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Supernilpotence need not imply nilpotence ☆

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ABSTRACT

Supernilpotence is a generalization of nilpotence using a recently developed theory of higher-arity commutators for universal algebras. Many important structural properties have been shown to be associated with supernilpotence, and the exact relationship between nilpotence and supernilpotence has been the subject of investigation. We construct an algebra which is not solvable (and hence not nilpotent) but which is supernilpotent, thereby showing that in general supernilpotence does not imply nilpotence. We also extend this construction to 'higher dimensions' to obtain similar results for (n)-step supernilpotence.
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1. Introduction

The topic of this manuscript is related to a broad generalization of commutator theory called higher commutator theory. Higher commutators are used to define a condition called *supernilpotence*, called such because it is usually a stronger condition than nilpo-

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tence. We construct algebras to demonstrate that supernilpotence and nilpotence are in general independent of one another and that this independence is preserved even if one considers 'higher dimensional' analogues of nilpotence.

Historically, a specific notion of commutator was used to study a specific variety of algebras, (e.g. a class of similar algebraic structures that satisfy some set of equational laws or identities), such as the variety of groups, rings, or Lie algebras. In each of these classes the notion of a commutator has led to important structural results, as it can be used to measure 'abelianness' and define generalizations of abelianness such as solvability and nilpotence. For example, a classical theorem of group theory states that a finite group is nilpotent if and only if it is the direct product of its Sylow subgroups.

Actually, each of these commutator theories is a special case of a commutator that may be formulated for any algebraic structure. The strength of the theory depends not on the similarity type of the algebra, but on the identities that it satisfies. The initial insight is due to Smith. In [18], he shows it is possible to define a commutator for any variety of algebras in which every member has a Mal'cev operation, that is, an operation $p(x, y, z)$ built from the basic operations that satisfies the identities

$$p(x, x, y) \approx p(y, x, x) \approx y,$$

and that this commutator retains all the essential features of the examples known at the time, all of which were for algebras with a Mal'cev operation.

Commutator theory for universal algebras has grown substantially since then and we do not attempt a survey in this introduction. We refer the reader to the text Freese and McKenzie [4] and the text Gumm [5] for two different approaches to commutator theory for congruence modular varieties of algebras. For the development of commutator theory outside of the context of congruence modularity, the reader is referred to the monograph Kiss and Kearnes [8].

Such a general commutator theory comes equipped with the naturally generalized versions of abelianness, solvability, and nilpotence. Under some additional assumptions, finite nilpotent algebras are very similar in their structure to finite nilpotent groups. For example, Lyndon [11] shows that the equational theory of a nilpotent group is finitely based and Freese and McKenzie [4] shows that if a finite algebra of finite type (belonging to a congruence modular variety) is nilpotent *and* is the direct product of nilpotent algebras of prime power order, then it has a finitely based equational theory. Such algebras are now known to be examples of *supernilpotent* algebras.

Supernilpotence is an analogue of abelianness that is definable with a higher arity commutator that generalizes the classical binary commutator. Such commutators were first introduced by Bulatov in [2]. In [1], Aichinger and Mudrinksi develop analogues of those properties shown to be essential for the binary commutator for the higher commutator (in a Mal'cev variety). In the same paper every supernilpotent algebra belonging to a Mal'cev variety is shown to be nilpotent. Using earlier results of Kearnes from [7], Aichinger and Mudrinksi go on to prove that every finite supernilpotent Mal'cev algebra of finite type is a product of prime power order nilpotent Mal'cev algebras, and vice versa.

Supernilpotent Mal'cev algebras of finite type share other properties with nilpotent groups. For example, Michael Kompatscher shows in [10] that there is a polynomial time algorithm that checks if equations over finite supernilpotent Mal'cev algebras of finite type have a solution. Equation solvability and related problems emphasize the need to understand the differences between nilpotence and supernilpotence, see Idziak and Krzaczkowski [6] for additional details.

The theory of the higher commutator has been recently extended to varieties that are not Mal'cev. In [13], the second author extends most of the theory of the higher commutator to congruence modular varieties. In [14], the second author develops a relational description of the modular ternary commutator and uses this to show that (2)-step supernilpotence implies (2)-step nilpotence in a congruence modular variety. In Wires [19], several properties of higher commutators are developed outside of the context of congruence modularity. Implicit in the results of Wires is that supernilpotence implies nilpotence for congruence modular varieties. More recently, Kearnes and Szendrei have announced that any *finite* supernilpotent algebra is nilpotent, which is to appear in [9]. It turns out that supernilpotence is a stronger condition than nilpotence for any variety of algebras that satisfies a nontrivial idempotent equational condition [15].

Each of the algebras we construct in this paper is therefore infinite and does not generate a Taylor variety. In Section 2 we develop notation and state definitions. In Section 3 we discuss different notions of nilpotence and solvability. In Section 4 we construct an algebra that is not solvable but is supernilpotent. The final section 5 generalizes this example to 'higher dimensions'.

2. Definitions

2.1. Notation

In this paper the set of natural numbers is denoted by ω and has as its least element the empty set, or 0. The finite ordinal n is the set of its predecessors and we will often write $i \in n$ instead of $0 \leq i \leq n-1$.

Some familiarity with the basics of Universal Algebra is assumed. Good references on the subject are [3] and [12]. An **algebra** is a set with some structure provided by a set of finitary operations. These two ingredients are usually written as a pair, e.g. $\mathbb{A} = \langle A; \{f_i\}_{i \in I}\rangle$. Product, subalgebra, and homomorphism are defined in the obvious way.

Let $\mathbb{A}$ be an algebra, $n \in \omega$, and $R \subseteq A^n$ be a set of tuples over A of length n. If R is a subalgebra of $\mathbb{A}^n$ we say that R is an $\mathbb{A}$**-invariant** relation, or just an invariant relation if there is no possibility for confusion.

The invariant equivalence relations of an algebra are called **congruences** and determine its possible homomorphic images. The lattice of all congruences of an algebra is denoted by $\mathrm{Con}(\mathbb{A})$, with the largest congruence and least congruence denoted by 1 and 0, respectively.

2.2. The higher commutator

The higher commutator is an operation on the lattice of congruences of an algebra and is usually defined via the so-called term condition, see [2] for the first instance in the literature. The main construction of this paper is most naturally presented by defining the commutator via a special invariant relation which we now describe. The commutator definition given here is equivalent to the usual one and the reader is referred to [13] or [17] for more details.

Let $\mathbb{A} = \langle A; \{f_i\}_{i\in I}\rangle$ be an algebra and $n \in \omega$ a natural number. An invariant relation

$$R \leq \mathbb{A}^{2^n}$$

is said to be an (n)**-dimensional invariant relation**. The reason for this terminology is that the set of functions 2^n is a natural coordinate system for the (n)-dimensional cube, where two functions are connected by an edge if and only if they differ in exactly one argument. A particular element

$$h \in A^{2^n}$$

is therefore thought of as a **vertex labeled** (n)**-dimensional cube**. Less formally, we will sometimes refer to h simply as an (n)-dimensional cube, or (when the dimension is clear) a just a cube.

A total function $f \in 2^n$ specifies the coordinates of a particular vertex of such an $h \in A^{2^n}$, and we denote the value of h at f by h_f. This notation may be extended to partial functions, and in doing so one may specify inside of h the location of lower dimensional vertex labeled cubes. That is, for $S \subseteq n$ and $f : S \to 2$ define

$$h_f := \left\{h_g : g \in 2^n \text{ and } f \subseteq g\right\}.$$

Less formally, a partial function $f : n \to 2$ determines some of the coordinates for a vertex in h. The coordinates that are not yet determined may be specified by those $g \in 2^n$ that extend f. We hope that the reader alarmed by the potential ambiguity of this notation will find no ambiguity in its use.

We distinguish for any domain S the function $\mathbf{1} : S \to 2$ that takes the constant value 1. Take some $h \in A^{2^n}$ and $i \neq j \in n$. Define

$$\begin{aligned}\mathrm{Lines}_i(h) &:= \left\{h_f : f \in 2^{n\setminus\{i\}}\right\}\\ &= \left\{h_f : f \in 2^{n\setminus\{i\}}, f \neq \mathbf{1}\right\} \cup \left\{h_{\mathbf{1}} : \mathbf{1} \in 2^{n\setminus\{i\}}\right\} \text{ and}\end{aligned}$$

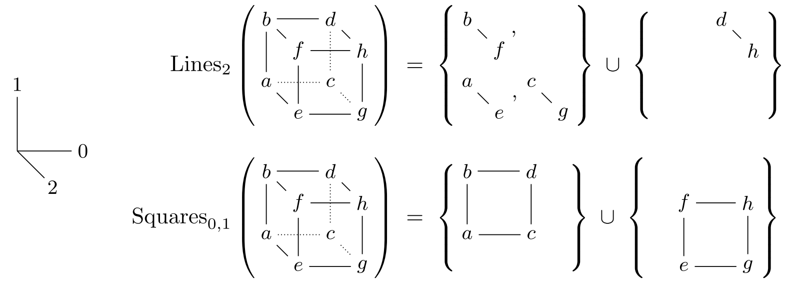

Fig. 1. (2)-cross section lines and (0,1)-cross section squares decomposed into support and pivot sets. Orientation of the labeled cube is given by the coordinate axes, $n = 3 = \{0, 1, 2\}$.

$$\begin{aligned}\mathrm{Squares}_{i,j}(h) &:= \left\{h_f : f \in 2^{n\setminus\{i,j\}}\right\} \\ &= \left\{h_f : f \in 2^{n\setminus\{i,j\}}, f \neq \mathbf{1}\right\} \cup \left\{h_{\mathbf{1}} : \mathbf{1} \in 2^{n\setminus\{i,j\}}\right\}.\end{aligned}$$

These sets are called the (i)**-cross section lines** and the (i,j)**-cross section squares** of h, respectively. The set of (i)-cross section lines is the disjoint union of two sets. The first set we denote by S-Lines$_i(h)$ and its members are called (i)**-support lines**; the single member of the second set is called the (i)**-pivot line**. Similarly, the set of (i,j)-cross section lines is composed of (i,j)**-support squares** and a single (i,j)**-pivot square**. See Fig. 1. We say that a line, square, or (generally) a cube is **constant** if all of the vertices have the same value. We call a set of lines, squares, or cubes **constant** if all of its members are.

Remark. We will often write equations in which terms are evaluated at vertex labeled cubes which are drawn as actual cubes. This notation is a different way of writing equations involving tuples in a product and is intended to emphasize the geometry of the relations that are being analyzed.

Let $(\theta_0, \ldots, \theta_{n-1}) \in \mathrm{Con}(\mathbb{A})^n$ be a sequence of congruences. The relation used to define the higher commutator is a certain (n)-dimensional invariant relation that is generated by special vertex labeled (n)-dimensional cubes. For each $i \in n$, let

$$\mathrm{gCube}^n_i(x, y) \in A^{2^n}$$

be the vertex labeled (n)-dimensional cube such that

$$\left(\mathrm{gCube}^n_i(x,y)\right)_f = \begin{cases} x & \text{if } f(i) = 0, \\ y & \text{if } f(i) = 1. \end{cases}$$

Now set

$$\mathrm{M}(\theta_0, \ldots, \theta_1) := \mathrm{Sg}_{\mathbb{A}^{2^n}}\left(\bigcup_{i\in n}\left\{\mathrm{gCube}^n_i(x,y) : \langle x, y\rangle \in \theta_i\right\}\right).$$

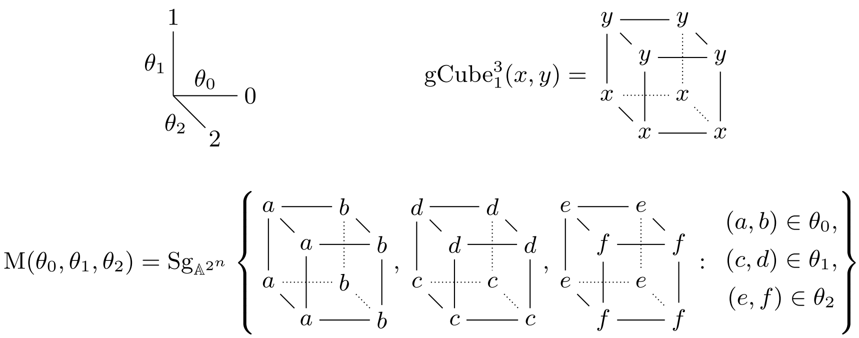

Fig. 2. Examples of a gCube and the generators of the (3)-dimensional $(\theta_0, \theta_1, \theta_2)$-matrix relation. Orientation is indicated by the coordinate axes. Elements of $\mathbb{A}$ connected by a line parallel to the i axis are members of θ_i.

This (n)-dimensional relation is called the algebra of $(\theta_0, \dots, \theta_{n-1})$**-matrices**. See Fig. 2. We can now formulate the centrality condition used to define the higher commutator.

Definition 2.1 *(Centrality).* Let $\mathbb{A}$ be an algebra, $2 \leq n \in \omega$, $\delta \in \mathrm{Con}(\mathbb{A})$ and $(\theta_0, \dots, \theta_{n-1}) \in \mathrm{Con}(\mathbb{A})^n$ a sequence of congruences with $\mathrm{M}(\theta_0, \dots, \theta_{n-1})$ defined as above. Let $\sigma \in S_n$ be a permutation of n. We say that $\theta_{\sigma(0)}, \dots, \theta_{\sigma(n-2)}$ **centralize** $\theta_{\sigma(n-1)}$ **modulo** δ provided the following condition holds:

> If $h \in \mathrm{M}(\theta_0, \dots, \theta_{n-1})$ is such that every $(\sigma(n-1))$-support line of h is a δ-pair, then the $(\sigma(n-1))$-pivot line of h is also a δ-pair.

This condition is abbreviated as $\mathrm{C}(\theta_{\sigma(0)}, \dots, \theta_{\sigma(n-2)}, \theta_{\sigma(n-1)}; \delta)$.

Definition 2.2 *(Higher Commutator).* Under the same assumptions given in Definition 2.2, set

$$\left[\theta_{\sigma(0)}, \dots, \theta_{\sigma(n-1)}\right] := \bigwedge \left\{\delta : \mathrm{C}(\theta_{\sigma(0)}, \dots, \theta_{\sigma(n-2)}, \theta_{\sigma(n-1)}; \delta)\right\}.$$

There is some potential for confusion with this definition, because there are several distinct algebras of matrices that can be used to define the same commutator. Each of these algebras of matrices can be obtained from the other by a permutation of coordinates, however. We therefore prefer, for a given *set* of congruences $\{\theta_i : i \in n\}$, to fix a coordinate system at the outset. All centrality conditions involving the n-many congruences belonging to $\{\theta_i : i \in n\}$ may then be formulated with respect to $\mathrm{M}(\theta_0, \dots, \theta_{n-1})$. This is best explained through example – see Figs. 3 and 4.

The following properties are immediate consequences of Definition 2.2.

Proposition 2.3. *Let $\mathbb{A}$ be an algebra and $\alpha \in \mathrm{Con}(\mathbb{A})$. The following hold:*

Fig. 3. The condition $\mathrm{C}(\theta_0, \theta_1, \theta_2; \delta)$. That is, (θ_0, θ_1) centralize θ_2 modulo δ. In Definition 2.1, $\sigma = \mathrm{id}$.

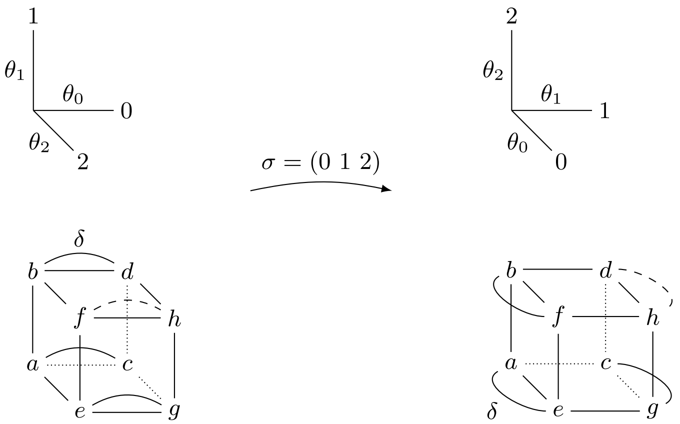

Fig. 4. The condition $\mathrm{C}(\theta_1, \theta_2, \theta_0; \delta)$. That is, (θ_1, θ_2) centralize θ_0 modulo δ. In Definition 2.1, $\sigma = (0\ 1\ 2)$. Applying σ to the coordinate axes gives a picture similar to Fig. 3.

(1) $[\alpha_0, \ldots, \alpha_{k-1}] \leq \bigwedge_{0 \leq i \leq k-1} \alpha_i$,

(2) *For* $\alpha_0 \leq \beta_0, \ldots, \alpha_{k-1} \leq \beta_{k-1}$ *in* $\mathrm{Con}(\mathbb{A})$, *we have*

$$[\alpha_0, \ldots, \alpha_{k-1}] \leq [\beta_0, \ldots, \beta_{k-1}].$$

That is, the commutator is monotone in each argument.

(3) $\underbrace{[\alpha_0, \ldots, \alpha_{k-1}]}_{k\text{-}ary} \leq \underbrace{[\alpha_1, \ldots, \alpha_{k-1}]}_{(k-1)\text{-}ary}$.

2.3. Nilpotence, supernilpotence, and solvability

Let $\mathbb{A}$ be an algebra and let $\alpha \in \mathrm{Con}(\mathbb{A})$. Recursively define over ω the congruences $[\alpha]_0 := \alpha =: (\alpha]_0$,

$$[\alpha]_{n+1} := \big[[\alpha]_n, [\alpha]_n\big], \qquad \text{and} \qquad (\alpha]_{n+1} := \big[\alpha, (\alpha]_n\big]$$

to produce two descending chains, called the **derived** and **lower central series** of α, respectively:

$$[\alpha_0] \geq [\alpha]_1 \geq \cdots \geq [\alpha]_n \geq \ldots \qquad \text{and} \qquad (\alpha_0] \geq (\alpha]_1 \geq \cdots \geq (\alpha]_n \geq \ldots.$$

If $[\alpha]_n = 0$ or $(\alpha]_n = 0$, then α is said to be (n)**-step solvable** or (n)**-step nilpotent**, respectively. Since the binary commutator is monotonic in each of its arguments, it follows that nilpotence is a stronger condition than solvability.

A congruence α of $\mathbb{A}$ is said to be (n)**-step supernilpotent** if it satisfies

$$\underbrace{[\alpha, \ldots, \alpha]}_{(n-1)\text{-ary}} = 0.$$

The reason for this terminology can be found in Aichinger and Mudrinksi [1], where it is shown that for a congruence permutable variety, all higher commutators of appropriate arity satisfy what they call HC8, which is an inequality involving nested commutators:

$$\big[\theta_0, \ldots, \theta_{m-1}, [\theta_m, \ldots, \theta_{n-1}]\big] \leq [\theta_0, \ldots, \theta_{n-1}]. \tag{HC8}$$

Therefore, for congruence permutable varieties an easy induction shows that if a congruence α is (n)-step supernilpotent then it must also be (n)-step nilpotent (and hence also (n)-step solvable).

If $\alpha = 1$ we simply say that the algebra $\mathbb{A}$ is (n)-step nilpotent, solvable, or supernilpotent, as the case may be. We conclude this section with a description of supernilpotence using the vocabulary that has been developed in this paper. The proof is only a translation of definitions and is therefore omitted.

Proposition 2.4. *Let $\mathbb{A}$ be an algebra, $n \geq 2$ a natural number, and $i \in n$. The algebra $\mathbb{A}$ is $(n-1)$-step supernilpotent if and only if there is no (n)-dimensional cube $h \in M(1, \ldots, 1)$ such that*

(1) *every line belonging to* $\text{S-Lines}_k(h)$ *is constant, and*
(2) *the (k)-pivot of h line is not constant.*

3. Generalized nilpotence and solvability

The main goal of this section is to demonstrate that the condition of nilpotence can be quite complicated and that, for our purposes, the condition of solvability is more useful. As noted in Section 1, the properties of nilpotence and solvability can be defined with the term condition commutator.

A choice was made in our definition of nilpotence to consistently evaluate the first argument of the binary commutator at α and the second argument at $(\alpha]_n$. If the commutator for $\mathbb{A}$ is symmetric then this choice is immaterial, but if the commutator fails to be symmetric then this choice is important. In the non-symmetric case, our definition of nilpotence is demoted to what we call **left nilpotence**. The notion of **right nilpotence** is defined in the obvious analogous way.

Left and right nilpotence are not the same, as demonstrated by the following example. Let G and $\{o\}$ be disjoint sets with G infinite. Let $A = G \cup \{o\}$ and fix some injection $s : A^2 \to G$. Let $\mathbb{A} = \langle A; t\rangle$ be the algebra with binary operation t defined by

$$t(x, y) = \begin{cases} o & \text{if } x = o, \\ s(x, y) & \text{otherwise.} \end{cases}$$

$\mathbb{A}$ is not left nilpotent, because for each $n \in \omega$ there is a $(1]_n$-class with infinitely many elements, namely $\{t(a, y) : y \in A\}$ for $a \in G$ via

$$t\left(\begin{array}{ccc} o & \text{—} & a \\ | & & | \\ o & \text{—} & a \end{array}, \begin{array}{ccc} y & \text{—} & y \\ | & & | \\ a & \text{—} & a \end{array}\right) = \begin{array}{ccc} o & \text{—} & t(a, y) \\ | & & | \\ o & \text{—} & t(a, a) \end{array}.$$

However, $\mathbb{A}$ is right nilpotent. To see this, let δ be the congruence with classes G and $\{o\}$. It is a routine exercise to show that $\mathrm{C}(1, 1; \delta)$ holds and that $[\delta, 1] = 0$. A consequence of this is that $[1, 1] \leq \delta$ and now (2) of Proposition 2.3 leads to the conclusion that $[[1, 1], 1] = 0$.

A moment's reflection will reveal that the situation can be complicated. Let $\mathcal{T}_{[\cdot,\cdot]}(\{x\})$ be the collection of all single-variable terms in the binary operation symbol $[\cdot, \cdot]$. The previous definitions of solvability and nilpotence are statements of the form

$$\mathrm{Con}(\mathbb{A}) \models \big(t(1) = 0\big)$$

for some special $t(x) \in \mathcal{T}_{[\cdot,\cdot]}(\{x\})$, and the example above shows that nilpotence witnessed by a particular term $t(x)$ need not imply that all terms of a particular depth evaluate to 0. The addition of higher arity commutators to the language allows for more complicated terms. That is, let

$$\mathcal{L}_n = \Big\{[\cdot, \cdot], \ldots, \underbrace{[\cdot, \ldots, \cdot]}_{n\text{-ary}}\Big\}$$

be the set of commutator operation symbols of arity at most n and $\mathcal{T}_{\mathcal{L}_n}(\{x\})$ be the set of all single variable terms in the operation symbols appearing in $\mathcal{L}_n$. We can now ask whether

$$\mathrm{Con}(\mathbb{A}) \models \big(t(1) = 0\big)$$

for some $t(x) \in \mathcal{T}_{\mathcal{L}_n}(\{x\})$.

Our aim in this article is not to explore these complexities in full detail, but rather to construct algebras $\mathbb{A}_n$ such that for all $t(x) \in \mathcal{T}_{\mathcal{L}_n}(\{x\})$,

$$\mathrm{Con}(\mathbb{A}) \not\models \big(t(1) = 0\big)$$

but

$$\mathrm{Con}(\mathbb{A}) \models \big(\underbrace{[1,\ldots,1]}_{(n+1)\text{-ary}} = 0\big).$$

These two conditions say that $\mathbb{A}_n$ fails to be nilpotent for any definition one could produce involving commutators up to arity n, but is nevertheless (n)-step supernilpotent.

We can simplify the problem by introducing a generalization of solvability. For $n \geq 2$ and $\alpha \in \mathrm{Con}(\mathbb{A})$, define $[\alpha]_0^n := \alpha$. Now recursively define over ω the descending chain of congruences

$$[\alpha]_{m+1}^n := \underbrace{\Big[[\alpha]_m^n,\ldots,[\alpha]_m^n\Big]}_{n\text{-ary}}.$$

If $[\alpha]_m^n = 0$ for some $m, n \in \omega$ we say that α is (m)**-step solvable in dimension** n.

Lemma 3.1. *Let $\mathbb{A}$ be an algebra, $\alpha \in \mathrm{Con}(\mathbb{A})$, and $n \geq 2$ be a natural number. For all $t(x) \in \mathcal{T}_{\mathcal{L}_n}(\{x\})$ there exists $m \in \omega$ such that*

$$\mathrm{Con}(\mathbb{A}) \models \Big([\alpha]_m^n \leq t(\alpha)\Big).$$

Proof. The proof proceeds by induction on the complexity of terms. It is clear that the Lemma holds when $t(x) = x$, establishing the basis. Suppose that

$$t(x) = [s_0(x),\ldots,s_{k-1}(x)]$$

for some terms $s_0,\ldots,s_{k-1}$, where $k \leq n$. By the inductive hypothesis there exist $m_0,\ldots,m_{k-1} \in \omega$ such that

$$\mathrm{Con}(\mathbb{A}) \models \Big([\alpha]_{m_i}^n \leq s_i(\alpha)\Big),$$

for each $i \in k$. Set m to be the maximum of $m_0,\ldots,m_{k-1}$. It follows from (2) and (3) of Proposition 2.3 that

$$\mathrm{Con}(\mathbb{A}) \models \Bigg(\underbrace{\big[[\alpha]_m^n,\ldots,[\alpha]_m^n\big]}_{n\text{-ary}} \leq \underbrace{\big[[\alpha]_m^n,\ldots,[\alpha]_m^n\big]}_{k\text{-ary}} \leq t(\alpha)\Bigg).$$

This completes the proof. $\square$

Proposition 3.2. *Let $\mathbb{A}$ be an algebra, $\alpha \in \mathrm{Con}(\mathbb{A})$, and let $n \geq 2$ be a natural number. For all $t(x) \in \mathcal{T}_{\mathcal{L}_n}(\{x\})$,*

$$\mathrm{Con}(\mathbb{A}) \not\models \big(t(\alpha) = 0\big)$$

if and only if α fails to be (m)-step solvable in dimension n for all $m \in \omega$.

Proof. One direction is obvious. For the other direction, suppose that there is some $t(x) \in \mathcal{T}_{\mathcal{L}_n}(\{x\})$ such that

$$\mathrm{Con}(\mathbb{A}) \models \big(t(\alpha) = 0\big).$$

By Lemma 3.1 there is an $m \in \omega$ such that

$$\mathrm{Con}(\mathbb{A}) \models \Big([\alpha]^n_m \leq t(\alpha)\Big).$$

This forces α to be (m)-step solvable in dimension n. $\square$

4. The algebra $\mathbb{A}_2$

Let O, R, G be disjoint countably infinite sets where the elements of O and R are indexed as follows:

$$O = \{o^j_i : i, j \in \omega\} \qquad \text{and} \qquad R = \{r^j_i : i, j \in \omega\}.$$

Define $A_2 = O \cup R \cup G$ and let $\mathbb{A}_2 = \langle A_2; t\rangle$ be the algebra with underlying set A_2 and a binary operation $t : (A_2)^2 \to A_2$ defined below.

(1) For all $i, j \in \omega$,

$$\begin{aligned} &t(r^j_{4i}, r^j_{4i}) = t(r^j_{4i}, r^j_{4i+2}) := o^j_i, \\ &t(r^j_{4i+2}, r^j_{4i}) := r^{j+1}_i, \\ &t(r^j_{4i+2}, r^j_{4i+2}) := r^{j+1}_{i+1}. \end{aligned}$$

This can be written compactly as

$$t\left(\begin{array}{ccc} r^j_{4i} & \text{---} & r^j_{4i+2} \\ \Big| & & \Big| \\ r^j_{4i} & \text{---} & r^j_{4i+2} \end{array}, \begin{array}{ccc} r^j_{4i} & \text{---} & r^j_{4i} \\ \Big| & & \Big| \\ r^j_{4i+2} & \text{---} & r^j_{4i+2} \end{array}\right) := \begin{array}{ccc} o^j_i & \text{---} & r^{j+1}_i \\ \Big| & & \Big| \\ o^j_i & \text{---} & r^{j+1}_{i+1} \end{array}.$$

(2) Otherwise, $t(x, y) := s(x, y)$ for some injective function $s : (A_2)^2 \to G$.

See Fig. 5.

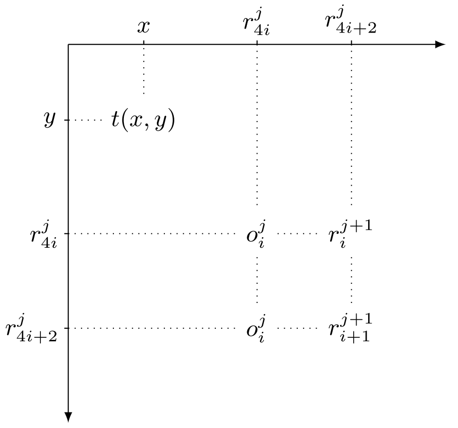

Fig. 5. Partial Multiplication Table for t.

4.1. $\mathbb{A}_2$ *is not solvable*

We will prove that the algebra $\mathbb{A}_2$ fails to be (n)-step solvable (in dimension 2) for all $n \in \omega$. Recall that the derived series of $\mathbb{A}_2$ is the sequence of congruences

$$1 = [1]_0 \geq \cdots \geq [1]_n \geq [1]_{n+1} \geq \ldots,$$

where $[1]_{n+1} = \big[[1]_n, [1]_n\big]$.

Lemma 4.1. *Let $\mathbb{A}_2 = \langle A_2; t\rangle$ be the algebra defined at the start of this section. For each $j \in \omega$, the set $R^j = \{r^j_i : i \in \omega\} \subseteq R$ is contained in a $[1]_j$-class.*

Proof. The proof proceeds by induction on j. The Lemma clearly holds for $j = 0$, establishing the basis. Suppose that $[1]_j$ has a class that contains the set $R^j = \{r^j_i : i \in \omega\}$. It follows that

$$\begin{array}{ccc} r^j_{4i} & \text{---} & r^j_{4i+2} \\ | & & | \\ r^j_{4i} & \text{---} & r^j_{4i+2} \end{array} \ , \quad \begin{array}{ccc} r^j_{4i} & \text{---} & r^j_{4i} \\ | & & | \\ r^j_{4i+2)} & \text{---} & r^j_{4i+2} \end{array} \ \in \mathrm{M}\left([1]_j, [1]_j\right)$$

for each $i \in \omega$. Therefore,

$$t\left(\begin{array}{ccc} r^j_{4i} & \text{---} & r^j_{4i+2} \\ | & & | \\ r^j_{4i} & \text{---} & r^j_{4i+2} \end{array} \ , \quad \begin{array}{ccc} r^j_{4i} & \text{---} & r^j_{4i} \\ | & & | \\ r^j_{4i+2} & \text{---} & r^j_{4i+2} \end{array}\right) = \begin{array}{ccc} o^j_i & \text{---} & r^{j+1}_i \\ | & & | \\ o^j_i & \text{---} & r^{j+1}_{i+1} \end{array} \ \in \mathrm{M}([1]_j, [1]_j).$$

Since $\left\langle o^j_i, o^j_i\right\rangle \in \big[[1]_j, [1]_j\big]$, we conclude that $\left\langle r^{j+1}_i, r^{j+1}_{i+1}\right\rangle \in [1]_{j+1}$ for each $i \in \omega$. Equivalence relations are transitively closed, so it follows that $\left\langle r^{j+1}_0, r^{j+1}_i\right\rangle \in [1]_{j+1}$ for each

$i \in \omega$. Therefore, $R^{j+1} = \{r_i^{j+1} : i \in \omega\}$ is a subset of the class of $[1]_{j+1}$ that is represented by r_0^{j+1}. This completes the induction and the proof. $\square$

Theorem 4.2. *The algebra $\mathbb{A}_2 = \langle A_2; t\rangle$ is not solvable (in dimension 2).*

Proof. If $\mathbb{A}_2$ were solvable then there would exist an $n \in \omega$ such that

$$[1]_n = 0.$$

In particular, every class of $[1]_n$ would contain exactly one element, but Lemma 4.1 ensures the existence of a class with infinitely many elements. $\square$

4.2. $\mathbb{A}_2$ is supernilpotent

We will now prove that the algebra $\mathbb{A}_2$ is (2)-step supernilpotent. The proof is an induction on the complexity of terms that generate the algebra of $(1,1,1)$-matrices (i.e. $\mathrm{M}(1,1,1)$). Before embarking on the proof, however, we must build up some of the necessary machinery. The following lemmas are proved in full generality in Section 5 at the start of Subsection 5.2. There is not a strong geometrical intuition that can be gained from examining the lower-dimension proofs, so we refer the reader to the next section for detailed justification of these lemmas.

Lemma 4.3. *Let $\mathbb{A}_2 = \langle A_2; t\rangle$ be the algebra defined at the start of this section.*

(1) *If $t(a,b) \in R \cup O$ then $a, b \in R$.*
(2) *If $t(a,b) = t(c,d) \notin O$ then $(a,b) = (c,d)$.*
(3) *If $t(a,b) = t(c,d)$ and $(a,b) \neq (c,d)$, then*
  - *$t(a,b) = t(c,d) = o_i^j$ for some $o_i^j \in O$,*
  - *$a = c = r_{4i}^j$, and*
  - *$\{b, d\} = \{r_{4i}^j, r_{4i+2}^j\}$.*

Lemma 4.4. *If*

$$h = \begin{array}{ccc} c & \text{——} & r_i^j \\ | & & | \\ c & \text{——} & r_k^\ell \end{array} \in M(1,1) \quad \textit{for some } c \in A_2,$$

then $j = \ell$ and $|i - k| \in \{0, 1\}$.

Lemma 4.5. *If*

$$h = \begin{array}{ccc} r_i^j & \text{——} & r_\ell^k \\ | & & | \\ r_u^v & \text{——} & a \end{array} \in M(1,1)$$

for some $r_i^j, r_l^k, r_u^v \in R$ *and* $a \in A_2$*, then*

$$h \in \left\{ \begin{array}{ccc} x & \text{---} & x \\ | & & | \\ y & \text{---} & y \end{array}, \begin{array}{ccc} x & \text{---} & y \\ | & & | \\ x & \text{---} & y \end{array} : x, y \in A_2 \right\}.$$

We are now ready to prove that $\mathbb{A}_2$ is not (2)-step supernilpotent. Although the proof of this theorem can be worked out from the proof of the higher-dimensional Theorem 5.7, we include it here in the hope that it will provide some geometrical intuition for the general case.

Theorem 4.6. *The algebra* $\mathbb{A}_2 = \langle A_2; t \rangle$ *is* (2)*-step supernilpotent.*

Proof. By Proposition 2.4, $\mathbb{A}_2$ is (2)-step supernilpotent if and only if

$$h = \begin{array}{cccc} a & & b & \\ & c & & e \\ a & & b & \\ & c & & d \end{array} \in \mathrm{M}(1,1,1)$$

implies $e = d$. In other words, if the vertical support lines are constant, then the vertical pivot line is constant as well. Set

$$X_0 = \left\{ \begin{array}{cccc} x & & y & \\ & x & & y \\ x & & y & \\ & x & & y \end{array}, \begin{array}{cccc} y & & y & \\ & y & & y \\ x & & x & \\ & x & & x \end{array}, \begin{array}{cccc} x & & x & \\ & y & & y \\ x & & x & \\ & y & & y \end{array} : x, y \in A_2 \right\} \quad \text{and}$$

$$X_{n+1} = X_n \cup \Big\{ t(a, b) : a, b \in X_n \Big\}.$$

By definition, $\mathrm{M}(1,1,1) = \mathrm{Sg}_{(A_2)^{2^3}}(X) = \bigcup_{n \in \mathbb{N}} X_n$. We proceed by induction on n.

For a cube $h \in X_0$ it is true that having constant vertical support lines implies a constant vertical pivot line, establishing the basis. Suppose now that this implication holds for X_n and that

$$h = \begin{array}{cccc} a & & b & \\ & c & & e \\ a & & b & \\ & c & & d \end{array} \in X_{n+1} \setminus X_n.$$

We will show that $d = e$. We have that

$$h = \begin{tikzcd}[row sep=tiny, column sep=tiny] a \arrow[rr, dash] \arrow[dr, dash] \arrow[dd, dash] & & b \arrow[dr, dash] \arrow[dd, dash, dotted] & \\ & c \arrow[rr, dash] \arrow[dd, dash] & & e \arrow[dd, dash] \\ a \arrow[rr, dash, dotted] \arrow[dr, dash] & & b \arrow[dr, dash, dotted] & \\ & c \arrow[rr, dash] & & d \end{tikzcd} = t\left(\begin{tikzcd}[row sep=tiny, column sep=tiny] a'_0 \arrow[rr, dash] \arrow[dr, dash] \arrow[dd, dash] & & b'_0 \arrow[dr, dash] \arrow[dd, dash, dotted] & \\ & c'_0 \arrow[rr, dash] \arrow[dd, dash] & & e_0 \arrow[dd, dash] \\ a_0 \arrow[rr, dash, dotted] \arrow[dr, dash] & & b_0 \arrow[dr, dash, dotted] & \\ & c_0 \arrow[rr, dash] & & d_0 \end{tikzcd}, \begin{tikzcd}[row sep=tiny, column sep=tiny] a'_1 \arrow[rr, dash] \arrow[dr, dash] \arrow[dd, dash] & & b'_1 \arrow[dr, dash] \arrow[dd, dash, dotted] & \\ & c'_1 \arrow[rr, dash] \arrow[dd, dash] & & e_1 \arrow[dd, dash] \\ a_1 \arrow[rr, dash, dotted] \arrow[dr, dash] & & b_1 \arrow[dr, dash, dotted] & \\ & c_1 \arrow[rr, dash] & & d_1 \end{tikzcd} \right),$$

where the two argument cubes are elements of X_n. From Lemma 4.3, it must be that $a_0 = a'_0$, $b_0 = b'_0$, and $c_0 = c'_0$. Applying the inductive hypothesis to the first argument cube now yields $e_0 = d_0$, so the first argument cube has its bottom face equal to its top face. Observe that we need only prove that $e_1 = d_1$ since $e_0 = d_0$ already. The situation is now

$$h = \begin{tikzcd}[row sep=tiny, column sep=tiny] a \arrow[rr, dash] \arrow[dr, dash] \arrow[dd, dash] & & b \arrow[dr, dash] \arrow[dd, dash, dotted] & \\ & c \arrow[rr, dash] \arrow[dd, dash] & & e \arrow[dd, dash] \\ a \arrow[rr, dash, dotted] \arrow[dr, dash] & & b \arrow[dr, dash, dotted] & \\ & c \arrow[rr, dash] & & d \end{tikzcd} = t\left(\begin{tikzcd}[row sep=tiny, column sep=tiny] a_0 \arrow[rr, dash] \arrow[dr, dash] \arrow[dd, dash] & & b_0 \arrow[dr, dash] \arrow[dd, dash, dotted] & \\ & c_0 \arrow[rr, dash] \arrow[dd, dash] & & d_0 \arrow[dd, dash] \\ a_0 \arrow[rr, dash, dotted] \arrow[dr, dash] & & b_0 \arrow[dr, dash, dotted] & \\ & c_0 \arrow[rr, dash] & & d_0 \end{tikzcd}, \begin{tikzcd}[row sep=tiny, column sep=tiny] a'_1 \arrow[rr, dash] \arrow[dr, dash] \arrow[dd, dash] & & b'_1 \arrow[dr, dash] \arrow[dd, dash, dotted] & \\ & c'_1 \arrow[rr, dash] \arrow[dd, dash] & & e_1 \arrow[dd, dash] \\ a_1 \arrow[rr, dash, dotted] \arrow[dr, dash] & & b_1 \arrow[dr, dash, dotted] & \\ & c_1 \arrow[rr, dash] & & d_1 \end{tikzcd} \right). \tag{4.1}$$

Let S be the set of vertical support lines of the second argument cube and let D be the set of constant lines:

$$S = \left\{ \begin{smallmatrix} a'_1 \\ | \\ a_1 \end{smallmatrix}, \begin{smallmatrix} b'_1 \\ | \\ b_1 \end{smallmatrix}, \begin{smallmatrix} c'_1 \\ | \\ c_1 \end{smallmatrix} \right\}, \qquad D = \left\{ \begin{smallmatrix} \alpha \\ | \\ \alpha \end{smallmatrix} : \alpha \in A_2 \right\}.$$

We will proceed with a case analysis of $S \cap D$.

Case $S \cap D \neq \emptyset$: In this case, there is a constant vertical support line of the second argument cube, say $c_1 = c'_1$. Suppose towards a contradiction that $a_1 \neq a'_1$. Since h has all constant vertical support lines, t evaluated at this line must be a failure of injectivity. By Lemma 4.3, it must be that (modulo vertical reflection) $a = o_i^j$, $a_0 = r_{4i}^j$, $a_1 = r_{4i}^j$, and $a'_1 = r_{4i+2}^j$. Equation (4.1) is now

$$h = \begin{tikzcd}[row sep=tiny, column sep=tiny] o_i^j \arrow[rr, dash] \arrow[dr, dash] \arrow[dd, dash] & & b \arrow[dr, dash] \arrow[dd, dash, dotted] & \\ & c \arrow[rr, dash] \arrow[dd, dash] & & e \arrow[dd, dash] \\ o_i^j \arrow[rr, dash, dotted] \arrow[dr, dash] & & b \arrow[dr, dash, dotted] & \\ & c \arrow[rr, dash] & & d \end{tikzcd} = t\left(\begin{tikzcd}[row sep=tiny, column sep=tiny] r_{4i}^j \arrow[rr, dash] \arrow[dr, dash] \arrow[dd, dash] & & b_0 \arrow[dr, dash] \arrow[dd, dash, dotted] & \\ & c_0 \arrow[rr, dash] \arrow[dd, dash] & & d_0 \arrow[dd, dash] \\ r_{4i}^j \arrow[rr, dash, dotted] \arrow[dr, dash] & & b_0 \arrow[dr, dash, dotted] & \\ & c_0 \arrow[rr, dash] & & d_0 \end{tikzcd}, \begin{tikzcd}[row sep=tiny, column sep=tiny] r_{4i+2}^j \arrow[rr, dash] \arrow[dr, dash] \arrow[dd, dash] & & b'_1 \arrow[dr, dash] \arrow[dd, dash, dotted] & \\ & c_1 \arrow[rr, dash] \arrow[dd, dash] & & e_1 \arrow[dd, dash] \\ r_{4i}^j \arrow[rr, dash, dotted] \arrow[dr, dash] & & b_1 \arrow[dr, dash, dotted] & \\ & c_1 \arrow[rr, dash] & & d_1 \end{tikzcd} \right).$$

If we apply Lemma 4.4 to the left face of the second cube, we obtain a contradiction, since $|4i + 2 - 4i| = 2 \notin \{0, 1\}$. It follows that $a_1 = a'_1$. We can continue around the cube in this manner to obtain all constant vertical support lines, forcing $e_1 = d_1$ by the inductive hypothesis. This, in turn, implies that $e = d$. At the start of this case we assumed that the (c_1, c'_1) line was the constant vertical support line, but the above argument works no matter which support line is constant.

Case $S \cap D = \emptyset$: In this case, from the definition of t and Lemma 4.3, equation 4.1 looks like

$$h = \begin{tikzcd}[row sep=small, column sep=small] o_k^\ell \arrow[rr, dash] \arrow[dr, dash] \arrow[dd, dash] & & o_m^n \arrow[dr, dash] \arrow[dd, dash, dotted] & \\ & o_i^j \arrow[rr, dash] \arrow[dd, dash] & & e \arrow[dd, dash] \\ o_k^\ell \arrow[rr, dash, dotted] \arrow[dr, dash] & & o_m^n \arrow[dr, dash, dotted] & \\ & o_i^j \arrow[rr, dash] & & d \end{tikzcd} = t\left(\begin{tikzcd}[row sep=small, column sep=small] r_{4k}^\ell \arrow[rr, dash] \arrow[dr, dash] \arrow[dd, dash] & & r_{4m}^n \arrow[dr, dash] \arrow[dd, dash, dotted] & \\ & r_{4i}^j \arrow[rr, dash] \arrow[dd, dash] & & d_0 \arrow[dd, dash] \\ r_{4k}^\ell \arrow[rr, dash, dotted] \arrow[dr, dash] & & r_{4m}^n \arrow[dr, dash, dotted] & \\ & r_{4i}^j \arrow[rr, dash] & & d_0 \end{tikzcd}, \begin{tikzcd}[row sep=small, column sep=small] r_{4k+\beta}^\ell \arrow[rr, dash] \arrow[dr, dash] \arrow[dd, dash] & & r_{4m+\delta}^n \arrow[dr, dash] \arrow[dd, dash, dotted] & \\ & r_{4i+\tau}^j \arrow[rr, dash] \arrow[dd, dash] & & e_1 \arrow[dd, dash] \\ r_{4k+\alpha}^\ell \arrow[rr, dash, dotted] \arrow[dr, dash] & & r_{4m+\gamma}^n \arrow[dr, dash, dotted] & \\ & r_{4i+\epsilon}^j \arrow[rr, dash] & & d_1 \end{tikzcd} \right),$$

where $\{0,2\} = \{\alpha,\beta\} = \{\gamma,\delta\} = \{\epsilon,\tau\}$. Applying Lemma 4.5 to the leftmost face, we obtain $r_{4k+\alpha}^\ell = r_{4i+\epsilon}^j$ and $r_{4k+\beta}^\ell = r_{4i+\tau}^j$. Similarly, the back face implies that $r_{4k+\alpha}^\ell = r_{4m+\gamma}^n$ and $r_{4k+\beta}^\ell = r_{4m+\delta}^n$. The situation is now (modulo vertically flipping the second cube)

$$h = \begin{tikzcd}[row sep=small, column sep=small] o_k^\ell \arrow[rr, dash] \arrow[dr, dash] \arrow[dd, dash] & & o_k^\ell \arrow[dr, dash] \arrow[dd, dash, dotted] & \\ & o_k^\ell \arrow[rr, dash] \arrow[dd, dash] & & e \arrow[dd, dash] \\ o_k^\ell \arrow[rr, dash, dotted] \arrow[dr, dash] & & o_k^\ell \arrow[dr, dash, dotted] & \\ & o_k^\ell \arrow[rr, dash] & & d \end{tikzcd} = t\left(\begin{tikzcd}[row sep=small, column sep=small] r_{4k}^\ell \arrow[rr, dash] \arrow[dr, dash] \arrow[dd, dash] & & r_{4k}^\ell \arrow[dr, dash] \arrow[dd, dash, dotted] & \\ & r_{4k}^\ell \arrow[rr, dash] \arrow[dd, dash] & & d_0 \arrow[dd, dash] \\ r_{4k}^\ell \arrow[rr, dash, dotted] \arrow[dr, dash] & & r_{4k}^\ell \arrow[dr, dash, dotted] & \\ & r_{4k}^\ell \arrow[rr, dash] & & d_0 \end{tikzcd}, \begin{tikzcd}[row sep=small, column sep=small] r_{4k+2}^\ell \arrow[rr, dash] \arrow[dr, dash] \arrow[dd, dash] & & r_{4k+2}^\ell \arrow[dr, dash] \arrow[dd, dash, dotted] & \\ & r_{4k+2}^\ell \arrow[rr, dash] \arrow[dd, dash] & & e_1 \arrow[dd, dash] \\ r_{4k}^\ell \arrow[rr, dash, dotted] \arrow[dr, dash] & & r_{4k}^\ell \arrow[dr, dash, dotted] & \\ & r_{4k}^\ell \arrow[rr, dash] & & d_1 \end{tikzcd} \right).$$

Applying Lemma 4.5 to the top and bottom faces of the first argument cube implies that $d_0 = r_{4k}^\ell$. Applying Lemma 4.5 to the top and bottom faces of the second argument cube implies that $d_1 = r_{4k}^\ell$ and $e_1 = r_{4k+2}^\ell$. Evaluating it all gives us $e = o_k^\ell = d$, as desired.

This completes the case analysis and the induction. $\square$

5. The algebra $\mathbb{A}_n$

Let $n \in \omega$. Let O, R, G be disjoint sets, indexed as follows

$$O = \left\{ o_{i,g}^j : i,j \in \omega,\ g \in 2^{n-1} \right\} \qquad \text{and} \qquad R = \left\{ r_i^j : i,j \in \omega \right\}.$$

Define $A_n = O \cup R \cup G$ and let $\mathbb{A}_n = \langle A_n; t \rangle$ be the algebra with underlying set A_n and an n-ary operation $t : (A_n)^n \to A_n$ with values given by the cube equation

$$\left(t\Big(\mathrm{gCube}_0^n(r_{4i}^j, r_{4i+2}^j), \dots, \mathrm{gCube}_{n-1}^n(r_{4i}^j, r_{4i+2}^j) \Big) \right)_f := \begin{cases} r_i^{j+1} & \text{if } f = (1,\dots,1,0) \in 2^n, \\ r_{i+1}^{j+1} & \text{if } f = (1,\dots,1,1) \in 2^n, \\ o_{i,g}^j & \text{if } f|_{n-1} = g \neq \mathbf{1} \in 2^{n \setminus \{n-1\}}, \end{cases} \tag{5.1}$$

and otherwise t takes the value of some fixed injection $s : (A_n)^n \to G$. See Fig. 6.

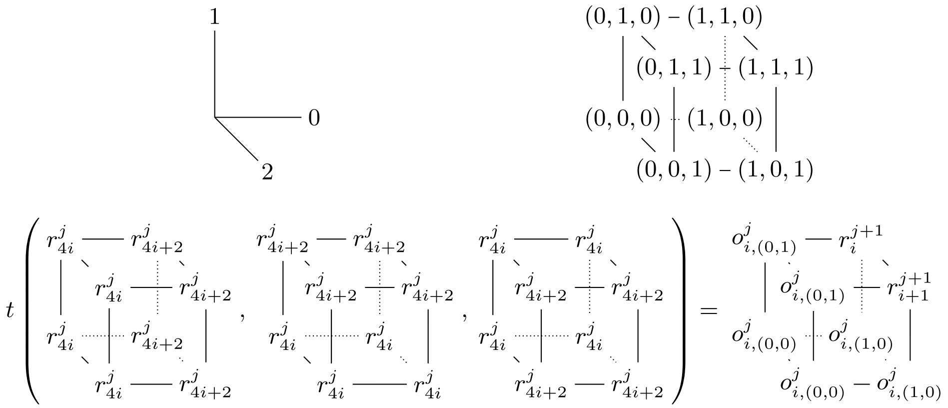

Fig. 6. Equation (5.1) with associated (n)-dimensional cube coordinate system for $n = 3$.

5.1. $\mathbb{A}_n$ is not solvable in dimension n

In this section we prove that the algebra $\mathbb{A}_n$ fails to be (j)-step solvable in dimension n for all $j \in \omega$. Recall that the (n)-dimensional generalization of the derived series of $\mathbb{A}_n$ is the sequence of congruences

$$1 = [1]^n_0 \geq \cdots \geq [1]^n_j \geq [1]^n_{j+1} \geq \dots,$$

where $[1]^n_{j+1} = \big[[1]^n_j, \dots, [1]^n_j\big]$ (n-ary). We now repeat the same analysis that we did in the previous section.

Lemma 5.1. *Let $\mathbb{A}_n = \langle A_n; t\rangle$ be the algebra defined above. For each $j \in \omega$, the set $R^j = \{r^j_i : i \in \omega\} \subseteq R$ is contained in a $[1]^n_j$-class.*

Proof. The proof proceeds by induction on j. The Lemma clearly holds for $j = 0$, establishing the basis. Suppose that $[1]^n_j$ has a class that contains the set $R^j = \{r^j_i : i \in \omega\}$. It follows that

$$\Big\{\mathrm{gCube}^n_k(r^j_{4i}, r^j_{4i+2}) : k \in n\Big\} \subseteq \mathrm{M}\,\big(\underbrace{[1]^n_j, \dots, [1]^n_j}_{n}\big)$$

for each $i \in \omega$. Therefore,

$$h = t\Big(\mathrm{gCube}^n_0(r^j_{4i}, r^j_{4i+2}), \dots, \mathrm{gCube}^n_{n-1}(r^j_{4i}, r^j_{4i+2})\Big) \in \mathrm{M}\,\big([1]^n_j, \dots, [1]^n_j\big).$$

By the definition of t, every element of $\text{S-Lines}_{n-1}(h)$ is constant. It follows that the pair determined by the $(n-1)$-pivot line of h must belong to $\big[[1]^n_j, \dots, [1]^n\big]$ (n-ary). The pair determined by the $(n-1)$-pivot line of h is $\big\langle r^{j+1}_i, r^{j+1}_{i+1}\big\rangle$, so we have shown that

$$\left\langle r_i^{j+1}, r_{i+1}^{j+1} \right\rangle \in [1]_{j+1}^n.$$

It follows that $\left\langle r_0^{j+1}, r_i^{j+1} \right\rangle \in [1]_{j+1}$ for each $i \in \omega$. Therefore, $R^{j+1} = \{r_i^{j+1} : i \in \omega\}$ is a subset of the class of $[1]_{j+1}^n$ that is represented by r_0^{j+1}. This completes the induction and the proof. □

Theorem 5.2. *The algebra $\mathbb{A}_n = \langle A_n; t \rangle$ is not solvable (in dimension n).*

Proof. If $\mathbb{A}_n$ were solvable in dimension n then there would exist an $m \in \omega$ such that

$$[1]_m^n = 0.$$

In particular, every class of $[1]_m^n$ would contain exactly one element, but Lemma 5.1 ensures the existence of a class with infinitely many elements. □

5.2. $\mathbb{A}_n$ is (n)-step supernilpotent

We now prove versions of Lemmas 4.3, 4.4, and 4.5 for $\mathbb{A}_n$. These lemmas describe in detail the exact manner in which the operation t fails to be injective, and the different kinds of squares that can appear in $\mathrm{M}(1,1)$. The following Lemma follows immediately from the definition of $\mathbb{A}_n$ and is therefore omitted.

Lemma 5.3. *Let $\mathbb{A}_n = \langle A_n; t \rangle$ be the algebra defined at the start of this section.*

(1) *If $t(\overline{a}) \in R \cup O$ then $\overline{a} \in R^n$.*
(2) *If $t(\overline{a}) = t(\overline{b}) \notin O$ then $\overline{a} = \overline{b}$.*
(3) *If $t(\overline{a}) = t(\overline{b})$ and $\overline{a} \neq \overline{b}$, then*
- $t(\overline{a}) = t(\overline{b}) = o_{i,g}^j$ *for some* $o_{i,g}^j \in O$,
- $a_k = b_k \in \left\{r_{4i}^j, r_{4i+2}^j\right\}$ *for all* $k \in (n-1)$, *and*
- $\left\{a_{n-1}, b_{n-1}\right\} = \left\{r_{4i}^j, r_{4i+2}^j\right\}$.

We are now ready to begin our analysis of the squares in $\mathrm{M}(1,1)$.

Lemma 5.4. *If*

$$h = \begin{array}{ccc} c & \text{——} & r_i^j \\ \Big| & & \Big| \\ c & \text{——} & r_k^\ell \end{array} \in M(1,1) \quad \textit{for some } c \in A_n,$$

then $j = \ell$ and $|i-k| \in \{0,1\}$.

Proof. The proof shall be by induction on the level at which h first appears during subalgebra generation. Let

$$X_0 = \left\{ \begin{array}{ccc} x & \text{---} & x \\ | & & | \\ y & \text{---} & y \end{array} , \begin{array}{ccc} x & \text{---} & y \\ | & & | \\ x & \text{---} & y \end{array} : x, y \in A_n \right\} \qquad \text{and}$$

$$X_{m+1} = X_m \cup \Big\{ t(\overline{a}) : \overline{a} \in (X_m)^n \Big\}.$$

By definition, $\mathrm{M}(1,1) = \mathrm{Sg}_{(\mathbb{A}_n)^{2^2}}(X_0) = \bigcup_{m \in \omega} X_m$. We will proceed by induction on m. The Lemma clearly holds for $h \in X_0$, establishing the basis. Suppose now that the Lemma holds for X_m and that

$$h = \begin{array}{ccc} c & \text{---} & r_i^j \\ | & & | \\ c & \text{---} & r_k^\ell \end{array} \in X_{m+1} \setminus X_m.$$

We will prove that $j = \ell$ and $|i - k| \in \{0, 1\}$.

From Lemma 5.3, it must be that

$$h = \begin{array}{ccc} c & \text{---} & r_i^j \\ | & & | \\ c & \text{---} & r_k^\ell \end{array} = t\Bigg(\underbrace{ \begin{array}{ccc} a_0 & \text{---} & r_{u_0}^{j_0 - 1} \\ | & & | \\ a_0 & \text{---} & r_{v_0}^{\ell_0 - 1} \end{array}, \dots, \begin{array}{ccc} a_{n-2} & \text{---} & r_{u_{n-2}}^{j_{n-2} - 1} \\ | & & | \\ a_{n-2} & \text{---} & r_{v_{n-2}}^{\ell_{n-2} - 1} \end{array}, \begin{array}{ccc} b_{n-1} & \text{---} & \beta \\ | & & | \\ a_{n-1} & \text{---} & \alpha \end{array} }_{\in X_m} \Bigg)$$

(note that the last square need not have equal vertical lines). Applying the inductive hypothesis to the first $(n-1)$ argument squares gives us $j_p = \ell_p$ and $|u_p - v_p| \in \{0, 1\}$ for all $p \in (n-1)$.

Consider the evaluation of t on the rightmost vertical lines of the argument squares:

$$t\left(\begin{array}{c} r_{u_0}^{j_0 - 1} \\ | \\ r_{v_0}^{\ell_0 - 1} \end{array}, \dots, \begin{array}{c} r_{u_{n-2}}^{j_{n-2} - 1} \\ | \\ r_{v_{n-2}}^{\ell_{n-2} - 1} \end{array}, \begin{array}{c} \beta \\ | \\ \alpha \end{array} \right) = \begin{array}{c} r_i^j \\ | \\ r_k^\ell \end{array}.$$

From Lemma 5.3 and the definition of t, the only way that this is possible is if there are some $\epsilon, \tau \in \{0, 1\}$ such that for all $p \in (n-1)$

$$\begin{aligned} j_p &= j, & u_p &= 4(i - \epsilon) + 2, & \alpha &= r^j_{4(i-\epsilon)+2\epsilon}, \\ \ell_p &= \ell, & v_p &= 4(k - \tau) + 2, & \beta &= r^\ell_{4(k-\tau)+2\tau}. \end{aligned}$$

The reader is encouraged to consult Fig. 6. Combining this with the conclusions from the end of the previous paragraph, we have that $j = \ell$ and

$$|u_p - v_p| = \Big| \big(4(i - \epsilon) + 2\big) - \big(4(k - \tau) + 2\big) \Big| = 4 \big| (i - k) - (\epsilon - \tau) \big| \in \{0, 1\}.$$

This implies that $i - k = \epsilon - \tau$. For all possibilities of $\epsilon, \tau \in \{0, 1\}$, we have $|i - k| \in \{0, 1\}$. This completes the induction, and finishes the proof. $\square$

Lemma 5.5. *If*

$$h = \begin{tikzcd} r_i^j \arrow[r, dash] \arrow[d, dash] & r_\ell^k \arrow[d, dash] \\ r_u^v \arrow[r, dash] & a \end{tikzcd} \in M(1,1)$$

for some $r_i^j, r_l^k, r_u^v \in R$ *and* $a \in A_n$, *then*

$$h \in \left\{ \begin{tikzcd} x \arrow[r, dash] \arrow[d, dash] & x \arrow[d, dash] \\ y \arrow[r, dash] & y \end{tikzcd}, \begin{tikzcd} x \arrow[r, dash] \arrow[d, dash] & y \arrow[d, dash] \\ x \arrow[r, dash] & y \end{tikzcd} : x, y \in A_n \right\}.$$

Proof. The proof is similar to the proof of Lemma 5.4, and we begin the same way. Let

$$X_0 = \left\{ \begin{tikzcd} x \arrow[r, dash] \arrow[d, dash] & x \arrow[d, dash] \\ y \arrow[r, dash] & y \end{tikzcd}, \begin{tikzcd} x \arrow[r, dash] \arrow[d, dash] & y \arrow[d, dash] \\ x \arrow[r, dash] & y \end{tikzcd} : x, y \in A_n \right\} \qquad \text{and}$$

$$X_{m+1} = X_m \cup \left\{ t(\overline{a}) : \overline{a} \in (X_m)^n \right\},$$

so that $\mathrm{M}(1,1) = \mathrm{Sg}_{(\mathbb{A}_n)^{2^2}}(X_0) = \bigcup_{m \in \omega} X_m$. We proceed by induction on m. The Lemma trivially holds for $h \in X_0$, establishing the basis. Suppose now that the Lemma holds for X_n and that

$$h = \begin{tikzcd} r_i^j \arrow[r, dash] \arrow[d, dash] & r_\ell^k \arrow[d, dash] \\ r_u^v \arrow[r, dash] & a \end{tikzcd} \in X_{m+1} \setminus X_m.$$

We will prove that $h \in X_0$. As in Lemma 5.4, this implies (from the definition of t and by Lemma 5.3) that

$$h = \begin{tikzcd} r_i^j \arrow[r, dash] \arrow[d, dash] & r_\ell^k \arrow[d, dash] \\ r_u^v \arrow[r, dash] & a \end{tikzcd}$$

$$= t\Bigg(\underbrace{ \begin{tikzcd} r^{j-1}_{4(i-\tau)+2} \arrow[r, dash] \arrow[d, dash] & r^{k-1}_{4(\ell-\sigma)+2} \arrow[d, dash] \\ r^{v-1}_{4(u-\epsilon)+2} \arrow[r, dash] & a_0 \end{tikzcd}, \dots, \begin{tikzcd} r^{j-1}_{4(i-\tau)+2} \arrow[r, dash] \arrow[d, dash] & r^{k-1}_{4(\ell-\sigma)+2} \arrow[d, dash] \\ r^{v-1}_{4(u-\epsilon)+2} \arrow[r, dash] & a_{n-2} \end{tikzcd}, \begin{tikzcd} r^{j-1}_{4(i-\tau)+2\tau} \arrow[r, dash] \arrow[d, dash] & r^{k-1}_{4(\ell-\sigma)+2\sigma} \arrow[d, dash] \\ r^{v-1}_{4(u-\epsilon)+2\epsilon} \arrow[r, dash] & a_{n-1} \end{tikzcd} }_{\in X_m} \Bigg)$$

for some $\epsilon, \tau, \sigma \in \{0, 1\}$. The reader is encouraged to consult Fig. 6. The inductive hypothesis applies to each of the argument squares, so for each square the columns are constant or the rows are constant.

By symmetry, we may assume without loss of generality that the last argument square has constant columns. This implies that $j = v$ and that $4(i - \tau) + 2\tau = 4(u - \epsilon) + 2\epsilon$. This last equation reduces to $2i - \tau = 2u - \epsilon$. Since $\epsilon, \tau \in \{0, 1\}$, we have that $\tau = \epsilon$ and thus $i = u$. This forces the first column of all the argument squares to be constant, which in turn (by the inductive hypothesis) forces the second columns of all the argument squares to be constant. Hence h has constant columns, and so $h \in X_0$, completing the induction. $\square$

In the previous section, the above $n = 2$ version of Lemma 5.5 above was sufficient to analyze the cubes in Theorem 4.6 since the faces of (3)-dimensional cubes are squares. The faces of $(n+1)$-dimensional cubes, however, are (n)-dimensional cubes. The analysis which must be performed is therefore aided by generalizing the above lemma to (n)-dimensional cubes rather than squares.

Lemma 5.6. *Let*

$$h \in M(\underbrace{1, \ldots, 1}_{n \geq 2})$$

be an (n)-dimensional cube for $n \geq 2$. If we have $\text{S-Lines}_{n-1}(h) \subseteq R^2$ *then* $h = \text{gCube}_i^n(r', r'')$ *for some $i \in n$ and $r', r'' \in R$.*

Proof. Observe that when $n = 2$, this is just Lemma 5.5. We first show that $h \in R^{2^n}$. Since $\text{S-Lines}_{n-1}(h) \subseteq R^2$, we need only show that the $(n-1)$-pivot line of h lies in R^2. The two vertices of this line are $h_{\mathbf{1}}$ and h_f where $\mathbf{1} \in 2^n$ and $f = (1, \ldots, 1, 0) \in 2^n$. The $(0, 1)$-pivot square and $(0, n-1)$-pivot square of h are

$$\underbrace{\begin{array}{ccc} r'' & \text{---} & h_{\mathbf{1}} \\ | & & | \\ r' & \text{---} & r''' \end{array}}_{(0,1)\text{-pivot}} \quad \text{and} \quad \underbrace{\begin{array}{ccc} r'' & \text{---} & h_{\mathbf{1}} \\ | & & | \\ r' & \text{---} & h_f \end{array}}_{(0,n-1)\text{-pivot}} .$$

Applying Lemma 5.5 to the first square and then the second yields $h_{\mathbf{1}}, h_f \in R$, proving that $h \in R^{2^n}$. Lemma 5.5 applied to all the cross section squares of h proves that each cross section square must be of the form $\text{gCube}_i^2(r', r'')$ for some $i \in 2$ and $r', r'' \in R$. The proof will be finished after we establish the following claim.

Claim. *Let $m \geq 2$ be an integer, S a set, and $h \in S^{2^m}$ an (m)-dimensional cube. If every cross section square is of the form* $\text{gCube}_i^2(a, b)$ *for some $i \in 2$ and $a, b \in S$ then* $h = \text{gCube}_j^m(c, d)$ *for some $j \in n$ and $c, d \in S$.*

Proof of claim. We proceed by induction on the dimension m. The claim is trivial if $m = 2$. Assume now that it holds for $m \geq 2$ and take $h \in S^{2^{m+1}}$ satisfying the hypotheses of the claim. Denote by $(m \mapsto 0)$ and $(m \mapsto 1)$ the functions from the singleton $\{m\}$ into 2 that assign m the value 0 and 1, respectively. The inductive assumption implies that

$$h_{(m\mapsto 0)} = \mathrm{gCube}^m_{j_0}(a_0, b_0) \qquad \text{and} \qquad h_{(m\mapsto 1)} = \mathrm{gCube}^m_{j_1}(a_1, b_1)$$

for some $j_0, j_1 \in m$ and $a_0, b_1, a_1, b_1 \in S$. If both $a_0 = b_0$ and $a_1 = b_1$ then $h = \mathrm{gCube}^{m+1}_m(a_0, a_1)$. We therefore assume that $a_0 \neq b_0$. A typical (m, j_0)-cross section square of h looks like

$$\begin{array}{ccc} b_0 & \text{——} & d \\ | & & | \\ a_0 & \text{——} & c \end{array},$$

where $\langle a_0, b_0\rangle$ and $\langle c, d\rangle$ are (j_0)-cross section lines of $h_{(m\mapsto 0)}$ and $h_{(m\mapsto 1)}$, respectively. By hypothesis, this square must be of the form $\mathrm{gCube}^2_i(a'_0, b'_0)$. Since $a_0 \neq b_0$, it must be that $i = 1$, $a'_0 = a_0 = c$, and $b'_0 = b_0 = d$. Applying the same argument to a (m, j_1)-cross section square yields $a_0 = a_1$ and $b_0 = b_1$. In turn, this now implies $j_1 = j_0$. Putting it all together, we have $h = \mathrm{gCube}^{m+1}_{j_0}(a_0, b_0)$, proving the claim. □

We are now ready to prove the general version of Theorem 4.6.

Theorem 5.7. *The algebra $\mathbb{A}_n = \langle A_n; t\rangle$ is (n)-step supernilpotent.*

Proof. By Proposition 2.4, we must show that for all $(n + 1)$-dimensional cubes

$$h \in \mathrm{M}(\underbrace{1, \ldots, 1}_{n+1}),$$

if $\text{S-Lines}_n(h)$ has all constant edges, then the (n)-pivot line is constant as well. Let

$$X_0 = \left\{ \mathrm{gCube}^{n+1}_i(x, y) : x, y \in A_n,\ i \in n+1 \right\} \text{ and}$$
$$X_{m+1} = X_m \cup \left\{ t(\overline{a}) : \overline{a} \in (X_m)^n \right\}.$$

Note that $\mathrm{M}(1, \ldots, 1) = \mathrm{Sg}_{(\mathbb{A}_n)^{2^{n+1}}}(X_0) = \bigcup_{m\in\omega} X_m$. We will proceed by induction on m.

For $h \in X_0$ it is true that having constant (n)-support lines implies having a constant (n)-pivot line, establishing the basis. Suppose now that this implication holds for X_m and that

$$h \in X_{m+1} \setminus X_m$$

has $\text{S-Lines}_n(h)$ constant. We will show that the (n)-pivot line must also be constant. Since $h \in X_{m+1} \setminus X_m$, there are cubes $c_0, \ldots, c_{n-1} \in X_m$ such that

$$h = t\big(c_0, \ldots, c_{n-2}, c_{n-1}\big).$$

Now, the (n)-support line of h_f for a particular $f \in 2^{(n+1)\setminus\{n\}}$ is of the form

$$h_f = t\left(\begin{matrix} a_0 \\ | \\ b_0 \end{matrix}, \ldots, \begin{matrix} a_{n-1} \\ | \\ b_{n-1} \end{matrix}\right),$$

where for each $d \in n$, $\langle a_d, b_d \rangle$ is the (n)-support line of $(c_d)_f$. Lemma 5.3 implies that $a_d = b_d$ for all $d \in (n-1)$ and either $a_{n-1} = b_{n-1}$ or $\{a_{n-1}, b_{n-1}\} = \{r^j_{4i}, r^j_{4i+2}\}$ for some $i, j \in \omega$. The inductive hypothesis applied to c_d for $d \in (n-1)$ implies that the (n)-pivot line of c_d (that is, $(c_d)_{\mathbf{1}}$) is constant. Succinctly, we have determined that

$$\mathrm{Lines}_n(c_d) \subseteq \left\{ \begin{matrix} c \\ | \\ c \end{matrix} : c \in A_n \right\} \text{ for all } d \in (n-1) \text{ and}$$

$$\text{S-Lines}_n(c_{n-1}) \subseteq \left\{ \begin{matrix} c \\ | \\ c \end{matrix} : c \in A_n \right\} \cup \left\{ \begin{matrix} r^j_{4i+\tau} \\ | \\ r^j_{4i+\epsilon} \end{matrix} : i, j \in \omega,\ \{\epsilon, \tau\} = \{0, 2\} \right\}.$$

Observe that if the (n)-pivot line of c_{n-1} is constant then the (n)-pivot line of h will be constant as well. Let D be the set of constant lines:

$$D = \left\{ \begin{matrix} c \\ | \\ c \end{matrix} : c \in A_n \right\}.$$

We now proceed with a case analysis of $\text{S-Lines}_n(c_{n-1}) \cap D$.

Case $\text{S-Lines}_n(c_{n-1}) \cap D \neq \emptyset$**:** In this case, there is some constant (n)-support line of c_{n-1}. This is enough to force every (n)-support line of c_{n-1} to be constant. To see this, notice that the hypercube 2^n is path connected, where a path connecting two functions $f, g \in 2^n$ is a sequence of 'bit flips', or functions

$$f = z_0, z_1, \ldots, z_{e-1} = g$$

such that two consecutive functions differ in exactly one argument.

Claim. *Let $f, g \in 2^n = 2^{(n+1)\setminus\{n\}}$ be functions that differ in exactly one argument. If the (n)-support line $(c_{n-1})_f$ is constant then the (n)-support line $(c_{n-1})_g$ is also constant.*

Proof of claim. Suppose that $k \in n$ is the unique argument such that $f(k) \neq g(k)$. We may assume without loss of generality that

$$f(k) = 0, \qquad g(k) = 1, \qquad (c_{n-1})_f = \begin{matrix} a \\ | \\ a \end{matrix}, \qquad \text{and} \qquad (c_{n-1})_g = \begin{matrix} r^j_{4i+\tau} \\ | \\ r^j_{4i+\epsilon} \end{matrix}$$

for some $a \in A_n$, $i, j \in \omega$, and $\epsilon, \tau \in \{0, 2\}$. Since f and g agree everywhere on $n \setminus \{k\}$, there is $h \in 2^{(n+1)\setminus\{k,n\}}$ such that f and g extend h. This means that $(c_{n-1})_f$ and $(c_{n-1})_g$ will be the columns of the (k, n)-cross section square

$$(c_{n-1})_h = \begin{array}{ccc} a & \text{—} & r^j_{4i+\tau} \\ | & & | \\ a & \text{—} & r^j_{4i+\epsilon} \end{array}.$$

Applying Lemma 5.4 to this we obtain $\epsilon = \tau$, which proves the claim. $\square$

An induction using the above claim shows that if $(c_{n-1})_f$ is a constant (n)-support line and g is connected to f by a path in 2^n then $(c_{n-1})_g$ is also a constant (n)-support line. Since 2^n is path connected, this forces every (n)-support line of c_{n-1} to be constant. The inductive hypothesis applied to c_{n-1} now implies that the (n)-pivot line of c_{n-1} is also constant, which finishes the proof in this case.

Case S-Lines$_n(c_{n-1}) \cap D = \emptyset$**:** The condition for this case is equivalent to the statement

$$\text{S-Lines}_n(c_{n-1}) \subseteq \left\{ \begin{array}{c} r^j_{4i+\tau} \\ | \\ r^j_{4i+\epsilon} \end{array} : i, j \in \omega,\ \{\epsilon, \tau\} = \{0, 2\} \right\}.$$

For $f \in 2^n$ the (n)-support line of h at f is therefore

$$h_f = t\big((c_0)_f, \dots, (c_{n-2})_f, (c_{n-1})_f\big) = t\left(\begin{array}{c} a_0 \\ | \\ a_0 \end{array}, \dots, \begin{array}{c} a_{n-2} \\ | \\ a_{n-2} \end{array}, \begin{array}{c} r^j_{4i+\tau} \\ | \\ r^j_{4i+\epsilon} \end{array} \right)$$

for some $a_0, \dots, a_{n-2} \in A_n$, $i, j \in \omega$, and $\{\epsilon, \tau\} = \{0, 2\}$. By assumption h_f is constant, so an application of Lemma 5.3 yields $a_0, \dots, a_{n-2} \in R$. This reasoning works for any $f \in 2^n \setminus \{\mathbf{1}\}$, so we conclude that

$$\text{S-Lines}_n(c_d) \subseteq \left\{ \begin{array}{c} r'' \\ | \\ r' \end{array} : r', r'' \in R \right\}$$

for all $d \in n$. Applying Lemma 5.6 to this we obtain that $c_d \in X_0$ for all $d \in n$.

The situation now is that all of the $c_0, \dots, c_{n-1}$ are generators of $\mathrm{M}(1, \dots, 1)$. For $d \neq n-1$ we know that S-Lines$_n(c_d)$ is constant, so it follows that

$$c_d = \mathrm{gCube}^{n+1}_i(a_d, b_d)$$

for some $i \neq n$ and $a_d, b_d \in A_n$. We have also assumed (in the paragraph before the case analysis) that the (n)-pivot line of c_{n-1} is not constant, so we also know that

$$c_{n-1} = \mathrm{gCube}^{n+1}_n(r^j_{4i+\epsilon}, r^j_{4i+\tau})$$

for some $i, j \in \omega$ and $\{\epsilon, \tau\} = \{0, 2\}$.

Each of the c_d with $d \neq n-1$ is an $(n+1)$-dimensional cube that is constant in all but a single dimension in $(n+1) \setminus \{n\}$. There are $n-1$ many such c_d and $n+1$ many dimensions, so there is at least one $k \in (n+1) \setminus \{n\}$ such that all of the c_d are constant in dimension k. It follows that, for each $d \neq n-1$, any (k, n)-cross section square of c_d is constant. In particular, for $\mathbf{1} \in 2^{(n+1)\setminus\{k,n\}}$

$$(c_d)_{\mathbf{1}} = \begin{array}{ccc} b_d & \text{---} & b_d \\ | & & | \\ b_d & \text{---} & b_d \end{array}$$

for some $b_d \in A_n$. Hence, the (k, n)-pivot square of h is

$$h_{\mathbf{1}} = t\left(\begin{array}{ccc} b_0 & \text{---} & b_0 \\ | & & | \\ b_0 & \text{---} & b_0 \end{array}, \ldots, \begin{array}{ccc} b_{n-2} & - & b_{n-2} \\ | & & | \\ b_{n-2} & - & b_{n-2} \end{array}, \begin{array}{ccc} r^j_{4i+\tau} & \text{---} & r^j_{4i+\tau} \\ | & & | \\ r^j_{4i+\epsilon} & \text{---} & r^j_{4i+\epsilon} \end{array} \right)$$

$$= \begin{array}{ccc} t(b_0, \ldots, b_{n-2}, r^j_{4i+\tau}) & \text{---} & t(b_0, \ldots, b_{n-2}, r^j_{4i+\tau}) \\ | & & | \\ t(b_0, \ldots, b_{n-2}, r^j_{4i+\epsilon}) & \text{---} & t(b_0, \ldots, b_{n-2}, r^j_{4i+\epsilon}) \end{array}.$$

One of the columns of the (k, n)-pivot square of h is an (n)-support line and the other is the (n)-pivot column. Since these columns are equal and we have assumed that the (n)-support lines of h are constant, it follows that the (n)-pivot line is constant as well.

This completes the case analysis. In all cases, we showed that if all the (n)-support lines of h are constant, then the (n)-pivot line of h is constant as well. From the remarks at the start of the proof, this is enough to show that $\mathbb{A}_n$ is (n)-step supernilpotent. □

6. Concluding remarks

In [15], the second author shows that supernilpotence implies nilpotence in varieties that satisfy a nontrivial idempotent equational condition. Such varieties are called Taylor varieties in the literature. In [16], Olšák produces a strong Mal'cev condition characterizing the class of Taylor varieties. We ask the question: If $[\mathcal{V}]$ is a chapter in the lattice of interpretability of types that does not lie in the interval above Olšák's term, is there a variety $\mathcal{W} \in [\mathcal{V}]$ with a supernilpotent algebra that is not nilpotent?

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