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# Isotypies for $p$-blocks with non-abelian metacyclic defect groups, $p$ odd

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ABSTRACT

Let $p$ be an odd prime, $G$ be a finite group and $b$ be a $p$-block of $G$ with non-abelian metacyclic defect group $P$. Then it is known that a hyperfocal subgroup $Q$ of $b$ is cyclic. In this study motivated by Rouquier's conjecture on blocks with abelian hyperfocal subgroups, we show that $b$ is isotypic to its Brauer correspondents in $N_G(P)$ and $N_G(Q)$.


## 1. Introduction and notation

Let $G$ be a finite group and $(\mathcal{K}, \mathcal{O}, k)$ be a sufficiently large $p$-modular system such that $k$ is algebraically closed where $p$ is a fixed prime. Let $b$ be a $p$-block of $\mathcal{O}G$ with a maximal $b$-Brauer pair $(P, b_P)$. Let $Q = \mathfrak{hyp}(b)$ be the hyperfocal subgroup of $b$ with respect to $(P, b_P)$ ([14]). A character-theoretic shadow of Rouquier's conjecture ([15] A.2) says that if $Q$ is abelian, then $b$ and $b_P^{N_G(Q)}$ are perfectly isometric ([4] 1.4).

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By the results in [20], and [19] or [7], if $p$ is odd and $P$ is non-abelian metacyclic, then $Q$ is cyclic. In this article we will prove the following:

**Theorem 1.1.** *Assume that $p$ is odd, and $P$ is metacyclic and either of the following holds.*

(i) *$P$ is non-abelian.*

(ii) *$P$ is abelian and $Q$ is cyclic.*

*Then $b$ is isotypic to its Brauer correspondents in $N_G(P)$ and $N_G(Q)$.*

Hence, a character-theoretic version of Rouquier's conjecture for blocks with non-abelian metacyclic defect groups is true when $p$ is odd. See [16] Chapter 8 on results on $p$-blocks with metacyclic defect groups including 2-blocks. Note that, in [22], an isotypy between $b$ and its Brauer correspondent in the case where $P$ is abelian (not necessarily rank 2) and $Q$ is cyclic is already proved in a different manner. Also note that if $b$ has a non-trivial cyclic hyperfocal subgroup $Q$, then $p$ is odd by [23] Lemma 3 and Lemma 4 (ii).

We give an outline of our proof of Theorem 1.1 using the notation mentioned below. We may assume $Q \neq 1$, since $Q = 1$ if and only if $b$ is nilpotent. By referring to ideas in [5], we will determine $\mathrm{Irr}(b)$ from $\mathrm{Irr}(\tilde{b})$ where $\tilde{b} = b_P^{\tilde{G}}$, $\tilde{G} = N_G(Q_1)$ and $Q_1$ is the minimal subgroup of $Q$. But $l(b)$ and $k(b)$ are already known by [23]. In determining $\mathrm{Irr}(b)$, Broué-Puig $*$-construction also plays a big role.

The block $c = b_P^{C_G(Q_1)}$ is nilpotent and $\tilde{b}$ covers $c$. Since $\mathrm{Irr}(c)$ is known ([3]), $\mathrm{Irr}(\tilde{b})$ is determined by Clifford theory for blocks and Fong-Reynolds correspondence (Theorem 2.6).

The induction from $\tilde{G}$ to $G$ induces the $\mathcal{K}$-*linear isometry* from $\sum_{u\in Q\setminus\{1\}} X_{\mathcal{K}}^{(u,\tilde{b}_u)}(\tilde{G},\tilde{b})$ to $\sum_{u\in Q\setminus\{1\}} X_{\mathcal{K}}^{(u,b_u)}(G,b)$ (Theorem 3.3), and any $\chi \in \mathrm{Irr}(b)$ appears in some element of $\sum_{u\in Q\setminus\{1\}} X_{\mathcal{K}}^{(u,b_u)}(G,b)$ (Proposition 3.4). This is a crucial key to determine $\mathrm{Irr}(b)$.

In §4, $\mathrm{Irr}(b)$ is determined by long calculations (Theorem 4.4). In particular, (4.17) below is an important equation expressing a connection between $\mathrm{Irr}(\tilde{b})$ and $\mathrm{Irr}(b)$. Then we also determine the Cartan matrix of $b$ with respect to a basic set (Proposition 4.6).

For any $(u, b_u) \in (P, b_P)$, $b_u$ is nilpotent, or $b_u$ has a metacyclic defect group and a cyclic hyperfocal subgroup and $l(b_u) = e$. That is, we can apply the results in previous sections to non-nilpotent $b_u$. In §5, we determine generalized decomposition numbers in $b$ from (4.17), the orthogonality relations for them and the Cartan matrix of $b_u$, $u \in P$ (Theorem 5.1 and Theorem 5.2).

In the final section, by applying [11] Theorem 2 for $b$ and $b' = b_P^{N_G(P,b_P)}$, we obtain a perfect isometry between them using the signs appearing in (4.17). We also obtain a perfect isometry $I^u$ between local blocks $b_u$ and $b'_u$ for any $u \in P$. The isometry in the local blocks is arranged by the sign appearing in Theorem 5.1 or Theorem 5.2. Then $\{I^u \mid u \in P\}$ defines an isotypy between $b$ and $b'$ (Theorem 6.5).

We denote by $\mathrm{Irr}(b)$ (resp. $\mathrm{IBr}(b)$) the set of ordinary (resp. Brauer) irreducible characters in $b$ and by $\mathrm{Irr}_i(b)$ the set of ordinary irreducible characters in $b$ of height $i$. We set $l(b) = |\mathrm{IBr}(b)|$, $k(b) = |\mathrm{Irr}(b)|$ and $k_i(b) = |\mathrm{Irr}_i(b)|$. We set $X(G,b) = \sum_{\chi\in\mathrm{Irr}(b)} \boldsymbol{Z}\chi$ and

$X_{\mathcal{K}}(G,b) = \sum_{\chi\in\mathrm{Irr}(b)} \mathcal{K}\chi$. For $\alpha, \beta \in X_{\mathcal{K}}(G,b)$, we denote by $(\alpha, \beta)$ the inner product of $\alpha$ and $\beta$. For each $u \in P$, let $(u, b_u)$ be the Brauer element belonging to $(P, b_P)$. For $\chi \in \mathrm{Irr}(b)$ and a basic set $\{\varphi_1^{(u)}, \varphi_2^{(u)}, \cdots, \varphi_{l(b_u)}^{(u)}\}$ for $b_u$, we denote by $d^u(\chi, \varphi_j^{(u)})$ the generalized decomposition number. For $\chi \in X_{\mathcal{K}}(G,b)$, $\chi^{(u,b_u)}$ is the class function of $G$ vanishing outside of the $p$-section of $u$ and which is such that $\chi^{(u,b_u)}(us) = \chi(usb_u)$ for $s \in C_G(u)_{p'}$ where $G_{p'}$ is the set of $p'$-elements of $G$. If $(u, b_u)$ and $(v, b_v)$ are not $G$-conjugate, then $(\chi^{(u,b_u)}, \chi'^{(v,b_v)}) = 0$ for any $\chi, \chi' \in X_{\mathcal{K}}(G,b)$ (cf. [13] Theorems 3.6.13 and 5.4.7). We define a $\mathcal{K}$-vector space

$$X_{\mathcal{K}}^{(u,b_u)}(G,b) = \{\chi^{(u,b_u)} \mid \chi \in X_{\mathcal{K}}(G,b)\}.$$

Then $\dim_{\mathcal{K}}\left(X_{\mathcal{K}}^{(u,b_u)}(G,b)\right) = l(b_u)$. For a normal subgroup $N$ of $G$ and a character $\zeta$ of $N$, we denote by $S_G(\zeta)$ the stabilizer of $\zeta$ in $G$. By inflation, $\mathrm{Irr}(G/N)$ will be regarded as a subset of $\mathrm{Irr}(G)$. For $x \in G$, we denote by $x^G$ the conjugacy class of $x$, and by $\widehat{x^G}$ the class sum. For a finite abelian group $X$, we denote by $\hat{X}$ the character group of $X$. For a subgroup $Y$ of $X$, we have $\hat{X}/Y^\perp \simeq \hat{Y}$ via restriction where $Y^\perp = \{\lambda \in \hat{X} \mid Y \subseteq \mathrm{Ker}(\lambda)\}$. We can regard $\hat{Y}$ as a subset of $\hat{X}$ via extension of linear characters to $X$, which is not uniquely determined.

**In this paper we assume $Q$ is cyclic.** Then the Brauer category $\mathcal{F}_{(P,b_P)}(G,b)$ is controlled by $N_G(P, b_P)$, see [23] Theorem 3. Any Brauer pair $(T, b_T)$ contained in $(P, b_P)$ is extremal in $(P, b_P)$, see [23] Lemma 5. Let $E$ be a complement of $PC_G(P)/C_G(P)$ in $N_G(P, b_P)/C_G(P)$, and $e = |E|$ be the inertial index of $b$. We have $l(b) = e$, see [23] Theorem 1. The group $E$ is cyclic of order dividing $p-1$ since $E \leq \mathrm{Aut}(Q)$, see [23] Lemma 3. We have $Q = [Q, E]$, and $e = 1$ if and only if $Q = 1$, see [23] Lemma 4 (ii). Set

$$L = P \rtimes E,$$

and define $\Pi$ as follows:

$$\Pi : \text{ a set of representatives for the } L\text{-conjugacy classes of } P.$$

Then $\{(u, b_u) \mid u \in \Pi\}$ is a set of representatives for the $G$-conjugacy classes of $b$-Brauer elements. Let $\eta$ be a $(G, b_P)$-stable generalized character of $P$, that is, if $(u, b_u), (u', b_{u'}) \in (P, b_P)$ are $G$-conjugate, then $\eta(u) = \eta(u')$. For $\chi \in X_{\mathcal{K}}(G,b)$, $\chi * \eta \in X_{\mathcal{K}}(G,b)$ is such that

$$\chi * \eta = \sum_{u\in\Pi} \eta(u)\chi^{(u,b_u)},$$

and $\chi * \eta \in X(G,b)$ whenever $\chi \in X(G,b)$, see [2]. We set

$$R = C_P(E).$$

**Moreover we assume $Q$ is non-trivial.** Then $p \neq 2$. By [23] Lemma 4 (i),

$$P = Q \rtimes R.$$

Note that any generalized character $\lambda$ of $R$ regarded as a generalized character of $P$ is $(G, b_P)$-stable as $N_G(P, b_P)$ controls $\mathcal{F}_{(P,b_P)}(G, b)$. For $u \in P$, $b_u$ is nilpotent if and only if $u \notin_P R$, see [23] Lemma 7. When $u \in_P R$, we have $l(b_u) = e$ by [23] Lemma 6 and Lemma 7. For $\mu \in \mathrm{Irr}(Q)$, we set

$$P_\mu = S_P(\mu), \quad R_\mu = R \cap P_\mu, \quad h_\mu \text{ is such that } p^{h_\mu} = |P : P_\mu| = |R : R_\mu|.$$

Then $P_\mu$ is normal in $N_G(P, b_P)$ and $E$-invariant as $\mathrm{Aut}(Q)$ is cyclic. We denote by $\hat{\mu}$ the extension of $\mu$ to $P_\mu$ with $R_\mu \subseteq \mathrm{Ker}\ \hat{\mu}$. Let $Q_1$ be the subgroup of $Q$ with order $p$. Note that $Q_1 \subseteq Z(P)$. Set

$$C = C_G(Q_1), \ \ c = b_{Q_1}, \ \ \tilde{N} = N_G(Q_1, c), \ \ \tilde{c} = c^{\tilde{N}}, \ \tilde{G} = N_G(Q_1), \ \ \tilde{b} = \tilde{c}^{\tilde{G}}.$$

The pair $(P, b_P)$ is a maximal $c$ (resp. $\tilde{c}$, $\tilde{b}$)-Brauer pair. The block $c$ is nilpotent. The block $\tilde{c}$ has an inertial group $E$, and has a hyperfocal subgroup $Q$ from $Q = [Q, E] \leq [P, E]$ and [23] Lemma 6. The block $\tilde{b}$ is the Clifford correspondent of $\tilde{c}$.

## 2. $\mathrm{Irr}(\tilde{b})$

In this section, we determine the irreducible characters in $\tilde{b}$.

Firstly, we have

$$\mathrm{Irr}(P) = \bigcup_{\mu \in \mathcal{R}} \{ \ (\hat{\mu}\lambda_\mu){\uparrow}_{P_\mu}^{P} \ \mid \lambda_\mu \in \mathrm{Irr}(R_\mu)\} \tag{2.1}$$

where $\mathcal{R}$ is a set of representatives for the $P$-conjugacy classes of $\mathrm{Irr}(Q)$.

**Proposition 2.1.** ([3] Theorem 1.2)

(i) $l(c) = 1$.

(ii) *For any $c$-Brauer element $(u, f)$, $f$ is nilpotent.*

(iii) *There is an irreducible character $\zeta_0$ in $c$ with height 0 such that $d^u(\zeta_0, \varphi_{(u,f)}) = \pm 1$ for any $c$-Brauer element $(u, f)$ and the unique irreducible Brauer character $\varphi_{(u,f)}$ in $f$.*

(iv) *Every generalized character of $P$ is $(C, b_P)$-stable and* $\mathrm{Irr}(c) = \{\zeta_0 * \nu \mid \nu \in \mathrm{Irr}(P)\}$.

We have $\tilde{N} = N_G(P, b_P)C$ since $\tilde{N} = N_{N_G(P,b_P)}(Q_1, c)C$ and $Q_1 \lhd N_G(P, b_P)$. We also have $N_C(P, b_P) = PC_G(P)$. In fact, we have $N_C(P, b_P) = \big(N_C(P, b_P) \cap P\big)\big(N_C(P, b_P) \cap F\big)$ for a lift $F$ in $N_G(P, b_P)$ of a suitable inertial quotient group of $b$, and $C \cap F$ acts trivially on $P$ since $F$ acts trivially on $P/Q$ and $C \cap F$ acts trivially on $Q$. Hence, we have $\tilde{N}/C \cong E$.

Since $\zeta_0$ is the unique irreducible character in $c$ such that it is $p$-rational, $\zeta_0$ is $\tilde{N}$-invariant. We set

$$\zeta_\nu = \zeta_0 * \nu$$

for $\nu \in \mathrm{Irr}(P)$. For $n \in N_G(P, b_P)$, we have

$$(\zeta_\nu)^n = (\zeta_0 * \nu)^n = \zeta_{\nu^n}. \tag{2.2}$$

Write $\nu = (\hat{\mu}\lambda)\uparrow_{P_\mu}^P$ where $\mu \in \mathrm{Irr}(Q)$ and $\lambda \in \mathrm{Irr}(R_\mu)$, see (2.1). If $\mu = 1_Q$, then $\nu = \lambda$, and we have $S_{\tilde{N}}(\zeta_\lambda) = \tilde{N}$ from $\tilde{N} = N_G(P, b_P)C$ and (2.2). Hence $\zeta_\lambda$ extends to $\tilde{N}$. On the other hand, if $\mu \neq 1_Q$, then we have $S_{\tilde{N}}(\zeta_\nu) = C$. In fact, if $\nu^n = \nu$ for $n \in N_G(P, b_P)$, then $\mu$ and $\mu^n$ are irreducible constituents of $\nu\downarrow_Q^P$. Hence $\mu = \mu^{nu}$ for some $u \in P$. Since a $p'$-automorphism of $\hat{Q}$ does not fix any element of $\hat{Q}\backslash\{1\}$, we have $nu \in PC_G(P) \subseteq C$, and so $n \in C$.

We define $\mathcal{M}$ as follows:

$$\mathcal{M}: \text{ a set of representatives for the } L\text{-conjugacy classes of } \mathrm{Irr}(Q)\backslash\{1\}.$$

Let

$$\tilde{\zeta}_{i,\lambda}\ (i = 1, 2, \cdots, e) \text{ be the extensions of } \zeta_\lambda \text{ to } \tilde{N}\ (\lambda \in \mathrm{Irr}(R)),$$

and set

$$\tilde{\zeta}_{\mu,\lambda_\mu} = (\zeta_{(\hat{\mu}\lambda_\mu)\uparrow_{P_\mu}^P})\uparrow_C^{\tilde{N}} \quad (\mu \in \mathcal{M},\ \lambda_\mu \in \mathrm{Irr}(R_\mu)).$$

Since $\tilde{c}$ is the unique block of $\tilde{N}$ covering $c$, the above implies the following:

**Theorem 2.2.**

$$\mathrm{Irr}(\tilde{c}) = \{\tilde{\zeta}_{i,\lambda} \mid \lambda \in \mathrm{Irr}(R),\ 1 \le i \le e\} \cup \bigcup_{\mu \in \mathcal{M}} \{\tilde{\zeta}_{\mu,\lambda_\mu} \mid \lambda_\mu \in \mathrm{Irr}(R_\mu)\}.$$

We note that $\tilde{\zeta}_{i,\lambda}$ is of height 0 and $\tilde{\zeta}_{\mu,\lambda_\mu}$ is of height $h_\mu$.

We denote by $(P, b_P)_c$ (resp. $(P, b_P)_{\tilde{c}}$, $(P, b_P)_{\tilde{b}}$) the pair $(P, b_P)$ regarded as a maximal $c$ (resp. $\tilde{c}$, $\tilde{b}$)-Brauer pair to avoid confusion. For each $S \le P$, let $(S, c_S) \subset (P, b_P)_c$, $(S, \tilde{c}_S) \subset (P, b_P)_{\tilde{c}}$ and $(S, \tilde{b}_S) \subset (P, b_P)_{\tilde{b}}$. Similarly, for each $u \in P$, let $(u, c_u) \in (P, b_P)_c$, $(u, \tilde{c}_u) \in (P, b_P)_{\tilde{c}}$ and $(u, \tilde{b}_u) \in (P, b_P)_{\tilde{b}}$.

**Lemma 2.3.** *For $S \le P$, $\tilde{c}_S$ is the unique block of $C_{\tilde{N}}(S)$ covering $c_S$ and $\tilde{b}_S$ is the unique block of $C_{\tilde{G}}(S)$ covering $\tilde{c}_S$.*

**Proof.** We show by induction on $|P:S|$. When $S=P$, the statement is clear. Let $S \lhd T \leq P$. Then $\mathrm{Br}_T(c_S\tilde{c}_S\tilde{b}_S)\overline{c_T\tilde{c}_T\tilde{b}_T} = \overline{c_T\tilde{c}_T\tilde{b}_T}$ by [21] Theorem 40.4. (Here $^-: \mathcal{O}G \to kG$ is the canonical epimorphism.) By the induction hypothesis we have $c_T\tilde{c}_T\tilde{b}_T \neq 0$, and so $c_S\tilde{c}_S\tilde{b}_S \neq 0$. Hence $\tilde{c}_S$ covers $c_S$ and $\tilde{b}_S$ covers $\tilde{c}_S$. For the uniqueness, note $Q_1 \trianglelefteq \tilde{G}$ and [13] Theorem 5.2.8 (ii). □

**Lemma 2.4.** *Let $\lambda$ be an $L$-invariant generalized character of $P$.*

(i) *For any $\tilde{\zeta} \in \mathrm{Irr}(\tilde{c})$,*

$$(\tilde{\zeta} * \lambda)\downarrow^{\tilde{N}}_{C} = (\tilde{\zeta}\downarrow^{\tilde{N}}_{C}) * \lambda.$$

(ii) *For any $\zeta \in \mathrm{Irr}(c)$,*

$$(\zeta * \lambda)\uparrow^{\tilde{N}}_{C} = (\zeta\uparrow^{\tilde{N}}_{C}) * \lambda.$$

(iii) *For any $\tilde{\zeta} \in \mathrm{Irr}(\tilde{c})$,*

$$(\tilde{\zeta} * \lambda)\uparrow^{\tilde{G}}_{\tilde{N}} = (\tilde{\zeta}\uparrow^{\tilde{G}}_{\tilde{N}}) * \lambda.$$

**Proof.** First of all we note that, for $u, v \in P$, if $u = v^h$ and $\tilde{c}_u$ covers $c_v^h$ for some $h \in \tilde{N}$, then $\lambda(v) = \lambda(u)$. In fact, the condition implies $(v, \tilde{c}_v)^h = (u, \tilde{c}_u)$ by Lemma 2.3, and hence $u$ and $v$ are $L$-conjugate.

(i) If $(\tilde{\zeta}^{(u,\tilde{c}_u)}\downarrow^{\tilde{N}}_{C})^{(v,c_v)} \neq 0$, then $\lambda(u) = \lambda(v)$. In fact, by the assumption there exists some $h \in \tilde{N}$ and $s \in C_C(v)_{p'}$ such that $u = v^h$ and

$$0 \neq |s^{C_C(v)}|(\tilde{\zeta}^{(u,\tilde{c}_u)}\downarrow^{\tilde{N}}_{C})^{(v,c_v)}(vs) = \tilde{\zeta}^{(u,\tilde{c}_u)}(v\widehat{s^{C_C(v)}}c_v) = \tilde{\zeta}\big(u(\widehat{s^{C_C(v)}}c_v)^h\tilde{c}_u\big).$$

Then we have

$$(\tilde{\zeta} * \lambda)\downarrow^{\tilde{N}}_{C} = \sum_{v\in\pi}\sum_{u\in\Pi}\lambda(u)(\tilde{\zeta}^{(u,\tilde{c}_u)}\downarrow^{\tilde{N}}_{C})^{(v,c_v)} = \sum_{v\in\pi}\lambda(v)(\tilde{\zeta}\downarrow^{\tilde{N}}_{C})^{(v,c_v)} = (\tilde{\zeta}\downarrow^{\tilde{N}}_{C}) * \lambda$$

where $\pi$ is a set of representatives for the conjugacy classes of $P$.

(ii) If $(\zeta^{(v,c_v)}\uparrow^{\tilde{N}}_{C})^{(u,\tilde{c}_u)} \neq 0$, then $\lambda(v) = \lambda(u)$. In fact, by the assumption there exists some $s \in C_{\tilde{N}}(u)_{p'}$ such that $(\zeta^{(v,c_v)}\uparrow^{\tilde{N}}_{C})(u\widehat{s^{C_{\tilde{N}}(u)}}\tilde{c}_u) \neq 0$, so we have $s \in C$ and there is $h \in \tilde{N}$ such that $v = u^h$ and

$$0 \neq \zeta^{(v,c_v)}\big((u\widehat{s^{C_{\tilde{N}}(u)}}\tilde{c}_u)^h\big) = \zeta\big(v(\widehat{s^{C_{\tilde{N}}(u)}}\tilde{c}_u)^h c_v\big).$$

Then we have

$$(\zeta * \lambda)\uparrow^{\tilde{N}}_{C} = \sum_{u\in\Pi}\sum_{v\in\pi}\lambda(v)(\zeta^{(v,c_v)}\uparrow^{\tilde{N}}_{C})^{(u,\tilde{c}_u)} = \sum_{u\in\Pi}\lambda(u)(\zeta\uparrow^{\tilde{N}}_{C})^{(u,\tilde{c}_u)} = (\zeta\uparrow^{\tilde{N}}_{C}) * \lambda.$$

(iii) Similar as (ii). □

Set

$$\tilde{\zeta}_i = \tilde{\zeta}_{i,1_R} \in \mathrm{Irr}(\tilde{c}) \;\; (i = 1, 2, \cdots, e), \;\; i.e., \; \tilde{\zeta}_i \text{ are the extensions of } \zeta_0 \text{ to } \tilde{N},$$
$$\tilde{\zeta}_\mu = \tilde{\zeta}_{\mu,1_{R_\mu}} = (\zeta_{\hat{\mu}\uparrow_{P_\mu}^P})\uparrow_C^{\tilde{N}} \in \mathrm{Irr}(\tilde{c}) \quad (\mu \in \mathcal{M}),$$
$$\tilde{\chi}_i = \tilde{\zeta}_i \uparrow_{\tilde{N}}^{\tilde{G}} \in \mathrm{Irr}(\tilde{b}) \quad (i = 1, 2, \cdots, e),$$
$$\tilde{\chi}_\mu = \tilde{\zeta}_\mu \uparrow_{\tilde{N}}^{\tilde{G}} \in \mathrm{Irr}(\tilde{b}) \quad (\mu \in \mathcal{M}).$$

**From now on we assume $R$ is abelian.** Since any $\lambda \in \hat{R} \subseteq \mathrm{Irr}(P)$ is $L$-invariant, $\hat{R}$ acts on $\mathrm{Irr}(\tilde{c})$, $\mathrm{Irr}(\tilde{b})$ and $\mathrm{Irr}(b)$ respectively, via $*$-construction. For $\chi \in \mathrm{Irr}(\tilde{b}) \cup \mathrm{Irr}(b)$, we denote by $\mathcal{O}(\chi)$ the $\hat{R}$-orbit of $\chi$.

**Proposition 2.5.** (i) *For any $\lambda \in \hat{R}$, $\tilde{\zeta}_i * \lambda$ $(i = 1, 2, \cdots, e)$ are the extensions of $\zeta_\lambda$.*

(ii) *For any $\mu \in \mathcal{M}$ and $\lambda \in \hat{R}$,*

$$\tilde{\zeta}_\mu * \lambda = \tilde{\zeta}_{\mu,\lambda\downarrow_{R_\mu}^R}.$$

*In particular, $\tilde{\zeta}_\mu * \lambda = \tilde{\zeta}_\mu$ if and only if $\lambda \in R_\mu^\perp$, and so $\mathcal{O}(\tilde{\zeta}_\mu) = \{\tilde{\zeta}_\mu * \lambda \mid \lambda \in \widehat{R_\mu}\}$.*

(iii) *For any $\mu \in \mathcal{M}$ and $\lambda \in \hat{R}$,*

$$(\tilde{\zeta}_\mu * \lambda)\uparrow_{\tilde{N}}^{\tilde{G}} = \tilde{\chi}_\mu * \lambda.$$

*In particular, $\tilde{\chi}_\mu * \lambda = \tilde{\chi}_\mu$ if and only if $\lambda \in R_\mu^\perp$, and so $\mathcal{O}(\tilde{\chi}_\mu) = \{\tilde{\chi}_\mu * \lambda \mid \lambda \in \widehat{R_\mu}\}$.*

**Proof.** (i) We have

$$\zeta_\lambda \uparrow_C^{\tilde{N}} = (\zeta_0 * \lambda)\uparrow_C^{\tilde{N}} = (\zeta_0 \uparrow_C^{\tilde{N}}) * \lambda = (\sum_{i=1}^{e} \tilde{\zeta}_i) * \lambda = \sum_{i=1}^{e} (\tilde{\zeta}_i * \lambda)$$

by Lemma 2.4 (ii). This implies (i).

(ii) By Lemma 2.4 (ii) and [13] Theorem 3.2.14 (i),

$$\tilde{\zeta}_\mu * \lambda = \left((\zeta_0 * \hat{\mu}\uparrow_{P_\mu}^P)\uparrow_C^{\tilde{N}}\right) * \lambda = \left(\zeta_0 * \left(\hat{\mu}(\lambda\downarrow_{P_\mu}^P)\right)\uparrow_{P_\mu}^P\right)\uparrow_C^{\tilde{N}} = \tilde{\zeta}_{\mu,\lambda\downarrow_{R_\mu}^R}.$$

(iii) This follows from Lemma 2.4 (iii) and (ii). □

By Theorem 2.2 and Proposition 2.5, we have the following:

**Theorem 2.6.**

$$\mathrm{Irr}(\tilde{b}) = \bigcup_{i=1}^{e} \{\tilde{\chi}_i * \lambda \;\mid \lambda \in \hat{R}\} \cup \bigcup_{\mu \in \mathcal{M}} \{\tilde{\chi}_\mu * \lambda_\mu \mid \lambda_\mu \in \widehat{R_\mu}\}.$$

We note that $\tilde{\chi}_i * \lambda$ is of height 0, and $\tilde{\chi}_\mu * \lambda_\mu$ is of height $h_\mu$.

## 3. A linear isometry from $\tilde{b}$ to $b$

All notation in previous sections are kept in the following sections. Set

$$X_{\mathcal{K}}(G,b;Q\backslash\{1\}) = \bigoplus_{u\in\Pi\cap(Q\backslash\{1\})} X_{\mathcal{K}}^{(u,b_u)}(G,b).$$

We shall obtain a linear isometry from $X_{\mathcal{K}}(\tilde{G},\tilde{b};Q\backslash\{1\})$ onto $X_{\mathcal{K}}(G,b;Q\backslash\{1\})$ in Theorem 3.3 below, which is a crucial tool to determine $\mathrm{Irr}(b)$. Note that $C_G(u)\le C$ and $b_u(=\tilde{b}_u)$ is nilpotent for $u\in Q\backslash\{1\}$.

We see

$$\begin{aligned}&T_u = \{x\in\tilde{G}\mid \text{the } p\text{-part of } x \text{ is } \tilde{G}\text{-conjugate to } u\,\}\quad (u\in Q\backslash\{1\})\\ &\text{is a T.I. set in } G \text{ with normalizer } \tilde{G}.\end{aligned}\tag{3.1}$$

**Lemma 3.1.** *Let $u\in\Pi\cap(Q\backslash\{1\})$ and $\tilde{\chi}\in\mathrm{Irr}(\tilde{b})$. We have*

$$\tilde{\chi}^{(u,\tilde{b}_u)}\uparrow_{\tilde{G}}^{G} = (\tilde{\chi}^{(u,\tilde{b}_u)}\uparrow_{\tilde{G}}^{G})^{(u,b_u)}.$$

**Proof.** Assume $(\tilde{\chi}^{(u,\tilde{b}_u)}\uparrow_{\tilde{G}}^{G})^{(v,f)}\neq 0$ for a Brauer element $(v,f)$ of $G$. Then we may assume $v=u$. Let $s\in C_G(u)_{p'}$ be such that $(\tilde{\chi}^{(u,\tilde{b}_u)}\uparrow_{\tilde{G}}^{G})^{(u,f)}(us)\neq 0$. From (3.1) we have

$$0\neq(\tilde{\chi}^{(u,\tilde{b}_u)}\uparrow_{\tilde{G}}^{G})(u\widehat{s^{C_G(u)}}f) = \tilde{\chi}^{(u,\tilde{b}_u)}(u\widehat{s^{C_G(u)}}f).$$

Hence $f=\tilde{b}_u=b_u$, and this completes the proof. □

For each $\mu\in\mathcal{M}$, we set

$$\tilde{\rho}_\mu = \sum_{\lambda_\mu\in\widehat{R_\mu}}\tilde{\chi}_\mu*\lambda_\mu - \sum_{i=1}^{e}\Big(\sum_{\lambda\in\hat{R}}\tilde{\chi}_i*\lambda\Big),\tag{3.2}$$

$$\rho_\mu = \tilde{\rho}_\mu\uparrow_{\tilde{G}}^{G}.\tag{3.3}$$

**Lemma 3.2.** *$\{\tilde{\rho}_\mu \mid \mu\in\mathcal{M}\}$ is a $\mathcal{K}$-basis of $X_{\mathcal{K}}(\tilde{G},\tilde{b};Q\backslash\{1\})$.*

**Proof.** From (3.2),

$$p^{h_\mu}\tilde{\rho}_\mu = \sum_{\lambda\in\hat{R}}\big(\tilde{\chi}_\mu - p^{h_\mu}\sum_{i=1}^{e}\tilde{\chi}_i\big)*\lambda = |R|\sum_{u\in\Pi\cap Q}\big(\tilde{\chi}_\mu - p^{h_\mu}\sum_{i=1}^{e}\tilde{\chi}_i\big)^{(u,\tilde{b}_u)}$$

by the second orthogonality relation for $R$. On the other hand, since $\tilde{\chi}_\mu = \big(\zeta_0*(\hat{\mu}\uparrow_{P_\mu}^{P})\big)\uparrow_{C}^{\tilde{G}}$ and $\sum_{i=1}^{e}\tilde{\chi}_i = \zeta_0\uparrow_{C}^{\tilde{G}}$, we have

$$\tilde{\chi}_\mu = p^{h_\mu} \sum_{i=1}^{e} \tilde{\chi}_i \text{ on } \tilde{G}_{p'}.$$

Hence $\tilde{\rho}_\mu \in X_{\mathcal{K}}(\tilde{G}, \tilde{b}; Q\backslash\{1\})$. Moreover, clearly $\tilde{\rho}_\mu$ $(\mu \in \mathcal{M})$ are linearly independent over $\mathcal{K}$ and $|\mathcal{M}| = |\Pi \cap (Q\backslash\{1\})|$. Therefore $\{\tilde{\rho}_\mu \mid \mu \in \mathcal{M}\}$ forms a $\mathcal{K}$-basis of $X_{\mathcal{K}}(\tilde{G}, \tilde{b}; Q\backslash\{1\})$. □

**Theorem 3.3.** *The induction from $\tilde{G}$ to $G$ gives a $\mathcal{K}$-linear isometry*

$$\vartheta : X_{\mathcal{K}}(\tilde{G}, \tilde{b}; Q\backslash\{1\}) \cong X_{\mathcal{K}}(G, b; Q\backslash\{1\}).$$

*Moreover $\{\rho_\mu \mid \mu \in \mathcal{M}\}$ is a $\mathcal{K}$-basis of $X_{\mathcal{K}}(G, b; Q\backslash\{1\})$.*

**Proof.** At first we note $\dim_{\mathcal{K}} X_{\mathcal{K}}(G, b; Q\backslash\{1\}) = |\Pi \cap (Q\backslash\{1\})| = \dim_{\mathcal{K}} X_{\mathcal{K}}(\tilde{G}, \tilde{b}; Q\backslash\{1\})$. By Lemma 3.1, $\vartheta$ is well-defined. From (3.1) we see that $\vartheta$ preserves the inner products (cf. [6] Theorem 12.1 (Brauer-Suzuki)). Set $X(\tilde{G}, \tilde{b}; Q\backslash\{1\}) = X(\tilde{G}, \tilde{b}) \cap X_{\mathcal{K}}(\tilde{G}, \tilde{b}; Q\backslash\{1\})$ and $X(G, b; Q\backslash\{1\}) = X(G, b) \cap X_{\mathcal{K}}(G, b; Q\backslash\{1\})$. Let $\vartheta_0$ be the restriction of $\vartheta$ to $X(\tilde{G}, \tilde{b}; Q\backslash\{1\})$. Then $\vartheta_0$ induces a map from $X(\tilde{G}, \tilde{b}; Q\backslash\{1\})$ into $X(G, b; Q\backslash\{1\})$ which is injective. Hence $\{\vartheta(\tilde{\rho}_\mu) \mid \mu \in \mathcal{M}\}$ is linearly independent over $\boldsymbol{Z}$ by Lemma 3.2. Since $X_{\mathcal{K}}(G, b) \cong \mathcal{K} \otimes_{\boldsymbol{Z}} X(G, b)$, $\{\vartheta(\tilde{\rho}_\mu) \mid \mu \in \mathcal{M}\}$ is linearly independent over $\mathcal{K}$. Hence $\vartheta$ is surjective and hence is a $\mathcal{K}$-linear isometry. This and (3.3) complete the proof. □

The following propositions will be used in the proof of Proposition 4.3 below.

**Proposition 3.4.** *For $\chi \in \mathrm{Irr}(b)$, there exists $\mu \in \mathcal{M}$ such that $(\rho_\mu, \chi) \neq 0$.*

**Proof.** We have $\chi^{(z,c)} \neq 0$ for $z \in Q_1\backslash\{1\}$ by [1] (4C) and we can write $\chi^{(z,c)} = \sum_{\mu \in \mathcal{M}} a_\mu \rho_\mu$ $(a_\mu \in \mathcal{K})$ by Theorem 3.3. Hence we have

$$\Big(\sum_{\mu \in \mathcal{M}} a_\mu \rho_\mu, \chi\Big) = \frac{1}{|C|} \sum_{a \in C_{p'}} \chi^{(z,c)}(za)\chi((za)^{-1}) = \frac{1}{|C|} \sum_{a \in C_{p'}} |\chi^{(z,c)}(za)|^2 \neq 0.$$

This completes the proof. □

**Proposition 3.5.** *For $\mu \in \mathcal{M}$, $\chi \in \mathrm{Irr}(b)$ and $\lambda \in \hat{R}$,*

$$(\rho_\mu,\ \chi * \lambda) = (\rho_\mu,\ \chi).$$

**Proof.** Since $\rho_\mu \in X_{\mathcal{K}}(G, b; Q\backslash\{1\})$ we have $\rho_\mu * \lambda = \rho_\mu$, and $(\rho_\mu, \chi) = (\rho_\mu * \lambda, \chi * \lambda) = (\rho_\mu, \chi * \lambda)$. □

## 4. Irreducible characters in a block with metacyclic defect group

**From now we consider the case where $p$ is odd and $P$ is metacyclic.** In the case where $P$ is non-abelian, using a theorem of fusion in [20], and an analysis of the automorphism group of $P$ in [19] or [7], we see that $Q$ is cyclic and the assumption $Q \neq 1$ implies $P$ is split. In the case where $P$ is abelian, recall that we are assuming $Q$ is non-trivial cyclic. Hence we may assume that

$$\begin{aligned} &P = \langle\, x, y \mid x^{p^m} = y^{p^n} = 1,\ yxy^{-1} = x^{1+p^l}\,\rangle,\ Q = \langle x \rangle,\ R = \langle y \rangle \\ &\text{where}\ \ m \geq 1,\ n \geq 1,\ l \geq 1,\ 0 \leq m - l \leq n. \end{aligned} \tag{4.1}$$

Our purpose of this section is to determine $\mathrm{Irr}(b)$ (Theorem 4.4 below).

Concerning the action of $y$ on $x$, we note that for an odd $p$ and an integer $c$ such that $p \nmid c$, we have

$$(1 + cp^l)^{p^i} = 1 + c'p^{l+i} \text{ for some } c' \text{ such that } c' \equiv c \pmod{p}.$$

The kernel of the action of $R$ on $Q$ is $\langle y^{p^{m-l}} \rangle$, that is, $R/\langle y^{p^{m-l}} \rangle$ is isomorphic to a subgroup of $\mathrm{Aut}(Q)$ of order $p^{m-l}$, and $R/\langle y^{p^{m-l}} \rangle$ is also isomorphic to a subgroup of $\mathrm{Aut}(\hat{Q})$ of order $p^{m-l}$ as $\mu^y = \mu^{1+p^l}$ for $\mu \in \hat{Q}$.

Set

$$\begin{gathered} R_i = \langle y^{p^i} \rangle \leq R, \quad P_i = Q \rtimes R_i \quad (0 \leq i \leq n), \\ \mathcal{M}_i = \{\mu \in \mathcal{M} \mid R_\mu = R_i\}, \quad m_i = |\mathcal{M}_i| \quad (0 \leq i \leq m - l). \end{gathered}$$

Then $\mathcal{M} = \bigcup_{i=0}^{m-l} \mathcal{M}_i$ and we see

$$m_0 = \frac{p^l - 1}{e} \quad \text{and} \quad m_i = \frac{p^l - p^{l-1}}{e} \ (1 \leq i \leq m - l). \tag{4.2}$$

We have $l(b) = e$ and $k(b) = k(b_0)$ where $b_0 = b_P^{N_G(P,b_P)}$, see [23] Theorem 1. Since $k_0(b_0) = (\frac{p^l-1}{e}+e)p^n$, $k_i(b_0) = \frac{p^l-p^{l-1}}{e}p^{n-i}$ $(1 \leq i \leq m-l)$ and $k_i(b_0) = 0$ $(i > m-l)$ from (4.2) and Theorem 2.6 in the case $G = N_G(P, b_P)$, we have $k(b_0) = (\frac{p^l+p^{l-1}-p^{2l-m-1}-1}{e} + e)p^n$. Therefore

$$k(b) = \Big(\frac{p^l + p^{l-1} - p^{2l-m-1} - 1}{e} + e\Big)p^n, \tag{4.3}$$

see [16] Theorem 8.8.

Set

$$\Pi_0 = \Pi \cap \big(P \backslash (Q\langle y^p \rangle)\big), \quad \Pi_i = \Pi \cap \big(Q\langle y^{p^i} \rangle \backslash Q\langle y^{p^{i+1}} \rangle\big)\ (1 \leq i \leq m - l).$$

We remark $\Pi_{m-l}$ is empty if $m - l = n$.

**Lemma 4.1.** (i) *Let* $u \in P$ *with* $u \notin_P R$. *Then* $(u^P)^a \neq u^P$ *for any* $a \in E \backslash \{1\}$.

(ii) $|\Pi_0| = (1 + \frac{p^l - 1}{e})(p^n - p^{n-1})$ *and* $\sum_{u \in \Pi_0} l(b_u) = (e + \frac{p^l - 1}{e})(p^n - p^{n-1})$.

(iii) *Assume that* $1 \leq i \leq m - l$ *and* $i < n$. *Then*

$$|\Pi_i| = (1 + \frac{p^l - 1}{e} + i\frac{p^l - p^{l-1}}{e})(p^{n-i} - p^{n-i-1}) \text{ and}$$
$$\sum_{u \in \Pi_i} l(b_u) = (e + \frac{p^l - 1}{e} + i\frac{p^l - p^{l-1}}{e})(p^{n-i} - p^{n-i-1}).$$

**Proof.** (i) Note that $(u^P)^a = (u^a)^P$ is a conjugacy class of $P$. Now suppose that $u$ and $u^a$ are $P$-conjugate for some $a \in E \backslash \{1\}$. Then $\langle a \rangle$ acts on $u^P$ by conjugation, and there is $u' \in u^P$ such that $u'^a = u'$ by a lemma of Glauberman. Hence we have $u^{v\hat{a}} = u^v$ and so $u^{v\hat{a}v^{-1}} = u$ for some $v \in P$ where $\hat{a}$ is an inverse image of $a$ in $N_G(P, b_P)$. This gives a contradiction by [23] Lemma 4(i).

(ii) For any $y' \in \langle y \rangle \backslash \langle y^p \rangle$, we have

$$Qy' = \bigcup_s x^s \langle x^{p^l} \rangle y'$$

where unions are disjoint and $s$ ranges over the integers such that $0 \leq s \leq p^l - 1$. Let $y' = y^j$ where $p \nmid j$. From the relation in (4.1), we have

$$(x^k y^{j'})(x^s y')(x^k y^{j'})^{-1} = x^{s(1+p^l)^{j'}} \cdot x^{k(1-(1+p^l)^j)} \cdot y' \in x^s \langle x^{p^l} \rangle y'$$

for any $k$ and $j'$. Note that $\langle x^{1-(1+p^l)^j} \rangle = \langle x^{p^l} \rangle$. Hence we have $(x^s y')^P = x^s \langle x^{p^l} \rangle y'$, and

$$\{y'\} \cup \{x^s y' \mid 1 \leq s \leq p^l - 1\}$$

is a set of representatives for the $P$-conjugacy classes of the $L$-invariant subset $Qy'$ of $P$. Then the statement follows from (i).

(iii) For any $y' \in \langle y^{p^i} \rangle \backslash \langle y^{p^{i+1}} \rangle$, we see

$$Qy' = \langle x^{p^i} \rangle y' \cup \bigcup_{v=0}^{i-1} \left( \langle x^{p^v} \rangle y' \setminus \langle x^{p^{v+1}} \rangle y' \right),$$

and

$$\langle x^{p^i} \rangle y' = \bigcup_s x^{sp^i} \langle x^{p^{l+i}} \rangle y', \quad \langle x^{p^v} \rangle y' \setminus \langle x^{p^{v+1}} \rangle y' = \bigcup_t x^{tp^v} \langle x^{p^{l+v}} \rangle y' \quad (0 \leq v \leq i - 1)$$

where unions are disjoint, $s$ ranges over the integers such that $0 \leq s \leq p^l - 1$ and $t$ ranges over the integers such that $0 \leq t \leq p^l - 1$ and $p \nmid t$. Let $y' = y^{p^i j}$ where $p \nmid j$. From

$$(x^k y^{j'})(x^{sp^i} y')(x^k y^{j'})^{-1} = x^{sp^i(1+p^l)^{j'}} \cdot x^{k(1-(1+p^l)^{p^i j})} \cdot y' \in x^{sp^i} \langle x^{p^{l+i}} \rangle y',$$

we have $(x^{sp^i} y')^P = x^{sp^i} \langle x^{p^{l+i}} \rangle y'$. Note that $\langle x^{(1-(1+p^l)^{p^i j})} \rangle = \langle x^{p^{l+i}} \rangle$. Also from

$$(x^k y^{j'})(x^{tp^v} y')(x^k y^{j'})^{-1} = x^{tp^v(1+p^l)^{j'}} \cdot x^{k(1-(1+p^l)^{p^i j})} \cdot y' \in x^{tp^v} \langle x^{p^{l+v}} \rangle y',$$

we have $(x^{tp^v} y')^P = x^{tp^v} \langle x^{p^{l+v}} \rangle y'$. Note that $\{x^{tp^v(1+p^l)^{j'}} \mid j' \text{ ranges over integers}\} = x^{tp^v} \langle x^{p^{l+v}} \rangle$ since we see $x^{tp^v(1+p^l)^{j'}} = x^{tp^v(1+p^l)^{j''}}$ if and only if $j' \equiv j'' \pmod{p^{m-l-v}}$. Hence

$$\{y'\} \cup \{x^{sp^i} y' \mid 1 \le s < p^l\} \cup \bigcup_{v=0}^{i-1} \{x^{tp^v} y' \mid 0 \le t < p^l,\ p \nmid t\}$$

is a set of representatives for the $P$-conjugacy classes of the $L$-invariant subset $Qy'$ of $P$. Then the statement follows from (i). $\square$

Let $z = y^{p^{m-l}} \in Z(P)$. Then $\chi^{(z,b_z)} \neq 0$ for any $\chi \in \mathrm{Irr}(b)$ by [1] (4C). Hence, if $\chi * \lambda = \chi$ for $\lambda \in \hat{R}$, then $\lambda \in R_{m-l}^{\perp}$ and so $|\mathcal{O}(\chi)| \ge p^{n-(m-l)}$. Let

$$\mathrm{Irr}'_i(b) = \{\chi \in \mathrm{Irr}(b) \mid |\mathcal{O}(\chi)| = p^{n-i}\} \text{ for } 0 \le i \le m-l.$$

(In fact, $\mathrm{Irr}'_i(b)$ coincides with the set $\mathrm{Irr}_i(b)$ of irreducible characters in $b$ with height $i$ by Proposition 5.8 below.) For $\chi \in \mathrm{Irr}(b)$ and $i$ where $0 \le i \le m-l$, $\chi \in \mathrm{Irr}'_i(b)$ if and only if $\chi^{(u,b_u)} = 0$ for all $u \in \cup_{j=0}^{i-1} \Pi_j$. Hence a table

$$\left(\chi^{(u,b_u)}\right)_{\chi \in \mathrm{Irr}(b), u \in \cup_{j=0}^{m-l} \Pi_j}$$

is of the form as follows:

| | $\Pi_0$ | $\Pi_1$ | $\Pi_2$ | $\cdots$ | $\Pi_{m-l-1}$ | $\Pi_{m-l}$ |
|---|---|---|---|---|---|---|
| $\mathrm{Irr}'_0(b)$ | $*$ | $*$ | $*$ | $\cdots$ | $*$ | $*$ |
| $\mathrm{Irr}'_1(b)$ | 0 | $*$ | $*$ | $\cdots$ | $*$ | $*$ |
| $\mathrm{Irr}'_2(b)$ | 0 | 0 | $*$ | $\cdots$ | $*$ | $*$ |
| $\vdots$ | 0 | 0 | 0 | $\cdots$ | $*$ | $*$ |
| $\mathrm{Irr}'_{m-l}(b)$ | 0 | 0 | 0 | $\cdots$ | 0 | $*$ |

(4.4)

Let

$$\tilde{\mathrm{Irr}}'_i(b) \text{ be a set of representatives for the elements of } \mathrm{Irr}'_i(b) \text{ under } \hat{R}\text{-action},$$

and let

$$n_i = n_i(b) = |\tilde{\mathrm{Irr}}'_i(b)|.$$

If $\mu \in \mathcal{M}_i$, then $|\mathcal{O}(\tilde{\chi}_\mu)| = p^{n-i}$ by Proposition 2.5(iii). Hence $n_0(\tilde{b}) = m_0 + e$ and $n_i(\tilde{b}) = m_i$ $(1 \leq i \leq m - l)$ by Theorem 2.6. This holds for $b$ too, see Proposition 4.3 below.

**Lemma 4.2.** (i) $n_0 \geq e + m_0$.

(ii) *Let $m - l \geq 1$. If $n_0 = e + m_0$, then $n_1 \geq m_1$.*

(iii) *Let $m - l \geq 2$ and $i$ be such that $2 \leq i \leq m - l$. If $n_0 = e + m_0$ and $n_j = m_j$ for any $j$ where $1 \leq j \leq i - 1$, then $n_i \geq m_i$.*

**Proof.** (i) By the table (4.4), we have $\dim_{\mathcal{K}}\big(\bigoplus_{u\in\Pi_0} X_{\mathcal{K}}^{(u,b_u)}(G,b)\big) \leq n_0 p^n$. In fact,

$$\dim_{\mathcal{K}}\big(\bigoplus_{u\in\Pi_0} X_{\mathcal{K}}^{(u,b_u)}(G,b)\big) \leq n_0 p^{n-1}(p-1)$$

since the $R_1^\perp$-orbit sum of $\chi \in \mathrm{Irr}'_0(b)$ vanishes on $\Pi_0$. On the other hand,

$$\dim_{\mathcal{K}}\big(\bigoplus_{u\in\Pi_0} X_{\mathcal{K}}^{(u,b_u)}(G,b)\big) = (e + \frac{p^l - 1}{e})(p^n - p^{n-1})$$

by Lemma 4.1(ii). Hence, we have

$$(e + m_0)(p^n - p^{n-1}) \leq n_0(p^n - p^{n-1}),$$

and so (i) follows.

(ii) At first we consider the case $n = 1$. Then $m - l = 1$. From (4.2) and (4.3), we have

$$n_0 p + n_1 = \big(\frac{p^l + p^{l-1} - p^{l-2} - 1}{e} + e\big)p = (e + m_0)p + m_1.$$

Hence by the assumption, we have $n_1 = m_1$.

Next assume $n > 1$. By the table (4.4), we have $\dim_{\mathcal{K}}\big(\bigoplus_{u\in\Pi_0\cup\Pi_1} X_{\mathcal{K}}^{(u,b_u)}(G,b)\big) \leq n_0 p^n + n_1 p^{n-1}$. In fact,

$$\dim_{\mathcal{K}}\big(\bigoplus_{u\in\Pi_0\cup\Pi_1} X_{\mathcal{K}}^{(u,b_u)}(G,b)\big) \leq n_0 p^{n-2}(p^2-1) + n_1 p^{n-2}(p-1)$$

since the $R_2^\perp$-orbit sum of $\chi \in \mathrm{Irr}'_j(b)$ $(j = 0, 1)$ vanishes on $\Pi_0 \cup \Pi_1$. On the other hand,

$$\begin{aligned}\dim_{\mathcal{K}}\big(\bigoplus_{u\in\Pi_0\cup\Pi_1} X_{\mathcal{K}}^{(u,b_u)}(G,b)\big) &= (e + \frac{p^l - 1}{e})(p^n - p^{n-1}) \\ &\quad + (e + \frac{p^l - 1}{e} + \frac{p^l - p^{l-1}}{e})(p^{n-1} - p^{n-2})\end{aligned}$$

by Lemma 4.1(ii) and Lemma 4.1(iii) for $i = 1$. Hence, we have

$$(e + m_0)(p^n - p^{n-2}) + m_1(p^{n-1} - p^{n-2}) \leq n_0(p^n - p^{n-2}) + n_1(p^{n-1} - p^{n-2}),$$

and so (ii) follows by the assumption.

(iii) We can show similarly. □

**Proposition 4.3.** (i)

$$n_0 = e + m_0 = e + \frac{p^l - 1}{e}.$$

(ii)

$$n_i = m_i = \frac{p^l - p^{l-1}}{e} \quad (1 \leq i \leq m - l).$$

**Proof.** In the proof, $\mu_i$, $\mu_i'$ and $\mu_i''$ are elements in $\mathcal{M}_i$ $(0 \leq i \leq m - l)$. Set

$$\rho_{\mu,\mu'} = \rho_\mu - \rho_{\mu'} \quad \text{for } \mu, \mu' \in \mathcal{M}.$$

From (3.2), (3.3) and Theorem 3.3,

$$(\rho_{\mu_i}, \rho_{\mu_j}) = \delta_{ij} p^{n-i} + ep^n.$$

If $\mu_i$, $\mu_i'$ and $\mu_i''$ are different from each other, then

$$(\rho_{\mu_i}, \rho_{\mu_i,\mu_i'}) = p^{n-i}, \; (\rho_{\mu_i,\mu_i'}, \rho_{\mu_i,\mu_i'}) = 2p^{n-i}, (\rho_{\mu_i,\mu_i'}, \rho_{\mu_i,\mu_i''}) = p^{n-i}.$$

If $i \neq j$, then

$$(\rho_{\mu_i,\mu_i'}, \rho_{\mu_j,\mu_j'}) = 0, \quad (\rho_{\mu_i,\mu_j}, \rho_{\mu_i,\mu_j}) = p^{n-i} + p^{n-j}.$$

Moreover, if $\mu_i \neq \mu_i'$ and $i \neq j$, then,

$$(\rho_{\mu_i,\mu_i'}, \rho_{\mu_i,\mu_j}) = p^{n-i}.$$

These equations are used repeatedly in the proof.

We note that $m_0 = 1$, if and only if $m_i = 1$ for all $0 \leq i \leq m - l$, if and only if $l = 1$ and $e = p - 1$ from (4.2).

At first, we consider the case $m_i > 1$.

For $\mu_0, \mu_0' \in \mathcal{M}_0$, $\rho_{\mu_0,\mu_0'}$ has at most two constituents in $\tilde{\mathrm{Irr}}_0'(b)$ from Proposition 3.5 and $(\rho_{\mu_0,\mu_0'}, \rho_{\mu_0,\mu_0'}) = 2p^n$. We show

$$\text{there exists } \mu_0, \mu_0' \in \mathcal{M}_0 \text{ such that } \rho_{\mu_0,\mu_0'} \text{ consists of two elements in } \tilde{\mathrm{Irr}}_0'(b). \tag{4.5}$$

Assume (4.5) does not hold and let $\mu_0 \in \mathcal{M}_0$. Since $(\rho_{\mu_0}, \rho_{\mu_0}) < n_0 p^n$ from $m_0 > 1$ and Lemma 4.2(i), there exists $\chi \in \tilde{\mathrm{Irr}}_0'(b)$ such that $\chi$ does not appear in $\rho_{\mu_0}$, and $\chi$ appears in $\rho_{\mu_0,\mu}$ for some $\mu \in \mathcal{M}\backslash\{\mu_0\}$ by Proposition 3.4. Then $\chi$ is the unique element of $\tilde{\mathrm{Irr}}_0'(b)$ appearing in $\rho_{\mu_0,\mu}$ by the assumption or the inequality $(\rho_{\mu_0,\mu}, \rho_{\mu_0,\mu}) < 2p^n$ where $\mu \in \mathcal{M}\backslash\mathcal{M}_0$. If some $\chi' \in \tilde{\mathrm{Irr}}_0'(b)\backslash\{\chi\}$ also does not appear in $\rho_{\mu_0}$, then similarly there is some $\mu' \in \mathcal{M}\backslash\{\mu_0, \mu\}$ such that $\chi'$ is the unique element of $\tilde{\mathrm{Irr}}_0'(b)$ appearing in $\rho_{\mu_0,\mu'}$, and then both the $\chi$ and $\chi'$ appear in $\rho_{\mu',\mu} = \rho_{\mu_0,\mu} - \rho_{\mu_0,\mu'}$, which gives a contradiction. Hence, any element of $\tilde{\mathrm{Irr}}_0'(b)\backslash\{\chi\}$ appears in $\rho_{\mu_0}$, and so $(n_0 - 1)p^n \leq (\rho_{\mu_0}, \rho_{\mu_0}) = (e+1)p^n$. Then by Lemma 4.2(i) we have $(\rho_0, \rho_0) = (n_0 - 1)p^n$ and $\rho_{\mu_0}$ consists of $n_0 - 1$ elements of $\tilde{\mathrm{Irr}}_0'(b)\backslash\{\chi\}$, which gives a contradiction since $(\rho_{\mu_0}, \rho_{\mu_0,\mu}) \neq 0$. Hence (4.5) holds. Below, let $\mu_0, \mu_0' \in \mathcal{M}_0$ be as in (4.5) and let $\rho_{\mu_0,\mu_0'}$ consist of two elements $\chi_{\mu_0}, \chi_{\mu_0'} \in \tilde{\mathrm{Irr}}_0'(b)$.

Set

$$A = \{\chi \in \tilde{\mathrm{Irr}}_0'(b) \,|\, \chi \text{ appears in } \rho_{\mu_0,\mu} \text{ for some } \mu \in \mathcal{M}\}.$$

Then we have

$$A = \{\chi \in \tilde{\mathrm{Irr}}_0'(b) \,|\, \chi \text{ appears in } \rho_{\mu_0,\mu} \text{ for some } \mu \in \mathcal{M}_0\}$$

since $(\rho_{\mu_0,\mu}, \rho_{\mu_0,\mu}) < 2p^n$ and $(\rho_{\mu_0,\mu}, \rho_{\mu_0,\mu_0'}) \neq 0$ for $\mu \in \mathcal{M}\backslash\mathcal{M}_0$. For $\mu \in \mathcal{M}_0\backslash\{\mu_0, \mu_0'\}$ $\rho_{\mu_0,\mu}$ has at most one constituent $\chi_\mu$ in $\tilde{\mathrm{Irr}}_0'(b)\backslash\{\chi_{\mu_0}, \chi_{\mu_0'}\}$ since $(\rho_{\mu_0,\mu}, \rho_{\mu_0,\mu}) = 2p^n$ and $(\rho_{\mu_0,\mu}, \rho_{\mu_0,\mu_0'}) = p^n$. Hence we have $|A| \leq m_0$. Next, set

$$B = \{\chi \in \tilde{\mathrm{Irr}}_0'(b) \,|\, \chi \text{ appears in } \rho_{\mu_0}\}.$$

Then we have $|B| \leq e+1$ since $(\rho_{\mu_0}, \rho_{\mu_0}) = (e+1)p^n$. We may assume $\chi_{\mu_0} \in A \cap B$ since $(\rho_{\mu_0,\mu_0'}, \rho_{\mu_0}) \neq 0$, and we have $|A \cap B| \geq 1$. Since $\tilde{\mathrm{Irr}}_0'(b) = A \cup B$, we have

$$n_0 \leq m_0 + (e+1) - 1 = e + m_0.$$

Therefore we have $n_0 = e + m_0$ by Lemma 4.2(i) and above inequalities are equalities. Hence we see there exist $e$ characters $\chi_1, \cdots, \chi_e \in \tilde{\mathrm{Irr}}_0'(b)\backslash\{\chi_\mu \,|\, \mu \in \mathcal{M}_0\}$ and some signs $\epsilon, \epsilon_1, \cdots, \epsilon_e$ such that

$$\rho_{\mu_0} = \sum_{\lambda \in \widehat{R}} \epsilon(\chi_{\mu_0} * \lambda) - \sum_{j=1}^{e} \sum_{\lambda \in \widehat{R}} \epsilon_j(\chi_j * \lambda).$$

Moreover we see $\rho_{\mu_0,\mu}$ consists of $\chi_{\mu_0}$ and $\chi_\mu$ for $\mu \in \mathcal{M}_0\backslash\{\mu_0, \mu_0'\}$. Then for $\mu \in \mathcal{M}_0\backslash\{\mu_0\}$ we have

$$\rho_{\mu_0,\mu} = \sum_{\lambda \in \widehat{R}} \delta(\chi_{\mu_0} * \lambda) - \sum_{\lambda \in \widehat{R}} \delta(\chi_\mu * \lambda)$$

for some sign $\delta$ as $\rho_{\mu_0,\mu}(1) = 0$. Since $\rho_\mu = \rho_{\mu_0} - \rho_{\mu_0,\mu}$ and $(\rho_\mu, \rho_\mu) = (e+1)p^n$, we have $\epsilon = \delta$. Therefore

$$\begin{aligned} &\tilde{\mathrm{Irr}}_0'(b) = \{\chi_j \mid 1 \le j \le e\} \cup \{\chi_\mu \mid \mu \in \mathcal{M}_0\} \text{ and} \\ &\rho_\mu = \sum_{\lambda \in \widehat{R}} \epsilon(\chi_\mu * \lambda) - \sum_{j=1}^{e} \sum_{\lambda \in \widehat{R}} \epsilon_j(\chi_j * \lambda) \quad \text{for } \mu \in \mathcal{M}_0. \end{aligned} \tag{4.6}$$

Below let $\mu \in \mathcal{M} \backslash \mathcal{M}_0$. We show

$$(\rho_\mu, \chi_{\mu'}) = 0 \text{ for any } \mu' \in \mathcal{M}_0 \text{ and } (\rho_\mu, \chi_j) = -\epsilon_j \text{ for any } j\ (1 \le j \le e). \tag{4.7}$$

Let $\mu \in \mathcal{M}_i$. Then we have $(\rho_{\mu',\mu}, \rho_{\mu',\mu}) = p^n + p^{n-i}$. On the right side of this equation, $p^{n-i}$ comes from a constituent of $\rho_\mu$ and $p^n$ comes from a constituent of $\rho_{\mu'}$ by (4.6) and $(\rho_{\mu'}, \rho_{\mu'}) > (\rho_\mu, \rho_\mu)$. Hence the multiplicities of elements of $\tilde{\mathrm{Irr}}_0'(b)$ in $\rho_{\mu'}$ and in $\rho_\mu$ are the same except one element of $\tilde{\mathrm{Irr}}_0'(b)$ and the exception is $\chi_{\mu'}$ or $\chi_j$ for some $j$. The multiplicities of $\chi_{\mu'}$ and $\chi_{\mu''}$ in $\rho_\mu$ are the same for any $\mu'' \in \mathcal{M}_0$ from $(\rho_\mu, \rho_{\mu',\mu''}) = 0$ and (4.6). Hence if the exception is $\chi_j$ for some $j$, then we have $(\rho_\mu, \rho_\mu) \ge \{(e-1) + m_0\}p^n \ge (e+1)p^n$, which is a contradiction. Therefore the exception is $\chi_{\mu'}$, and (4.7) follows from $(\rho_\mu, \rho_\mu) = p^{n-1} + ep^n$.

Let $\chi \in \mathrm{Irr}_i'(b)$ where $i \ne 0$ and assume $\chi$ appears in $\rho_\mu$ for $\mu \in \mathcal{M}_j$. Note $j \ne 0$ by (4.6). If $j \ge i+1$, then $|\mathcal{O}(\chi)| \le p^{n-j} \le p^{n-(i+1)}$ from $(\rho_\mu, \rho_\mu) = p^{n-j} + ep^n$ and (4.7), which is a contradiction. Hence we have

$$\text{for } \chi \in \mathrm{Irr}_i'(b)\ (i \ne 0), \text{ there exists some } \mu \in \bigcup_{j=1}^{i} \mathcal{M}_j \text{ such that } \chi \text{ appears in } \rho_\mu. \tag{4.8}$$

For $\mu \in \mathcal{M}_1$, $\rho_\mu$ has at most one constituent $\chi_\mu$ in $\tilde{\mathrm{Irr}}_1'(b)$ and the multiplicity (when $\chi_\mu$ appears) is $\epsilon$ from $(\rho_\mu, \rho_\mu) = p^{n-1} + ep^n$, (4.7) and $\rho_{\mu,\mu_0}(1) = 0$. For any $\chi \in \tilde{\mathrm{Irr}}_1'(b)$ there exists some $\mu \in \mathcal{M}_1$ such that $\chi$ appears in $\rho_\mu$ by (4.8). Hence we have $n_1 \le m_1$. Therefore by Lemma 4.2(ii) we have $n_1 = m_1$ and the following:

$$\begin{aligned} &\tilde{\mathrm{Irr}}_1'(b) = \{\chi_\mu \mid \mu \in \mathcal{M}_1\} \text{ and} \\ &\rho_\mu = \sum_{\lambda_1 \in \widehat{R_1}} \epsilon(\chi_\mu * \lambda_1) - \sum_{j=1}^{e} \sum_{\lambda \in \widehat{R}} \epsilon_j(\chi_j * \lambda) \quad \text{for } \mu \in \mathcal{M}_1. \end{aligned} \tag{4.9}$$

Similarly, for $\mu \in \mathcal{M}_2$, $\rho_\mu$ has at most one constituent $\chi_\mu$ in $\tilde{\mathrm{Irr}}_2'(b)$ and the multiplicity (when $\chi_\mu$ appears) is $\epsilon$. For any $\chi \in \tilde{\mathrm{Irr}}_2'(b)$ there exists some $\mu \in \mathcal{M}_2$ such that $\chi$ appears in $\rho_\mu$ by (4.8) and (4.9). Hence we have $n_2 \le m_2$. Therefore by Lemma 4.2(iii) for $i = 2$ we have $n_2 = m_2$ and the following:

$$\tilde{\mathrm{Irr}}'_2(b) = \{\chi_\mu \mid \mu \in \mathcal{M}_2\} \text{ and}$$
$$\rho_\mu = \sum_{\lambda_2 \in \widehat{R_2}} \epsilon(\chi_\mu * \lambda_2) - \sum_{j=1}^{e} \sum_{\lambda \in \widehat{R}} \epsilon_j(\chi_j * \lambda) \quad \text{for } \mu \in \mathcal{M}_2. \tag{4.10}$$

Continuing this procedure, Proposition 4.3 in the case $m_i > 1$ follows.

We show

$$\text{if } \chi \in \mathrm{Irr}(b) \text{ appears in } \rho_{\mu_i,\mu_j}, \text{ then } |\mathcal{O}(\chi)| \leq \max\{p^{n-i}, p^{n-j}\}. \tag{4.11}$$

Assume (4.11) does not hold. Then $\max\{p^{n-i}, p^{n-j}\} < |\mathcal{O}(\chi)| \leq (\rho_{\mu_i,\mu_j}, \rho_{\mu_i,\mu_j}) = p^{n-i} + p^{n-j} \leq 2\max\{p^{n-i}, p^{n-j}\} < p\max\{p^{n-i}, p^{n-j}\}$. Since $|\mathcal{O}(\chi)|$ is a $p$-power, this gives a contradiction, so (4.11) holds.

Below, we assume $m_i = 1$ for all $i$ where $0 \leq i \leq m - l$. Let $\mathcal{M}_i = \{\mu_i\}$. We have $\rho_{\mu_1} = \rho_{\mu_0} + \rho_{\mu_1,\mu_0}$, $\rho_{\mu_2} = \rho_{\mu_0} + \rho_{\mu_1,\mu_0} + \rho_{\mu_2,\mu_1}, \cdots, \rho_{\mu_{m-l}} = \rho_{\mu_0} + \rho_{\mu_1,\mu_0} + \rho_{\mu_2,\mu_1} + \cdots + \rho_{\mu_{m-l},\mu_{m-l-1}}$, and by Proposition 3.4 we have

$$\text{any } \chi \in \mathrm{Irr}(b) \text{ appears in at least one of } \rho_{\mu_0}, \rho_{\mu_1,\mu_0}, \rho_{\mu_2,\mu_1}, \cdots, \rho_{\mu_{m-l},\mu_{m-l-1}}. \tag{4.12}$$

Firstly, we consider the case $m-l = 0$. Any $\chi \in \mathrm{Irr}'_0(b) (= \mathrm{Irr}(b))$ appears in $\rho_{\mu_0}$. Hence $n_0 p^n \leq (\rho_\mu, \rho_\mu) = (e+1)p^n$. Therefore by Lemma 4.2(i) we have $n_0 = e + 1 = e + m_0$ and we can write

$$\tilde{\mathrm{Irr}}'_0(b) = \{\chi_{\mu_0}, \chi_1, \cdots, \chi_e\} \text{ and}$$
$$\rho_{\mu_0} = \sum_{\lambda \in \widehat{R}} \epsilon(\chi_{\mu_0} * \lambda) - \sum_{j=1}^{e} \sum_{\lambda \in \widehat{R}} \epsilon_j(\chi_j * \lambda) \quad \text{for some signs } \epsilon_{\mu_0}, \epsilon_1, \cdots, \epsilon_e. \tag{4.13}$$

Below, we consider the case $m - l \geq 1$.

Any $\chi \in \tilde{\mathrm{Irr}}'_0(b)$ appears in $\rho_{\mu_0}$ or $\rho_{\mu_1,\mu_0}$ by (4.11) and (4.12). The number of constituents of $\rho_{\mu_0}$ in $\tilde{\mathrm{Irr}}'_0(b)$ is $e$ or $e+1$ from $(\rho_{\mu_0}, \rho_{\mu_0}) = (e+1)p^n$, $(\rho_{\mu_1,\mu_0}, \rho_{\mu_1,\mu_0}) = p^{n-1} + p^n$ and Lemma 4.2(i). If $e$ elements of $\tilde{\mathrm{Irr}}'_0(b)$ appear in $\rho_{\mu_0}$, then from Lemma 4.2(i) and $(\rho_{\mu_1,\mu_0}, \rho_{\mu_1,\mu_0}) = p^{n-1} + p^n$ there exists just one element of $\tilde{\mathrm{Irr}}'_0(b)$ which appears in $\rho_{\mu_1,\mu_0}$ and does not appear in $\rho_{\mu_0}$, and so $\rho_{\mu_1}$ would have $e+1$ constituents in $\tilde{\mathrm{Irr}}'_0(b)$, contradicting to $(\rho_{\mu_1}, \rho_{\mu_1}) = p^{n-1} + ep^n$. Hence, $\rho_{\mu_0}$ consists of $e+1$ elements in $\tilde{\mathrm{Irr}}'_0(b)$. We also have $n_0 = e+1 = e+m_0$ from $(\rho_{\mu_1,\mu_0}, \rho_{\mu_1,\mu_0}) = p^{n-1} + p^n$ and $(\rho_{\mu_1,\mu_0}, \rho_{\mu_0}) = p^n$. Therefore we have (4.13) in this case too.

From (4.13), $(\rho_{\mu_1,\mu_0}, \rho_{\mu_0}) = p^n$ and $(\rho_{\mu_1}, \rho_{\mu_1}) = p^{n-1} + ep^n$, we see

$$(\rho_{\mu_1}, \chi_{\mu_0}) = 0 \text{ and } (\rho_{\mu_1}, \chi_j) = -\epsilon_j \text{ for any } j\ (1 \leq j \leq e), \tag{4.14}$$

changing the notations in (4.13) appropriately. Note $n_1 \geq m_1 = 1$ by Lemma 4.2(ii). We show

$$\text{any } \chi \in \mathrm{Irr}'_1(b) \text{ appears in } \rho_{\mu_1}. \tag{4.15}$$

When $m - l = 1$, this is clear from Proposition 3.4 and (4.13). Let $m - l \geq 2$. From (4.11) and (4.12) any $\chi \in \mathrm{Irr}'_1(b)$ appears in $\rho_{\mu_0}$ or $\rho_{\mu_1,\mu_0}$ or $\rho_{\mu_2,\mu_1}$, and hence in $\rho_{\mu_1}$ or $\rho_{\mu_2}$ by (4.13). From $(\rho_{\mu_2,\mu_1}, \rho_{\mu_2,\mu_1}) < p^n$, the multiplicities of the elements of $\mathrm{Irr}'_0(b)$ in $\rho_{\mu_2}$ and $\rho_{\mu_1}$ are the same. Hence no element of $\chi \in \mathrm{Irr}'_1(b)$ appears in $\rho_{\mu_2}$ from $(\rho_{\mu_2}, \rho_{\mu_2}) = p^{n-2} + ep^n$ and (4.14). So (4.15) holds. Then from $(\rho_{\mu_1}, \rho_{\mu_1}) = p^{n-1} + ep^n$ and (4.14), $\rho_{\mu_1}$ has just one constituent $\chi_{\mu_1}$ in $\tilde{\mathrm{Irr}}'_1(b)$ and $n_1 = 1 = m_1$. Also from $\rho_{\mu_1,\mu_0}(1) = 0$ we have

$$\begin{gathered}\tilde{\mathrm{Irr}}'_1(b) = \{\chi_{\mu_1}\} \quad \text{and} \\ \rho_{\mu_1} = \sum_{\lambda_1 \in \widehat{R_1}} \epsilon(\chi_{\mu_1} * \lambda_1) - \sum_{j=1}^{e} \sum_{\lambda \in \hat{R}} \epsilon_j(\chi_j * \lambda).\end{gathered} \tag{4.16}$$

Continuing this procedure, Proposition 4.3 in the case $m_i = 1$ follows. □

In the proof of Proposition 4.3, the following theorem is proved:

**Theorem 4.4.**

$$\mathrm{Irr}(b) = \bigcup_{i=1}^{e} \{\chi_i * \lambda \mid \lambda \in \hat{R}\} \cup \bigcup_{\mu \in \mathcal{M}} \{\chi_\mu * \lambda_\mu \mid \lambda_\mu \in \widehat{R_\mu}\}$$

*where* $\chi_i$ $(i = 1, 2, \cdots, e)$ *and* $\chi_\mu$ $(\mu \in \mathcal{M})$ *satisfy*

$$\rho_\mu = \sum_{\lambda_\mu \in \widehat{R_\mu}} \epsilon(\chi_\mu * \lambda_\mu) - \sum_{i=1}^{e} \sum_{\lambda \in \hat{R}} \epsilon_i(\chi_i * \lambda) \quad \textit{for some signs } \epsilon, \epsilon_1, \cdots, \epsilon_e. \tag{4.17}$$

We remark that $\mathcal{O}(\chi_\mu) = \{\chi_\mu * \lambda_\mu \mid \lambda_\mu \in \widehat{R_\mu}\}$ from the proof of Proposition 4.3. We call $\chi_i$ $(1 \leq i \leq e)$ *non-exceptional irreducible characters* of $b$.

**Proposition 4.5.** *For* $i$ $(1 \leq i \leq e)$, *the* $\hat{R}$*-orbit* $\mathcal{O}(\chi_i)$ *contains a unique* $p$*-rational character.*

**Proof.** Following [1] §6 (cf. [13] Chapt. V §4), we consider the action of the Galois group $\Gamma = \mathrm{Gal}\big(\boldsymbol{Q}(\sqrt[|G|]{1})/\boldsymbol{Q}(\sqrt[|G|_{p'}]{1})\big)$. Note that $\Gamma$ is cyclic since $p \neq 2$. For $\gamma \in \Gamma$, $\chi^\gamma(g) = \chi(g)^\gamma$ where $g \in G$ and $\chi \in \mathrm{Irr}(b)$, and $\Gamma$ acts on $X_{\mathcal{K}}(G, b)$ by the $\mathcal{K}$-linear extension. There exists a rational integer $(\gamma)$ such that $p \nmid (\gamma)$, $(\gamma) \equiv 1 \pmod{|G|_{p'}}$ and $\chi^\gamma(us) = \chi(u^{(\gamma)}s)$ where $u$ is a $p$-element of $G$ and $s \in C_G(u)_{p'}$. $\Gamma$ also acts on the $b$-Brauer elements by $(u, b_u)^\gamma = (u^{(\gamma)}, b_u)$. This action is compatible with the $G$-conjugation, and $\Gamma$ acts on the $G$-conjugacy classes of $b$-Brauer elements. In the proof, $G$-conjugate $b$-Brauer elements will be identified. Note $d^u(\chi^\gamma, \varphi) = d^u(\chi, \varphi)^\gamma = d^{u^{(\gamma)}}(\chi, \varphi)$ where

$\varphi \in \mathrm{IBr}(b_u)$. Then $(\chi * \eta)^\gamma = \chi^\gamma * \eta^\gamma$ for a $(G, b_P)$-stable character $\eta$ of $P$. Hence $\mathcal{O}(\chi)^\gamma = \mathcal{O}(\chi^\gamma)$ and $\Gamma$ acts on the set of $\hat{R}$-orbits of $\mathrm{Irr}(b)$. Note that there exists at most one $p$-rational character in $\mathcal{O}(\chi)$.

Assume $(u, b_u)$ is fixed by $\Gamma$. Then $(u^{(\gamma)}, b_u)$ and $(u, b_u)$ are $G$-conjugate for any $\gamma \in \Gamma$ and there exists some $a \in L$ such that $u^{(\gamma)} = u^a$. Hence $|N_L(\langle u\rangle)/C_L(u)| = |u^\Gamma| = p^{s-1}(p-1)$ where $u^\Gamma = \{u^{(\gamma)} \mid \gamma \in \Gamma\}$ and $p^s$ is the order of $u$. Then $e = p-1$ and $u \notin_P R$. In particular, $l(b_u) = 1$. Moreover, from Theorem 4.4 we see $u \in Q$, since the column of the generalized decomposition matrix of $b$ corresponding to $(u, b_u)$ consists of rational integers by the assumption. Set

$$\begin{aligned} W &= \{u \in \Pi \cap (Q\backslash\{1\}) \mid (u, b_u) \text{ is fixed by } \Gamma\} \\ &= \{u \in \Pi \cap (Q\backslash\{1\}) \mid u^{(\gamma)} =_L u \text{ for any } \gamma \in \Gamma\} \end{aligned}$$

and $w = |W|$. Applying Brauer's permutation lemma ([13] Lemma 3.2.18) to the generalized decomposition matrix of $b$, we see $b$ has exactly $(e + w)$ $p$-rational irreducible characters.

Here we consider the condition that an element of $Q$ belongs to $W$. Let $u \in Q$. Since $\langle u\rangle$ is stabilized by $L$, we have $u^L \subseteq u^\Gamma$. Therefore $u \in W$ if and only if $|u^L| = |u^\Gamma|$. Assume $W$ is non-empty. Then $e = p-1$. Let $u \in W$ and suppose $u \in \langle x^{p^{i-1}}\rangle \backslash \langle x^{p^i}\rangle$ for some $i$ $(1 \le i \le m)$. Then the order of $u$ is $p^{m-(i-1)}$ and $|u^\Gamma| = (p-1)p^{m-i}$. On the other hand, since $yxy^{-1} = x^{1+p^l}$, we have $|u^L| = e \cdot p^{m-l-(i-1)}$ when $i \le m-l$, and $|u^L| = e$ when $i > m-l$. Thus, we have $w = m$ when $l = 1$, and we have $w = 1$ when $l > 1$.

Return to the proof, set

$$W' = \{\mu \in \mathcal{M} \mid \mu^\gamma = \mu^{(\gamma)} =_L \mu \text{ for any } \gamma \in \Gamma\}$$

and $w' = |W'|$. Then $w = w'$ since $\mu^y = \mu^{1+p^l}$ for $\mu \in \hat{Q}$. When $|\mathcal{M}| = 1$, clearly $w = w' = 1$ and $b$ has exactly $(e+1)$ $p$-rational irreducible characters. Therefore each of $\mathcal{O}(\chi_i)$ and $\mathcal{O}(\chi_\mu)$ contains a $p$-rational character. Suppose $|\mathcal{M}| \ge 2$. Since $\zeta_0$ is $p$-rational, $\sum_{i=1}^{e} \tilde{\chi}_i = \zeta_0 \uparrow_C^{\tilde{G}}$ is fixed by $\Gamma$ and we have $(\tilde{\chi}_\mu)^\gamma = ((\zeta_0 * (\hat{\mu} \uparrow_{P_\mu}^P)) \uparrow_C^G)^\gamma = \tilde{\chi}_{\mu^\gamma}$ for $\mu \in \mathcal{M}$ and $\gamma \in \Gamma$. Hence $(\rho_\mu)^\gamma = \rho_{\mu^\gamma}$. Since $\rho_\mu - \rho_{\mu'} = \sum_{\lambda_\mu \in \widehat{R_\mu}} \epsilon(\chi_\mu * \lambda_\mu) - \sum_{\lambda_{\mu'} \in \widehat{R_{\mu'}}} \epsilon(\chi_{\mu'} * \lambda_{\mu'})$ for $\mu, \mu' \in \mathcal{M}$ such that $\mu \ne \mu'$, we have $(\mathcal{O}(\chi_\mu))^\gamma = \mathcal{O}(\chi_{\mu^\gamma})$ for $\mu \in \mathcal{M}$. In particular $\mathcal{O}(\chi_\mu)$ is stabilized by $\Gamma$ if and only if $\mu \in W'$. Hence there exist at most $w$ $p$-rational characters in $\{\mathcal{O}(\chi_\mu) \mid \mu \in \mathcal{M}\}$. Therefore each of $\mathcal{O}(\chi_i)$ and $\mathcal{O}(\chi_\mu)$ stabilized by $\Gamma$ contains a $p$-rational character. This completes the proof. $\square$

Below, we will assume that $\chi_i$ $(i = 1, 2, \cdots, e)$ is $p$-rational.

**Proposition 4.6.** *Keeping our notations, set*

$$\varphi_j = \epsilon_j \chi_j \downarrow_{G_{p'}} \quad (j = 1, 2, \cdots, e).$$

*Then*

$$Bs(b) = \{\varphi_j \mid j = 1, 2, \cdots, e\}$$

*is a basic set for $b$ and the decomposition numbers $d(\chi, \varphi_j)$ of $\chi$ with respect to $Bs(b)$ are given as follows:*

$$d(\chi_i * \lambda,\, \varphi_j) = \epsilon_i \delta_{ij},$$
$$d(\chi_\mu * \lambda_\mu,\, \varphi_j) = \epsilon p^{h_\mu}$$

*where $i = 1, 2, \cdots, e$, $\lambda \in \hat{R}$, $j = 1, 2, \cdots, e$, $\mu \in \mathcal{M}$ and $\lambda_\mu \in \widehat{R_\mu}$. (Here, $\delta_{ij}$ is Kronecker delta.) Moreover the Cartan matrix of $b$ with respect to $Bs(b)$ is of the form*

$$C = |R| \begin{pmatrix} t+1 & t & \cdots & t \\ t & t+1 & \cdots & t \\ \vdots & \vdots & \vdots & \vdots \\ t & t & \cdots & t+1 \end{pmatrix}_{e \times e}$$

*where*

$$t = \frac{|Q| - 1}{e}.$$

**Proof.** From (4.17) and $\rho_{\mu_i} \in X_{\mathcal{K}}(G, b; Q\backslash\{1\})$, we have

$$\chi_{\mu_i} \downarrow_{G_{p'}} = \epsilon p^i \sum_{j=1}^{e} \epsilon_j (\chi_j \downarrow_{G_{p'}})$$

for $\mu_i \in \mathcal{M}_i$ $(0 \leq i \leq m - l)$. Then the statement follows from $l(b) = e$, Theorem 4.4 and (4.2). $\square$

**Remark.** When $t = 1$, we have $m - l = 0$ and $m_0 = 1$, and let $\mathcal{M} = \mathcal{M}_0 = \{\mu\}$. In this case $\epsilon_j \chi_j$ and $-\epsilon \chi_\mu$ are interchangeable with respect to Theorem 4.4 for any $j$. Also note that $-\epsilon \chi_\mu \downarrow_{G_{p'}} = -\sum_{j=1}^{e} \epsilon_j \chi_j \downarrow_{G_{p'}}$.

## 5. Generalized decomposition numbers in a block with metacyclic defect group

In this section we determine the generalized decomposition numbers of $b$ with respect to a basic set obtained by the $p'$-restriction of irreducible characters with signs.

For $(u, b_u) \in (P, b_P)$, $b_u$ has a defect group $C_P(u)$. The block $b_u$ is nilpotent if and only if $u \notin_P R$. Let

$\varphi_u$ be the unique irreducible Brauer character in $b_u$ when $u \notin_P R$.

When $u \in R$, $E$ can be viewed as an inertial quotient group of $b_u$, and $b_u$ has a hyperfocal subgroup $C_Q(u)$ from $[C_Q(u), E] = C_Q(u)$. Also $C_P(u) = C_Q(u) \rtimes R$. For the above, see [23] Lemma 5, Lemma 6 and Lemma 7. Note that, when $u \in_P R$, we can apply results in previous sections for $b_u$. We denote by $e_u$ the inertial index of $b_u$ for $u \in P$.

For an $E$-invariant subgroup $T$ of $P$ containing $Q$ and $\nu \in \mathrm{Irr}(T)$, we define

$$\eta_\nu = \sum_{a \in E} \nu^a.$$

Note that we have $\eta_{\nu\uparrow_T^P} = \eta_\nu \uparrow_T^P$ and $\eta_\nu$ does not depend on the choice of $E$.

We will prove the following two theorems.

**Theorem 5.1.** *Let $u \in P$ be such that $u \notin_P R$, that is, $e_u = 1$. Then there exists a sign $\delta_u$ such that*

$$d^u(\chi_i * \lambda, \varphi_u) = \epsilon_i \delta_u \lambda(u),$$
$$d^u(\chi_\mu * \lambda_\mu, \varphi_u) = \epsilon \delta_u \eta_{\hat{\mu}\uparrow_{P_\mu}^P}(u) \lambda_\mu(u)$$

*where $i = 1, 2, \cdots, e$, $\lambda \in \hat{R}$, $\mu \in \mathcal{M}$ and $\lambda_\mu \in \widehat{R_\mu}$.*

**Theorem 5.2.** *Let $u \in P$ be such that $u \in_P R$, that is, $e_u = e$. Then there exists a sign $\delta_u$ such that for the basic set $Bs(b_u) = \{\varphi_j^{(u)} \mid j = 1, 2, \cdots, e\}$ for $b_u$ (see Proposition 4.6 for $Bs(b_u)$)*

$$d^u(\chi_i * \lambda, \varphi_j^{(u)}) = \epsilon_i \delta_u \delta_{ij} \lambda(u),$$
$$d^u(\chi_\mu * \lambda_\mu, \varphi_j^{(u)}) = \epsilon \delta_u (\hat{\mu} \uparrow_{P_\mu}^P)(u) \lambda_\mu(u)$$

*under suitable choice of the notations of* $\mathrm{Irr}(b_u)$ *where $i = 1, 2, \cdots, e$, $\lambda \in \hat{R}$, $j = 1, 2, \cdots, e$, $\mu \in \mathcal{M}$ and $\lambda_\mu \in \widehat{R_\mu}$.*

For the proof of the above theorems, firstly we collect some lemmas.

**Lemma 5.3.** *Let $u \notin_P R$. Then*

$$\sum_{\mu \in \mathcal{M}} \frac{1}{p^{h_\mu}} |\eta_{\hat{\mu}\uparrow_{P_\mu}^P}(u)|^2 = \frac{|C_P(u)|}{p^n} - e.$$

**Proof.** We have

$$\begin{aligned} &\sum_{\mu \in \mathcal{M}} |R_\mu| \, |\eta_{\hat{\mu}\uparrow_{P_\mu}^P}(u)|^2 \\ &= \sum_{a, a' \in E} \sum_{\mu \in \mathcal{M}} |R_\mu| (\hat{\mu}\uparrow_{P_\mu}^P)^a(u^{-1}) (\hat{\mu}\uparrow_{P_\mu}^P)^{a'}(u) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{a'\in E}\sum_{\mu\in\mathcal{M}}\sum_{a\in E}|R_\mu|(\hat{\mu}{\uparrow}_{P_\mu}^{P})^a(u^{-1})(\hat{\mu}{\uparrow}_{P_\mu}^{P})^a(u^{a'}) \\
&= \sum_{a'\in E}\sum_{\lambda\in\hat{R}}\sum_{\mu\in\mathcal{M}}\sum_{a\in E}\frac{1}{p^{h_\mu}}(\hat{\mu}{\uparrow}_{P_\mu}^{P})^a(u^{-1})\lambda(u^{-1})(\hat{\mu}{\uparrow}_{P_\mu}^{P})^a(u^{a'})\lambda(u^{a'}) \\
&= \sum_{a'\in E}\sum_{\lambda\in\widehat{R_\mu}}\sum_{\mu\in\mathcal{M}}\sum_{a\in E}(\hat{\mu}{\uparrow}_{P_\mu}^{P})^a(u^{-1})\lambda(u^{-1})(\hat{\mu}{\uparrow}_{P_\mu}^{P})^a(u^{a'})\lambda(u^{a'}) \\
&= \sum_{a'\in E}\Big(\sum_{\nu\in\mathrm{Irr}(P)}\nu(u^{-1})\nu(u^{a'}) - \sum_{\lambda\in\hat{R}}(1_P\lambda)(u^{-1})(1_P\lambda)(u^{a'})\Big) \\
&= |C_P(u)| - ep^n.
\end{aligned}
$$

Note (2.1) for $\mathrm{Irr}(P)$, and that $u^{a'}$ is not $P$-conjugate to $u$ for $a' \in E\backslash\{1\}$ by Lemma 4.1(i). $\square$

**Lemma 5.4.** *Let $u \notin_P R$. Then*

$$\sum_{\mu\in\mathcal{M}}\frac{1}{p^{h_\mu}}\eta_{\hat{\mu}\uparrow_{P_\mu}^{P}}(u)(1_{R_\mu}{\uparrow}_{R_\mu}^{R})(u) = -1.$$

**Proof.** Let $\overline{Q} = Q/[Q,u]$ and write $u = qr$ where $q \in Q$ and $r \in R$. We have $q \notin [Q,u]$, since if $q \in [Q,u] = \{[q',r] \mid q' \in Q\}$, then we would have $u \in r^Q$. Note that $u \in P_\mu$ if and only if $\mu \in [Q,u]^\perp \simeq \widehat{\overline{Q}}$. Then we have

$$\sum_{\mu\in\mathcal{M}}\frac{1}{p^{h_\mu}}\eta_{\hat{\mu}\uparrow_{P_\mu}^{P}}(u)(1_{R_\mu}{\uparrow}_{R_\mu}^{R})(u) = \sum_{\mu\in\mathcal{M}\cap[Q,u]^\perp}\eta_{\hat{\mu}}{\uparrow}_{P_\mu}^{P}(u) = \sum_{\mu\in\widehat{\overline{Q}}\backslash\{1\}}\mu(q) = -1. \quad \square$$

**Lemma 5.5.** *Let $r \in R$. Then*

$$\sum_{\mu\in\mathcal{M}}\frac{1}{p^{h_\mu}}\big((1_{R_\mu}{\uparrow}_{R_\mu}^{R})(r)\big)^2 = \frac{|C_Q(r)|-1}{e}.$$

**Proof.** Note $r \in R_\mu$ if and only if $\mu \in [Q,r]^\perp$, and $Q/C_Q(r) \simeq [Q,r]$. Then we have

$$
\begin{aligned}
\sum_{\mu\in\mathcal{M}}\frac{1}{p^{h_\mu}}\big((1_{R_\mu}{\uparrow}_{R_\mu}^{R})(r)\big)^2 &= \sum_{\mu\in\mathcal{M}\cap[Q,r]^\perp}p^{h_\mu} = \sum_{\mu\in\mathcal{M}\cap[Q,r]^\perp}\frac{|\mu^L|}{e} \\
&= \frac{|[Q,r]^\perp|-1}{e} = \frac{|C_Q(r)|-1}{e}. \quad \square
\end{aligned}
$$

Next, we consider the generalized decomposition numbers when $u \in Q\backslash\{1\}$ and then the heights of irreducible characters in $b$. For $u \in Q\backslash\{1\}$, note that $C_{\tilde{G}}(u) = C_{\tilde{N}}(u) = C_G(u)$ and $\tilde{b}_u = \tilde{c}_u = c_u$, and let $\delta_u = d^u(\zeta_0,\varphi_u)$ (a sign), see [3].

**Lemma 5.6.** *Let $u \in Q\backslash\{1\}$. Then*
(i) $d^u(\tilde{\chi}_i, \varphi_u) = \delta_u$ *for* $i = 1, 2, \cdots, e$.
(ii) $d^u(\tilde{\chi}_\mu, \varphi_u) = \delta_u \eta_{\hat{\mu}\uparrow^P_{P_\mu}}(u)$ *for* $\mu \in \mathcal{M}$.

**Proof.** Let $s \in C_{\tilde{G}}(u)_{p'} = C_C(u)_{p'}$. Then we have

$$\tilde{\chi}_i(us\tilde{b}_u) = \tilde{\zeta}_i(us\tilde{c}_u) = \zeta_0(usc_u) = d^u(\zeta_0, \varphi_u)\varphi_u(s).$$

This implies (i). Since $\zeta_0$ is $\tilde{N}$-invariant and $\tilde{N}/C \cong E$, we have

$$\begin{aligned}\tilde{\chi}_\mu(us\tilde{b}_u) = \tilde{\zeta}_\mu(us\tilde{c}_u) = \big(\zeta_0 * (\hat{\mu}\uparrow^P_{P_\mu})\big)\uparrow^{\tilde{N}}_C (us\tilde{c}_u) = \sum_{a\in E} \hat{\mu}\uparrow^P_{P_\mu} (u^a)\zeta_0(usc_u) \\ = \eta_{\hat{\mu}\uparrow^P_{P_\mu}}(u)\zeta_0(usc_u).\end{aligned}$$

This implies (ii). □

**Proposition 5.7.** *Let $u \in Q\backslash\{1\}$. Then*
(i) $d^u(\chi_i, \varphi_u) = \epsilon_i\delta_u$ *for* $i = 1, 2, \cdots, e$.
(ii) $d^u(\chi_\mu, \varphi_u) = \epsilon\delta_u \eta_{\hat{\mu}\uparrow^P_{P_\mu}}(u)$ *for* $\mu \in \mathcal{M}$.

**Proof.** From (3.2), (3.3), (4.17) and Lemma 3.1, we have

$$\Big(\sum_{\lambda\in\hat{R}} \big(\tilde{\chi}_\mu - p^{h_\mu}\sum_{i=1}^{e}\tilde{\chi}_i\big) * \lambda\Big)^{(u,\tilde{b}_u)}\uparrow^G_{\tilde{G}} = \Big(\sum_{\lambda\in\hat{R}} \big(\epsilon\chi_\mu - p^{h_\mu}\sum_{i=1}^{e}\epsilon_i\chi_i\big) * \lambda\Big)^{(u,b_u)}.$$

This implies

$$\delta_u(\eta_{\hat{\mu}\uparrow^P_{P_\mu}}(u) - p^{h_\mu}e) = \epsilon d^u(\chi_\mu, \varphi_u) - p^{h_\mu}\sum_{i=1}^{e}\epsilon_j d^u(\chi_i, \varphi_u)$$

by Lemma 5.6. Hence we have

$$\epsilon d^u(\chi_\mu, \varphi_u) = \delta_u \eta_{\hat{\mu}\uparrow^P_{P_\mu}}(u) + p^{h_\mu}X \tag{5.1}$$

where

$$X = \sum_{i=1}^{e}\epsilon_i d^u(\chi_i, \varphi_u) - \delta_u e.$$

Since

$$\sum_{i=1}^{e} d^u(\chi_i, \varphi_u)^2 + \sum_{\mu\in\mathcal{M}} \frac{1}{p^{h_\mu}}|d^u(\chi_\mu, \varphi_u)|^2 = \frac{|C_P(u)|}{p^n}$$

from [13] Theorem 5.4.11, and

$$\sum_{\mu\in\mathcal{M}} \frac{1}{p^{h_\mu}} |d^u(\chi_\mu, \varphi_u)|^2 = (\frac{|C_P(u)|}{p^n} - e) - 2\delta_u X + \frac{p^m - 1}{e} X^2$$

from Lemma 5.3, the second orthogonality relation for $Q$ and (4.2), we have

$$(\sum_{i=1}^{e} \epsilon_i d^u(\chi_i, \varphi_u) - \delta_u)^2 = -\frac{p^m - 1}{e} X^2.$$

Hence (i) holds, and (ii) also holds by (5.1). □

**Proposition 5.8.** (i) *$\chi_i$ is of height 0 for $i = 1, 2, \cdots, e$.*

(ii) *$\chi_\mu$ is of height $h_\mu$ for $\mu \in \mathcal{M}$.*

**Proof.** From [1] (4C) and Proposition 5.7 for $u \in Q_1\backslash\{1\}$, we see the statements (i) and (ii). For (ii), note also that $\eta_{\hat{\mu}}(u) \equiv e \not\equiv 0 \bmod J(\mathcal{O})$. □

By Theorem 4.4 and Proposition 5.8, (4.3) is refined to the following proposition, which is a generalization of [9] Theorem 5.21, [10] Theorem, [8] Theorem 1.1 and [17] Theorem 2.3:

**Proposition 5.9.** (i) $k_0(b) = (e + \frac{p^l - 1}{e})p^n$

(ii) $k_i(b) = \frac{p^l - p^{l-1}}{e} p^{n-i} \quad (1 \leq i \leq m - l)$

(iii) $k_i(b) = 0 \quad (i > m - l)$

(iv) $k(b) = (\frac{p^l + p^{l-1} - p^{2l-m-1} - 1}{e} + e)p^n$

Now we will show equations on generalized characters in $b$, see Proposition 5.11 below. It is used in the proofs of Theorem 5.1 and Theorem 5.2.

**Lemma 5.10.** (i) *Let $\mu \in \mathcal{M}_0$. Then*

$$\eta_{\widehat{\mu^{-1}}}\eta_{\hat{\mu}} = \sum_{s=1}^{e-1} \eta_{\widehat{\mu_s}} + e1_P$$

*where $\mu_s \in \langle\mu\rangle\backslash\langle\mu^p\rangle$.*

(ii) *Assume $1 \leq i \leq m - l$ and let $\mu \in \mathcal{M}_i$. Then*

$$\eta_{\widehat{\mu^{-1}}\uparrow_{P_\mu}^{P}} \eta_{\hat{\mu}\uparrow_{P_\mu}^{P}} = \sum_{s=1}^{p^i(e-1)} \eta_{\widehat{\mu_s}\uparrow_{P_\mu}^{P}} + \sum_{t=1}^{p^i-1} \eta_{(\widehat{\nu_t}\downarrow_{P_\mu}^{P_{\nu_t}})\uparrow_{P_\mu}^{P}} + e(1_{P_\mu}\uparrow_{P_\mu}^{P})$$

*where $\mu_s \in \langle\mu\rangle\backslash\langle\mu^p\rangle$ and $\nu_t \in \langle\mu^p\rangle\backslash\{1\}$.*

**Proof.** Note that $\mu \mapsto \mu^{-1}(\mu^r)^a$ gives an automorphism of $\langle\mu\rangle$, that is, $\mu^{-1}(\mu^r)^a \in \langle\mu\rangle\backslash\langle\mu^p\rangle$, if and only if $a \neq 1$ where $\mu \in \mathcal{M}$, $r \in R/R_\mu$ and $a \in E$.

(i) We have

$$\eta_{\widehat{\mu^{-1}}}\eta_{\widehat{\mu}} = \sum_{a,a' \in E} (\widehat{\mu}^{-1})^a \widehat{\mu}^{a'} = \sum_{a,a' \in E} (\widehat{\mu}^{-1}\widehat{\mu}^a)^{a'} = \sum_{a \in E\backslash\{1\}} \eta_{\widehat{\mu}^{-1}\widehat{\mu}^a} + e1_P.$$

(ii) We have

$$\eta_{\widehat{\mu^{-1}}\uparrow_{P_\mu}^P}\eta_{\widehat{\mu}\uparrow_{P_\mu}^P} = (\eta_{\widehat{\mu^{-1}}}\uparrow_{P_\mu}^P)(\eta_{\widehat{\mu}}\uparrow_{P_\mu}^P) = \Big(\eta_{\widehat{\mu^{-1}}}\big((\eta_{\widehat{\mu}}\uparrow_{P_\mu}^P)\downarrow_{P_\mu}^P\big)\Big)\uparrow_{P_\mu}^P$$

and

$$\begin{aligned}\eta_{\widehat{\mu^{-1}}}\big((\eta_{\widehat{\mu}}\uparrow_{P_\mu}^P)\downarrow_{P_\mu}^P\big) &= \sum_{a \in E}\sum_{r \in R/R_\mu} \eta_{\widehat{\mu}^{-1}(\widehat{\mu}^r)^a} \\ &= \sum_{a \in E\backslash\{1\}}\sum_{r \in R/R_\mu} \eta_{\widehat{\mu}^{-1}(\widehat{\mu}^r)^a} + \sum_{r \in (R/R_\mu)\backslash\{1\}} \eta_{\widehat{\mu}^{-1}\widehat{\mu}^r} + e1_{P_\mu}. \quad \square\end{aligned}$$

**Proposition 5.11.** *For $\mu \in \mathcal{M}$, we have*

$$(\epsilon_1\chi_1) * \eta_{\hat{\mu}\uparrow_{P_\mu}^P} = (e-1)(\epsilon_1\chi_1) * (1_{R_\mu}\uparrow_{R_\mu}^R) - \sum_{i=2}^{e}(\epsilon_i\chi_i) * (1_{R_\mu}\uparrow_{R_\mu}^R) + \epsilon\chi_\mu$$

*by replacing $\chi_\mu$ by an element of $\mathcal{O}(\chi_\mu)$ if necessary.*

**Proof.** From $\rho_\mu \in X_{\mathcal{K}}(G, b; Q\backslash\{1\})$, Proposition 5.7 and the second orthogonal relation for $R$ and $R_\mu$, we see

$$\begin{aligned}&\sum_{\lambda_\mu \in \widehat{R_\mu}} (\epsilon_1\chi_1) * (\eta_{\hat{\mu}\uparrow_{P_\mu}^P}\lambda_\mu) \\ &\qquad = (e-1)\sum_{\lambda \in \hat{R}}(\epsilon_1\chi_1) * \lambda - \sum_{i=2}^{e}\sum_{\lambda \in \hat{R}}(\epsilon_i\chi_i) * \lambda + \sum_{\lambda_\mu \in \widehat{R_\mu}}(\epsilon\chi_\mu) * \lambda_\mu.\end{aligned} \tag{5.2}$$

From (5.2) at least one element of $\mathcal{O}(\chi_\mu)$ appears in $(\epsilon_1\chi_1) * \eta_{\hat{\mu}\uparrow_{P_\mu}^P}$. On the other hand, since $(\chi_1 * \eta_{\hat{\mu}\uparrow_{P_\mu}^P}) * \lambda = \chi_1 * \eta_{\hat{\mu}\uparrow_{P_\mu}^P}$ for $\lambda \in R_\mu^\perp$, we can set

$$\begin{aligned}(\epsilon_1\chi_1) * \eta_{\hat{\mu}\uparrow_{P_\mu}^P} &= \sum_{i=1}^{e} c_{i,1_{R_\mu}}\big((\epsilon_i\chi_i) * (1_{R_\mu}\uparrow_{R_\mu}^R)\big) \\ &+ \sum_{i=1}^{e}\sum_{\nu(\neq 1_{R_\mu}) \in \widehat{R_\mu}} c_{i,\nu}\big((\epsilon_i\chi_i) * (\nu\uparrow_{R_\mu}^R)\big) + c\chi_\mu + \cdots,\end{aligned} \tag{5.3}$$

where $c_{i,\nu}$ $(\nu \in \widehat{R_\mu})$ and $c$ are integers by [2] Theorem. We may assume $c \neq 0$ by replacing $\chi_\mu$ by $\chi_\mu * \lambda$ $(\lambda \in \hat{R})$ if necessary. Since $\sum_{\lambda_\mu \in \widehat{R_\mu}} (\nu \uparrow_{R_\mu}^{R}) \lambda_\mu = \sum_{\lambda \in \hat{R}} \lambda$ for any $\nu \in \widehat{R_\mu}$, we have

$$e - 1 = \sum_{\nu \in \widehat{R_\mu}} c_{1,\nu}, \quad -1 = \sum_{\nu \in \widehat{R_\mu}} c_{i,\nu} \quad (2 \leq i \leq e)$$

from (5.2) and (5.3).

Let $\Gamma = \mathrm{Gal}\big(\boldsymbol{Q}(\sqrt[|G|]{1})/\boldsymbol{Q}(\sqrt[|G|_{p'}]{1})\big)$ be the Galois group as in the proof of Proposition 4.5, and let $\sigma$ be an element of $\Gamma$ of order $e$. Note $\langle \sigma \rangle$ acts on $\widehat{R_\mu} \backslash \{1_{R_\mu}\}$ fixed-point freely and $\eta_{\hat{\mu}} = \sum_{t=0}^{e-1} \hat{\mu}^{\sigma^t}$. Then $\big((\epsilon_1\chi_1) * \eta_{\hat{\mu}\uparrow_{P_\mu}^{P}}\big)^\sigma = (\epsilon_1\chi_1)^\sigma * (\eta_{\hat{\mu}\uparrow_{P_\mu}^{P}})^\sigma = (\epsilon_1\chi_1) * \eta_{\hat{\mu}\uparrow_{P_\mu}^{P}}$ and hence

$$c_{1,1_{R_\mu}} \equiv e - 1 \ (\mathrm{mod}\ e), \quad c_{i,1_{R_\mu}} \equiv -1 \ (\mathrm{mod}\ e) \quad (2 \leq i \leq e).$$

In particular, $c_{i,1_{R_\mu}} \neq 0$. Considering the action of $\Gamma$ on $(\epsilon_1\chi_1) * \eta_{\hat{\mu}\uparrow_{P_\mu}^{P}}$, we see $c_{1,1_{R_\mu}}$ does not depend on $\mu^l$ with $p \nmid l$. Set $X = c_{1,1_{R_\mu}}$.

Now let $\mu \in \mathcal{M}_i$ $(0 \leq i \leq m - l)$. We will prove the statement by induction on $i$.

Suppose that $i = 0$. Then from (5.3) we have

$$\begin{aligned}
&\big(\epsilon_1\chi_1,\ (\epsilon_1\chi_1) * (\eta_{\widehat{\mu^{-1}}}\eta_{\hat{\mu}})\big) = \big((\epsilon_1\chi_1) * \eta_{\hat{\mu}},\ (\epsilon_1\chi_1) * \eta_{\hat{\mu}}\big) \\
&\geq X^2 + \sum_{i=2}^{e} {c_{i,1_{R_\mu}}}^2 + c^2 \geq X^2 + (e-1) + 1 = X^2 + e.
\end{aligned}$$

On the other hand, by Lemma 5.10(i) we have

$$\big(\epsilon_1\chi_1,\ (\epsilon_1\chi_1) * (\eta_{\widehat{\mu^{-1}}}\eta_{\hat{\mu}})\big) = (e-1)X + e.$$

Hence we have $(e-1)X \geq X^2$. From this and $X \equiv e - 1$ (mod $e$), we have $X = e - 1$ and above inequalities are equalities. Therefore we have $c_{i,\nu} = 0$ $(1 \leq i \leq e,\ \nu \neq 1_{R_\mu})$ and $c_{i,1_{R_\mu}} = -1$ $(2 \leq i \leq e)$. Moreover, (5.2) and (5.3) imply $c = \epsilon$. Hence the statement holds for $\mu \in \mathcal{M}_0$.

Next suppose that $\mu \in \mathcal{M}_i$ and $i \geq 1$ assuming $m - l \geq 1$. Then from (5.3) we have

$$\begin{aligned}
&\big(\epsilon_1\chi_1,\ (\epsilon_1\chi_1) * (\eta_{\widehat{\mu^{-1}}\uparrow_{P_\mu}^{P}}\eta_{\hat{\mu}\uparrow_{P_\mu}^{P}})\big) = \big((\epsilon_1\chi_1) * \eta_{\hat{\mu}\uparrow_{P_\mu}^{P}},\ (\epsilon_1\chi_1) * \eta_{\hat{\mu}\uparrow_{P_\mu}^{P}}\big) \\
&\geq p^i X^2 + p^i \sum_{i=2}^{e} {c_{i,1_{R_\mu}}}^2 + c^2 \geq p^i X^2 + p^i(e-1) + 1.
\end{aligned}$$

On the other hand, by the induction hypothesis and Lemma 5.10(ii) we have

$$\big(\epsilon_1\chi_1,\ (\epsilon_1\chi_1) * (\eta_{\widehat{\mu^{-1}}\uparrow_{P_\mu}^{P}}\eta_{\hat{\mu}\uparrow_{P_\mu}^{P}})\big) = Xp^i(e-1) + (p^i - 1)(e-1) + e.$$

Here note that we have

$$\eta_{(\widehat{\nu_t}\downarrow^{P_{\nu_t}}_{P_\mu})\uparrow^P_{P_\mu}} = \sum_{\lambda' \in \widehat{P_{\nu_t}/P_\mu}} \eta_{\widehat{\nu_t}\uparrow^P_{P_{\nu_t}}} \lambda' \quad \text{for } \nu_t \in \langle \mu^p \rangle \backslash \{1\}$$

where we view $\lambda'$ as a character of $P$ by extension and inflation, and that $\epsilon_1\chi_1 * \lambda'^{-1}$ does not appear in $\epsilon_1\chi_1 * \eta_{\widehat{\nu_t}\uparrow^P_{P_{\nu_t}}}$ by the induction hypothesis. Hence we have $Xp^i(e-1-X) \geq 0$, and as in the case $i = 0$, we have the statement in this case too. □

We choose $\chi_\mu$ ($\mu \in \mathcal{M}$) so that the relation in Proposition 5.11 is satisfied.

Now, we prove Theorem 5.1 and Theorem 5.2.

**Proof of Theorem 5.1.** Clearly it suffices to show the case $\lambda = 1$. Set $x_i = \epsilon_i d^u(\chi_i, \varphi_u)$. Note $x_i$ is a rational integer by Proposition 4.5. Also note $x_i$ is non-zero by Proposition 5.8 and [3] Theorem 1.5. From Proposition 5.11 we have

$$\epsilon d^u(\chi_\mu, \varphi_u) = \eta_{\hat{\mu}\uparrow^P_{P_\mu}}(u)x_1 + (1_{R_\mu}\uparrow^R_{R_\mu})(u)\sum_{i=2}^{e}(x_i - x_1) \quad \text{for } \mu \in \mathcal{M}. \tag{5.4}$$

Hence, for the proof it suffices to show

$$\text{there is some sign } \delta_u \text{ depending on } u \text{ such that } x_1 = x_2 = \cdots = x_e = \delta_u. \tag{5.5}$$

From [13] Theorem 5.4.11 and (5.4) we have

$$\sum_{i=1}^{e} x_i^2 + \sum_{\mu \in \mathcal{M}} \frac{1}{p^{h_\mu}} \left| \eta_{\hat{\mu}\uparrow^P_{P_\mu}}(u)x_1 + (1_{R_\mu}\uparrow^R_{R_\mu})(u)\sum_{i=2}^{e}(x_i - x_1)\right|^2 = \frac{|C_P(u)|}{p^n}.$$

By Lemma 5.3 and Lemma 5.4, this equation can be translated to

$$(x_1^2 - 1)\frac{|C_P(u)|}{p^n} + \sum_{i=2}^{e}(x_i - x_1)^2 + \sum_{\mu \in \mathcal{M}} \frac{1}{p^{h_\mu}} \left((1_{R_\mu}\uparrow^R_{R_\mu})(u)\sum_{i=2}^{e}(x_i - x_1)\right)^2 = 0.$$

Then (5.5) follows from this equation. □

Note that the notation $\delta_u$ in Proposition 5.7 is consistent with the notation $\delta_u$ in Theorem 5.1.

**Proof of Theorem 5.2.** We may assume $u \in R = C_P(E)$. We will consider the basic set $\mathrm{Bs}(b_u) = \{\varphi_j^{(u)} \mid 1 \leq j \leq e\}$ for $b_u$ as described in Proposition 4.6 and the generalized decomposition numbers $d^u(\chi_i, \varphi_j^{(u)})$ $(1 \leq i \leq e,\ 1 \leq j \leq e)$ with respect to the basic set $\mathrm{Bs}(b_u)$. Set $x_{ij} = \epsilon_i d^u(\chi_i, \varphi_j^{(u)})$. Note $x_{ij}$ is a rational integer.

Since $\eta_{\hat{\mu}\uparrow_{P_\mu}^{P}}(u) = e(1_{R_\mu}\uparrow_{R_\mu}^{R})(u)$, we have

$$\epsilon d^u(\chi_\mu, \varphi_j^{(u)}) = (1_{R_\mu}\uparrow_{R_\mu}^{R})(u)\sum_{i=1}^{e} x_{ij} \quad (\mu \in \mathcal{M},\ 1 \le j \le e) \tag{5.6}$$

from Proposition 5.11. Hence, we have

$$\sum_{i=1}^{e} x_{ij}x_{ik} + t_u\Big(\sum_{i=1}^{e} x_{ij}\Big)\Big(\sum_{i=1}^{e} x_{ik}\Big) = t_u + \delta_{jk} \tag{5.7}$$

from [13] Theorem 5.4.11 and Lemma 5.5 where $t_u = \dfrac{|C_Q(u)| - 1}{e}$. Since $\chi_{\mu_0}$ $(\mu_0 \in \mathcal{M}_0)$ is of height 0, there is some $j_0$ such that $\sum_{i=1}^{e} x_{ij_0} \neq 0$ by (5.6) and [3] Theorem 1.5.

At first, assume $t_u \ge 2$. Since

$$\sum_{i=1}^{e} x_{ij_0}^2 + t_u\Big(\sum_{i=1}^{e} x_{ij_0}\Big)^2 = t_u + 1$$

by (5.7), we have $\sum_{i=1}^{e} x_{ij_0}^2 = 1$. Hence there exists some $i_0$ such that $x_{i_0j_0} = \pm 1$ and $x_{ij_0} = 0$ for any $i$ different from $i_0$. Set $\delta_u = x_{i_0j_0}$. Let $j_1$ be different from $j_0$. Then we have

$$\begin{aligned}
&\sum_{i=1}^{e} x_{ij_1}^2 + t_u\Big(\sum_{i=1}^{e} x_{ij_1}\Big)^2 = t_u + 1\\
&\sum_{i=1}^{e} x_{ij_1}x_{ij_0} + t_u\Big(\sum_{i=1}^{e} x_{ij_1}\Big)\Big(\sum_{i=1}^{e} x_{ij_0}\Big) = t_u
\end{aligned}$$

by (5.7). From this we see there exists $i_1\,(\neq i_0)$ such that $x_{i_1j_1} = \delta_u$ and $x_{ij_1} = 0$ for any $i$ different from $i_1$. Let $j_2$ be different from $j_0$ and $j_1$. Then we have

$$\begin{aligned}
&\sum_{i=1}^{e} x_{ij_2}^2 + t_u\Big(\sum_{i=1}^{e} x_{ij_2}\Big)^2 = t_u + 1\\
&\sum_{i=1}^{e} x_{ij_2}x_{ij_0} + t_u\Big(\sum_{i=1}^{e} x_{ij_2}\Big)\Big(\sum_{i=1}^{e} x_{ij_0}\Big) = t_u\\
&\sum_{i=1}^{e} x_{ij_2}x_{ij_1} + t_u\Big(\sum_{i=1}^{e} x_{ij_2}\Big)\Big(\sum_{i=1}^{e} x_{ij_1}\Big) = t_u
\end{aligned}$$

by (5.7). From this we see there exists $i_2\,(\neq i_0, i_1)$ such that $x_{i_2j_2} = \delta_u$ and $x_{ij_2} = 0$ for any $i$ different from $i_2$. Continuing this procedure, if we choose the index $j$ of non-exceptional irreducible characters of $b_u$ so that $i_0 = j_0$, $i_1 = j_1$, $\cdots$, then we have $\epsilon_i d^u(\chi_i, \varphi_j^{(u)}) = x_{ij} = \delta_u\delta_{ij}$ and so $\epsilon d^u(\chi_\mu, \varphi_j^{(u)}) = (1_{R_\mu}\uparrow_{R_\mu}^{R})(u)\sum_{i=1}^{e} x_{ij} = \delta_u(\hat{\mu}\uparrow_{P_\mu}^{P})(u)$ by (5.6). Hence we have the statement in the case $t_u \ge 2$.

Next, assume $t_u = 1$. Then since

$$\sum_{i=1}^{e} x_{ij}^2 + \big(\sum_{i=1}^{e} x_{ij}\big)^2 = 2,$$

we have

$$\sum_{i=1}^{e} x_{ij}^2 = 1 \quad \text{and} \quad (\sum_{i=1}^{e} x_{ij})^2 = 1$$
$$\text{or}$$
$$\sum_{i=1}^{e} x_{ij}^2 = 2 \quad \text{and} \quad (\sum_{i=1}^{e} x_{ij})^2 = 0.$$

When $j = j_0$, the former case occurs and there exists some $i_0$ such that $x_{i_0 j_0} = \pm 1$ and $x_{ij_0} = 0$ for any $i$ different from $i_0$.

Assume there exists $j_1$ ($\neq j_0$) such that $\sum_{i=1}^{e} x_{ij}^2 = 1$ and $(\sum_{i=1}^{e} x_{ij})^2 = 1$. Then since

$$\sum_{i=1}^{e} x_{ij_1} x_{ij_0} + \big(\sum_{i=1}^{e} x_{ij_1}\big)\big(\sum_{i=1}^{e} x_{ij_0}\big) = 1,$$

we see there exists $i_1$ ($\neq i_0$) such that $x_{i_1 j_1} = x_{i_0 j_0}$ and $x_{ij_1} = 0$ for any $i$ different from $i_1$. Set $\delta_u = x_{i_0 j_0}$. Let $j_2$ be different from $j_0$ and $j_1$. Since we have

$$\begin{aligned}
&\sum_{i=1}^{e} x_{ij_2}^2 + \big(\sum_{i=1}^{e} x_{ij_2}\big)^2 = 2 \\
&\sum_{i=1}^{e} x_{ij_2} x_{ij_0} + \big(\sum_{i=1}^{e} x_{ij_2}\big)\big(\sum_{i=1}^{e} x_{ij_0}\big) = 1 \\
&\sum_{i=1}^{e} x_{ij_2} x_{ij_1} + \big(\sum_{i=1}^{e} x_{ij_2}\big)\big(\sum_{i=1}^{e} x_{ij_1}\big) = 1,
\end{aligned}$$

we see there exists $i_2$ ($\neq i_0, i_1$) such that $x_{i_2 j_2} = \delta_u$ and $x_{ij_2} = 0$ for any $i$ different from $i_2$. Continuing this procedure, under suitable choice of the index $j$, we have $\epsilon_i d^u(\chi_i, \varphi_j^{(u)}) = \delta_u \delta_{ij}$ and $\epsilon d^u(\chi_\mu, \varphi_j^{(u)}) = \delta_u(\hat{\mu} \uparrow_{P_\mu}^{P})(u)$ as before. Hence we have the statement in the case where $t_u = 1$ and there exists $j_1$ ($\neq j_0$) such that $\sum_{i=1}^{e} x_{ij}^2 = 1$ and $(\sum_{i=1}^{e} x_{ij})^2 = 1$.

Finally, we consider the case where $t_u = 1$, $\sum_{i=1}^{e} x_{ij}^2 = 2$ and $(\sum_{i=1}^{e} x_{ij})^2 = 0$ for any $j$ different from $j_0$. Let $j_1$ be different from $j_0$. Then since

$$\sum_{i=1}^{e} x_{ij_1} x_{ij_0} + \big(\sum_{i=1}^{e} x_{ij_1}\big)\big(\sum_{i=1}^{e} x_{ij_0}\big) = 1,$$

we see $x_{i_0 j_1} = x_{i_0 j_0}$ and there exists $i_1$ ($\neq i_0$) such that $x_{i_1 j_1} = -x_{i_0 j_1}$ and $x_{ij_1} = 0$ for any $i$ different from $i_0$ and $i_1$. Set $\delta_u = -x_{i_0 j_0}$. Let $j_2$ be different from $j_0$ and $j_1$. Since we have

$$\sum_{i=1}^{e} x_{ij_2}x_{ij_0} + \Big(\sum_{i=1}^{e} x_{ij_2}\Big)\Big(\sum_{i=1}^{e} x_{ij_0}\Big) = 1$$
$$\sum_{i=1}^{e} x_{ij_2}x_{ij_1} + \Big(\sum_{i=1}^{e} x_{ij_2}\Big)\Big(\sum_{i=1}^{e} x_{ij_1}\Big) = 1,$$

we see $x_{i_0j_2} = x_{i_0j_0}$ and there exists $i_2\,(\neq i_0, i_1)$ such that $x_{i_2j_2} = -x_{i_0j_2}$ and $x_{ij_2} = 0$ for any $i$ different from $i_0$ and $i_2$. Continuing this procedure, under suitable choice of the index $j$, we have

$$\epsilon_i d^u(\chi_i, \varphi_j^{(u)}) = \delta_u\delta_{ij}\ (i \neq i_0), \quad \epsilon_{i_0} d^u(\chi_{i_0}, \varphi_j^{(u)}) = -\delta_u,$$
$$\epsilon d^u(\chi_\mu, \varphi_j^{(u)}) = 0\ (j \neq j_0), \quad \epsilon d^u(\chi_\mu, \varphi_{j_0}^{(u)}) = -\delta_u(\hat{\mu}\uparrow_{P_\mu}^{P})(u).$$

If we take an alternative basic set

$$\{\varphi_1^{(u)}, \cdots, \varphi_{j_0-1}^{(u)},\ -\sum_{j=1}^{e}\varphi_j^{(u)},\ \varphi_{j_0+1}^{(u)}, \cdots, \varphi_e^{(u)}\}$$

of $b_u$, then the generalized decomposition numbers with respect to this basic set are

$$\epsilon_i d^u(\chi_i, \varphi_j^{(u)}) = \delta_u\delta_{ij}\ (j \neq j_0), \quad \epsilon_i d^u(\chi_i, -\sum_{j=1}^{e}\varphi_j^{(u)}) = 0\ (i \neq i_0),$$
$$\epsilon_{i_0} d^u(\chi_{i_0}, -\sum_{j=1}^{e}\varphi_j^{(u)}) = \delta_u, \quad \epsilon d^u(\chi_\mu, \varphi_j^{(u)}) = \delta_u(\hat{\mu}\uparrow_{P_\mu}^{P})(u)\ (j \neq j_0),$$
$$\epsilon d^u(\chi_\mu, -\sum_{j=1}^{e}\varphi_j^{(u)}) = \delta_u(\hat{\mu}\uparrow_{P_\mu}^{P})(u).$$

Then changing the notations of $\mathrm{Irr}(b_u)$ as in Remark after Proposition 4.6 for $j = j_0$, we have the statement in this case too. □

## 6. Perfect isometries and isotypies

In this section we prove Theorem 1.1. It suffices to construct an isotypy between $b$ and $b_P^{N_G(P,b_P)}$ (see Theorem 6.5 below). For the notions of perfect isometry and isotypy introduced by Broué ([4] 1.4, 4.6), we follow Definition 2.1 and Definition 2.2 in [12]. The $\mathcal{K}$-vector space $X_{\mathcal{K}}(G, b)$ coincides with $\mathrm{CF}(G, b, \mathcal{K})$ defined in [12].

**Lemma 6.1.** *Assume that $G = N_G(P, b_P)$. Then we have*

$$\epsilon = 1,\ \epsilon_i = 1\ (i = 1, 2, \cdots, e),\ \delta_u = 1\ (u \in P).$$

**Proof.** By the assumption we have $G = \tilde{N}$. Then we have $\epsilon' = 1$ and $\epsilon'_i = 1$ by (3.2), (3.3) and (4.17).

We have $C = PC_G(P)$, $\zeta_0$ is a canonical character of $b$, $\chi_i$ $(i = 1, 2, \cdots, e)$ are the extensions of $\zeta_0$, and $\mathrm{Bs}(b) = \{\varphi_j \,|\, j = 1, 2, \cdots, e\} = \{\chi_j {\downarrow_{G_{p'}}} \,|\, j = 1, 2, \cdots, e\}$ coincides with $\mathrm{IBr}(b)$. For $u \in P$, $C_G(u)$ normalizes a maximal $b_u$-Brauer pair $(C_P(u), b_{C_P(u)})$. Hence the same situation as $b$ occurs for $b_u$ when $u \in_P R$. In particular, $\mathrm{Bs}(b_u) = \mathrm{IBr}(b_u)$. Since $\chi_i(us) = \varphi_i {\downarrow_{C_G(u)}} (s)$ for $u \in P$ and $s \in C_G(u)_{p'}$, generalized decomposition numbers for $\chi_i$ in Theorem 5.1 and Theorem 5.2 are non-negative integers. Hence, $\delta_u = 1$ for $u \in P$. $\square$

Let $G'$ be a finite group and $b'$ be a block of $G'$. Let $I : X_{\mathcal{K}}(G, b) \to X_{\mathcal{K}}(G', b')$ be a perfect isometry. Then we have

$$I(\alpha^{(1,b)}) = \big(I(\alpha)\big)^{(1,b')} \quad \text{for } \alpha \in X_{\mathcal{K}}(G, b)$$

by the "separation condition" ([12] Definition 2.1(b)) of the perfect isometry, and let

$$I_{p'} : X_{\mathcal{K}}^{(1,b)}(G, b) \to X_{\mathcal{K}}^{(1,b')}(G', b')$$

be the $\mathcal{K}$-linear map induced by $I$. A class function on $G_{p'}$ belonging to $b$ will be viewed as an element of $X_{\mathcal{K}}^{(1,b)}(G, b)$. Under this convention, we have

$$I_{p'}(\alpha {\downarrow_{G_{p'}}}) = \big(I(\alpha)\big){\downarrow_{G'_{p'}}} . \tag{6.1}$$

**From now we set**

$$G' = N_G(P, b_P), \quad b' = b_P^{G'} .$$

We use $'$ for the notations concerning to $b'$. Then $\mathcal{F}_{(P,b_P)}(G, b) \simeq \mathcal{F}_{(P,b_P)}(G', b')$, and $Q$ is the hyperfocal subgroup of $b'$. We may take $L' = L$, $R' = R$, $\Pi' = \Pi$, $\mathcal{M}' = \mathcal{M}$ and so on. Note $\epsilon' = \epsilon'_i = \delta'_u = 1$ for $i = 1, 2, \cdots, e$ and $u \in P$ by Lemma 6.1.

**Proposition 6.2.** *The $\mathcal{K}$-linear map*

$$I^1 : X_{\mathcal{K}}(G, b) \to X_{\mathcal{K}}(G', b')$$

*such that*

$$I^1(\chi_i * \lambda) = \epsilon_i \chi'_i * \lambda,$$
$$I^1(\chi_\mu * \lambda_\mu) = \epsilon \chi'_\mu * \lambda_\mu,$$

*where $i = 1, 2, \cdots, e$, $\lambda \in \hat{R}$, $\mu \in \mathcal{M}$ and $\lambda_\mu \in \widehat{R_\mu}$, is a perfect isometry.*

**Proof.** This follows from [11] Theorem 2 (see also [18] Theorem 6.1) and Theorem 5.1 and Theorem 5.2 for $b$ and $b'$. In fact, for $u \in \Pi\backslash\{1\}$ take

$$\{\delta_u\varphi_u\} \text{ and } \{\varphi'_u\} \text{ when } u \notin_P R$$
$$\{\delta_u\varphi_j^{(u)} \mid j=1,2,\cdots,e\} \text{ and } \{{\varphi'}_j^{(u)} \mid j=1,2,\cdots,e\} \text{ when } u \in_P R$$

as $\mathrm{Bs}(b_u)$ and $\mathrm{Bs}(b'_u)$ in [11] Theorem 2 (iv), where $\varphi_j^{(u)}$ and ${\varphi'}_j^{(u)}$ are taken so that the generalized decomposition numbers are described as in Theorem 5.2. □

In Proposition 6.2, the numbering of the non-exceptional irreducible characters of $b$ is arbitrary, and in the situation of Remark after Proposition 4.6, the choice of $\chi_1, \cdots, \chi_e$ is also arbitrary. Similar for $b'$.

Next, we consider the perfect isometries in the local blocks. These isometries are arranged by the sign $\delta_u$ in Theorem 5.1 and Theorem 5.2.

By [3] Theorem 1.2 we have the following ([4] 5.2):

**Proposition 6.3.** *Let $u \in P$ be such that $u \notin_P R$, that is, $e_u = e'_u = 1$. Then the $\mathcal{K}$-linear map*

$$I^u : X_{\mathcal{K}}(C_G(u), b_u) \to X_{\mathcal{K}}(C_{G'}(u), b'_u)$$

*such that*

$$I^u(\zeta_u * \lambda_u) = \delta_u\zeta'_u * \lambda_u,$$

*where $\zeta_u$ and $\zeta'_u$ are the unique $p$-rational irreducible characters of $b_u$ and $b'_u$ respectively and $\lambda_u \in \mathrm{Irr}(C_P(u))$, is a perfect isometry.*

When $u \in_P R$, we can apply the results in previous sections to $b_u$ and $b'_u$. We use $^{(u)}$ for the notations concerning to $b_u$ and $'^{(u)}$ for the notations concerning to $b'_u$. When $u \in R^v$ for $v \in P$, we may take $L^{(u)} = L'^{(u)} = C_P(u) \rtimes E^v$, $R^{(u)} = R'^{(u)} = R^v$ and so on. Note $\epsilon'^{(u)} = {\epsilon'}_i^{(u)} = 1$ for $i = 1,2,\cdots,e$. From Theorem 5.1 and Theorem 5.2 for $b_u$ and $b'_u$ and [11] Theorem 2, we have the following:

**Proposition 6.4.** *Let $u \in P$ be such that $u \in_P R$, that is, $e_u = e'_u = e$. Then the $\mathcal{K}$-linear map*

$$I^u : X_{\mathcal{K}}(C_G(u), b_u) \to X_{\mathcal{K}}(C_{G'}(u), b'_u)$$

*such that*

$$I^u(\chi_i^{(u)} * \lambda^{(u)}) = \delta_u\epsilon_i^{(u)}{\chi'}_i^{(u)} * \lambda^{(u)},$$
$$I^u(\chi_{\mu^{(u)}}^{(u)} * \lambda_{\mu^{(u)}}^{(u)}) = \delta_u\epsilon^{(u)}{\chi'}_{\mu^{(u)}}^{(u)} * \lambda_{\mu^{(u)}}^{(u)},$$

*where* $i = 1, 2, \cdots, e$, $\lambda^{(u)} \in \widehat{R^{(u)}}$, $\mu^{(u)} \in \mathcal{M}^{(u)}$ *and* $\lambda^{(u)}_{\mu^{(u)}} \in \widehat{R^{(u)}_{\mu^{(u)}}}$, *is a perfect isometry.*

A similar remark as stated after Proposition 6.2 holds for Proposition 6.4.

Since $\chi_i$ is $p$-rational, we have $\delta_u = \delta_v$ and hence $I^u = I^v$ for $u, v \in P$ such that $\langle u\rangle = \langle v\rangle$. So, for a non-trivial cyclic subgroup $S$ of $P$, we have a perfect isometry

$$I^S : X_{\mathcal{K}}(C_G(S), b_S) \to X_{\mathcal{K}}(C_{G'}(S), b'_S)$$

defined by $I^S = I^u$ where $u$ is any generator of $S$.

For $u \in P$, let

$$d_G^{(u,b_u)} : X_{\mathcal{K}}(G, b) \to X_{\mathcal{K}}^{(1,b_u)}(C_G(u), b_u)$$

be the $\mathcal{K}$-linear map defined by $d_G^{(u,b_u)}(\chi)(s) = \chi^{(u,b_u)}(us)$ where $\chi \in \mathrm{Irr}(b)$ and $s \in C_G(u)_{p'}$.

**Theorem 6.5.** *$b$ and $b'$ are isotypic with a local system $\{I^S \mid S$ : cyclic subgroup of $P\}$.*

**Proof.** For the proof, it suffices to confirm

$$I^u_{p'} \circ d_G^{(u,b_u)} = d_{G'}^{(u,b'_u)} \circ I^1$$

for any $u \in P$.

Let $u \in P$ be such that $u \notin_P R$. For $i = 1, 2, \cdots, e$ and $\lambda \in \hat{R}$,

$$I^u_{p'} \circ d_G^{(u,b_u)}(\chi_i * \lambda) = I^u_{p'}\big(\epsilon_i\delta_u\lambda(u)\varphi_u\big) = \epsilon_i\delta_u\lambda(u)(\delta_u\varphi'_u) = \epsilon_i\lambda(u)\varphi'_u$$

by Theorem 5.1 for $b$ and (6.1), and

$$d_{G'}^{(u,b'_u)} \circ I^1\big(\chi_i * \lambda\big) = d_{G'}^{(u,b'_u)}(\epsilon_i\chi'_i * \lambda) = \epsilon_i\lambda(u)\varphi'_u$$

by Theorem 5.1 for $b'$. Hence we have $I^u_{p'} \circ d_G^{(u,b_u)}(\chi_i * \lambda) = d_{G'}^{(u,b'_u)} \circ I^1\big(\chi_i * \lambda\big)$. Similarly we have

$$I^u_{p'} \circ d_G^{(u,b_u)}(\chi_\mu * \lambda_\mu) = \epsilon\eta_{\hat{\mu}\uparrow^P_{P_\mu}}(u)\lambda_\mu(u)\varphi'_u = d_{G'}^{(u,b'_u)} \circ I^1\big(\chi_\mu * \lambda_\mu\big)$$

for $\mu \in \mathcal{M}$ and $\lambda_\mu \in \widehat{R_\mu}$.

Next let $u \in_P R$. By Theorem 5.2 for $b$ and (6.1)

$$I^u_{p'} \circ d_G^{(u,b_u)}(\chi_i * \lambda) = I^u_{p'}(\epsilon_i\delta_u\lambda(u)\varphi_i^{(u)}) = \epsilon_i\lambda(u)\varphi_i'^{(u)}.$$

On the other hand, by Theorem 5.2 for $b'$

$$d_{G'}^{(u,b'_u)} \circ I^1(\chi_i * \lambda) = d_{G'}^{(u,b'_u)}(\epsilon_i\chi'_i * \lambda) = \epsilon_i\lambda(u)\varphi_i'^{(u)}.$$

Similarly,

$$I_{p'}^{u} \circ d_{G}^{(u,b_u)}(\chi_\mu * \lambda_\mu) = I_{p'}^{u}\big(\sum_{j=1}^{e} \epsilon\delta_u \eta_{\hat{\mu}\uparrow_{P_\mu}^{P}}(u)\lambda_\mu(u)\varphi_j^{(u)}\big) = \epsilon\eta_{\hat{\mu}\uparrow_{P_\mu}^{P}}(u)\lambda_\mu(u)\sum_{j=1}^{e}\varphi_j'^{(u)}$$

and

$$d_{G'}^{(u,b'_u)} \circ I^1(\chi_\mu * \lambda_\mu) = d_{G'}^{(u,b'_u)}(\epsilon\chi'_\mu * \lambda_\mu) = \epsilon\eta_{\hat{\mu}\uparrow_{P_\mu}^{P}}(u)\lambda_\mu(u)\sum_{j=1}^{e}\varphi_j'^{(u)}.$$

This completes the proof. □

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