x_i = x'_i$ and $\overline{x}'_j = \overline{x}_j$ in $\mathrm{gr}(BC^{\mathrm{aff}}_{r,t})$.

Definition 3.11. We say that $\mathbf{m}$ is a regular monomial of $BC^{\mathrm{aff}}_{r,t}$ if $\mathbf{m} = x^\alpha d\overline{x}^\beta \prod_{n\in\mathbb{Z}^{>0}} \omega_{2n+1}^{a_{2n+1}}$, for some $d \in \mathcal{S}$, $a_{2n+1} \in \mathbb{N}$ and $(\alpha,\beta) \in \mathbb{N}^r \times \mathbb{N}^t$, where $\mathcal{S}$ is given in (2.12), $x^\alpha = \prod_{i=1}^r x_i^{\alpha_i}$, $\overline{x}^\beta = \prod_{i=1}^t \overline{x}_i^{\beta_i}$ and all but finitely many a_{2n+1}'s are zero.

Proposition 3.12. *As an R-module, $BC^{\mathrm{aff}}_{r,t}$ is spanned by all regular monomials in Definition 3.11.*

Proof. Let M be the R-submodule of $BC^{\mathrm{aff}}_{r,t}$ spanned by all regular monomials $\boldsymbol{m}$ in Definition 3.11. We want to prove

$$h\boldsymbol{m} \in M \quad \text{for any generator } h \text{ of } BC^{\mathrm{aff}}_{r,t}. \tag{3.2}$$

If so, we have $M = BC^{\mathrm{aff}}_{r,t}$ since $1 \in M$. In the following, we omit $\prod_{n\in\mathbb{Z}^{>0}} \omega_{2n+1}^{a_{2n+1}}$ in $\boldsymbol{m}$ since any ω_{2n+1} is central in $BC^{\mathrm{aff}}_{r,t}$.

We prove (3.2) by induction on $|\alpha|$. If $|\alpha| = 0$, i.e., $\alpha_i = 0$ for all possible i's, then (3.2) follows from Theorem 2.4 unless $h = \overline{x}_1$. In the later case, by (2.7), (2.8) and Lemma 3.8, we need to compute $\overline{x}_k e^f$ when $1 \le k \le t$ and $f > 0$. If $k \in \{1, 2, \ldots, f\}$, by Lemma 3.9(4), we use $-x_k$ instead of $\overline{x}_k$ since we work on the graded superalgebra $\mathrm{gr}(BC^{\mathrm{aff}}_{r,t})$. So, $h\boldsymbol{m} \in M$. Otherwise, $k > f$. By Lemma 3.10(2), we can use $e^f\overline{x}_k$ instead of $\overline{x}_k e^f$. So, (3.2) follows from Lemma 3.8 and Theorem 2.4.

Suppose $|\alpha| > 0$. By (2.7), (2.8), Lemma 3.8 and Theorem 2.4, we see that (3.2) holds unless $h \in \{\overline{x}_1, e_1\}$. Suppose $h = \overline{x}_1$. By Lemma 3.9(1) and (2), $\overline{x}_1 x_i = x_i \overline{x}_1$ in $\mathrm{gr}(BC^{\mathrm{aff}}_{r,t})$. So, we need to deal with $\overline{x}_k e^f$ when $1 \le k \le t$. They are the cases that we have dealt with. So, $\overline{x}_1\mathbf{m} \in M$.

Finally, we assume $h = e_1$. If $\alpha_i \ne 0$ for some i with $2 \le i \le r$, then $e_1 x_i = x_i e_1$ in $\mathrm{gr}(BC^{\mathrm{aff}}_{r,t})$ (see Lemma 3.10(1)). By inductive assumption on $|\alpha|$, we have (3.2). In order to finish the proof, it remains to consider the case that $x^\alpha = x_1^{\alpha_1}$ such that $\alpha_1 > 0$. In this case,

$$\mathbf{m} = x_1^{\alpha_1} c^\gamma d_1^{-1} e^f w d_2 \overline{c}^\delta \overline{x}^\beta \in M, \tag{3.3}$$

where $d_1, d_2 \in \mathscr{D}^f_{r,t}$ and $\beta \in \mathbb{N}^t$ and $(\gamma,\delta) \in \mathbb{Z}_2^r \times \mathbb{Z}_2^t$. Write $d_1 e_1 d_1^{-1} = e_{i,j}$ for some i, j. By (2.7) and inductive assumption on $|\alpha|$, we can use $d_1^{-1} x_i^{\alpha_1}$ to replace $x_1^{\alpha_1} d_1^{-1}$ in (3.3). So, we need to verify

$$e_{i,j} x_i^{\alpha_1} c^\gamma e^f w d_2 \overline{c}^\delta \overline{x}^\beta \in M. \tag{3.4}$$

By Lemma 3.9(3) and inductive assumption, it is enough to verify

$$e_{i,j} \overline{x}_j^{\alpha_1} c^\gamma e^f w d_2 \overline{c}^\delta \overline{x}^\beta \in M. \tag{3.5}$$

If $j \geq f+1$, then (3.5) follows from Lemma 3.10(2) and Theorem 2.4. Otherwise, $j \leq f$. If $i = j$, by inductive assumption, we use $(\overline{x}_i + \overline{L}_i)^{\alpha_1}$ instead of $\overline{x}_i^{\alpha_1}$ in $e_i \overline{x}_i^{\alpha_1} c^\gamma e_i$. If $\gamma_i = 0$, then $e_i \overline{x}_i^{\alpha_1} e_i = \overline{\omega}_{\alpha_1} e_1$ in $\mathrm{gr}(BC_{r,t}^{\mathrm{aff}})$ (see Lemma 3.10(4)). If $\gamma_i \neq 0$, then $e_i \overline{x}_i^{\alpha_1} c_i e_i = 0$ in $\mathrm{gr}(BC_{r,t}^{\mathrm{aff}})$ (see Lemma 3.9(5)). In any case, (3.5) follows from inductive assumption on $|\alpha|$. Finally, we assume $i \neq j$. If $i \neq k$, then $e_{i,j} c_k = c_k e_{i,j}$. By inductive assumption, we need to consider $e_{i,j} x_i^{\alpha_1} c_i^{\gamma_i} e_j = e_{i,j} x_i^{\alpha_1} e_j c_i^{\gamma_i}$. Since

$$e_{i,j} x_i^{\alpha_1} e_j = e_{i,j} e_j x_i^{\alpha_1} = (i,j) x_i^{\alpha_1} e_j = x_j^{\alpha_1} (i,j) e_j$$

in $\mathrm{gr}(BC_{r,t}^{\mathrm{aff}})$, by inductive assumption and our previous results on $h \in \{s_1, \ldots, s_{r-1}, c_1, \ldots, c_r, x_1\}$, we have (3.5). So (3.4) is true. This completes the proof. □

Definition 3.13. Let I be the two-sided ideal of $BC_{r,t}^{\mathrm{aff}}$ generated by $\omega_{2k+1} - \tilde{\omega}_{2k+1}$, where $\tilde{\omega}_{2k+1} \in R$ for all $k \in \mathbb{Z}^{>0}$. Let $\widetilde{BC}_{r,t} = BC_{r,t}^{\mathrm{aff}}/I$.

Definition 3.14. Let I be the two-sided ideal of $\widetilde{BC}_{r,t}$ generated by $f(x_1)$ and $g(\overline{x}_1)$, where

$$f(x_1) = x_1^k \prod_{i=1}^{m} (x_1^2 - u_i^2), \text{ and } g(\overline{x}_1) = \overline{x}_1^{k_1} \prod_{j=1}^{m_1} (\overline{x}_1^2 - \overline{u}_j^2), \tag{3.6}$$

for some non-zero $u_1, \ldots, u_m, \overline{u}_1, \ldots, \overline{u}_{m_1} \in R$ such that $\ell = k + 2m = k_1 + 2m_1$ and

$$e_1 f(x_1) = (-1)^k e_1 g(\overline{x}_1). \tag{3.7}$$

The *level ℓ or cyclotomic walled Brauer–Clifford superalgebra $BC_{\ell,r,t}$* is the quotient algebra $\widetilde{BC}_{r,t}/I$.

In section 6, we will explain the reason why $f(x_1)$ and $g(\overline{x}_1)$ have to satisfy (3.6) and (3.7).

Definition 3.15. We say that $\mathbf{m}$ is a regular monomial of $\widetilde{BC}_{r,t}$ (resp., $BC_{\ell,r,t}$) if it is of form $x^\alpha d \overline{x}^\beta$, for some $d \in \mathcal{S}$, and $(\alpha, \beta) \in \mathbb{N}^r \times \mathbb{N}^t$ (resp. $\mathbb{Z}_\ell^r \times \mathbb{Z}_\ell^t$), where $\mathcal{S}$ is given in (2.12).

Corollary 3.16. *As R-modules, both $\widetilde{BC}_{r,t}$ and $BC_{\ell,r,t}$ are spanned by their regular monomials.*

Proof. By Proposition 3.12, $\widetilde{BC}_{r,t}$ is spanned by all its regular monomials. Let $\phi_\ell : \widetilde{BC}_{r,t} \twoheadrightarrow BC_{\ell,r,t}$ be the canonical epimorphism. It is enough to verify that the image of a regular monomial $\mathbf{m}$ of $\widetilde{BC}_{r,t}$ can be expressed as a linear combination of regular monomials of $BC_{\ell,r,t}$. If $(\alpha, \beta) \in \mathbb{Z}_\ell^r \times \mathbb{Z}_\ell^t$, then the images of $\mathbf{m}$ is a regular monomial of $BC_{\ell,r,t}$. Otherwise, either $\alpha_i \geq \ell$ or $\beta_j \geq \ell$ for some possible i or j. Since $\widetilde{BC}_{r,t}$ inherits the graded structure of $BC^{\mathrm{aff}}_{r,t}$, it results in a graded structure on $BC_{\ell,r,t}$. So, either $x_i^{\alpha_i}$ or $\overline{x}_j^{\beta_j}$ can be expressed as a linear combination of elements in $BC_{\ell,r,t}$ with lower degrees. Using these elements to replace either $x_i^{\alpha_i}$ or $\overline{x}_j^{\beta_j}$ in the image of $\mathbf{m}$ and considering the inverse images of such elements in $\widetilde{BC}_{r,t}$, we see that the image of $\mathbf{m}$ can be expressed as a linear combination of regular monomials of $BC_{\ell,r,t}$, as required. $\square$

4. A basis of $BC_{2,r,t}$ with special parameters

Let $\mathfrak{g} = \mathfrak{q}(n)$ be the queer Lie superalgebra of rank n over $\mathbb{C}$. Then $\mathfrak{g}$ has a basis $\mathsf{e}_{i,j} = E_{i,j} + E_{-i,-j}$ (even element), $\mathsf{f}_{i,j} = E_{i,-j} + E_{-i,j}$ (odd element) for $i,j \in I^+ = \{1,2,...,n\}$, where $E_{i,j}$ is the $2n \times 2n$ matrix with entry 1 at (i,j) position and zero otherwise for $i,j \in I = I^+ \cup I^-$, and $I^- = -I^+$. Let $V = \mathbb{C}^{n|n} = V_{\overline{0}} \oplus V_{\overline{1}}$ be the natural $\mathfrak{g}$-module (and the natural $\mathfrak{gl}_{n|n}$-module) with basis $\{v_i \,|\, i \in I\}$, where $\mathfrak{gl}_{n|n}$ is the general linear Lie superalgebra with basis $\{E_{i,j} \mid i,j \in I\}$. Then v_i has the parity $[v_i] = [i] \in \mathbb{Z}_2$, where $[i] = 0$ and $[-i] = 1$ for $i \in I^+$. Let V^* be the linear dual space of V with dual basis $\{\overline{v}_i \,|\, i \in I\}$. Thus V^* is a left $\mathfrak{g}$-module with action

$$E_{a,b}\overline{v}_i = -(-1)^{[a]([a]+[b])}\delta_{i,a}\overline{v}_b \text{ for } a,b,i \in I. \tag{4.1}$$

Let $\mathfrak{h} = \mathfrak{h}_{\overline{0}} \oplus \mathfrak{h}_{\overline{1}}$ be a Cartan subalgebra of $\mathfrak{g}$ with even part $\mathfrak{h}_{\overline{0}} = \mathrm{span}\{\mathsf{e}_{i,i} \mid i \in I^+\}$ and odd part $\mathfrak{h}_{\overline{1}} = \mathrm{span}\{\mathsf{f}_{i,i} \mid i \in I^+\}$. Let $\mathfrak{h}^*_{\overline{0}}$ be the dual space of $\mathfrak{h}_{\overline{0}}$ with $\{\varepsilon_i \,|\, i \in I^+\}$ being the dual basis of $\{\mathsf{e}_{i,i} \mid i \in I^+\}$. Then an element $\lambda \in \mathfrak{h}^*$ (called a *weight*) can be written as

$$\lambda = \sum_{i \in I^+} \lambda_i \varepsilon_i = (\lambda_1, ..., \lambda_n) \text{ with } \lambda_i \in \mathbb{C}. \tag{4.2}$$

Let M be any $\mathfrak{g}$-module. For any $r,t \in \mathbb{Z}^{\geq 0}$, set $M^{r,t} = M \otimes V^{\otimes r} \otimes (V^*)^{\otimes t}$. For convenience we denote the ordered set

$$J = \{0\} \cup J_1 \cup J_2, \quad \text{where } J_1 = \{1,...,r\},\ J_2 = \{\overline{1},...,\overline{t}\}, \tag{4.3}$$

and $0 \prec 1 \prec \ldots \prec r \prec \overline{1} \prec \ldots \prec \overline{t}$. We write $M^{r,t}$ as

$$M^{r,t} = \bigotimes_{i \in J} V_i, \quad \text{where } V_0 = M,\ V_i = V \text{ if } 0 \prec i \prec \overline{1}, \text{ and } V_i = V^* \text{ if } i \succ r, \tag{4.4}$$

(hereafter all tensor products will be taken according to the order in J), which is a left $U(\mathfrak{g})^{\otimes(r+t+1)}$-module (where $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$), with the action given by

$$\Big(\bigotimes_{i \in J} g_i\Big)\Big(\bigotimes_{i \in J} x_i\Big) = (-1)^{\sum\limits_{i \in J}[g_i]\sum\limits_{j \prec i}[x_j]} \bigotimes_{i \in J}(g_i x_i) \text{ for } g_i \in U(\mathfrak{g}),\ x_i \in V_i.$$

In order to prove a basis theorem for level 2 walled Brauer–Clifford superalgebra, we take $n = 2m$ to be an even integer. We denote $I_1^+ = \{1, ..., m\}$, $I_2^+ = m + I_1^+$. Thus $I^+ = I_1^+ \cup I_2^+$. For $i \in I_1^+$, we denote $i_\bullet = i + m \in I_2^+$. For $i \in I_2^+$, we denote $i_\circ = i - m \in I_1^+$. Let $M = L_\lambda$ be the finite dimensional simple $\mathfrak{g}$-module of type M with dominant highest weight

$$\lambda = (p, p-1, ..., p-n+1) \text{ for some } p\in\mathbb{C} \text{ such that } p\in\mathbb{C}\backslash\mathbb{Z} \text{ or } p\in\mathbb{Z} \text{ with } p>n \text{ or } p<0. \tag{4.5}$$

Then $\mathrm{End}_{\mathfrak{g}}(L_\lambda)$ is one dimensional. Denote by v_λ a fixed highest weight vector of L_λ with even parity, and $(L_\lambda)_\lambda$ the highest weight space of L_λ, which is 2^m-dimensional with a basis

$$B_1 = \{b^\theta v_\lambda \,|\, b^\theta \in B_0\} \text{ with } B_0 = \Big\{ b^\theta := \prod_{i\in I_1^+} \mathsf{f}_{i,i}^{\theta_i} \,\Big|\, \theta = (\theta_1, ..., \theta_m) \in \mathbb{Z}_2^m \Big\}, \tag{4.6}$$

where the products are taken in any fixed order (changing the order only changes the vectors by a factor ± 1). For $i \in I_2^+$, we have

$$\mathsf{f}_{i,i} v_\lambda = \sqrt{-\frac{p+i-1}{p+i_\circ-1}}\, \mathsf{f}_{i_\circ,i_\circ} v_\lambda. \tag{4.7}$$

Let C be the PBW monomial basis of $U(\mathfrak{g}^- \oplus \mathfrak{h})$. We say a basis element $a \in C$ has *length* $\ell(a) := k$ if a contains k factors; for instance, $\ell(b^\theta) = |\theta|$. For $i \in \mathbb{Z}^{\geq 0}$, let $C_i = \{a \in C \,|\, \ell(a) = i\}$. Set

$$D = \Big\{ u^\sigma := \prod_{i\in I_1^+} \mathsf{f}_{i_\bullet,i}^{\sigma_i} \,\Big|\, \sigma = (\sigma_1, ..., \sigma_m) \in \mathbb{Z}_2^m \Big\} \subset C, \text{ and } D_i = D \cap C_i. \tag{4.8}$$

Let $(L_\lambda)_i$ be the subspace of L_λ spanned by cv_λ for $c \in C$ with $\ell(c) \leq i$. Set $(L_\lambda)_{-1} = 0$. Note that elements of $\mathfrak{g}^+$ acting on L_λ send $(L_\lambda)_i$ to $(L_\lambda)_{i-1}$.

Lemma 4.1. *For $i \in \mathbb{Z}^{\geq 0}$, the set $D_i v_\lambda$ is $\mathbb{C}$-linear independent under modulo $(L_\lambda)_{i-1}$.*

Proof. Assume $c := \sum_{\sigma\in\mathbb{Z}_2^m:|\sigma|=i} a_\sigma u^\sigma v_\lambda \in (L_\lambda)_{i-1}$ for some $a_\sigma \in \mathbb{C}$ with at least one $a_\sigma \neq 0$. Take a $\tilde\sigma \in \mathbb{Z}_2^m$ such that $a_{\tilde\sigma} \neq 0$. Assume $\tilde\sigma_\ell \neq 0$ for some $\ell \in I_1^+$. Applying $\mathsf{f}_{\ell,\ell_\bullet} \in \mathfrak{g}^+$ to c, by moving $\mathsf{f}_{\ell,\ell_\bullet}$ to the right until it meets v_λ, using the commutation relation $[\mathsf{f}_{\ell,\ell_\bullet}, \mathsf{f}_{j_\bullet,j}] = \delta_{\ell\, j}(\mathsf{e}_{\ell,\ell} + \mathsf{e}_{\ell_\bullet,\ell_\bullet})$ (which is a Cartan element commuting with $\mathsf{f}_{i_\bullet,i}$ for $i \neq \ell$), we can easily obtain $\mathsf{f}_{\ell,\ell_\bullet} c = \sum_{\sigma\in\mathbb{Z}_2^m:\sigma_\ell\neq 0} a'_\sigma (2p - (\ell + \ell_\bullet)) u^{\sigma - 1_\ell} v_\lambda \in (L_\lambda)_{i-2}$, where $a'_\sigma = \pm a_\sigma$, $1_\ell = (\delta_{1,\ell}, ..., \delta_{m,\ell}) \in \mathbb{Z}_2^m$. Note that $2p - (\ell + \ell_\bullet) \neq 0$ by (4.5). Now induction on $|\theta|$ gives that $a_{\tilde{\sigma}} = 0$, a contradiction with the assumption. □

For each $i = 1, 2, ...$, by Lemma 4.1, we can choose a maximal subset $\hat{C}_i$ of C_i satisfying the following conditions (i.e., we extend D_i to a basis $\hat{C}_i$ of $(L_\lambda)_i$ modulo $(L_\lambda)_{i-1}$, thus $\#\hat{C}_i = \dim(L_\lambda)_i/(L_\lambda)_{i-1}$):

(C1) $\hat{C}_i \supset D_i$;
(C2) $\{uv_\lambda \,|\, u \in \hat{C}_i\}$ is a $\mathbb{C}$-linear independent subset of L_λ.

Then we have the following basis of L_λ,

$$B^e := \{wv_\lambda \,|\, w \in \hat{C}\}, \text{ where } \hat{C} = \bigcup_{i=0}^{\infty} \hat{C}_i. \tag{4.9}$$

We say the basis element wv_λ has *length* $\ell(wv_\lambda) := \ell(w)$. Then from our choice of $\hat{C}_i$, we immediately have the following.

Lemma 4.2. *Let $\alpha \in C$ be a monomial basis element of length j. Then αv_λ is a combination of basis elements in B^e with length $\leq j$.*

Take a basis B_M of $M^{r,t}$, where

$$B_M = \Big\{ b_M = b \otimes \bigotimes_{i \in J_1} v_{k_i} \otimes \bigotimes_{i \in J_2} \overline{v}_{k_i} \,\Big|\, b \in B^e,\ k_i \in I \Big\}. \tag{4.10}$$

Introduce the following elements,

$$\begin{aligned} &\bar{\mathsf{e}}_{i,j} = E_{i,j} - E_{-i,-j}, \quad \bar{\mathsf{f}}_{i,j} = E_{-i,j} - E_{i,-j} \quad \in \quad \mathfrak{gl}_{n|n}, \\ &\Omega_0 = \sum_{i,j \in I} (-1)^{[j]} E_{i,j} \otimes E_{j,i} \in \mathfrak{gl}_{n|n}^{\otimes 2}, \\ &\Omega_1 = \sum_{i,j \in I^+} \mathsf{e}_{i,j} \otimes \bar{\mathsf{e}}_{j,i} - \sum_{i,j \in I^+} \mathsf{f}_{i,j} \otimes \bar{\mathsf{f}}_{j,i} \in \mathfrak{g} \otimes \mathfrak{gl}_{n|n}. \end{aligned} \tag{4.11}$$

For $a, b \in J$ with $a \prec b$, we define $\pi_{a,b} : U(\mathfrak{g}) \otimes U(\mathfrak{gl}_{n|n}) \to U(\mathfrak{g}) \otimes U(\mathfrak{gl}_{n|n})^{\otimes(r+t)}$ by

$$\pi_{a,b}(x \otimes y) = 1 \otimes \cdots \otimes 1 \otimes x \otimes 1 \otimes \cdots \otimes 1 \otimes y \otimes 1 \otimes \cdots \otimes 1, \tag{4.12}$$

where x and y are in the a-th and b-th tensors respectively. Similarly we have $\pi_a : U(\mathfrak{g}) \to U(\mathfrak{g})^{\otimes(r+t+1)}$ which sends x to the a-th tensor.

Definition 4.3. We can use (4.12) to define the following elements in the endomorphism algebra $\mathrm{End}_{U(\mathfrak{g})}(M^{r,t})^{\mathrm{op}}$,

$$s_i = \pi_{i,i+1}(\Omega_0)|_{M^{r,t}}\ (1\le i<r), \quad \overline{s}_j = \pi_{\overline{j},\overline{j+1}}(\Omega_0)|_{M^{r,t}}\ (1\le j<t),$$

$$x'_i = -\pi_{0,i}(\Omega_1)|_{M^{r,t}}\ (1\le i\le r), \quad \overline{x}'_j = -\pi_{0,\overline{j}}(\Omega_1)|_{M^{r,t}}\ (1\le j\le t),$$

$$e_i = -\pi_{i,\overline{i}}(\Omega_0)|_{M^{r,t}}\ (1\le i\le \min\{r,t\}), \quad c_i = \pi_i(c)\ (1\le i\le r), \quad \overline{c}_i = \pi_{\overline{i}}(\overline{c})\ (1\le i\le t), \tag{4.13}$$

where $c : V \to V$ (resp., $\overline{c} : V^* \to V^*$) is the automorphism such that $c(v_{\pm i}) = \pm v_{\mp i}$ (resp., $\overline{c}(\overline{v}_{\pm i}) = \overline{v}_{\mp i}$). Set $x_1 = x'_1$, $\overline{x}_1 = \overline{x}'_1$.

Observe that $c^2 = -1$ and $\overline{c}^2 = 1$, and $c, \overline{c}$ correspond to maps $c, \overline{c} : I \to I$ such that

$$c(\pm i) = \overline{c}(\pm i) = \mp i \text{ for } i \in I^+, \text{ and } c(v_i) = [i] v_{c(i)}, \quad \overline{c}(\overline{v}_i) = \overline{v}_{\overline{c}(i)} \text{ for } i \in I. \tag{4.14}$$

Lemma 4.4.

(a) *The minimal polynomial $f(x)$ of x_1 with respect to $M^{r,t}$ is $x^2 - p(p+1)$.*
(b) *The minimal polynomial $g(x)$ of $\overline{x}_1$ with respect to $M^{r,t}$ is $x^2 - (p-n+1)(p-n)$.*
(c) *We have $e_1 x_1 e_1 = -n(2p-n+1)e_1$ with respect to $M^{r,t}$.*

Proof. (a) We may assume $r = 1$, $t = 0$. Note that the only possible highest weight in the finite dimensional $\mathfrak{g}$-module $L_\lambda \otimes V$ is $\mu = \lambda + \varepsilon_1$, which is a typical dominant weight. Thus $L_\lambda \otimes V$ must be completely reducible, and thus a direct sum of finite copies of L_μ. Observe that the set $\{u^\theta \otimes v_{\pm 1} \mid u^\theta \in B_1\}$, with 2^{m+1} elements, is a maximal set of $\mathbb{C}$-linear independent highest weight vectors of weight μ. Since L_μ occupies 2^m $\mathbb{C}$-linear independent highest weight vectors, we see that $L_\lambda \otimes V = L_\mu^{\oplus 2}$, which as a $\mathfrak{g}$-module is generated by $v_\mu^\pm := v_\lambda \otimes v_{\pm 1}$. One can easily verify that $v_\mu^\pm x_1 = \mp(p+1)v_\mu^\pm \pm \mathsf{f}_{1,1} v_\mu^\mp$. Thus $v_\mu^+, \mathsf{f}_{1,1} v_\mu^-$ (resp., $v_\mu^-, \mathsf{f}_{1,1} v_\mu^+$) span a 2-dimensional x_1-invariant subspace of $L_\lambda \otimes V$, and the minimal polynomial $f(x)$ of x_1 in this subspace is $x^2 - p(p+1)$. Since x_1 commutes with the $\mathfrak{g}$-action and $L_\lambda \otimes V$ is generated by $v_\mu^\pm$, we see $f(x)$ is also the minimal polynomial of x_1 in $M^{r,t}$.

(b) We can assume $r = 0$, $t = 1$. Similar to the arguments in (a), we have $L_\lambda \otimes V^* = L_\nu^{\oplus 2}$ with highest weight $\nu = \lambda - \varepsilon_n$ (which is again a typical dominant weight) and two highest weight vectors $v_\nu^\pm := v_\lambda \otimes \overline{v}_{\pm n}$. In addition, $v_\nu^\pm \overline{x}_1 = \pm(p-n)v_\nu^\pm + \mathsf{f}_{n,n} v_\nu^\mp$. Thus the minimal polynomial $g(x)$ of $\overline{x}_1$ is $x^2 - (p-n+1)(p-n)$.

(c) We can assume $r = t = 1$. Then for $a, b \in I$, we have

$$\begin{aligned}(v_\lambda \otimes v_a \otimes \overline{v}_b) e_1 x_1 e_1 &= (-1)^{[a]} \delta_{ab} \sum_{i\in I} (v_\lambda \otimes v_i \otimes \overline{v}_i) x_1 e_1 \\ &= (-1)^{[a]} \delta_{ab} \sum_{i\in I} (-1)^{[i]+1} \big((p+1-|i|) v_\lambda \otimes v_i \otimes \overline{v}_i\big) e_1 = -n(2p-n+1)(v_\lambda \otimes v_a \otimes \overline{v}_b) e_1.\end{aligned}$$

Since $L_\lambda \otimes V \otimes V^*$ is generated by $v_\lambda \otimes v_a \otimes \overline{v}_b$ for $a, b \in I$, and e_1, x_1 commute with the $\mathfrak{g}$-action, we obtain (c). $\square$

Lemma 4.5. *For $k \in I^+$, we have*

$$(v_\lambda \otimes v_{\pm k})x_1 = \mp\lambda_k v_\lambda \otimes v_{\pm k} \pm \mathsf{f}_{k,k} v_\lambda \otimes v_{\mp k} \mp \sum_{j<k} \mathsf{e}_{k,j} v_\lambda \otimes v_{\pm j} \mp \sum_{j<k} \mathsf{f}_{k,j} v_\lambda \otimes v_{\mp j},$$

$$(v_\lambda \otimes \overline{v}_{\pm k})\overline{x}_1 = \pm\lambda_k v_\lambda \otimes \overline{v}_{\pm k} + \mathsf{f}_{k,k} v_\lambda \otimes \overline{v}_{\mp k} \pm \sum_{i>k} \mathsf{e}_{i,k} v_\lambda \otimes \overline{v}_{\pm i} + \sum_{i>k} \mathsf{f}_{i,k} v_\lambda \otimes \overline{v}_{\mp i}.$$

Proof. The result follows from the definitions of x_1 and $\overline{x}_1$. □

For any $a \in I$, we set $a^+ = |a| \in I^+$. Then (4.14) gives

$$(c(a))^+ = (\overline{c}(a))^+ = a^+ \text{ for } a \in I. \tag{4.15}$$

Now we assume $BC_{2,r,t}$ is the level two walled Brauer–Clifford superalgebra such that $x_1, \overline{x}_1$ satisfy the degree 2 polynomials in Lemma 4.4, and parameters satisfy

$$\omega_0 = 0,\ \omega_1 = -2m(2p - 2m + 1),\ \omega_i = p(p+1)\omega_{i-2} \text{ for } i \geq 2, \tag{4.16}$$

where $p \in \mathbb{C}\backslash\mathbb{Z}$, or $p \in \mathbb{Z}$ with $p > 2m$ or $p < 0$, and $m \in \mathbb{Z}^{>0}$ satisfies $m \geq 2(r+t)$. By Lemma 4.4 and Definition 3.1(1)(3), we have $e_1 f(x_1) = e_1 g(\overline{x}_1)$. We take $n = 2m$. Take the weight λ as in (4.5), then we can define the space $M^{r,t}$ as in (4.4).

Proposition 4.6. *There is a superalgebra homomorphism $\varphi : BC_{2,r,t} \to \mathrm{End}_{\mathfrak{g}}(M^{r,t})^{op}$ such that φ sends the generators e_1, x_1, $\overline{x}_1$, s_i's, $\overline{s}_j$'s, c_m's, $\overline{c}_n$'s to the same symbols defined in Definition 4.3.*

Proof. By Lemma 4.4(a)–(b), we need to show the images of the generators satisfy the relations in Definition 3.1. First it is easy to see that (2.1)–(2.3) and Definition 2.1 (1)–(10) are satisfied (cf. [9] or [3, Theorem 1.4]). Moreover, (2), (4), (8)–(11) in Definition 3.1 follows from Definition 4.3.

Let $\Omega_{0,i} = \pi_{0,i}(\Omega_1)|_{M^{r,t}}$, $S_{i,j} = \pi_{i,j}(\Omega_0)|_{M^{r,t}}$ and $C_i = \pi_i(C)|_{M^{r,t}}$ for $i, j \in J_1 \cup J_2$, where $C = \sum_{i \in I^+} \overline{f}_{i,i}$. Then $c_i = C_i$, $\overline{c}_i = C_{\overline{i}}$, $s_i = S_{i,i+1}$, $\overline{s}_i = S_{\overline{i},\overline{i+1}}$, $e_1 = -S_{1,\overline{1}}$, $x_1 = -\Omega_{0,1}$ and $\overline{x}_1 = -\Omega_{0,\overline{1}}$. Direct calculations as in the proof of [8, Theorem 7.4.1] show that

(a) $\Omega_{0,i} C_i = -C_i \Omega_{0,i}$, for $i \in J_1 \cup J_2$,
(b) $S_{i,i+1}\Omega_{0,i}S_{i,i+1} = \Omega_{0,i+1}$, for $1 \leq i \leq r-1$ or $\overline{1} \leq i \leq \overline{t-1}$,
(c) $\Omega_{0,i}\Omega_{0,j} - \Omega_{0,j}\Omega_{0,i} = (\Omega_{0,j} - \Omega_{0,i})S_{i,j} + (\Omega_{0,j} + \Omega_{0,i})C_i C_j S_{i,j}$ for $i, j \in J_1$,
(d) $\Omega_{0,\overline{i}}\Omega_{0,\overline{j}} - \Omega_{0,\overline{j}}\Omega_{0,\overline{i}} = (\Omega_{0,\overline{j}} - \Omega_{0,\overline{i}})S_{\overline{i},\overline{j}} - (\Omega_{0,\overline{j}} + \Omega_{0,\overline{i}})C_{\overline{i}} C_{\overline{j}} S_{\overline{i},\overline{j}}$ for $\overline{i}, \overline{j} \in J_2$,
(e) $\Omega_{0,1}\Omega_{0,\overline{1}} - \Omega_{0,\overline{1}}\Omega_{0,1} = (S_{1,\overline{1}} - C_1 S_{1,\overline{1}} C_1)\Omega_{0,1} - \Omega_{0,1}(S_{1,\overline{1}} - C_1 S_{1,\overline{1}} C_1)$.

Thanks to (a)–(e), (2.7)–(2.8) and Definition 3.1(3) are satisfied. By definition of e_1, we have $e_1^2 = 0$. Moreover, from Lemma 4.4(a), (c), $e_1 x_1^a e_1 = \omega_a e_1$ and (4.16) follows. So, Definition 3.1(5)–(6) are satisfied. Direct calculation by using Lemma 4.5 shows that Definition 3.1(1) is satisfied. Finally, it follows from Lemma 4.4(b) and Definition 3.1(1) that $e_1\overline{x}_1^a e_1 = \overline{\omega}_a e_1$ for some $\overline{\omega}_a \in \mathbb{C}$. So, Definition 3.1(7) is satisfied. $\square$

By Proposition 4.6, $M^{r,t}$ is a right $BC_{2,r,t}$-module. For any $\alpha, \beta \in \mathbb{Z}_2^r$, $\overline{\alpha}, \overline{\beta} \in \mathbb{Z}_2^t$, we define the following elements of $BC_{2,r,t}$:

$$c^\alpha = \prod_{i=1}^{r} c_i^{\alpha_i}, \quad x'^{\,\beta} = \prod_{j=1}^{r} x_j'^{\,\beta_j}, \quad \overline{c}^{\overline{\alpha}} = \prod_{i=1}^{t} \overline{c}_i^{\overline{\alpha}_i}, \quad \overline{x}'^{\,\overline{\beta}} = \prod_{j=1}^{t} \overline{x}_j'^{\,\overline{\beta}_j}, \tag{4.17}$$

where the product in $x'^{\,\beta}$ is written in the order $x_r'^{\,\beta_r} \cdots x_1'^{\,\beta_1}$ (thus x_1' acts first on $M^{r,t}$) and the like for $\overline{x}'^{\,\overline{\beta}}$.

Theorem 4.7. *The monomials*

$$\boldsymbol{m} := d_1^{-1} c^\alpha x'^{\,\beta} e^f \overline{x}'^{\,\overline{\beta}} \overline{c}^{\overline{\alpha}} w d_2, \tag{4.18}$$

with $\alpha, \beta \in \mathbb{Z}_2^r$, $\overline{\alpha}, \overline{\beta} \in \mathbb{Z}_2^t$ and d_1, e^f, w, d_2 as in (2.12), are $\mathbb{C}$-linearly independent elements of $BC_{2,r,t}$.

Proof. Suppose there is a nonzero $\mathbb{C}$-combination $\boldsymbol{c} := \sum_{\boldsymbol{m}} r_{\boldsymbol{m}} \boldsymbol{m}$ of monomials (4.18) being zero. We fix a monomial $\tilde{\boldsymbol{m}} := \tilde{d}_1^{-1} c^{\tilde{\alpha}} x'^{\,\tilde{\beta}} e^{\tilde{f}} \overline{x}'^{\,\tilde{\overline{\beta}}} \overline{c}^{\tilde{\overline{\alpha}}} \tilde{w} \tilde{d}_2$ in $\boldsymbol{c}$ with nonzero coefficient $r_{\tilde{\boldsymbol{m}}} \neq 0$ which satisfies the following conditions:

(i) $|\tilde{\beta}| + |\tilde{\overline{\beta}}|$ is maximal;
(ii) $\tilde{f}$ is minimal among all monomials satisfying (i).

We take the basis element $v = v_\lambda \otimes \otimes_{i\in J_1} v_{k_i} \otimes \otimes_{i \in J_2} \overline{v}_{k_i} \in B_M$ (cf. (4.10)) such that (note that here is the place where we require condition that $2(r+t) \leq m$)

(1) for $1 \leq i \leq r$, $k_i = i$ if $\tilde{\beta}_i = 0$ and $k_i = -i_\bullet$ if $\tilde{\beta}_i = 1$;
(2) for $1 \leq i \leq \tilde{f}$, $k_{\overline{i}} = i$;
(3) for $\tilde{f} < i \leq t$, $k_{\overline{i}} = r + i$ if $\tilde{\overline{\beta}}_i = 0$ and $k_{\overline{i}} = -(r + i_\bullet)$ if $\tilde{\overline{\beta}}_i = 1$.

Take

$$z := (v) c^{\tilde{\alpha}} \tilde{d}_1 \boldsymbol{c}\, \tilde{d}_2^{-1} \tilde{w}^{-1} \overline{c}^{\tilde{\overline{\alpha}}} \in M^{r,t}, \tag{4.19}$$

$$\tilde{u} = \prod_{i=1}^{r} \mathsf{f}_{i_\bullet, i}^{\tilde{\beta}_i} \prod_{i=1}^{t} \mathsf{f}_{r+i_\bullet, r+i}^{\tilde{\overline{\beta}}_i} v_\lambda \in B^e, \tag{4.20}$$

$$\tilde{v} = \tilde{u} \otimes v_{\mathbf{i}} \otimes \bigotimes_{i \in J_2} \overline{v}_{r+i} \in B_M, \tag{4.21}$$

where the product in (4.20) is in the same order as in (4.8) and $v_{\mathbf{i}} = v_{i_1} \otimes v_{i_2} \otimes \ldots \otimes v_{i_r}$ such that

$$i_j = \begin{cases} j, & \text{if either } \tilde{f} < j \le r \text{ or } \overline{\tilde{\beta}}_j = 0 \text{ and } j \le \tilde{f}; \\ -(r + j_\bullet), & \text{if } j \le \tilde{f} \text{ and } \overline{\tilde{\beta}}_j = 1. \end{cases}$$

We want to prove that when write z as a combination of basis B_M in (4.10), the coefficient $\chi^z_{\tilde{v}}$ of $\tilde{v}$ is nonzero. Thus assume a monomial $\boldsymbol{m}$ in (4.18) appears in the expression of $\boldsymbol{c}$ with $r_{\boldsymbol{m}} \neq 0$ and consider the following element,

$$\begin{aligned} z_1 :=& (v)c^{\tilde{\alpha}}\tilde{d}_1 \boldsymbol{m}\, \tilde{d}_2^{-1}\tilde{w}^{-1}\overline{c}^{\overline{\tilde{\alpha}}} = (v)c^{\tilde{\alpha}}\tilde{d}_1 d_1^{-1} c^{\alpha} x'^{\,\beta} e^f \overline{x}'^{\,\overline{\beta}}\overline{c}^{\overline{\alpha}} w d_2 \tilde{d}_2^{-1}\tilde{w}^{-1}\overline{c}^{\overline{\tilde{\alpha}}} \\ =& \Big(v_\lambda \otimes \bigotimes_{i\in J_1} v_{c^{\gamma_i + \tilde{\alpha}_i}(k_{(i)\tilde{d}_1 d_1^{-1}})} \otimes \bigotimes_{i\in J_2} \overline{v}_{k_{(i)\tilde{d}_1 d_1^{-1}}} \Big) x'^{\,\beta} e^f \overline{x}'^{\,\overline{\beta}}\overline{c}^{\overline{\alpha}} w d_2 \tilde{d}_2^{-1}\tilde{w}^{-1}\overline{c}^{\overline{\tilde{\alpha}}}, \end{aligned} \tag{4.22}$$

where $\gamma_i = \alpha_{(i)\tilde{d}_1 d_1^{-1}}$, and where the last equation is understood as "equal up to a sign" (cf. (4.14)), which follows by noting that elements in $\Sigma_r \times \overline{\Sigma}_t$ have natural right actions on $J_1 \cup J_2$ by permutations and c acts on I by (4.14). Write z_1 as a $\mathbb{C}$-combination of basis B_M. If $\tilde{v}$ appears as a term with a nonzero coefficient in the combination, then we say that z_1 produces $\tilde{v}$.

Note that $\tilde{u}$ has length $|\beta| + |\overline{\beta}|$. By Definition 4.3 and from our choice of B^e in (4.9), we see that factors of $\tilde{u}$ can be only contributed by the actions of x'_i's and $\overline{x}'_i$'s, and each x'_i or $\overline{x}'_i$ can at most contribute one length of $\tilde{u}$ by observing the following: if the first factor of a term in $\pi_{0i}(\Omega_1)$ for $i \in J_1 \cup J_2$ acting on the first factor of an element in B_M changes the first factor to a basis element in B^e then this $\pi_{0i}(\Omega_1)$ may contribute one length, otherwise the first factor is changed to a combination of basis elements with length not increasing by Lemma 4.2. We see that z_1 cannot produce a basis element with degree higher than $|\beta| + |\overline{\beta}|$. Thus $\tilde{v}$ cannot be produced if $|\beta| + |\overline{\beta}| < |\tilde{\beta}| + |\overline{\tilde{\beta}}|$. So by condition (i), we can assume

$$|\beta| + |\overline{\beta}| = |\tilde{\beta}| + |\overline{\tilde{\beta}}|. \tag{4.23}$$

Then $f \geq \tilde{f}$ by condition (ii).

For any basis element b_M written as in (4.10), we say k_i the *i-th label* of b_M for $i \in J_1 \cup J_2$. Note from (4.20) that all factors of $\tilde{u}$ have the following form

$$\mathsf{f}_{i_\bullet, i} \text{ with } i \in I_1^+. \tag{4.24}$$

Thus when $x'_i = -\pi_{0i}(\Omega_1)$ for $1 \le i \le r$ is applied to the element inside the bracket, it can only change its i-th label, say $\pm i_\bullet$, to $\mp i$. Since $\tilde{d}_1 d_1^{-1}$ only permutes labels and $c^{\gamma_i + \tilde{\alpha}_i}$ only changes labels up to a sign, in order for a term in (4.22) to contribute to $\chi^{z_1}_{\tilde{v}}$, we need at least f pairs $(i, \overline{j}) \in J_1 \times J_2$ such that the i-th label k_i and j-th label $k_{\overline{j}}$ satisfy the condition $k_i^+ = k_{\overline{j}}^+$ or $k_i^+ = k_{\overline{j}}^+ + m$. From our choice of the vector v, we must have $f \le \tilde{f}$. Thus we can suppose $\tilde{f} = f$ by the fact that $f \ge \tilde{f}$.

Set $J_f = \{i, \bar{i} \,|\, 1 \leq i \leq \tilde{f} = f\} \subset J_1 \cup J_2$ (cf. (4.3)). If $d_1 \neq \tilde{d}_1$, then by definition (2.11), we have

$$\tilde{j} := (j)\tilde{d}_1 d_1^{-1} \notin J_f \quad \text{for some } j \in J_f. \tag{4.25}$$

Say $\tilde{j} \in J_1$ (the proof is similar if $\tilde{j} \in J_2$), then $f < \tilde{j} \leq r$. Condition (1) shows that $k^+_{\tilde{j}} = \tilde{j}$ or $k^+_{\tilde{j}} = m + \tilde{j}$. Then conditions (2) and (3) show that there is no $\bar{\ell} \in J_2$ with $k^+_{\tilde{j}} = k^+_{\bar{\ell}}$ or $k^+_{\tilde{j}} = k^+_{\bar{\ell}} + m$. Since all factors of $\tilde{u}$ have the form (4.24), we see that z_1 cannot produce the basis element $\tilde{v}$. Thus we can suppose $\tilde{d}_1 = d_1$. Then $c^{\gamma_i + \tilde{\alpha}_i}(k_{(i)\tilde{d}_1 d_1^{-1}}) = c^{\alpha_i - \tilde{\alpha}_i}(k_i)$ (note that $c^2 = \mathrm{id}$ acting on I). If $\alpha_i \neq \tilde{\alpha}_i$ for some $1 \leq i \leq f$, then $c^{\alpha_i - \tilde{\alpha}_i}(k_i) = -k_i$ and after applying $x'^{\,\beta}$ to the element inside the bracket in (4.22), we obtain an element which satisfies the condition that either its i-th label is not i (in this case after we apply e^f we obtain the zero element) or else its zero-th factor cannot contain the factor $\prod_{i=1}^r \mathsf{f}_{i_\bullet, i}^{\tilde{\beta}_i}$. In any case we cannot obtain the element $\tilde{v}$. Thus we can assume $\alpha_i = \tilde{\alpha}_i$ for $1 \leq i \leq f$. Similarly, we can assume $\alpha_i = \tilde{\alpha}_i$ for $f < i \leq r$, i.e., $\alpha = \tilde{\alpha}$.

By conditions (1) and (2), we see that if $\beta_i \neq \tilde{\beta}_i$ for some i with $1 \leq i \leq f$, or $\beta_i = 1 > \tilde{\beta}_i$ for some $i \in J_1$, then again z_1 cannot produce the basis element $\tilde{v}$. Thus we suppose: $\beta_i = \tilde{\beta}_i$ if $1 \leq i \leq f$, and $\beta_i \leq \tilde{\beta}_i$ for $i \in J_1$. If $\tilde{\beta}_i = 1$ but $\beta_i = 0$ for some $i \in J_1$, then by (4.24), z_1 can only produce some basis elements which have at least a tensor factor, say v_ℓ, with $\ell = -(m+i)$ for some $1 \leq i \leq r$, and thus $\tilde{v}$ cannot be produced. Hence we can suppose $\beta = \tilde{\beta}$. Dually, we can suppose $\overline{\beta} = \overline{\tilde{\beta}}$.

We have $wd_2\tilde{d}_2^{-1}\tilde{w}^{-1} = d_{20}\tilde{d}_{20}^{-1}w'$, where $d_{20} = wd_2w^{-1}$, $\tilde{d}_{20} = w\tilde{d}_2w^{-1}$ and $w' = w\tilde{w}^{-1}$. Note that $w' \in \mathfrak{S}_{r-f} \times \overline{\mathfrak{S}}_{t-f}$, which only permutes elements of $(J_1 \cup J_2) \backslash J_f$. We see that if $d_{20} \neq \tilde{d}_{20}$, then as in (4.25), there exists some $j \in J_f$ with $\tilde{j} := (j)d_{20}\tilde{d}_{20}^{-1}w' \notin J_f$, thus $\tilde{v}$ cannot be produced. So assume $d_{20} = \tilde{d}_{20}$. Similarly we can suppose $w' = 1$. Then the same arguments after (4.25) show that we can assume $\overline{\alpha} = \overline{\tilde{\alpha}}$.

The above has in fact proven that if the coefficient $\chi_{\tilde{v}}^{z_1}$ is nonzero then z_1 in (4.22) must satisfy $(d_1, \alpha, \beta, f, \overline{\beta}, \overline{\alpha}, w, d_2) = (\tilde{d}_1, \tilde{\alpha}, \tilde{\beta}, \tilde{f}, \overline{\tilde{\beta}}, \overline{\tilde{\alpha}}, \tilde{w}, \tilde{d}_2)$, i.e., $z_1 = (v)\tilde{\boldsymbol{m}}$. In this case, one can easily verify that $\chi_{\tilde{v}}^{z_1} = \pm 1$. This proves that z defined in (4.19) is nonzero, a contradiction. The theorem is proven. □

Corollary 4.8. *$BC_{2,r,t}$ has a $\mathbb{C}$-basis which consists of all regular monomials of it.*

Proof. We have the result immediately from Corollary 3.16 and Theorem 4.7. □

5. Homomorphisms between $\widetilde{BC}_{r,t}$ and $BC_{2,r+k,t+k}$

In this section, we generalize Theorem 3.7 so as to establish infinitely many homomorphisms from $BC^{\mathrm{aff}}_{r,t}$ to $BC_{2,r+k,t+k}$ for all positive integers k, where $BC_{2,r+k,t+k}$ are level two walled Brauer–Clifford superalgebras which appear in the higher level mixed Schur–Weyl–Sergeev duality in section 4. As an application, we prove that $BC^{\mathrm{aff}}_{r,t}$

has R-basis which consists of all regular monomials in Definition 3.11. Recall $x'_i, \overline{x}'_j$ in (2.9).

Lemma 5.1. *For all admissible i, j, we have $s_j\overline{x}'_i = \overline{x}'_i s_j$, $\overline{s}_j x'_i = x'_i\overline{s}_j$, $x'_i\overline{c}_j = \overline{c}_j x'_i$ and $\overline{x}'_i c_j = c_j\overline{x}'_i$ in $BC^{\mathrm{aff}}_{r,t}$.*

Proof. Easy exercises. □

Lemma 5.2. *Recall y_i and $\overline{y}_j$ in Definition 2.8. The following results hold in $BC^{\mathrm{aff}}_{r,t}$ for all admissible i, j:*

(1) $x'_i\overline{y}_i = \overline{y}_i x'_i$,
(2) $\overline{x}'_i y_i = y_i\overline{x}'_i$,
(3) $x'_{i+1}y_i = y_i x'_{i+1}$,
(4) $\overline{x}'_{i+1}\overline{y}_i = \overline{y}_i\overline{x}'_{i+1}$,
(5) $(\overline{x}'_{i+1} + \overline{y}_{i+1})x'_j = x'_j(\overline{x}'_{i+1} + \overline{y}_{i+1})$, *if* $j \le i$,
(6) $(x'_{i+1} + y_{i+1})x'_j = x'_j(x'_{i+1} + y_{i+1})$, *if* $j \le i$,
(7) $(\overline{x}'_{i+1} + \overline{y}_{i+1})\overline{x}'_j = \overline{x}'_j(\overline{x}'_{i+1} + \overline{y}_{i+1})$, *if* $j \le i$,
(8) $(x'_{i+1} + y_{i+1})\overline{x}'_j = \overline{x}'_j(x'_{i+1} + y_{i+1})$, *if* $j \le i$.

Proof. By symmetry, it is enough to prove (1), (3), (5) and (6).

(1) If $j \le i-1$, then $x'_i e_{j,i} = e_{j,i}x'_i$, $x'_i\overline{c}_j = \overline{c}_j x'_i$, and $\overline{s}_j x'_i = x'_i\overline{s}_j$ by Lemmas 3.10(1) and 5.1. So,

$$x'_i\overline{y}_i = x'_i\Big(\sum_{j=1}^{i-1}(e_{j,i} - \overline{e}_{j,i}) - \overline{L}_i\Big) = \overline{y}_i x'_i.$$

One can check (3) via Definition 2.8 similarly.

(5) By Lemmas 3.9(1), 3.10(1)–(2), $x'_j(\overline{x}'_{i+1}+e_{j,i+1}-\overline{e}_{j,i+1}) = (\overline{x}'_{i+1}+e_{j,i+1}-\overline{e}_{j,i+1})x'_j$ and $x'_j e_{s,i+1} = e_{s,i+1}x'_j$ and $x'_j\overline{e}_{s,i+1} = \overline{e}_{s,i+1}x'_j$ whenever $j \ne s$. Since $x'_j\overline{L}_{i+1} = \overline{L}_{i+1}x'_j$, we have

$$\begin{aligned}(\overline{x}'_{i+1} + \overline{y}_{i+1})x'_j &= \Big(\overline{x}'_{i+1} + e_{j,i+1} - \overline{e}_{j,i+1} + \sum_{1\le s\le i, s\ne j}(e_{s,i+1} - \overline{e}_{s,i+1}) - \overline{L}_{i+1}\Big)x'_j \\ &= x'_j\Big(\overline{x}'_{i+1} + e_{j,i+1} - \overline{e}_{j,i+1} + \sum_{1\le s\le i, s\ne j}(e_{s,i+1} - \overline{e}_{s,i+1}) - \overline{L}_{i+1}\Big) \\ &= x'_j(\overline{x}'_{i+1} + \overline{y}_{i+1}).\end{aligned}$$

(6) By (2.7), (2.10) and Lemma 3.10(1), we have

$$\begin{aligned}x_1(x'_{i+1} + y_{i+1}) &= x_1\Big(x_{i+1} + \sum_{j=1}^{i}(e_{i+1,j} + \overline{e}_{i+1,j})\Big) = \Big(x_{i+1} + \sum_{j=1}^{i}(e_{i+1,j} + \overline{e}_{i+1,j})\Big)x_1 \\ &= (x'_{i+1} + y_{i+1})x_1.\end{aligned}$$

Applying $(1, j)$ on both sides of the above equation yields (6). □

For the simplification of notation, we define

$$z_i = x'_i + y_i \text{ and } \overline{z}_j = \overline{x}'_j + \overline{y}_j \text{ for all admissible } i \text{ and } j. \tag{5.1}$$

Lemma 5.3. *The following results hold in $BC^{\mathrm{aff}}_{r,t}$ for all admissible i, j:*

(1) $s_j z_i = z_i s_j$, $\overline{s}_j \overline{z}_i = \overline{z}_i \overline{s}_j$, *if* $j \neq i-1, i$,
(2) $s_j \overline{z}_i = \overline{z}_i s_j$, $\overline{s}_j z_i = z_i \overline{s}_j$, *if* $j \neq i-1$,
(3) $z_i c_i = -c_i z_i$, $\overline{z}_i \overline{c}_i = -\overline{c}_i \overline{z}_i$,
(4) $z_i c_j = c_j z_i$, $\overline{z}_i \overline{c}_j = \overline{c}_j \overline{z}_i$, *if* $i \neq j$,
(5) $z_i \overline{c}_j = \overline{c}_j z_i$, $\overline{z}_i c_j = c_j \overline{z}_i$, *if* $i \leq j$,
(6) $z_i(e_i + \overline{z}_i - \overline{e}_i) = (e_i + \overline{z}_i - \overline{e}_i) z_i$,
(7) $e_i \overline{z}_i = -e_i(x_i + \overline{L}_i)$, $e_i z_i = e_i(x_i + \overline{L}_i)$,
(8) $e_i s_i z_i s_i = s_i z_i s_i e_i$, $e_j \overline{s}_j \overline{z}_j \overline{s}_j = \overline{s}_j \overline{z}_j \overline{s}_j e_j$,
(9) $z_i \tilde{z}_i = \tilde{z}_i z_i$,
(10) $\overline{z}_i \tilde{\overline{z}}_i = \tilde{\overline{z}}_i \overline{z}_i$,
(11) $e_i z_i e_i = e_i x'_i e_i = \omega_1 e_i$,
(12) $e_i z_i^k c_i e_i = 0$, $\forall k \in \mathbb{N}$,
(13) $e_i z_i^{2n} e_i = 0$, $e_i(\overline{z}_i)^{2n} e_i = 0$ $\forall n \in \mathbb{N}$,

where $\tilde{z}_i = (s_i z_i s_i - (1 - c_i c_{i+1}) s_i)$ *and* $\tilde{\overline{z}}_i = (\overline{s}_i \overline{z}_i \overline{s}_i - (1 + \overline{c}_i \overline{c}_{i+1}) \overline{s}_i)$.

Proof. (1)–(5) follow from Lemma 2.9(1)–(5), Lemma 5.1, (2.7) and (2.8). (6) follows from Lemmas 2.9(7), 3.9(1), 5.2(1)–(2). (7) follows from Lemmas 2.9(8), 3.9(3)–(4). (8) follows from Lemmas 2.9(9), 3.10(1). Multiplying $(2, i+1)(1, i)$ on both sides of $x_1 x_2 = x_2 x_1$ (see (2.7)) yields

$$x'_i(x'_{i+1} - (1 - c_i c_{i+1}) s_i) = (x'_{i+1} - (1 - c_i c_{i+1}) s_i) x'_i. \tag{5.2}$$

Now, (9) follows from (5.2), Lemmas 2.9(10), 5.2(3)–(4). We leave (10) to the reader since it can be verified, similarly. (11) follows from Lemmas 2.9(13), 3.10(3). Via (6), one can prove (12) by arguments similar to those for Lemma 2.9(11). Finally, one can verify (13) by arguments similar to those for Lemma 2.9(12). □

From here to the end of Theorem 5.13, we assume that $R = \mathbb{C}$. Also, $BC_{2,r,t}$ is one of those which appear in the higher level mixed Schur–Weyl–Sergeev duality in section 4.

Lemma 5.4. *The $e_k BC_{2,k,k}$ is the left $BC_{2,k-1,k-1}$-module generated by $e_k c_k^{\sigma_1} x_k'^{\sigma_2} s_{k,j} \overline{s}_{k,l}$ for all $\sigma_1, \sigma_2 \in \mathbb{Z}_2$ and $1 \leq j, l \leq k$, where $BC_{2,k-1,k-1}$ is the subalgebra of $BC_{2,k,k}$ generated by $e_1, c_1, \overline{c}_1$, $x_1, \overline{x}_1$, $s_1, \ldots, s_{k-2}$ and $\overline{s}_1, \ldots, \overline{s}_{k-2}$.*

Proof. This result, which is a counterpart of Lemma 2.6, can be proven similarly. □

Proposition 5.5. *Recall z_k and $\overline{z}_k$ in (5.1). We have:*

(1) $e_k BC_{2,k,k} e_k = e_k BC_{2,k-1,k-1}$,
(2) *There is a unique $\xi_{a,k}$ (resp., $\overline{\xi}_{a,k}$) in $BC_{2,k-1,k-1}$ such that $e_k z_k^a e_k = \xi_{a,k} e_k$ (resp., $e_k \overline{z}_k^a e_k = \overline{\xi}_{a,k} e_k$). Moreover, $\xi_{2n,k} = \overline{\xi}_{2n,k} = 0$, and $\xi_{1,k} = \omega_1$.*

Proof. (1) follows from Lemma 5.4 (see the proof of Proposition 2.7) and (2) follows from (1), Corollary 4.8 for $BC_{2,k,k}$ and Lemma 5.3(11), (13). □

Lemmas 5.6 and 5.7 can be proven by arguments similar to those for Lemmas 2.11–2.12.

Lemma 5.6. *For any* $n \in \mathbb{N}$, $e_i \overline{z}_i^{2n+1} = \sum_{j=0}^{n} a_{2n+1,j}^{(i)} e_i z_i^{2j+1}$ *for some* $a_{2n+1,j}^{(i)} \in R[\xi_{3,i}, \ldots, \xi_{2n-1,i}]$ *such that*

(1) $a_{2n+1,n}^{(i)} = -1$,
(2) $a_{2n+1,j}^{(i)} = a_{2n-1,j-1}^{(i)}$ *for all* $1 \leq j \leq n-1$,
(3) $a_{2n+1,0}^{(i)} = \sum_{j=0}^{n-1} a_{2n-1,j}^{(i)} \xi_{2j+1,i}$.

Lemma 5.7. *For any positive integer* n, $e_i \overline{z}_i^{2n} = \sum_{j=0}^{n} a_{2n,j}^{(i)} e_i z_i^{2j}$ *for some* $a_{2n,j}^{(i)} \in R[\xi_{3,i}, \ldots, \xi_{2n-1,i}]$ *such that*

(1) $a_{2n,n}^{(i)} = 1$,
(2) $a_{2n,j}^{(i)} = a_{2n-1,j-1}^{(i)}$ *for all* $1 \leq j \leq n-1$,
(3) $a_{2n,0}^{(i)} = \sum_{j=0}^{n-1} a_{2n-2,j}^{(i)} \xi_{2j+1,i}$.

In Lemma 5.8, we can assume $k \geq 2$ and $n \geq 1$ since $z_k = x'_k + y_k$, $\overline{z}_k = \overline{x}'_k + \overline{y}_k$, $\xi_{1,k} = \omega_1$ and $\overline{\xi}_{1,k} = -\omega_1$ by Lemma 5.3(11).

Lemma 5.8. *If* $(k,n) \in \mathbb{Z}^{\geq 2} \times \mathbb{Z}^{\geq 1}$, *then* $\overline{\xi}_{2n+1,k} \in R[\xi_{3,k}, \ldots, \xi_{2n+1,k}]$. *Furthermore, both* $\xi_{2n+1,k}$ *and* $\overline{\xi}_{2n+1,k}$ *are central in* $BC_{2,k-1,k-1}$.

Proof. The first statement follows from Lemma 5.6. We have $s_j z_k = z_k s_j$ and $c_j z_k = z_k c_j$ for all $j \leq k-1$ by Lemma 5.3(1)–(4). Since $y_k = \sum_{i=1}^{k-1}(e_{k,i} + \overline{e}_{k,i}) - L_k$, $z_k = x'_k + y_k = x_k + \sum_{i=1}^{k-1}(e_{k,i} + \overline{e}_{k,i})$. By (2.7) and Lemma 3.10(1) and (2), $x_1 z_k = z_k x_1$ for $k \geq 2$. Obviously, e_1 commutes with x'_k and $e_{k,i}, \overline{e}_{k,i}, c_k, (k,i)$ whenever $i \neq 1$ and $k \geq 2$. Since $e_1 e_{k,1} = e_1(k,1)$, we have $e_1 z_k = z_k e_1$.

We have proven that h commutes with e_k, z_k for any $h \in \{e_1, s_i, c_j, x_1 \mid 1 \leq i \leq k-2\ , 1 \leq j \leq k-1\}$. So $e_k(h\xi_{a,k}) = e_k(\xi_{a,k}h)$. By Corollary 4.8 and Proposition 5.5, $h\xi_{2n+1,k} = \xi_{2n+1,k}h$. Finally, we need to check $e_k(h\xi_{a,k}) = e_k(\xi_{a,k}h)$ for any $h \in \{\overline{s}_i, \overline{c}_j, \overline{x}_1 \mid 1 \leq i \leq k-2\ , 1 \leq j \leq k-1\}$. In this case, we use Lemma 5.6 so as to use $\overline{z}_k$ instead of z_k in $e_k z_k^{2n+1} e_k$. Therefore, $h\xi_{2n+1,k} = \xi_{2n+1,k}h$, as required. □

Lemma 5.9. *For* $k, a \in \mathbb{Z}^{\geq 1}$, *we have*

$$s_k z_{k+1}^a = h_k^a s_k - \sum_{b=0}^{a-1} h_k^{a-1-b} z_{k+1}^b + \sum_{b=0}^{a-1} (-1)^{a-b} c_k c_{k+1} h_k^{a-b-1} z_{k+1}^b, \tag{5.3}$$

where $h_k = z_k + e_k + \overline{e}_k$.

Proof. The result can be easily checked by induction on a. $\square$

Recall $z_i, \overline{z}_j$ in (5.1) for all admissible i and j. For all $1 \le j \le k-1$, define

$$
\begin{aligned}
p_{j,k} &= s_{j,k-1}(z_{k-1} + e_{k-1} + \overline{e}_{k-1})s_{k-1,j}, \quad \text{and} \\
\overline{p}_{j,k} &= \overline{s}_{j,k-1}(\overline{z}_{k-1} + e_{k-1} - \overline{e}_{k-1})\overline{s}_{k-1,j}.
\end{aligned}
\tag{5.4}
$$

Note that $\xi_{0,k} = 0$, and $e_k h = 0$ for $h \in BC_{2,k-1,k-1}$ if and only if $h = 0$. We will use this fact freely in the proof of Lemma 5.10, where we use the terminology that a monomial in $p_{j,k+1}$'s, $\overline{p}_{j,k+1}$'s, x'_j's and $\overline{x}'_j$'s is a *leading term* in an expression if it has the highest degree by defining $\deg p_{i,j} = \deg \overline{p}_{i,j} = \deg x'_j = \deg \overline{x}'_j = 1$.

Lemma 5.10. *For any positive integer n, $\xi_{2n+1,k+1}$ can be written as an R-linear combination of monomials in $p_{j,k+1}$, $\overline{p}_{j,k+1}$, x'_j and $\overline{x}'_j$ for $1 \le j \le k$ such that the summation of leading terms of $\xi_{2n+1,k+1}$ is $-2\sum_{j=1}^{k}(p_{j,k+1}^{2n} + p_{j,k+1}^{2n-1}x'_j - \overline{p}_{j,k+1}^{2n} - \overline{p}_{j,k+1}^{2n-1}\overline{x}'_j)$.*

Proof. We have $x_1^2 = p(p+1)$ and $\overline{x}_1^2 = (p-n+1)(p-n)$ in $BC_{2,r,t}$ by Lemma 4.4. So,

$$
\begin{aligned}
\xi_{2n+1,k+1}e_{k+1} &\overset{\text{Proposition 5.5}}{=} e_{k+1}z_{k+1}^{2n+1}e_{k+1} = e_{k+1}y_{k+1}z_{k+1}^{2n}e_{k+1} + e_{k+1}x'_{k+1}z_{k+1}^{2n}e_{k+1} \\
=& e_{k+1}y_{k+1}z_{k+1}^{2n}e_{k+1} + e_{k+1}(x'_{k+1})^2 z_{k+1}^{2n-1}e_{k+1} + e_{k+1}x'_{k+1}y_{k+1}z_{k+1}^{2n-1}e_{k+1} \\
=& e_{k+1}y_{k+1}z_{k+1}^{2n}e_{k+1} + p(p+1)e_{k+1}z_{k+1}^{2n-1}e_{k+1} - e_{k+1}\overline{x}'_{k+1}y_{k+1}z_{k+1}^{2n-1}e_{k+1} \\
&\quad (\text{by Lemma 3.9(3)}) \\
=& e_{k+1}y_{k+1}z_{k+1}^{2n}e_{k+1} + p(p+1)e_{k+1}z_{k+1}^{2n-1}e_{k+1} + e_{k+1}\overline{y}_{k+1}\overline{x}'_{k+1}z_{k+1}^{2n-1}e_{k+1} \\
&\quad (\text{Lemmas 2.9(8), 5.2(2)}) \\
=& e_{k+1}(-L_{k+1} + \overline{L}_{k+1})z_{k+1}^{2n}e_{k+1} + p(p+1)\xi_{2n-1,k+1}e_{k+1} \\
&\quad + e_{k+1}(L_{k+1} - \overline{L}_{k+1})\overline{x}'_{k+1}z_{k+1}^{2n-1}e_{k+1}.
\end{aligned}
\tag{5.5}
$$

Recall $h_k = z_k + e_k + \overline{e}_k$ in (5.3). Considering the right-hand side of (5.5) and expressing L_{k+1} by (2.8), using $(j, k+1) = s_{j,k}s_k s_{k,j}$ and the fact that $s_{j,k}, s_{k,j}$ commute with $x'_{k+1}, y_{k+1}, e_{k+1}$, we see that a term in the linear combination of $e_{k+1}L_{k+1}z_{k+1}^{2n}e_{k+1}$ becomes

$$
s_{j,k}e_{k+1}s_k z_{k+1}^{2n}e_{k+1}s_{k,j} + s_{j,k}e_{k+1}c_{k+1}s_k c_{k+1}z_{k+1}^{2n}e_{k+1}s_{k,j}. \tag{5.6}
$$

Since $c_{k+1}x'_{k+1} = -x'_{k+1}c_{k+1}$ and $c_{k+1}y_{k+1} = -y_{k+1}c_{k+1}$, we have $c_{k+1}z_{k+1}^{2n} = z_{k+1}^{2n}c_{k+1}$. Note that $e_{k+1}c_{k+1} = e_{k+1}\overline{c}_{k+1}$, and $\overline{c}_{k+1}$ commutes with s_k, x'_{k+1} and y_{k+1}. So

$$\begin{aligned}
&s_{j,k}e_{k+1}c_{k+1}s_kc_{k+1}z_{k+1}^{2n}e_{k+1}s_{k,j} = s_{j,k}e_{k+1}\left(s_kz_{k+1}^{2n}\right)e_{k+1}s_{k,j}\\
&\overset{(5.3)}{=} s_{j,k}e_{k+1}\left(h_k^{2n}s_k - \sum_{b=0}^{2n-1}h_k^{2n-b-1}z_{k+1}^b + (-1)^b\sum_{b=0}^{2n-1}c_kc_{k+1}h_k^{2n-b-1}z_{k+1}^b\right)e_{k+1}s_{k,j}\\
&= s_{j,k}h_k^{2n}s_{k,j}e_{k+1} - s_{j,k}\sum_{b=0}^{2n-1}h_k^{2n-b-1}s_{k,j}\xi_{b,k+1}e_{k+1} \text{ (by Lemma 5.3(12)).}
\end{aligned} \tag{5.7}$$

By induction assumption, the leading terms of $\xi_{b,k+1}$ are of degree $b-1$. So, the leading term of $s_{j,k}e_{k+1}\left(s_kz_{k+1}^{2n}\right)e_{k+1}s_{k,j}$ is $p_{j,k+1}^{2n}$ and hence $-e_{k+1}L_{k+1}z_{k+1}^{2n}e_{k+1}$ contributes to the leading terms $-2\sum_{j=1}^{k}p_{j,k+1}^{2n}$.

We compute $e_{k+1}L_{k+1}\overline{x}'_{k+1}z_{k+1}^{2n-1}e_{k+1}$. A term of it becomes $2s_{j,k}e_{k+1}s_k\overline{x}'_{k+1}z_{k+1}^{2n-1}\times e_{k+1}s_{k,j}$ since

$$e_{k+1}c_{k+1}s_kc_{k+1}\overline{x}'_{k+1}z_{k+1}^{2n-1}e_{k+1} = e_{k+1}s_k\overline{x}'_{k+1}z_{k+1}^{2n-1}e_{k+1}.$$

Thus, it is enough to compute the leading terms of $s_{j,k}e_{k+1}s_k\overline{x}'_{k+1}z_{k+1}^{2n-1}e_{k+1}s_{k,j}$. Since x'_k commutes with z_{k+1} and e_{k+1}, and $e_{k+1}s_k\overline{x}'_{k+1} = e_{k+1}\overline{x}'_{k+1}s_k = -e_{k+1}x'_{k+1}s_k = -e_{k+1}s_kx'_k$, we have

$$s_{j,k}e_{k+1}s_k\overline{x}'_{k+1}z_{k+1}^{2n-1}e_{k+1}s_{k,j} = -s_{j,k}e_{k+1}s_kz_{k+1}^{2n-1}e_{k+1}s_{k,j}x'_j.$$

By (5.7), $s_{j,k}e_{k+1}s_k\overline{x}'_{k+1}z_{k+1}^{2n-1}e_{k+1}s_{k,j}$ contributes to the leading term $p_{j,k+1}^{2n-1}x'_j$ whose degree is $2n$. Finally, we need to compute the leading terms of $e_{k+1}\overline{L}_{k+1}z_{k+1}^{2n}e_{k+1}$ and $-e_{k+1}\overline{L}_{k+1}\overline{x}'_{k+1}z_{k+1}^{2n-1}e_{k+1}$. By Lemma 5.7, one can use $\overline{z}_{k+1}^{2n}$ to replace z_{k+1}^{2n} in $e_{k+1}\overline{L}_{k+1}z_{k+1}^{2n}e_{k+1}$. Thus, $e_{k+1}\overline{L}_{k+1}z_{k+1}^{2n}e_{k+1}$ contributes to the leading terms $\overline{p}_{j,k+1}^{2n}$, $1\le j\le k$. Similarly, a term of $e_{k+1}\overline{L}_{k+1}\overline{x}'_{k+1}z_{k+1}^{2n-1}e_{k+1}$ is of form $\overline{x}'_je_{k+1}(\overline{j},\overline{k+1})z_{k+1}^{2n-1}e_{k+1}$ whose leading term $\overline{p}_{j,k+1}^{2n-1}\overline{x}'_j$ is of degree $2n$. The proof is completed. □

Lemma 5.11. *For $a\in\mathbb{Z}^{\ge 0}$, $k\in\mathbb{Z}^{\ge 1}$, both $\xi_{a,k+1}$ and $\overline{\xi}_{a,k+1}$ commute with $x'_{k+1}+y_{k+1}$ and $\overline{x}'_{k+1}+\overline{y}_{k+1}$.*

Proof. By Proposition 5.5, we can assume that $a=2n+1$ and $n\ge 1$. By Lemma 5.10, $\xi_{2n+1,k}$ can be written as linear combinations of monomials in $p_{j,k+1}$, $\overline{p}_{j,k+1}$, x'_j, and $\overline{x}'_j$ for $1\le j\le k$. From (5.4),

$$p_{j,k+1} = x'_j + z_{j,k+1}, \text{ and } \overline{p}_{j,k+1} = \overline{x}'_j + \overline{z}_{j,k+1}, \tag{5.8}$$

where $z_{j,k+1}, \overline{z}_{j,k+1}$ are defined in Lemma 2.15. So, it is enough to prove that $z_{j,k+1}$, $\overline{z}_{j,k+1}$, x'_j and $\overline{x}'_j$ commute with $\overline{x}'_{k+1}+\overline{y}_{k+1}$ if $1\le j\le k$. By Lemma 5.2(5)–(6), both x'_j and $\overline{x}'_j$ commute with $\overline{x}'_{k+1}+\overline{y}_{k+1}$. Finally, $z_{j,k+1}$, $\overline{z}_{j,k+1}$ commute with $\overline{x}'_{k+1}+\overline{y}_{k+1}$ since

- both $z_{j,k+1}$ and $\overline{z}_{j,k+1}$ commute with y_{k+1} and $\overline{y}_{k+1}$ (see the proof of Lemma 2.17),
- $z_{j,k+1}$ and $\overline{z}_{j,k+1}$ are linear combinations of elements which commute with both $\overline{x}'_{k+1}$ and x'_{k+1}, by Lemmas 2.15, 3.10(1)–(2), and 5.1 and (2.7) and (2.8).

This proves that $\xi_{a,k+1}$ commutes with $x'_{k+1}+y_{k+1}$ and $\overline{x}'_{k+1}+\overline{y}_{k+1}$ and so is $\overline{\xi}_{a,k+1}$ by Lemma 5.6. $\square$

Lemma 5.12. *For $a\in\mathbb{Z}^{\geq 0}$, $k\in\mathbb{Z}^{\geq 1}$, both $\xi_{2a+1,k+1}$ and $\overline{\xi}_{2a+1,k+1}$ commute with c_j and $\overline{c}_j$ if $j\geq k+1$.*

Proof. We have proven that both $z_{j,k+1}$ and $\overline{z}_{j,k+1}$ commute with c_j and $\overline{c}_j$ if $j\geq k+1$ in the proof of Lemma 2.17. If $j\leq k$, then x'_j and $\overline{x}_j$ commute with c_l and $\overline{c}_l$ for $l\geq k+1$, by Lemma 5.1, (2.7) and (2.8). Now, the result follows from (5.8) and Lemma 5.10, immediately. $\square$

In Theorem 5.13, we assume the ground field is $\mathbb{C}$ since we use level two walled Brauer–Clifford superalgebras in section 4. After we have proven the freeness of cyclotomic walled Brauer–Clifford superalgebras in section 6, we know that Theorem 5.13 is available over an integral domain R containing $1/2$.

Theorem 5.13. *For any $k\in\mathbb{Z}^{>0}$, there is a superalgebra homomorphism $\phi_k: BC^{\mathrm{aff}}_{r,t}\to BC_{2,r+k,t+k}$ sending $\omega_{2n+1},\overline{\omega}_{2n+1},e_1,s_i,\overline{s}_j,c_l,\overline{c}_m,x_1,\overline{x}_1$ to $\xi_{2n+1,k+1},\overline{\xi}_{2n+1,k+1},e_{k+1}$, $s_{k+i},\overline{s}_{k+j},c_{k+l},\overline{c}_{k+m},z_{k+1}$, $\overline{z}_{k+1}$ for all admissible i,j,l,m,n, respectively.*

Proof. It is enough to verify the images of generators of $BC^{\mathrm{aff}}_{r,t}$ satisfy the defining relations for $BC^{\mathrm{aff}}_{r,t}$ in Definition 3.1. If so, then ϕ_k is an algebra homomorphism. Since ϕ_k sends even (resp., odd) generators to even (resp., odd) elements in $BC_{2,r+k,t+k}$, ϕ_k is a superalgebra homomorphism.

By Corollary 2.5, ϕ_k satisfies (2.1)–(2.3) and Definition 2.1(1)–(10). By Lemma 5.3(1), (3), (9), (10), ϕ_k satisfies (2.7) and (2.8). Applying the anti-involution σ on Lemma 2.9(7), we see that ϕ_k satisfies the Definition 3.1(1). By Lemma 5.3(8), (6) (resp., (11)–(13)), ϕ_k satisfies Definition 3.1(2)–(4) (resp., (5)–(7)). Finally, ϕ_k satisfies Definition 3.1(8)–(11) by Lemma 5.3(2), (5). $\square$

The following result is a counterpart of [14, Theorem 4.14].

Theorem 5.14. *Suppose R is a domain which contains 2^{-1} and ω_1. Then $BC^{\mathrm{aff}}_{r,t}$ is free over R spanned by all regular monomials in Definition 3.11. In particular, $BC^{\mathrm{aff}}_{r,t}$ is of infinite rank.*

Proof. By Proposition 3.12, it is enough to prove that M, the set of all regular monomials of $BC^{\mathrm{aff}}_{r,t}$, is linear independent over $\mathbb{Z}[\omega_1,2^{-1}]$, where ω_1 is an indeterminate. By fundamental theorem on algebras, it suffices to prove it for sufficiently many ω_1's. This can be done by choosing ω_1 as in (4.16). So, it is enough to prove that M is linear independent over $\mathbb{C}$ for infinitely many ω_1's in (4.16).

By Lemma 3.8(1)–(3) and Definition 3.11, we assume that a regular monomial $\mathbf{m}$ of $BC^{\mathrm{aff}}_{r,t}$ is of form

$$\mathbf{m} = c^\alpha x^\beta d_1^{-1} e^f w d_2 \overline{x}^\gamma \overline{c}^\delta \prod_{n\in\mathbb{Z}^{>0}} \omega_{2n+1}^{a_{2n+1}}, \tag{5.9}$$

where $(\alpha,\delta)\in\mathbb{Z}_2^r\times\mathbb{Z}_2^t$, $(\beta,\gamma)\in\mathbb{N}^r\times\mathbb{N}^t$, $d_1,d_2\in D^f_{r,t}$, $w\in\Sigma_{r-f}\times\Sigma_{\overline{t-f}}$, $0\le f\le\min\{r,t\}$ and all but finitely many a_{2n+1}'s are zero. So, it is equivalent to prove that the above regular monomials are linear independent. If it were false, then there is a finite subset $S\subset M$ such that $\sum_{\mathbf{m}\in S} r_{\mathbf{m}}\mathbf{m}=0$ and $r_{\mathbf{m}}\neq 0$ for all $\mathbf{m}\in S$. For each S, we set

$$\begin{aligned}\tilde{k} &= \max\big\{|\beta| + \sum_n (2n+1)a_{2n+1} \mid c^\alpha x^\beta d_1^{-1} e^f w d_2 \overline{x}^\gamma \overline{c}^\delta \prod_{n\in\mathbb{Z}^{>0}} \omega_{2n+1}^{a_{2n+1}} \in S\big\},\\ \hat{k} &= \max\big\{|\gamma| + \sum_n (2n+1)a_{2n+1} \mid c^\alpha x^\beta d_1^{-1} e^f w d_2 \overline{x}^\gamma \overline{c}^\delta \prod_{n\in\mathbb{Z}^{>0}} \omega_{2n+1}^{a_{2n+1}} \in S\big\}.\end{aligned} \tag{5.10}$$

If $\tilde{k}\ge\hat{k}$, we define $k=\tilde{k}$ and

$$\begin{aligned} f_0 &= \min\big\{f \mid c^\alpha x^\beta d_1^{-1} e^f w d_2 \overline{x}^\gamma \overline{c}^\delta \prod_{n\in\mathbb{Z}^{>0}} \omega_{2n+1}^{a_{2n+1}} \in S, |\beta| + \sum_n (2n+1)a_{2n+1} = k\big\},\\ k_1 &= \max\big\{|\gamma| \mid c^\alpha x^\beta d_1^{-1} e^{f_0} w d_2 \overline{x}^\gamma \overline{c}^\delta \prod_{n\in\mathbb{Z}^{>0}} \omega_{2n+1}^{a_{2n+1}} \in S, |\beta| + \sum_n (2n+1)a_{2n+1} = k\big\}.\end{aligned} \tag{5.11}$$

If $\tilde{k}<\hat{k}$, we define $k=\hat{k}$ and

$$\begin{aligned} f_0 &= \min\big\{f \mid c^\alpha x^\beta d_1^{-1} e^f w d_2 \overline{x}^\gamma \overline{c}^\delta \prod_{n\in\mathbb{Z}^{>0}} \omega_{2n+1}^{a_{2n+1}} \in S, |\gamma| + \sum_n (2n+1)a_{2n+1} = k\big\},\\ k_1 &= \max\big\{|\beta| \mid c^\alpha x^\beta d_1^{-1} e^{f_0} w d_2 \overline{x}^\gamma \overline{c}^\delta \prod_{n\in\mathbb{Z}^{>0}} \omega_{2n+1}^{a_{2n+1}} \in S, |\gamma| + \sum_n (2n+1)a_{2n+1} = k\big\}.\end{aligned} \tag{5.12}$$

Let $\phi_k : BC^{\mathrm{aff}}_{r,t} \to BC_{2,r+k,t+k}(\omega_1)$ be the superalgebra homomorphism in Theorem 5.13. Since $x_i = x'_i - L_i$ and $\overline{x}_i = \overline{x}'_i - \overline{L}_i$,

$$\begin{aligned}\phi_k(x_i) &= \sum_{j=1}^k (e_{k+i,j} + \overline{e}_{k+i,j}) + x'_{k+i} - L_{k+i}, \text{ and}\\ \phi_k(\overline{x}_i) &= \sum_{j=1}^k (e_{j,k+i} - \overline{e}_{j,k+i}) + \overline{x}'_{k+i} - \overline{L}_{k+i}.\end{aligned} \tag{5.13}$$

Using Lemma 5.10 to express $\xi_{2n+1,k+1}$ for $n \in \mathbb{Z}^{\geq 1}$, we see that some terms of $\phi_k(\mathbf{m})$ are of forms (we will see in the next paragraph that other terms of $\phi_k(\mathbf{m})$ will not contribute to our computations)

$$\prod_{i=1}^{r} c_{i+k}^{\alpha_i} \prod_{i=1}^{r}(k+i, i_1)\cdots(k+i, i_{\beta_i})\phi_k(d_1^{-1}e^f wd_2)\prod_{j=1}^{t}(\overline{k+j}, \overline{j}_1)\cdots(\overline{k+j}, \overline{j}_{\gamma_j})\prod_{j=1}^{t}\overline{c}_{j+k}^{\delta_j}\prod_{n\geq 1}\mathbf{c}_{2n+1}, \tag{5.14}$$

where $\mathbf{c}_{2n+1}$, which comes from $\xi_{2n+1,k+1}$, ranges over products of a_{2n+1} disjoint cycles in Σ_k (or $\overline{\Sigma}_k$) such that each cycle is of length $2n+1$.

By Theorem 4.7, $BC_{r,t}$ is a subalgebra of $BC_{2,r,t}$ and so is the walled Brauer algebra, say $B_{r,t}(0)$ which is isomorphic to the subalgebra generated by $e_1, s_1, \ldots, s_{r-1}$ and $\overline{s}_1, \ldots, \overline{s}_{t-1}$. Similarly, we have the subalgebra $B_{r+k,t+k}(0)$ of $BC_{2,r+k,t+k}$. It is known that $B_{r,t}(0)$ can be defined by so called (r,t)-walled Brauer diagrams. Each of them is a diagram with $(r+t)$ vertices on the top and bottom rows, and vertices on both rows are labeled from left to right by $r, \ldots, 2, 1, \overline{1}, \overline{2}, \ldots, \overline{t}$. Every vertex $i \in \{1, 2, \ldots, r\}$ (resp., $\overline{i} \in \{\overline{1}, \overline{2}, \ldots, \overline{t}\}$) on each row must be connected to a unique vertex $\overline{j}$ (resp., j) on the same row or a unique vertex j (resp., $\overline{j}$) on the other row. The pairs $[i,j]$ and $[\overline{i}, \overline{j}]$ are called *vertical edges*, and the pairs $[\overline{i}, j]$ and $[i, \overline{j}]$ are called *horizontal edges*. By definition, a $\phi_k(d_1^{-1}e^f wd_2)$ in (5.14) corresponds to a unique $(r+k, t+k)$-walled Brauer diagram such that $[i,i]$ and $[\overline{j}, \overline{j}]$ are its vertical edges for all $1 \leq i, j \leq k$ (see e.g. [14]). We call the terms of the form (5.14) the *leading terms* if

(i) $k = |\beta| + \sum_n (2n+1)a_{2n+1}$ if $\tilde{k} \geq \hat{k}$ and $k = |\gamma| + \sum_n (2n+1)a_{2n+1}$ if $\tilde{k} < \hat{k}$. (cf. (5.10)),
(ii) the corresponding f in (5.14) is f_0 in (5.12),
(iii) $|\gamma| = k_1$ if $\tilde{k} \geq \hat{k}$ and $|\beta| = k_1$ if $\tilde{k} < \hat{k}$,
(iv) in the first case of (i), the juxtapositions of the sequences $i_1, i_2, \ldots, i_{\beta_i}$ for $1 \leq i \leq r$ and $\mathbf{c}_{2n+1}$, $n \geq 1$ run through all permutations of the sequences in $1, 2, \ldots, k$ and the sequences $\overline{j}_1, \overline{j}_2, \ldots, \overline{j}_{\gamma_j}$, $1 \leq j \leq t$ run through all permutations of the sequence $\overline{1}, \overline{2}, \ldots, \overline{k}_1$; while in the second case of (i), the juxtapositions of the sequences $\overline{j}_1, \overline{j}_2, \ldots, \overline{j}_{\gamma_j}$ for $1 \leq j \leq t$ and $\mathbf{c}_{2n+1}$, $n \geq 1$ run through all permutations of the sequences in $\overline{1}, \overline{2}, \ldots, \overline{k}$ and the sequence of $i_1, i_2, \ldots, i_{\beta_i}$, $1 \leq i \leq r$ run through all permutations of sequence $1, 2, \ldots, k_1$.

By Theorem 4.18, all $\tilde{\mathbf{m}} = c^{\tilde{\alpha}} x'^{\tilde{\beta}} \tilde{d}_1^{-1} e^{\tilde{f}} \tilde{w} \tilde{d}_2 \overline{x}'^{\tilde{\gamma}} \overline{c}^{\tilde{\delta}} \in BC_{2,r+k,t+k}$ consist of a basis of $BC_{2,r+k,t+k}$ over $\mathbb{C}$, where $\tilde{\alpha}, \tilde{\beta} \in \mathbb{Z}_2^{r+k}$, $\tilde{\gamma}, \tilde{\delta} \in \mathbb{Z}_2^{t+k}$, $\tilde{d}_1, \tilde{d}_2 \in D^{\tilde{f}}_{r+k,t+k}$, $\tilde{w} \in \Sigma_{r+k-\tilde{f}} \times \Sigma_{\overline{t+k-\tilde{f}}}$ and $0 \leq \tilde{f} \leq \min\{r+k, t+k\}$. Such monomials will be called normal monomials. Moreover, $\tilde{\mathbf{m}}$ is called an admissible monomial if

(a) $\tilde{\alpha}_i = \tilde{\delta}_j = 0$ for all $1 \leq i \leq k$, $1 \leq j \leq k_1$ if $\tilde{k} \geq \hat{k}$; or $\tilde{\alpha}_i = \tilde{\delta}_j = 0$ for all $1 \leq i \leq k_1$ and $1 \leq j \leq k$ if $\tilde{k} < \hat{k}$,

(b) the corresponding walled Brauer diagram of $\tilde{d}_1^{-1}e^{\tilde{f}}\tilde{w}\tilde{d}_2$ satisfies (1)–(5) as follows:
  (1) $\tilde{f}=f_0$,
  (2) no vertical edge of form $[i,i]$ and $[\overline{j},\overline{j}]$, $1\le i\le k$, $1\le j\le k_1$ if $\tilde{k}\ge\hat{k}$,
  (3) no vertical edge of form $[i,i]$ and $[\overline{j},\overline{j}]$, $1\le i\le k_1$, $1\le j\le k$ if $\tilde{k}<\hat{k}$,
  (4) no horizontal edge of form $[i,\overline{j}]$, $1\le i\le k$, at the bottom row if $\tilde{k}\ge\hat{k}$,
  (5) no horizontal edge of form $[i,\overline{j}]$, $1\le j\le k$, at the top row if $\tilde{k}<\hat{k}$,

(c) $\tilde{\beta}_i=\tilde{\gamma}_j=0$, for all $1\le i\le r+k$ and $1\le j\le t+k$.

In the following, we assume that $\tilde{k}<\hat{k}$ (the case $\tilde{k}\ge\hat{k}$ can be dealt with in a similar way). A $\phi_k(\mathbf{m})$ contributes admissible monomials of $BC_{2,k+r,k+t}$ only when $\mathbf{m}\in S$ is given in (5.9) such that $k=|\gamma|+\sum_n(2n+1)a_{2n+1}$, $f=f_0$ and $k_1=|\beta|$. More explicitly, the leading terms exactly appear in $\phi_k(\mathbf{m})$ which are admissible monomials of $BC_{2,k+r,k+t}$. We claim that other terms in $\phi_k(\sum_{\mathbf{m}\in\mathcal{S}} r_{\mathbf{m}}\mathbf{m})$ are obtained from (5.14) by

(1) using the terms $e_{k+i,i_j},\overline{e}_{k+i,i_j}$ of $\phi_k(x_i)$ to replace some $(k+i,i_j)$,
(2) using the terms $e_{i,j},\overline{e}_{i,j}$ $1\le i,j\le k$ (resp., $\overline{e}_{j_i,k+j},e_{j_i,k+j}$ of $\phi_k(\overline{x}_j)$) to replace $(\overline{i},\overline{j})$ in $\mathbf{c}_{2n+1}$ (resp., $(\overline{k+j},\overline{j}_i)$) if (1) does not occur,
(3) using the term x'_{k+i} of $\phi_k(x_i)$ to replace $(k+i,i_j)$ or using the term $\overline{x}'_{k+i}$ of $\phi_k(\overline{x}_i)$ to replace $(\overline{k+i},\overline{i}_j)$, or using either x'_j or $\overline{x}'_j$ to replace some $(\overline{i},\overline{j})$ in $\mathbf{c}_{2n+1}$, provided that neither (1) nor (2) occurs,
(4) using some $(\overline{k+j},\overline{s})$ or $\overline{c}_s(\overline{k+j},\overline{s})\overline{c}_s$, $s>k$ to replace $(\overline{k+j},\overline{j}_i)$; or using $(i,j),c_j(i,j)c_j$ to replace $(\overline{i},\overline{j})$ in $\mathbf{c}_{2n+1}$; or using $(k+i,s)$, $c_s(k+i,s)c_s$, $s>k_1$, to replace $(k+i,i_j)$, provided that (1)–(3) do not occur,
(5) using $\overline{c}_{j_i}(\overline{k+j},\overline{j}_i)\overline{c}_{j_i}$ to replace $(\overline{k+j},\overline{j}_i)$, or using $\overline{c}_j(\overline{i},\overline{j})\overline{c}_j$ to replace $(\overline{i},\overline{j})$ in $\mathbf{c}_{2n+1}$ or using $c_{i_j}(k+i,i_j)c_{i_j}$ to replace $(k+i,i_j)$, provided that (1)–(4) do not occur.

In the case (1), we use defining relations for $BC_{2,r+k,t+k}$ to rewrite the corresponding monomial as a linear combination of normal monomials. Each of these normal monomials corresponds to a unique walled Brauer diagram, say D, in which there is a horizontal edge $[i,\overline{j}]$ at the top row of D such that $1\le j\le k$. Such a monomial does not satisfy (b)(5). Similarly, in case (2) (resp., (3), (4), (5)), the corresponding monomials of $BC_{2,k+r,k+t}$ can be written as linear combinations of normal monomials which do not satisfy (b)(1) or (b)(3) (resp., (c) or (b)(3), (b)(3), (a)). This verifies our claim.

We assume that $\mathbf{m}_1,\mathbf{m}_2,\dots,\mathbf{m}_p$ are all monomials in S which contribute leading terms. Write

$$\mathbf{m}_i=c^{\alpha(\mathbf{m}_i)}x^{\beta(\mathbf{m}_i)}d_1(\mathbf{m}_i)^{-1}e^{f(\mathbf{m}_i)}w(\mathbf{m}_i)d_2(\mathbf{m}_i)\overline{x}^{\gamma(\mathbf{m}_i)}\overline{c}^{\delta(\mathbf{m}_i)}\prod_{n\in\mathbb{Z}^{>0}}\omega_{2n+1}^{a_{2n+1}(\mathbf{m}_i)}. \quad (5.15)$$

Then $k=|\gamma(\mathbf{m}_i)|+\sum_n(2n+1)a_{2n+1}(\mathbf{m}_i)$, $f(\mathbf{m}_i)=f_0$ and $k_1=|\beta(\mathbf{m}_i)|$. Let A_i be the set of all leading terms contributed by $\phi_k(\mathbf{m}_i)$. These leading terms are admissible monomials of $BC_{2,k+r,k+t}$. We have proven that other terms of $\sum_{\mathbf{m}\in S} r_{\mathbf{m}}\phi_k(\mathbf{m})$ will not contribute admissible monomials of $BC_{2,k+r,k+t}$. So,

$$\sum_{i=1}^{p} \tilde{r}_{\mathbf{m}_i} \sum_{\mathbf{n}\in A_i} \mathbf{n} = 0, \tag{5.16}$$

where $\tilde{r}_{\mathbf{m}_i}$ is $ar_{\mathbf{m}}$ and a is a power of ± 2, which comes from the coefficients of leading terms of $\xi_{2n+1,k+1}$. In the following, we explain that (5.16) does not hold. If so, then S is linear independent and the result will follow.

Suppose $g_i = (k+i,1)(k+i,2)\cdots(k+i,k_1)$ and $\overline{g}_j = (\overline{k+j},\overline{1})(\overline{k+j},\overline{2})\cdots(\overline{k+j},\overline{k})$ for $1 \le i \le r$ and $1 \le j \le t$. Note that each element in A_s contains factors $(k+i,i_1)\cdots(k+i,i_{\beta_i})$ contributed by $\phi_k(x_i)^{\beta_i}$, such that $i_1,\cdots,i_{\beta_i}$, $1 \le i \le r$, is a permutation of some elements in $1,2,\ldots,k_1$. So, $(k+i,k_1)\cdots(k+i,k_1-\beta_i+1)$ is one of such factors and hence $g_i(k+i,k_1)(k+i,k_1-1)\cdots(k+i,k_1-\beta_i+1)$ fixes $k_1,\ldots,k_1-\beta_i+1$. So, there is an element in $g_i(A_s)$ whose walled Brauer diagram contains vertical edges $[j,j]$, where j ranges $\beta(\mathbf{m}_s)_i$ numbers in $\{1,2,\ldots,k_1\}$. If $g_i(A_{s'})$ contains an element such that the corresponding walled Brauer diagram contains vertical edges $[j,j]$ for $\beta(\mathbf{m}_s)_i$ numbers in $\{1,2,\ldots,k_1\}$, then $\beta(\mathbf{m}_{s'})_i = \beta(\mathbf{m}_s)_i$. So, we can assume $\beta(\mathbf{m}_1) = \beta(\mathbf{m}_j)$, $2 \le j \le p$. Similarly, we use $\overline{g}_j$ instead of g_i to obtain $\gamma(\mathbf{m}_1) = \gamma(\mathbf{m}_j)$, $2 \le j \le p$. Note that $\mathbf{c}_{2n+1}$ is the product of a_{2n+1} disjoint cycles with length $2n+1$. So, different $\prod_n \omega_{2n+1}^{a_{2n+1}}$ gives product of disjoint cycles with different lengths and thus, we can assume $a_{2n+1}(\mathbf{m}_i)$ is independent of $\mathbf{m}_i$. Since any leading term is of form in (5.14), by Theorem 4.8 or Theorem 2.4, we can also assume that $\alpha(\mathbf{m}_i), \delta(\mathbf{m}_i)$ are independent of $\mathbf{m}_i$. Since $c_i^2 = -1$ and $\overline{c}_j^2 = 1$, we can assume $\alpha(\mathbf{m}_i) = 0^r \in \mathbb{Z}_2^r$ and $\delta(\mathbf{m}_i) = 0^t \in \mathbb{Z}_2^t$. Using (5.14), we see that there exists a leading term in $A_i \cap A_j$ if and only if $\mathbf{m}_i = \mathbf{m}_j$. So, $\tilde{r}_{\mathbf{m}_i} = 0$ for all $1 \le i \le p$, a contradiction. So, S is linear independent over $\mathbb{C}$ and hence over $\mathbb{Z}[2^{-1},\omega_1]$. In general, using arguments on base change yields the result over an arbitrary integral domain R containing ω_1 and 2^{-1}. $\square$

The following result follows from Theorem 5.14, immediately.

Theorem 5.15. *Suppose R is a domain containing 2^{-1} and ω_{2n+1}, for all $n \in \mathbb{N}$. Then $\widetilde{BC}_{r,t}$ is free over R spanned by all of its regular monomials. In particular, $\widetilde{BC}_{r,t}$ is of infinite rank.*

6. A basis of the cyclotomic Brauer–Clifford superalgebra

In this section, we assume that R is a domain containing 2^{-1} and parameters $\{\omega_{2n+1} \in R \mid n \in \mathbb{N}\}$. The *affine walled Brauer–Clifford superalgebra* $\widetilde{BC}_{r,t}$ with respect to the defining parameters ω_{2n+1}'s can be also defined in a simpler way as follows. As a free R-superspace,

$$\widetilde{BC}_{r,t} = R[\mathbf{x}_r] \otimes BC_{r,t} \otimes R[\overline{\mathbf{x}}_t], \tag{6.1}$$

the tensor product of the walled Brauer–Clifford superalgebra $BC_{r,t}$ with two polynomial algebras $R[\mathbf{x}_r] := R[x_1, x_2, \ldots, x_r]$ and $R[\overline{\mathbf{x}}_t] := R[\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_t]$. The multiplication of

$\widetilde{BC}_{r,t}$ is defined such that $R[\mathbf{x}_r]\otimes 1\otimes 1$, $1\otimes 1\otimes R[\overline{\mathbf{x}}_t]$, $1\otimes BC_{r,t}\otimes 1$, $R[\mathbf{x}_r]\otimes HC_r\otimes 1$ and $1\otimes \overline{HC}_t\otimes R[\overline{\mathbf{x}}_t]$ are subalgebras isomorphic to $R[\mathbf{x}_r]$, $R[\overline{\mathbf{x}}_t]$, $BC_{r,t}$, HC_r^{aff}, and $\overline{HC}_t^{\,\mathrm{aff}}$ respectively, and (for simplicity, without confusion we identify elements $x_i\otimes 1\otimes 1$, $1\otimes s_i\otimes 1$, $1\otimes e_i\otimes 1$, $1\otimes \overline{s}_i\otimes 1$, $1\otimes c_i\otimes 1$, $1\otimes \overline{c}_i\otimes 1$, $1\otimes 1\otimes \overline{x}_i$ in (6.1) with $x_i, s_i, e_i, \overline{s}_i, c_i, \overline{c}_i, \overline{x}_i$, respectively)

$$e_1(x_1+\overline{x}_1)=(x_1+\overline{x}_1)e_1=0,\quad e_1s_1x_1s_1=s_1x_1s_1e_1,\quad e_1\overline{s}_1\overline{x}_1\overline{s}_1=\overline{s}_1\overline{x}_1\overline{s}_1e_1, \tag{6.2}$$

$$s_i\overline{x}_1=\overline{x}_1s_i,\quad \overline{s}_ix_1=x_1\overline{s}_i,\quad x_1(e_1-c_1e_1c_1+\overline{x}_1)=(e_1-c_1e_1c_1+\overline{x}_1)x_1, \tag{6.3}$$

$$e_1x_1^ke_1=\omega_ke_1,\quad e_1\overline{x}_1^ke_1=\overline{\omega}_ke_1, \tag{6.4}$$

where $\overline{\omega}_{2k+1}$ determined by $\omega_1,\dots,\omega_{2k+1}$ as in Corollary 3.5. Further, $\omega_{2n}=\overline{\omega}_{2n}=0$.

We hope to classify finite dimensional simple $\widetilde{BC}_{r,t}$-modules over an algebraically closed field F with characteristic not 2. This leads us to introduce cyclotomic Brauer–Clifford superalgebras as follows. Let $f(x)$ be the minimal polynomial of x_1 with respect to a finite dimensional simple $\widetilde{BC}_{r,t}$-module M. Then

$$f(x)=x^k\prod_{i=1}^{n}(x-u_i), \tag{6.5}$$

where $u_1,\dots,u_n$ are nonzero in F. Let $\langle f(x_1)\rangle$ be the two-sided ideal of $\widetilde{BC}_{r,t}$ generated by $f(x_1)$. Since M is simple, $\langle f(x_1)\rangle\neq \widetilde{BC}_{r,t}$. Let $\varepsilon\in\{-1,1\}$.

Lemma 6.1. *We have $c_1f(x_1)=\varepsilon f(x_1)c_1$, where $f(x)$ is given in (6.5).*

Proof. We prove the result by induction on $\deg f(x)$. If $\deg f(x)=1$, then $f(x_1)=x_1-u$. When $u\neq 0$, we have $\langle x_1-u\rangle=\widetilde{BC}_{r,t}$, a contradiction, since $2u=c_1(x_1-u)c_1-(x_1-u)\in\langle x_1-u\rangle$. So, $f(x_1)=x_1$ and $c_1f(x_1)=-f(x_1)c_1$. In general, if $f(x_1)c_1\in\{c_1f(x_1),-c_1f(x_1)\}$, there is nothing to prove. Otherwise, by Definition 3.1(2), $f(-x_1)=-c_1f(x_1)c_1$ and $f(-x_1)\neq f(x_1)$. In this case, we choose $h(x_1)\in\{f(-x_1)+f(x_1), f(-x_1)-f(x_1)\}$ such that $\deg h(x)<\deg f(x)$. Define $d(x)=\mathrm{g.c.d}\,(h(x),f(x))$, the greatest common divisor of $h(x)$ and $f(x)$. Then $\langle d(x_1)\rangle=\langle f(x_1)\rangle$ and hence the irreducible $\widetilde{BC}_{r,t}$-module M is killed by $d(x_1)$. This is a contradiction since $f(x_1)$ is the minimal polynomial of x_1 with respect to M. $\square$

Recall that $f(x_1)$ in (6.5). Since $c_1f(x_1)c_1=-\varepsilon f(x_1)$, and $c_1f(x_1)c_1=\pm x_1^k\prod_{i=1}^n(x_1+u_i)$, $f(x)$ is the minimal polynomial of x_1 with respect to M if and only if $x_1^k\prod_{i=1}^n(x_1+u_i)$ is the minimal polynomial of x_1 with respect to M. In other words, u_i and $-u_i$ appear simultaneously if $u_i\neq 0$. Thus, we can assume

$$f(x_1)=x_1^k\prod_{i=1}^{m}(x_1^2-u_i^2), \tag{6.6}$$

where $0 \neq u_i \in F$, $1 \leq i \leq m$. Moreover, by Lemmas 3.3–3.4, there is a monic polynomial $g(\overline{x}_1)$ with degree $l = k + 2m$ such that

$$e_1 f(x_1) = (-1)^k e_1 g(\overline{x}_1). \tag{6.7}$$

Lemma 6.2. *Let $g(\overline{x}_1)$ be given such that (6.7) is satisfied. Then $\overline{c}_1 g(\overline{x}_1) = \varepsilon g(\overline{x}_1)\overline{c}_1$.*

Proof. Since $l = k + 2m$, we have

$$\begin{aligned}(-1)^k e_1 g(\overline{x}_1)\overline{c}_1 &= e_1 f(x_1)\overline{c}_1 = e_1\overline{c}_1 f(x_1) = e_1 c_1 f(x_1) = \varepsilon e_1 f(x_1) c_1 \\ &= (-1)^k \varepsilon e_1 g(\overline{x}_1) c_1 = (-1)^k \varepsilon e_1 c_1 g(\overline{x}_1) = (-1)^k \varepsilon e_1 \overline{c}_1 g(\overline{x}_1).\end{aligned}$$

By Theorem 5.15, $\overline{c}_1 g(\overline{x}_1) = \varepsilon g(\overline{x}_1)\overline{c}_1$. □

In the remaining part of this paper, we assume that

$$g(\overline{x}_1) = \overline{x}_1^{k_1} \prod_{j=1}^{m_1} (\overline{x}_1^2 - \overline{u}_j^2), \tag{6.8}$$

such that $k_1 + 2m_1 = k + 2m$ and $0 \neq \overline{u}_j \in F$, $1 \leq j \leq m_1$. This is reasonable by Lemma 6.2. Since the finite dimensional simple $\widetilde{BC}_{r,t}$-module M is killed by $f(x_1)$, by (6.7), it is killed by $e_1 g(\overline{x}_1)$, too. We want to consider simple $\widetilde{BC}_{r,t}$-modules M such that e_1 acts on M nontrivially, it is necessary to assume that M is killed by $g(\overline{x}_1)$. That is the reason why we introduce cyclotomic walled Brauer–Clifford superalgebras as in the Definition 3.14.

From here to the end of this section, we assume both $\widetilde{BC}_{r,t}$ and $BC_{l,r,t}$ are defined over a domain R containing 2^{-1} and parameters ω_{2n+1} for all $n \in \mathbb{N}$.

Lemma 6.3. *Write $f(x_1) = x_1^{k+2m} + \sum_{i=1}^{2m} a_i x_1^{k+2m-i}$, where $f(x_1)$ is given in (6.6). Then e_1 is an R-torsion element of $BC_{k+2m,r,t}$ unless*

$$\omega_\ell = -(a_1\omega_{\ell-1} + \ldots + a_{2m}\omega_{\ell-2m}) \quad \textit{for all} \quad \ell \geq k + 2m. \tag{6.9}$$

Proof. Let $b_\ell = \omega_\ell + a_1\omega_{\ell-1} + \ldots + a_{2m}\omega_{\ell-2m} \in R$. By (6.4), $b_\ell e_1 = e_1 f(x_1) x_1^{\ell-2m-k} e_1$ in $\widetilde{BC}_{r,t}$ and $b_\ell e_1 = 0$ in $BC_{k+2m,r,t}$. Thus, e_1 is an R-torsion element if $b_\ell \neq 0$ for some $\ell \geq k + 2m$. □

Definition 6.4. The superalgebras $\widetilde{BC}_{r,t}$ and $BC_{k+2m,r,t}$ are called *admissible* (with respect to $f(x_1)$) if (6.9) holds.

Lemma 6.5. *For $1 \leq i \leq r$, $1 \leq j \leq t$, define $f_i = f(x'_i)$ and $g_i = g(\overline{x}'_i)$, where $f(x_1)$ and $g(\overline{x}_1)$ satisfy (6.6)–(6.8). Then the following equations hold for all admissible i, j:*

(1) $c_j f_i = \varepsilon f_i c_j$,
(2) $\overline{c}_j g_i = \varepsilon g_i \overline{c}_j$,
(3) $\overline{c}_j f_i = f_i \overline{c}_j$,
(4) $c_j g_i = g_i c_j$,
(5) $\overline{s}_j f_i = f_i \overline{s}_j$,
(6) $s_j g_i = g_i s_j$,
(7) $f_i f_j = f_j f_i$ *in* $\mathrm{gr}(\widetilde{BC}_{r,t})$,
(8) $g_i g_j = g_j g_i$ *in* $\mathrm{gr}(\widetilde{BC}_{r,t})$.

Proof. These equations can be easily verified by using Lemmas 6.1–6.2 and Definition 3.1. □

Note that the affine Hecke–Clifford superalgebra HC_r^{aff} (resp., $\overline{HC}_t^{\mathrm{aff}}$) is isomorphic to the sub-superalgebra of $\widetilde{BC}_{r,t}$ generated by $x_1, s_1, \dots, s_{r-1}$ and c_1 (resp., $\overline{x}_1, \overline{s}_1, \dots, \overline{s}_{t-1}$ and $\overline{c}_1$).

Lemma 6.6. *For any $a \in \mathbb{Z}^{>0}$, we have*

(1) $(x'_i)^a f(x'_\ell) - f(x'_\ell) {x'_i}^a \in \sum_{b<a} \sum_{h,h_1=1}^{\max\{i,\ell\}} f(x'_h)(x'_{h_1})^b HC_r$,
(2) $g(\overline{x}'_\ell)(\overline{x}'_i)^a - (\overline{x}'_i)^a g(\overline{x}'_\ell) \in \sum_{b<a} \sum_{h,h_1=1}^{\max\{i,\ell\}} \overline{HC}_t (\overline{x}'_{h_1})^b g(\overline{x}'_h)$.

Proof. We have $x_1 x_2 = x_2 x_1$, where $x_2 = x'_2 - s_1 - c_2 s_1 c_2$ (see (2.7)). By Lemma 6.5(1),

$$x'_2 f(x_1) = f(x_1)(x'_2 - s_1 - c_2 s_1 c_2) + f(x'_2) s_1 + \varepsilon f(x'_2) c_2 s_1 c_2. \tag{6.10}$$

Considering $s_{i,2} x'_2 f(x_1) s_{2,i}$ yields the result when $a = 1$ and $\ell = 1$. If $\ell > 1$, then

$$x'_i f(x'_\ell) = x'_i s_{\ell-1} f(x'_{\ell-1}) s_{\ell-1} = s_{\ell-1} x'_{(i)s_{\ell-1}} f(x'_{\ell-1}) s_{\ell-1}.$$

So, the result follows from inductive assumption on $\ell - 1$. This is (1) when $a = 1$. The general case follows from arguments on induction on a. Finally, (2) can be verified, similarly. □

Proposition 6.7. *Define $J_L = \sum_{i=1}^t \widetilde{BC}_{r,t}\, g_i$ and $J_R = \sum_{i=1}^r f_i\, \widetilde{BC}_{r,t}$. Let I be the two-sided ideal of $\widetilde{BC}_{r,t}$ generated by $f(x_1)$ and $g(\overline{x}_1)$. We have*

(1) *J_R is a left $HC_r^{\mathrm{aff}} \otimes \overline{HC}_t$-module,*
(2) *J_L is a right $HC_r \otimes \overline{HC}_t^{\mathrm{aff}}$-module,*
(3) *$I = J_L + J_R$ if $\widetilde{BC}_{r,t}$ is admissible.*

Proof. It is easy to see that J_R is stable under the left action of $HC_r \otimes \overline{HC}_t$. By Lemma 6.6(1), it is stable under the left action of HC_r^{aff}. One can check (2) via Lemma 6.6(2), similarly.

Obviously, $J_L + J_R \subseteq I$. So, (3) follows if we can prove $I \subseteq J_L + J_R$. Since $f(x_1), g(\overline{x}_1) \in J_L + J_R$, it suffices to verify that $J_L + J_R$ is a two-sided ideal of $\widetilde{BC}_{r,t}$. We claim

$$hJ_R \subset J_L + J_R, \tag{6.11}$$

for any generator h of $\widetilde{BC}_{r,t}$. If so, $h(J_L + J_R) \subset J_L + J_R$ and hence $J_L + J_R$ is a left ideal.

In fact, by (1), it is enough to verify (6.11) when $h \in \{\overline{x}_1, e_1\}$. If we have $e_1 J_R \subset J_L + J_R$, then $c_1 e_1 c_1 J_R \subset J_L + J_R$. Since $(\overline{x}_1 + e_1 - \overline{e}_1) f(x_1) = f(x_1)(\overline{x}_1 + e_1 - \overline{e}_1) \in J_R$, we have $\overline{x}_1 f(x_1) \in J_L + J_R$. Multiplying $(1, i)$ on both sides of $\overline{x}_1 f(x_1)$ yields $\overline{x}_1 f_i \in J_L + J_R$. So, we need to verify $e_1 J_R \subset J_L + J_R$. By (6.2), $e_1 f_i = f_i e_1$ for $i \geq 2$. So, $e_1 J_L \subset J_L + J_R$ if $e_1 f(x_1) \widetilde{BC}_{r,t} \subset J_L + J_R$. This will be verified by checking

$$e_1 f(x_1)\mathbf{m} \in J_L + J_R, \tag{6.12}$$

for each regular monomial $\mathbf{m}$ of $\widetilde{BC}_{r,t}$ in Definition 3.15. Using arguments on graded structure of $\widetilde{BC}_{r,t}$, we can write $\mathbf{m} = c^\alpha x^\beta e_{i_1,j_1} \cdots e_{i_f,j_f} w \overline{x}^\gamma \overline{c}^\delta$ for some $(\alpha, \delta) \in \mathbb{Z}_2^r \times \mathbb{Z}_2^t$ and $(\beta, \gamma) \in \mathbb{N}^r \times \mathbb{N}^t$, and $w \in \Sigma_r \times \Sigma_{\overline{t}}$ and $1 \leq i_1, \ldots, i_f \leq r$ and $1 \leq j_1, \ldots, j_f \leq t$ such that $\{i_k, j_k\} \cap \{i_l, j_l\} = \emptyset$ if $k \neq l$. In the following, we write $e_{\vec{i},\vec{j}} = e_{i_1,j_1} \cdots e_{i_f,j_f}$. We prove (6.12) by induction on $|\beta|$.

Case 1: $|\beta| = 0$.

If $f = 0$, then $e_1 f(x_1) c^\alpha w \overline{x}^\gamma \overline{c}^\delta = (-1)^k e_1 g(\overline{x}_1) c^\alpha w \overline{x}^\gamma \overline{c}^\delta \subseteq J_L$. The last inclusion follows from (2). Suppose $1 \leq f \leq \min\{r, t\}$. Since $\widetilde{BC}_{r,t}$ is admissible, $e_1 f(x_1) e_1 = 0$. On the other hand, we have $e_1 x_1^k c_1 e_1 = 0$ for all k. So, $e_1 f(x_1)\mathbf{m} = 0$ if e_1 is a factor of $e_{\vec{i},\vec{j}}$. If e_1 is not a factor of $e_{\vec{i},\vec{j}}$, there are three cases we need to discuss.

- If $e_{p,1}$ is a factor of $e_{\vec{i},\vec{j}}$, and $p \neq 1$, then we assume that $i_1 = p$ and $j_1 = 1$ since any two factors of $e_{\vec{i},\vec{j}}$ commute with each other. We have $e_1 f(x_1) c^\alpha e_{p,1} = \prod_{i=2}^r c_i^{\alpha_i} e_1 f(x_1) e_{p,1} c_1^{\alpha_1}$. Since

$$e_1 f(x_1) e_{p,1} = s_{p,2} e_1 f(x_1) s_1 e_1 s_{1,p} = s_{p,2} e_1 s_1 f(x_2') e_1 s_{1,p} = s_{p,2} f(x_2') e_1 s_{1,p} \in J_R,$$

  we have $e_1 f(x_1) c^\alpha e_{1,p} \in J_R$ by (1). So, $e_1 f(x_1)\mathbf{m} \in J_R$.
- If $e_{1,p}$ is a factor of $e_{\vec{i},\vec{j}}$ and $p \neq 1$, we assume $i_1 = 1$ and $j_1 = p$. We have

$$\begin{aligned} e_1 f(x_1) e_{1,p} &= (-1)^k \overline{s}_{p,2} e_1 g(\overline{x}_1) \overline{s}_1 e_1 \overline{s}_{1,p} = (-1)^k \overline{s}_{p,2} e_1 \overline{s}_1 g(\overline{x}_2') e_1 \overline{s}_{1,p} \\ &= (-1)^k \overline{s}_{p,2} e_1 g(\overline{x}_2') \overline{s}_{1,p} = (-1)^k \overline{s}_{p,2} e_1 \overline{s}_{1,p} g(\overline{x}_1) \in J_L. \end{aligned}$$

  So, $e_1 f(x_1) c^\alpha e_{1,p} = \prod_{i=2}^r c_i^{\alpha_i} e_1 f(x_1) c_1^{\alpha_1} e_{1,p} = \prod_{i=2}^r c_i^{\alpha_i} \overline{c}_p^{\alpha_1} e_1 f(x_1) e_{1,p} \in J_L$. By (2) and the equation $g(\overline{x}_1) \prod_{k=2}^f e_{i_k,j_k} = \prod_{k=2}^f e_{i_k,j_k} g(\overline{x}_1)$, $e_1 f(x_1)\mathbf{m} \in J_L$.
- Finally, suppose $\{i_l, j_l\} \cap \{1\} = \emptyset$ for all possible l, then $e_1 f(x_1)\mathbf{m} \in J_L$ follows from (2) and the following fact

$$e_1 \boldsymbol{f}(x_1) \prod_{l=1}^f e_{i_f,j_f} = \prod_{l=1}^f e_{i_f,j_f} e_1 \boldsymbol{f}(x_1) = (-1)^k \prod_{l=1}^f e_{i_f,j_f} e_1 \boldsymbol{g}(\overline{x}_1) \in J_L.$$

Case 2: $|\beta| > 0$.

Suppose $\beta_i \neq 0$ for some $2 \leq i \leq r$. We have $x_i\mathbf{m}' = \varepsilon\mathbf{m}$ by Lemma 3.8(1)–(2), where $\mathbf{m}'$ is obtained from $\mathbf{m}$ by removing one x_i and $\varepsilon = \pm 1$. So $e_1 f(x_1)\mathbf{m} \in J_L + J_R$ if $e_1 f(x_1)x_i\mathbf{m}' \in J_L + J_R$. Since $f(x_1)x_i = x_i f(x_1)$, it suffices to prove $e_1 L_i f(x_1)\mathbf{m}' \in J_L + J_R$ and $e_1(x_i - L_i)f(x_1)\mathbf{m}' \in J_L + J_R$.

In the first case, since $e_1(j,i) = (j,i)e_1$ if $j \neq 1$ and $e_1 c_i = c_i e_1$ and $c_i f(x_1) = f(x_1)c_i$, by inductive assumption on $|\beta|$ and (1)–(2), it is enough to prove $e_1(1,i)f(x_1)c_i^l\mathbf{m}' \in J_R$ for $l \in \mathbb{Z}_2$. In fact, it is the case since

$$e_1(1,i)f(x_1)c_i^l\mathbf{m}' = e_1 f(x_i')(1,i)c_i^l\mathbf{m}' = f(x_i')e_1(1,i)c_i^l\mathbf{m}' \in J_R.$$

In the second case, since $e_1(x_i - L_i)f(x_1)\mathbf{m}' = (x_i - L_i)e_1 f(x_1)\mathbf{m}'$, by induction on $|\beta|$, $e_1 f(x_1)\mathbf{m}' \in J_L + J_R$. By (1), we have $(x_i - L_i)e_1 f(x_1)\mathbf{m}' \in J_L + J_R$. So, $e_1 f(x_1)\mathbf{m} \in J_L + J_R$.

If $\beta_i = 0$, $2 \leq i \leq r$, then $x^\beta = x_1^{\beta_1}$ with $\beta_1 > 0$. In this case, $\mathbf{m} = c^\alpha x_1^{\beta_1} e_{\vec{i},\vec{j}} w \overline{x}^\gamma \overline{c}^\delta$. We want to prove $v = e_1 f(x_1)\mathbf{m} \in J_L + J_R$. If $j_\ell \neq 1$, $1 \leq \ell \leq f$, then by inductive assumption,

$$\begin{aligned} v &= e_1 f(x_1)c^\alpha x_1^{\beta_1} e_{\vec{i},\vec{j}} w\overline{x}^\gamma \overline{c}^\delta = (-1)^k e_1 g(\overline{x}_1)c^\alpha x_1^{\beta_1} e_{\vec{i},\vec{j}} w\overline{x}^\gamma \overline{c}^\delta \\ &\equiv (-1)^k e_1 c^\alpha x_1^{\beta_1} g(\overline{x}_1) e_{\vec{i},\vec{j}} w\overline{x}^\gamma \overline{c}^\delta \\ &= (-1)^k e_1 c^\alpha x_1^{\beta_1} e_{\vec{i},\vec{j}}\, g(\overline{x}_1) w\overline{x}^\gamma \overline{c}^\delta \in J_L w\overline{x}^\gamma \overline{c}^\delta \subset J_L + J_R, \end{aligned}$$

where the "$\equiv$" is modulo $J_L + J_R$. Finally, if $j_\ell = 1$ for some ℓ, without loss of any generality, we assume $j_1 = 1$. If $i_1 = 1$, we have $e_1 f(x_1)c^\alpha x_1^\beta e_1 = 0$ no matter whether $\alpha_1 = 1$ or $\alpha_1 = 0$. In the first case, this result follows from the equation $e_1 x_1^k c_1 e_1 = 0$ for all $k \in \mathbb{N}$. In the second case, this result follows from the fact that $\widetilde{BC}_{r,t}$ is admissible. It remains to deal with the cases when $i_l \neq 1$ for all l. Define $i' = (i_2, \ldots, i_f)$ and $j' = (j_2, \ldots, j_f)$. Then

$$\begin{aligned} v &= e_1 f(x_1)c^\alpha x_1^{\beta_1} e_{i_1,1} e_{\vec{i}',\vec{j}'} w\overline{x}^\gamma \overline{c}^\delta = \prod_{i=2}^r c_i^{\alpha_i} e_1 f(x_1) e_{i_1,1} c_1^{\alpha_1} x_1^{\beta_1} e_{\vec{i}',\vec{j}'} w\overline{x}^\gamma \overline{c}^\delta \\ &= \prod_{i=2}^r c_i^{\alpha_i} e_1 e_{i_1,1} f(x_1) c_1^{\alpha_1} x_1^{\beta_1} e_{\vec{i}',\vec{j}'} w\overline{x}^\gamma \overline{c}^\delta = \prod_{i=2}^r c_i^{\alpha_i} e_1 (1,i_1) f(x_1) c_1^{\alpha_1} x_1^{\beta_1} e_{\vec{i}',\vec{j}'} w\overline{x}^\gamma \overline{c}^\delta \\ &= \prod_{i=2}^r c_i^{\alpha_i} e_1 f(x_{i_1}')(1,i) c_1^{\alpha_1} x_1^{\beta_1} e_{\vec{i}',\vec{j}'} w\overline{x}^\gamma \overline{c}^\delta \in \prod_{i=2}^r c_i^{\alpha_i} J_R \subset J_R, \text{ by (1).} \end{aligned}$$

This completes the proof of (6.12) and hence $hJ_R \subset J_L + J_R$. One can similarly check $J_L h \subset J_L + J_R$. □

For $(\alpha, \beta) \in \mathbb{N}^r \times \mathbb{N}^t$, let $\boldsymbol{f}(x')^\alpha = f_1^{\alpha_1} \cdots f_r^{\alpha_r}$ and $\boldsymbol{g}(\overline{x}')^\beta = g_1^{\beta_1} \cdots g_t^{\beta_t}$. Recall that $l = k + 2m$.

Lemma 6.8. *The affine walled Brauer–Clifford superalgebra $\widetilde{BC}_{r,t}$ is a free R-module with $\mathcal{N}$ as its R-basis, where*

$$\mathcal{N} = \bigcup_{f=0}^{\min\{r,t\}} \{\boldsymbol{f}(x')^{\alpha} c^{\tilde{\gamma}} x^{\gamma} d_1^{-1} e^f w d_2 \overline{x}^{\delta} \overline{c}^{\tilde{\delta}} \boldsymbol{g}(\overline{x}')^{\beta} \mid (\alpha,\beta) \in \mathbb{N}^r \times \mathbb{N}^t, \\ (\gamma,\delta,\tilde{\gamma},\tilde{\delta}) \in \mathbb{Z}_l^r \times \mathbb{Z}_l^t \times \mathbb{Z}_2^r \times \mathbb{Z}_2^t, d_1, d_2 \in D_{r,t}^f, w \in \Sigma_{r-f} \times \overline{\Sigma}_{t-f}\}. \tag{6.13}$$

Proof. The result follows from Theorem 5.15 since the transition matrix between $\mathcal{N}$ and the basis in Theorem 5.15 is upper-unitriangular. $\square$

Lemma 6.9. *Let I be the two-sided ideal of $\widetilde{BC}_{r,t}$ generated by $f(x_1)$ and $g(\overline{x}_1)$ satisfying (6.6)–(6.8). If $\widetilde{BC}_{r,t}$ is admissible, then S is an R-basis of I, where*

$$S = \{\boldsymbol{f}(x')^{\alpha} c^{\tilde{\gamma}} x^{\gamma} d_1^{-1} e^f w d_2 \overline{x}^{\delta} \overline{c}^{\tilde{\delta}} \boldsymbol{g}(\overline{x}')^{\beta} \in \mathcal{N} \mid \alpha_i + \beta_j \neq 0 \text{ for some } i, j\}. \tag{6.14}$$

Proof. Let M be the R-module spanned by S. Obviously, $M \subseteq I$. If $J_L \subseteq M$ and $J_R \subseteq M$, by Lemma 6.7(3), $M = I$, proving the result. By symmetry, we verify $J_R \subseteq M$. By Lemma 6.8, we need to verify $f(x_i')\mathbf{m} \in M$ for any basis element $\mathbf{m}$ in (6.13). In fact, we have

$$f(x_i')f(x_\ell') \in f(x_\ell')f(x_i') + \sum_{j=1}^{\ell-1} f(x_j')\widetilde{BC}_{r,t}, \text{ by Lemma 6.6.}$$

Using induction on degrees, we have $f(x_i')\mathbf{m} \in M$. Finally, one can check $J_L \subset M$, similarly. $\square$

Theorem 6.10. *The cyclotomic walled Brauer–Clifford superalgebra $BC_{k+2m,r,t}$ is free over R with rank $2^{r+t}(k+2m)^{r+t}(r+t)!$ if and only if $BC_{k+2m,r,t}$ is admissible.*

Proof. By Corollary 3.16, $BC_{k+2m,r,t}$ is spanned by all of its regular monomials. If $BC_{k+2m,r,t}$ is not admissible, e_1 is an R-torsion element by Corollary 3.5. Since $e_1 \in M$, either $BC_{k+2m,r,t}$ is not free or the rank of $BC_{k+2m,r,t}$ is strictly less than $2^{r+t}(k+2m)^{r+t}(r+t)!$, the number of all regular monomials of $BC_{k+2m,r,t}$. If $BC_{k+2m,r,t}$ is admissible, by Lemmas 6.8–6.9, all regular monomials of $BC_{k+2m,r,t}$ are R-linear independent. So $BC_{k+2m,r,t}$ is free over R with rank $2^{r+t}(k+2m)^{r+t}(r+t)!$. $\square$

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