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Pairs of commuting nilpotent matrices, and Hilbert function

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Abstract

Let K be an infinite field. There has been recent study of the family $\mathcal{H}(n, K)$ of pairs of commuting nilpotent $n \times n$ matrices, relating this family to the fibre $H^{[n]}$ of the punctual Hilbert scheme $A^{[n]} = \mathrm{Hilb}^n(\mathbb{A}^2)$ over the point np of the symmetric product $\mathrm{Sym}^n(\mathbb{A}^2)$, where p is a point of the affine plane $\mathbb{A}^2$ [V. Baranovsky, The variety of pairs of commuting nilpotent matrices is irreducible, Transform. Groups 6 (1) (2001) 3–8; R. Basili, On the irreducibility of commuting varieties of nilpotent matrices, J. Algebra 268 (1) (2003) 56–80; A. Premet, Nilpotent commuting varieties of reductive Lie algebras, Invent. Math. 154 (3) (2003) 653–683]. In this study a pair of commuting nilpotent matrices (A, B) is related to an Artinian algebra $K[A, B]$. There has also been substantial study of the stratification of the local punctual Hilbert scheme $H^{[n]}$ by the Hilbert function as [J. Briançon, Description de $\mathrm{Hilb}^n \mathbf{C}[x, y]$, Invent. Math. 41 (1) (1977) 45–89], and others. However these studies have been hitherto separate.

We first determine the stable partitions: i.e. those for which P itself is the partition $Q(P)$ of a generic nilpotent element of the centralizer of the Jordan nilpotent matrix J_P. We then explore the relation between $\mathcal{H}(n, K)$ and its stratification by the Hilbert function of $K[A, B]$. Suppose that $\dim_K K[A, B] = n$, and that K is algebraically closed of characteristic 0 or large enough p. We show that a generic element of the pencil $A + \lambda B, \lambda \in K$ has Jordan partition the maximum partition $P(H)$ whose diagonal lengths are the Hilbert function H of $K[A, B]$.

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1. Pairs of commuting nilpotent matrices

1.1. Introduction

There has been substantial study of the family $\mathcal{H}(n, K)$ of pairs of commuting nilpotent $n \times n$ matrices with entries in a field K: see [Gur,GurSe,Hes,NeuSa,NeuSe,Bas1] and work referred to in these papers. There has been recent work by V. Baranovsky, R. Basili, and A. Premet relating $\mathcal{H}(n, K)$ to the fibre $H^{[n]}$ of the punctual Hilbert scheme $A^{[n]} = \mathrm{Hilb}^n(\mathbb{A}^2)$ over the point np of the symmetric product $\mathrm{Sym}^n(\mathbb{A}^2)$, where p is a point of the affine plane $\mathbb{A}^2$ [Bar,Bas2,Prem]. In this recent work, a pair of commuting nilpotent matrices (A, B) is related to an Artinian algebra $K[A, B]$. We include a brief summary of this connection in Section 2. There has also been substantial study of the stratification of the local punctual Hilbert scheme $H^{[n]}$ by the Hilbert function, begun by J. Briançon [Br] and the second author [I2], with contributions by many, including [ElS,Ch,Gö,Gu,Hui,KW,IY,Yam1,Yam2]. However these studies of $H(n, K)$ and the Hilbert function stratification of $H^{[n]}$ have been hitherto separate.

We first determine the stable partitions: i.e. those for which P itself is the partition $Q(P)$ of a generic nilpotent element of the centralizer of the Jordan nilpotent matrix J_P. We then explore the relation between $\mathcal{H}(n, K)$ and its stratification by the Hilbert function of $K[A, B]$. Suppose that $\dim_K K[A, B] = n$, and that K is algebraically closed of characteristic 0 or large enough p. We show that a generic element of the pencil $A + \lambda B, \lambda \in K$ has Jordan partition the maximum partition $P(H)$ whose diagonal lengths are the Hilbert function H of $K[A, B]$. These results were announced in the talk notes [Bas3,I4], and have been used by T. Košir and P. Oblak in their proof that $Q(P)$ is itself stable [KoOb]. We finally comment on a potential relation between their work and the original proof by J. Briançon that $H^{[n]}$ is irreducible.

Main results are outlined in Section 1.2. In Section 1.3 we prove those concerning stability—they depend primarily on results concerning ranks of powers of a general nilpotent matrix commuting with a given nilpotent matrix in [Bas2]. We study the Hilbert function strata in Section 2.1, and we prove our main results concerning pencils of nilpotent matrices in Section 2.2.

1.2. Main results

We assume throughout Section 1 that K is an infinite field. Further assumptions on K, when needed, will be explicitly stated in each result. Given $B = J_P \in M_n(K)$, a nilpotent $n \times n$ matrix in Jordan form corresponding to the partition P of n, we denote by $\mathcal{C}_B$ the centralizer of B,

$$\mathcal{C}_B = \big\{ A \in M_n(K) \,\big|\, [A, B] = 0 \big\}, \tag{1.1}$$

and by $\mathcal{N}_B$ the set of nilpotent elements of $\mathcal{C}_B$. They each have a natural scheme structure. It is well known that $\mathcal{N}_B$ is an irreducible algebraic variety ([Bas2, Lemma 2.3], see also Lemma 1.5 below). Thus there is a Jordan partition that we will denote $Q(P)$ of a generic matrix $A \in \mathcal{N}_B$. Several have studied the problem of determining $Q(P)$ given P [Ob1,Ob2,KoOb,Pan]. We here first determine the "stable" partitions P under $P \to Q(P)$—that is, those P for which $Q(P) = P$—using results from [Bas2] (see Theorem 1.12 below).

Theorem 1. *P is stable if and only if the parts of P differ pairwise by at least two.*

We next in Section 2 consider a pair of commuting $n \times n$ nilpotent matrices (A, B) such that $\dim_K K[A, B] = n$. The ring $\mathcal{A} = K[A, B] \cong R/I_{A,B}$, where $R = K\{x, y\}$ is the power series ring (see Definition 2.1), has a Hilbert function $H = H(\mathcal{A})$ satisfying

$$H = (1, 2, \ldots, \nu, t_\nu, \ldots, t_j, 0) \quad \text{where } \nu \geqslant t_\nu \geqslant \cdots \geqslant t_j > 0,$$

where j is the socle degree of H. We denote by $P(H)$ the dual partition to the partition of n given by H: thus, the entries of $P(H)$ are the lengths of the rows of the bar graph of H (Definition 2.7). We denote by $\mathcal{U}_B \subset \mathcal{N}_B$ the dense subset $\{A \in \mathcal{N}_B \mid \dim_K K[A, B] = n\}$. Considering an element of the pencil $C_\lambda = A + \lambda B$, $\lambda \in K$, and the multiplication endomorphism $\times(A + \lambda B)$ it induces on $K[A, B]$, we have (Theorems 2.16 and 2.21).

Theorem 2.

(A) *Suppose $A \in \mathcal{U}_B$, let $H = H(K[A, B])$ of socle degree j, and let K be an algebraically closed field with* $\operatorname{char} K = 0$ *or* $\operatorname{char} K > j$. *Then for generic $\lambda \in \mathbb{P}^1$ the Jordan block sizes of the action of $A + \lambda B$ on $K[A, B]$ are given by the parts of $P(H)$.*
(B) *Assume further* $\operatorname{char} K = 0$ *or* $\operatorname{char} K > n$. *Then the partition $Q(P)$ satisfies*

$$Q(P) = \max_{A \in \mathcal{U}_B} P\big(H\big(K[A, B]\big)\big),$$

and has decreasing parts.

These results were announced in the talk notes [I4], and have been used by T. Košir and P. Oblak in their proof that $Q(P)$ is itself stable [KoOb]. We state their result in Theorem 2.27.

1.3. Stable partitions P

We denote by $P = (p_1, \ldots, p_t)$, $p_1 \geqslant \cdots \geqslant p_t \geqslant 1$ a partition P with t parts (so the Jordan nilpotent matrix of partition P has rank $n - t$); we let $n(i) =$ # parts of P at least i. Then the dual partition $\hat{P}$ (switch rows and columns in the Ferrers graph of P) satisfies $\hat{P} = (n(1), n(2), \ldots)$. The following lemma is well known and motivates Definition 1.3.

Lemma 1.1 *(Jordan blocks of ${J_P}^i$). Consider the $n \times n$ Jordan matrix J_P of partition P. Then*

(i) *For $P = [n]$, a single block, the partition of $(J_P)^i$ for $i \leqslant n$ is the unique partition of n having i parts of sizes differing by at most* 1. *For $P = [n]$ and $i > n$ the partition of $(J_P)^i$ has n parts of size* 1.
(ii) *For an arbitrary P, the Jordan partition of $(J_P)^i$ is the union of the partitions for $(J_{[p_k]})^i$, $k = 1, \ldots, t$.*

(iii) *The rank of* $(J_P)^i$ *satisfies*

$$\operatorname{rank}(J_P)^i = n - \big(n(1) + \cdots + n(i)\big). \tag{1.2}$$

(iv) *Let* A *be nilpotent* $n \times n$. *The difference sequence* Δ *of* $(n, rk(A^1), rk(A^2), \ldots)$ *is the dual partition* $\hat{P_A}$ *to* P_A, *the Jordan partition of* A.

Proof. Here (iv) follows from (1.2). □

Example 1.2. For $P = [7]$, $(J_P)^2$ has blocks $(4, 3)$, $(J_P)^3$ has blocks $(3, 2, 2)$, $(J_P)^4$ has blocks $(2, 2, 2, 1)$.

Definition 1.3. We term a partition P whose largest and smallest part differ by at most one, a "string." Such a P is termed "almost rectangular" in [KoOb], since its Ferrer's graph (Definition 2.7) is obtained by removing a portion of the last column from that of a rectangular partition. Each partition P is the union $P = P(1) \cup \cdots \cup P(r)$ of strings $P(i)$. We let r_P be the minimum number r of subpartitions $P(i)$ in any such decomposition of P.

Example 1.4. For $P = (5, 4, 4, 3, 2)$ we may subdivide $P = (5, 4, 4) \cup (3, 2)$, which gives $r_P = 2$. For $P = (8, 7, 7, 7, 5, 5, 4, 2, 1)$, $r_P = 3$. The subdivision into r_P strings need not be unique: for $P = (5, 4, 3, 2, 1) = (5, 4) \cup (3, 2) \cup (1)$ or $(5, 4) \cup (3) \cup (2, 1)$, with $r_P = 3$.

Recall that we denote by $\mathcal{N}_B$ the set of nilpotent elements of the centralizer $\mathcal{C}_B$, endowed with its natural structure as a scheme [Bo]. R. Basili showed in [Bas2, Lemma 2.3] based on [TuAi], that the nilpotent commutator $\mathcal{N}_B$ of a nilpotent matrix B is an irreducible variety. For completeness we include a proof suggested by the referee of the following more general statement.

Lemma 1.5. *If* $\mathfrak{A}$ *is a finite-dimensional algebra over an infinite field* K, *then the scheme* $\mathcal{N}(\mathfrak{A})$ *of nilpotent elements of* $\mathfrak{A}$ *is an irreducible variety.*

Proof. Since irreducibility is a geometric property, we may make a base change and assume that K is algebraically closed. Let $\mathfrak{J}$ be the Jacobson radical of $\mathfrak{A}$. Then Wedderburn's theorem yields a semisimple subalgebra $\mathfrak{L} \subset \mathfrak{A}$ such that $\mathfrak{A} = \mathfrak{L} \oplus \mathfrak{J}$, an internal direct sum as vector spaces; and the restriction of the natural projection $p : \mathfrak{A} \to \mathfrak{A}/\mathfrak{J}$ gives an isomorphism

$$p_{|\mathfrak{L}} : \mathfrak{L} \to \mathfrak{A}/\mathfrak{J}. \tag{1.3}$$

Now $\mathcal{J}$ is a nilpotent ideal, and $\mathfrak{L} \cong \mathfrak{A}/\mathfrak{J}$ is a (split) semisimple algebra over K. Thus $\mathfrak{L}$ is a direct product of matrix algebras $\operatorname{Mat}_{r_u}(K)$ for certain r_u, by another theorem of Wedderburn. It is well known that the set of nilpotent elements $\mathcal{N}(\mathfrak{L})$ is irreducible, since the unit group $\mathfrak{L}^*$ of $\mathfrak{L}$ is a connected algebraic group, and has a dense orbit on $\mathfrak{L}$.

Now, an element $(\ell, j) \in \mathfrak{L} \oplus \mathfrak{J} = \mathfrak{A}$ is nilpotent when ℓ is nilpotent, and $\mathfrak{J}$ is an ideal, so as a variety is just a copy of an affine space, so is irreducible. Thus the nilpotent commutator $\mathcal{N}(\mathfrak{L}) \times \mathfrak{J}$ is the product of irreducible varieties, hence is irreducible. □

Note that the proof that $\mathcal{N}_B$ is irreducible given in [Bas2, Lemma 2.3] is essentially an application of the above proof to the special case $\mathfrak{A} = \mathcal{C}_B$, the centralizer of B. R. Basili uses there

a specific parametrization of $\mathcal{N}_B$: certain matrices $\tilde{A}_{u,u}$ appearing there for $\mathcal{C}_B$, whose nilpotence defines $\mathcal{N}_B$, are the elements of the matrix algebras $\mathrm{Mat}_{r_u}(K)$ in the above proof. Here r_u is the multiplicity of the uth distinct part of P.

Recall that, given a partition P, we denote by $B = J_P$ the Jordan nilpotent matrix of partition P. It follows from Lemma 1.5 that there is a unique partition $Q(P)$ that occurs for a generic element of $\mathcal{N}_B$.

We recall the natural majorization partial order on the partitions P (we assume $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_t$),

$$P \geqslant P' \quad \text{if and only if for each } i, \quad \sum_{1 \leqslant u \leqslant i} p_i \geqslant \sum_{1 \leqslant u \leqslant i} p'_i. \tag{1.4}$$

From Lemma 1.1 it is easy to see that

$$P \geqslant P' \quad \Leftrightarrow \quad \forall i,\ \mathrm{rank}\big(J_P{}^i\big) \geqslant \mathrm{rank}\big(J_{P'}{}^i\big). \tag{1.5}$$

We let O_P denote the $\mathrm{Gl}(n)$ orbit of J_P. We have [Hes]

$$\overline{O_P} \supset O_{P'} \quad \Leftrightarrow \quad P \geqslant P'. \tag{1.6}$$

Lemma 1.6. *The partition $Q(P)$ determined by the Jordan block sizes of a generic element of $\mathcal{N}_B$ satisfies $Q(P) \geqslant P_A$ for each $A \in \mathcal{N}_B$.*

Proof. This follows from the irreducibility of $\mathcal{N}_B$, from (1.2), and the semicontinuity of the ranks of powers of A. □

Before the present work was announced [I4], there were several results known about $Q(P)$.

Theorem 1.7. *(See [Bas2, Proposition 2.4].) The rank of a generic element $A \in \mathcal{N}_B$ is $n - r_P$. Equivalently, the partition $Q(P)$ has r_P parts.*

Also, P. Oblak had determined the "index" or largest part of $Q(P)$ using graph theory [Ob1]. We subsequently have given another proof of Oblak's result (see [Bas-I]).

We use the notation $|P| = n$, the integer partitioned by P.

Definition 1.8. Let $\mathcal{P} = (P_1, \ldots, P_{r_P})$ be a decomposition of P into r_P non-overlapping strings:

$$\bigcup_i P_i = P, \quad \text{and} \quad P_i \cap P_j = \emptyset \quad \text{if } i \neq j. \tag{1.7}$$

Given such a decomposition $\mathcal{P}$ of P, we denote by $\tilde{\mathcal{P}}$ the partition $(|P_1|, \ldots, |P_{r_P}|)$, rearranged in decreasing order.

For $P = (3, 3, 3, 2, 2, 1)$ two such decompositions into strings are $\mathcal{P} = ((3, 3, 3), (2, 2, 1))$ and $\mathcal{P}' = ((3, 3, 3, 2, 2), (1))$. We have $\tilde{\mathcal{P}} = (9, 5)$ and $\tilde{\mathcal{P}}' = (13, 1)$. Here $r_P = 2$.

Lemma 1.9. *Suppose that the partition P of n contains two parts that are equal, or that differ by one. Then $Q(P) > P$.*

Proof. Assume that P has two parts that are the same or that differ by one. Choose a decomposition $\mathcal{P}$ into r_P strings $P_1, \ldots, P_{r_P}$. We claim that some nilpotent matrix $\tilde{B}$ of partition $\tilde{\mathcal{P}}$ commutes with J_P. To show this, we may first reduce to the case that P has two parts, which differ by 0 or 1. We have by Lemma 1.1(i) that the partition of $A = (J_n)^2$ is P. Then $gAg^{-1} = J_P$ for some $g \in \mathrm{Gl}_n(K)$, so the nilpotent matrix gJ_ng^{-1} centralizes J_P and has partition $P' = |P|$. This proves the claim. Also P' is different from P since at least one string of P has length greater than one, and $P' > P$. We have by Lemma 1.6 that $Q(P) \geqslant P'$, so $Q(P) > P$. □

Note that when $P = (2, 2)$, then $P' = (4)$, and $J_{P'}$ does not itself commute with J_P.

We now determine the "stable" partitions P, for which $Q(P) = P$. We need the following result of R. Basili. Given a partition P, let s_P be the length of the longest string in P,

$$s_P = \max\bigl\{i \bigm| \exists k \bigm| (p_k \geqslant p_{k+1} \geqslant \cdots \geqslant p_{k+i-1}) \subset P \text{ and } p_k - p_{k+i-1} \leqslant 1\bigr\}.$$

For $P = (5, 4, 4, 3, 2)$ we have $s_P = 3$. Note that $s_P = 1$ iff the parts of P differ by at least two.

The next theorem shows that the Jordan partition $P_{A^{s_P}}$ of the s_P power of any element $A \in \mathcal{N}_B$ satisfies $P_{A^{s_P}} \leqslant P = P_B$.

Theorem 1.10. *(See [Bas2, Proposition 3.5].) Let $B \cong J_P$ be nilpotent of Jordan partition P, and let $A \in \mathcal{N}_B$, the nilpotent commutator of B. Then*

$$\mathrm{rank}\bigl(A^{s_P}\bigr)^m \leqslant \mathrm{rank}\bigl(B^m\bigr). \tag{1.8}$$

Theorem 1.11. *Suppose that P has a decomposition $\mathcal{P}$ into r_P strings, each of length s_P. Then $Q(P) = \tilde{\mathcal{P}}$.*

Proof. The assumption is equivalent to there being a unique decomposition of P into r_P strings, and also that these strings have equal length. Let $B = J_P$ and $s = s_P$. The proof of Lemma 1.9 implies there exists $\tilde{B} \in N_B$ of partition $\tilde{\mathcal{P}}$. By Lemma 1.1(ii) $\tilde{B}^s$ has Jordan blocks given by the partition P, so for $A = \tilde{B}$ there is equality in (1.8) of Theorem 1.10. Hence, by semicontinuity of rank, for an open dense subset of $A \in \mathcal{N}_B$, the Jordan partition of A^s is P. By the unicity of the decomposition of P into r_P strings, and using Theorem 1.7 we conclude that the partition of A is constant on that subset, implying $Q(P) = \tilde{\mathcal{P}}$. □

Thus, for $P = (5, 4, 2, 2)$, we have $Q(P) = (9, 4)$; for $P = (8, 7, 7, 5, 5, 4, 2, 2, 2)$, $Q(P) = (22, 14, 6)$.

Given a positive integer c, we denote by cP the partition obtained by repeating c times each part of P. For $P = (3, 1, 1)$, $2P = (3, 3, 1, 1, 1, 1)$.

Theorem 1.12 *(Stable partitions). The following are equivalent.*

(i) *The parts of the partition P differ pairwise by at least two*:

$$P = (p_1, \ldots, p_t), \quad p_1 \geqslant \ldots \geqslant p_t,$$
$$\text{and for} \quad 1 \leqslant u \leqslant t-1, \quad p_u - p_{u+1} \geqslant 2. \tag{1.9}$$

(ii) $Q(P) = P$.
(iii) *For some positive integer* c, $Q(cP) = (cp_1, cp_2, \ldots, cp_t)$.

Proof. Theorem 1.11 shows (i) $\Rightarrow$ (ii), and (i) $\Rightarrow$ (iii). (ii) $\Rightarrow$ (i) is from Lemma 1.9. To show (iii) $\Rightarrow$ (i), suppose, by way of contradiction, that P has two parts that differ by at most one: it suffices to assume $P = (p_1 \geqslant p_2)$. For any $c \geqslant 1$ one can use Lemma 1.1(i) in a way analogous to the proof of Lemma 1.9 to find a nilpotent matrix A whose $2c$th power is J_{cP} and whose partition has the single part $c(p_1 + p_2)$; but then $Q(cP) \neq (cp_1, cp_2)$. □

We note that D.I. Panyushev has recently determined the "self-large" (what we call "stable") nilpotent orbits in a quite general context of the Lie algebra of a connected simple algebraic group over an algebraically closed field K of characteristic zero [Pan, Theorem 2.1]. When the Lie algebra $\mathfrak{g}$ of G is $sl(V)$ his result restricts to Theorem 1.12 above for such K [Pan, Example 2.5 1(a)].

One of our goals is to show links between the study of nilpotent commuting matrices, and that of the punctual Hilbert scheme. Thus, we have taken pains to refer to some of the relevant literature that we were aware of. Nevertheless, the inclusion of proofs of Lemmas 1.5 and 2.14 make the main results of the paper relatively self contained. We do use in an essential way the standard bases for ideals in $K\{x, y\}$ from [Br,I2], in the proof of Theorem 2.16.

2. Pair of nilpotent matrices and the Hilbert scheme

We denote by $R = K\{x, y\}$ the power series ring, i.e. the completed local ring at $(0, 0)$ of the polynomial ring $K[x, y]$, over a field K. We assume throughout Section 2 that K is an algebraically closed field, unless we state otherwise for a particular result. We denote by $M = (x, y)$ the maximal ideal of R, and by V the n-dimensional vector space over the field K upon which $M_n(K)$ acts.

Definition 2.1. We denote by $\mathcal{N}(n, K)$ the set of nilpotent matrices in $M_n(K)$, with its natural structure as irreducible variety. We define $\mathcal{H}(n, K)$

$$\mathcal{H}(n, K) = \big\{(A, B) \;\big|\; A, B \in \mathcal{N}(n, K) \text{ and } AB - BA = 0\big\}.$$

Given an element $(A, B) \in \mathcal{H}(n, K)$, we denote by $\mathcal{A}_{A,B} \cong K[A, B]$ the Artinian quotient of R,

$$\mathcal{A} = \mathcal{A}_{A,B} = R/I, \quad I = I_{A,B} = \ker(\theta),$$
$$\theta : R \to k[A, B], \quad \theta(x) = A, \quad \theta(y) = B.$$

We let $\mathcal{U}(n, K) \subset \mathcal{H}(n, K)$ be the open subset such that $\dim_K(\mathcal{A}_{A,B}) = n$.

The Hilbert scheme $A^{[n]} = \mathrm{Hilb}^n(\mathbb{A}^2)$ parametrizes length-n subschemes of $\mathbb{A}^2$, and is a desingularization of the symmetric product $A^{(n)} = \mathrm{Sym}^n(\mathbb{A}^2)$. Given a point $s \in \mathbb{A}^2$, we denote by $H^{[n]}$ the fiber of $A^{[n]}$ over the point (ns) of $A^{(n)}$: roughly speaking, the local punctual Hilbert scheme $H^{[n]}$ parametrizes the length-n Artinian quotients of R.[2] J. Briançon and subsequently

[2] Work of R. Skjelnes et al. shows that this rough viewpoint is inaccurate, see [LST]; the fiber definition is accurate.

M. Granger of the Nice school, showed that the scheme $H^{[n]}$ is irreducible in characteristic zero [Br,Gr]; it was a slight extension to show $H^{[n]}$ is irreducible for char $K > n$ [I2], but further progress awaited a connection to $\mathcal{H}(n, K)$.

V. Baranovsky, R. Basili, and A. Premet related this problem of irreducibility to that of the irreducibility of $\mathcal{H}(n, K)$ [Bar,Bas2,Prem]. Following H. Nakajima and V. Baranovsky, we set

$$\mathfrak{W} \subset \mathcal{H}(n,k) \times V\colon \big\{(B, A, v) \in \mathcal{H}(n, K) \times V \;\big|\; v \text{ is a cyclic vector for } (B, A)\big\}. \tag{2.1}$$

That is, $(B, A, v) \in \mathfrak{W}$ if any (B, A)-invariant subspace of V containing v is all of V. The group $\mathrm{Gl}(V)$ acts on $H(n, K) \times V$ by conjugation of the matrices, and action on the vector.

Lemma 2.2. *(See [Nak, Theorem 1.9], [Bar, Lemma 6].) The action of* $\mathrm{Gl}(V)$ *on* $\mathfrak{W}$ *is free, and, taking* $x \to Av$, $y \to Bv$, x, y *local parameters at* $s \in \mathbb{A}^2$ *we have a morphism,*

$$\pi : \mathfrak{W} \to H^{[n]}, \tag{2.2}$$

whose fibers are the $\mathrm{Gl}(V)$ *orbits in* $\mathfrak{W}$.

Theorem 2.3. *(See [Bar, Theorem 4].*[3]*) The subset* $\mathfrak{W} \subset \mathcal{H}(n, K) \times V$ *is dense.*

See also Lemma 2.14ff. As a consequence of Lemma 2.2 and Theorem 2.3, the irreducibility of $H(n, K)$ is equivalent to that of $H^{[n]}$.

V. Baranovsky used this and Briançon's theorem to prove the irreducibility of $\mathcal{H}(n, K)$, for char $K = 0$ and char $K > n$. R. Basili gave a direct "elementary" proof of the irreducibility of $\mathcal{H}(n, K)$, that is valid also for char $K \geqslant n/2$. A. Premet later gave a Lie algebra proof of the irreducibility of $\mathcal{H}(n, K)$ that is valid in all characteristics. The Basili and Premet results gave new (and different) proofs of the irreducibility of $H^{[n]}$ when K is algebraically closed, for char $K > n/2$ (R. Basili) or arbitrary characteristic (A. Premet). Note that the space of $\mathbb{R}$ (real) points of $\mathrm{Hilb}^n(R)$ has at least $\lfloor n/2 \rfloor$ components [I2, §5B]. These results showed that there is a strong connection between $\mathcal{H}(n, K)$ and $H^{[n]}$.

2.1. Hilbert function strata

Let $\mathcal{A} = R/I$ be an Artinian quotient of $R = K\{x, y\}$ of length $\dim_K(\mathcal{A}) = n \geqslant 1$, and recall that $M = (x, y)$ denotes the maximum ideal. The *associated graded algebra* $\mathcal{A}^* = \mathrm{Gr}_M(\mathcal{A}) = \bigoplus_0^j \mathcal{A}_i$ of $\mathcal{A}$ satisfies (here $j =$ socle degree $\mathcal{A}$: $\mathcal{A}_j \neq 0$, $\mathcal{A}_{j+1} = 0$)

$$\mathcal{A}_i = \big\langle M^i \cap I + M^{i+1} \big\rangle / M^{i+1}.$$

The *Hilbert function* $H(\mathcal{A})$ is the sequence

$$H(\mathcal{A}) = (h_0, \ldots, h_j), \quad h_i = \dim_K \mathcal{A}_i.$$

We denote by $n = |H| = \sum_i h_i$ the length of H, satisfying $n = \dim_K(\mathcal{A})$.

[3] V. Baranovsky communicates in the MathSciNet review MR 1825165 of [Bar] that a parenthetical remark in the proof of Lemma 3, in (a) "i.e. B_1 has Jordan canonical form in this basis" is incorrect.

Example 2.4. Let $\mathcal{A} = R/I$, $I = (y^2 + x^4, xy + x^4)$. Then

$$\mathcal{A}^* = R/\big(y^2, xy, x^5\big), \quad \text{and} \quad H(\mathcal{A}) = (1, 2, 1, 1, 1), \tag{2.3}$$

since $x(\underline{y^2 + x^4}) - (y - x^3)(\underline{xy + x^4}) = x^5 + x^7 \in I \Rightarrow x^5 \in I$.

Let H be a fixed Hilbert function sequence of length n. We now study the connection between the Hilbert function strata $Z_H = \mathrm{Hilb}^H(R) \subset H^{[n]}$, parametrizing all Artinian quotients of R having Hilbert function H, and the analogous subscheme of commuting pairs of matrices,

$$\mathcal{H}^H(n, K) = \theta^{-1}(Z_H) = \big\{\text{pairs } (A, B) \;\big|\; H(\mathcal{A}_{A,B}) = H\big\}.$$

Here Z_H is locally closed in $H^{[n]}$ [I2, Proposition 1.6], and likewise so is $\mathcal{H}^H(n, K)$ in $\mathcal{U}(n, K)$. We have the projection

$$\tau : Z_H \to G_H, \quad \mathcal{A} \to \mathcal{A}^*$$

to the irreducible projective variety G_H parametrizing graded quotients of R having Hilbert function H. Each of Z_H, G_H have covers by opens in affine spaces of known dimension [Br,I2]; also τ makes Z_H a locally trivial bundle over G_H with fibres opens in an affine space, and having a global section [I2, Theorems 3.13, 3.14], but Z_H is not in general a vector bundle over G_H [I1]. When $\mathrm{char}\, K = 0$ or $\mathrm{char}\, K = p > n$ the fiber is an affine space and the covers are by affine spaces [I2, Theorems 2.9, 2.11]. The Nice school studied specializations of Z_H, see work of M. Granger [Gr] and J. Yaméogo [Yam1,Yam2], but the problem of understanding the intersection $\overline{Z_H} \cap Z_{H'}$ is in general difficult and quite unsolved (see [Gu,DB] for some recent progress). Let $Z_{v,n}$ parametrize order v colength n ideals I in $R = K\{x, y\}$: that is

$$Z_{v,n} = \big\{I \;\big|\; M^v \supset I,\ M^{v+1} \not\supseteq I, \text{ and } \dim_K R/I = n\big\}.$$

J. Briançon's irreducibility result can be stated, denoting by $\overline{X}$ the Zariski closure of X,

$$H^{[n]} = \overline{Z_{1,n}}.$$

M. Granger showed, more generally

Theorem 2.5. *(See [Gr].) For $v \geqslant 1$ we have*

$$\overline{Z_{v,n}} \supset Z_{v+1,n}. \tag{2.4}$$

We let $\mathcal{U}_{v,n} = \theta^{-1}(Z_{v,n}) \subset \mathcal{U}(n, k)$.

Corollary 2.6. *Fix n. Then for $v \geqslant 1$ we have*

$$\overline{\mathcal{U}_{v,n}} \supset \mathcal{U}_{v+1,n}. \tag{2.5}$$

Proof. This is an immediate consequence of Granger's theorem and Lemma 2.2. □

Recall that when an Artinian algebra A has embedding dimension at most two ($h_1 \leqslant 2$) its Hilbert function $H(A)$ satisfies (see [Mac2,Br,I2]),

$$H = (1, 2, \ldots, \nu, h_\nu, \ldots, h_j), \quad \nu \geqslant h_\nu \geqslant \cdots \geqslant h_j > 0. \tag{2.6}$$

Here, writing $A = R/I$, $R = K\{x, y\}$, we have that ν is the *order* $\nu(I)$ of the ideal I, namely the smallest initial degree of any element of I. (When $\nu(I) = 1$, $H = (1, 1, \ldots, 1)$: we regard this as also a sequence satisfying (2.6).)

Henceforth, by *Hilbert function* we will mean one of codimension at most two, so a sequence satisfying (2.6). The *length* of a Hilbert function is $n = \sum_0^j h_i$. The *socle degree* of H is the integer j from (2.6), and it is also the socle degree—maximum non-zero power of the maximal ideal—of any Artinian algebra of Hilbert function H.

Definition 2.7. Recall that we arrange the Ferrer's graph (Young diagram) of the partition $P = (p_1 \geqslant \cdots \geqslant p_t)$ with the largest row of length p_1 at the top. The *diagonal lengths* H_P of a partition P are the lengths of the lower left to upper right diagonals of the Ferrer's graph of P. The *dual partition* $\hat{P}$ to P is obtained by switching rows and columns in the Ferrer's graph:

$$\hat{P}_i = \#\{p_k \in P \mid p_k \geqslant i\}. \tag{2.7}$$

Given a Hilbert function H as in (2.6), we denote by $P(H)$ the unique partition having diagonal lengths H and ν strictly decreasing parts. It satisfies $P(H) = (p_1, \ldots)$ with p_i the length of the ith row of the bar graph of H. In other words, were the sequence H rearranged in descending order, then $P(H)$ would be the dual partition to H.

Example 2.8. For $H = (1, 2, 3, 2, 1)$, $P(H) = (5, 3, 1)$. The partitions $P = (4, 2, 1, 1, 1)$ and $P' = (3, 3, 3)$ also have diagonal lengths $H = (1, 2, 3, 2, 1)$, but are incomparable in the partial order (1.4). We show below that $P(H)$ is maximum among the partitions of diagonal lengths H.

Remark 2.9. Fixing a Hilbert function H, the elements of the set $\mathcal{P}(H)$ of partitions having diagonal lengths H with a certain grading (by the number of difference-one hooks), correspond bijectively to the cells in a cellular decomposition of the projective variety G_H, graded by dimension (see [IY]). The Hilbert function H determines a certain product $B(H)$ of rectangular partitions; and the elements of $\mathcal{P}(H)$ correspond bijectively to sequences of subpartitions of $B(H)$, in what is termed a "hook code" in [IY, Section 3D]. Thus, $\mathcal{P}(H)$ is enumerated by a certain product of binomial coefficients [IY, Theorem 3.30].

There is a natural partial order on the set $\mathcal{H}(n)$ of Hilbert functions of codimension at most two, having length n (see (2.6)), given by

$$H \leqslant H' \quad \Leftrightarrow \quad \forall u,\ 0 \leqslant u < n, \quad \sum_{k \leqslant u} H_k \leqslant \sum_{k \leqslant u} H'_k. \tag{2.8}$$

For example, $(1, 1, 1, 1, 1) < (1, 2, 1, 1) < (1, 2, 2)$.

The maximality of $P(H)$ in the next Lemma 2.10(ii) follows from the irreducibility of G_H and considering the cells corresponding to each partition P of diagonal lengths H in G_H (see [IY, Theorem 3.12ff]). We include a simple direct proof of (ii).

Lemma 2.10.

(i) *The assignment $H \to P(H)$ determines an order-reversing bijection between the partially ordered set (POS) of Hilbert functions of length n (see* (2.8)*), and the POS of partitions of n having decreasing parts (see* (1.4)*).*
(ii) *Let P have diagonal lengths H. Then $P(H) \geqslant P$ in the partial order* (1.4)*.*

Proof. We first show (i). It is well known that the correspondence taking P to $\hat{P}$ is an order reversing involution on the POS of partitions of n [CoM, Lemma 6.3.1]. It takes a partition $P = (p_1, p_2, \ldots, p_\nu)$ having ν decreasing parts, to a partition $Q = \hat{P}$ having no gaps among the integers $(1, 2, \ldots, \nu)$. Given P we let $H_P = (1, 2, \ldots, \nu, h_\nu, \ldots, h_j)$, be the sequence obtained by rearranging $\hat{P}$, so that H_P begins $(1, 2, \ldots, \nu)$, and ends with the rest of the parts of $\hat{P}$ in non-increasing order. Then H_P satisfies (2.6), and $P(H_P) = P$. Evidently, $P \to H_P$ is a bijection as stated in (i).

It remains to show that if two partitions $Q = \hat{P}$, and $Q' = \hat{P}'$ with maximum parts ν, ν' respectively, and having no gaps among $(1, 2, \ldots, \nu)$ and $(1, 2, \ldots, \nu')$, respectively, satisfy $Q \leqslant Q'$ then the rearranged sequences H, H' satisfy $H \leqslant H'$ in the order (2.8), and vice versa. It is well known that the partial order between two partitions of n is preserved by the operation of either removing (or, respectively, adding) a common part a to each, forming partitions of $n - a$ (or, respectively $n + a$). Removing in this way the parts $(1, 2, \ldots, \nu)$, and placing those parts first leaves remainder partitions $\alpha(Q) \leqslant \alpha(Q')$. Now the first sequence is $H = (1, 2, \ldots, \nu, \alpha(Q))$ and we have $H \leqslant (1, 2, \ldots, \nu, \alpha(Q'))$ in the partial order obtained by formally extending that of (2.8) to arbitrary sequences. Finally, rearranging the parts $(\nu + 1, \ldots, \nu')$ (if any) of $\alpha(Q')$ first, (so just after ν), puts the second sequence in the form H'; we have $H \leqslant H'$ since each of $\nu + 1, \ldots, \nu'$ are larger than every part of $\alpha(Q)$. This argument reverses, showing that the mapping $H \to P(H)$ inverts the partial order. This completes the proof of (i).

To show (ii), let P: $p_1 \geqslant \cdots \geqslant p_s$ have diagonal lengths H and consider the partition $P(u) = (p_1, \ldots, p_u)$ comprised of the first u rows of P. Rearranging the rows of the Ferrer's graph of $P(u)$ in staggered fashion, by advancing the vth row from the top (longest) by $v - 1$, and forming an adjusted Ferrer's graph AFG($P(u)$) we see that the sequence $H(u)$ given by the diagonal lengths of $P(u)$ is given by

$$H(u)_i = \text{the length of the } i\text{th column of AFG}\big(P(u)\big). \tag{2.9}$$

The partition $P(H(u))$ is obtained by pushing all squares in AFG($P(u)$) upward, so that in each column there are no gaps. Thus, $P(H(u))$ partitions the same number $|P(u)|$ as $P(u)$. Since $H_i \geqslant H(u)_i$ for each i, the Ferrer's graph of the partition $P(H)$ includes that of $P(H(u))$ (strictly if $p_{u+1} \neq 0$). Thus for each u

$$\sum_{k \leqslant u} p_k = \sum_{k \leqslant u} P\big(H(u)\big)_k = \big|P(u)\big| \leqslant \sum_{k \leqslant u} P(H)_k, \tag{2.10}$$

showing that $P \leqslant P(H)$ in the partial order (1.4). □

Example 2.11. The partitions $P = (6,4,3)$, $P' = (6,4,2,1)$ with decreasing parts satisfy $P \geqslant P'$, so their duals $\hat{P} = (3,3,3,2,1,1)$, $\hat{P}' = (4,3,2,2,1,1)$ satisfy $Q = \hat{P} \leqslant Q' = \hat{P}'$. Since $\nu = 3$ this implies that $\alpha(Q) = (3,3,1) \leqslant \alpha(Q') = (4,2,1)$ in the POS of Eq. (1.4), implying $H_P = (1,2,3,3,3,1) \leqslant H_{P'} = (1,2,3,4,2,1)$ in the POS of (2.8).

Let I be an ideal of colength n in $R = K\{x,y\}$ and let $H = H(\mathcal{A})$, $\mathcal{A} = R/I$. Recall $\nu =$ order of I; so $M^\nu \supset I$, $M^{\nu+1} \not\supseteq I$, where $M = (x,y)$. Consider the deg lex partial order,

$$1 < y < x < y^2 < yx < x^2 \cdots$$

and denote by $E = E(I)$ the monomial initial ideal of I in this order. The monomial cobasis $E(I)^c = \mathbf{N}^2 - E(I)$ may be seen as the Ferrer's graph of a partition $P = P(E)$ of diagonal lengths H. Conversely, given a partition $P = (k_0, \ldots, k_{\nu-1})$ with ν non-zero parts (the notation is from the standard bases introduced just below in Definition 2.12), we define the monomial ideal E_P

$$E_P = \big(x^{k_0}, yx^{k_1}, y^2x^{k_2}, \ldots, y^{\nu-1}x^{k_{\nu-1}}, y^\nu\big), \tag{2.11}$$

whose cobasis E_P^c is the complementary set, of monomials $E_P^c = \mathbf{N}^2 - E_P$ (where the pair of non-negative integers $(a,b) \in \mathbf{N}^2$ denotes $x^a y^b$).

Definition 2.12. The ideal $I \subset R = K\{x,y\}$ has *standard basis* $(f_\nu, \ldots, f_0)$ in the direction x if I has a (not necessarily minimal) generating set $(f_0, \ldots, f_\nu)$ of the following form:

$$\begin{aligned} &\big(f_\nu = g_\nu, f_{\nu-1} = x^{k_{\nu-1}} g_{\nu-1}, \ldots, f_0 = x^{k_0} g_0\big), \quad \text{where} \\ &g_i = y^i + h_i, \quad h_i \in M^i \cap k[x]\big\langle y^{i-1}, \ldots, y, 1\big\rangle \end{aligned} \tag{2.12}$$

and $k_0 \geqslant k_1 \geqslant \cdots \geqslant k_{\nu-1}$ [IY, Definition 3.9ff]. We term the basis *normal* if $k_0 > k_1 > \cdots > k_{\nu-1}$ [Br,I2]. We will sometimes refer to these as "standard generators," or "normal generators," respectively.

Then we have $E = E(I)$ is the monomial ideal of (2.11) and E^c is the set of monomials

$$E^c = \big\langle 1, x, \ldots, x^{k_0-1}; y, yx, \ldots, yx^{k_1-1}; \ldots; y^{\nu-1}, \ldots, y^{\nu-1}x^{k_{\nu-1}-1}\big\rangle. \tag{2.13}$$

The existence of a normal basis in the direction x does not depend on the choice of $y \in R_1$, such that $\langle y, x\rangle = R_1$. Note also that for a normal basis the decreasing sequence $P = (k_0, k_1, \ldots, k_{\nu-1})$ satisfies $P = P(H)$, where $H = H(R/I)$ is the Hilbert function of $\mathcal{A} = R/I$.

The following result is standard, see for example [I2, Lemma 1.4]. We denote by $\langle E^c\rangle$ the K-vector space spanned by E^c.

Lemma 2.13. *The condition* (2.12) *is equivalent to*

$$\forall i \geqslant 0, \quad \langle E^c\rangle \cap M^i \oplus I \cap M^i = M^i, \text{ an internal direct sum.} \tag{2.14}$$

This notion of standard basis is stronger than just "E^c is a complementary basis to I in R," used in [BaH,NeuSa].

The following lemma is well known, for example [Bar, Lemma 3] shows that for a generic A in $\mathcal{N}_B$ the pair (A, B) has a cyclic vector, and by [NeuSa] this implies $\dim_K K[A, B] = n$. We thank A. Sethuraman and T. Košir for discussions of these topics that led to our proof below.

Lemma 2.14.

(i) *Let B be an $n \times n$ nilpotent Jordan matrix of partition P and let A be generic in $\mathcal{N}_B$. Let K be an infinite field. Then*

$$\dim_K K[A, B] = n.$$

(ii) *(See [Bar, Lemma 3].) Let B be nilpotent, and $C \in \mathcal{N}_B$ and assume K is algebraically closed. Then there exists $A \in \mathcal{N}_B$ such that the pencil $A + tC \subset \mathcal{N}_B$, and the pair (A, B) has a cyclic vector.*

Proof. For (i) consider the monomial ideal E_P; then the matrix of $B = \times x$ acting on the basis E_P^c of (2.13) is the Jordan matrix of partition P; the matrix of $A = \times y$ has the conjugate Jordan partition $\hat{P}$, and $\dim_K K[A, B] = n$. Now $\dim_K K[A, B]$ is upper semicontinuous on $A \in \mathcal{N}_B$, an irreducible variety (Lemma 1.5), and the dimension of the algebra generated by any two commuting $n \times n$ matrices is less or equal n ([Ge], see also [Gur,GurSe]).

For (ii), since K is closed, w.l.o.g. we may assume that B is in Jordan form. By [Bas2, Lemma 2.3] there is an element $g \in \mathcal{C}_{\mathcal{B}}{}^*$ such that $gCg^{-1} \in S_B$, where S_B is a maximal nilpotent subalgebra of $\mathcal{N}_B$. Let A' be a general enough element of $\mathcal{S}_B$, and take $A = g^{-1}A'g$. Then the pencil $A + tC \subset g^{-1}\mathcal{S}_B g \subset \mathcal{N}_B$; and (i) implies $\dim_K K[A, B] = n$. □

V. Baranovsky uses (ii) to show that the subset $\mathfrak{W}$ of $\mathcal{H}(n, K) \times V$ consisting of triples (A, B, v) for which v is a cyclic vector for the pair (A, B) is dense (Theorem 2.3).

2.2. Pencil of matrices and Jordan form

We first give an example illustrating the connection between Hilbert function strata Z_H of Artinian algebras and those of commuting nilpotent matrices. Here are some features. Assume $K[A, B] \in \mathcal{H}^H(n, K)$. Then

(i) The ideals that occur in writing $K[A, B] \cong R/I$ are in general non-graded.
(ii) The partition P need not have diagonal lengths $H = H(K[A, B])$, and $P(H) \geqslant P$ in (1.4).
(iii) The partition P_λ arising from the action of $A + \lambda B$ satisfies $P_\lambda = P(H)$ for a generic λ, all but a finite number (Theorem 2.16).
(iv) The closure of the orbit of P includes a partition of diagonal lengths $P(H)$ (Theorem 2.20).

Example 2.15 *(Pencil and specialization).* Take for B the Jordan matrix of partition $(3,1,1)$. It is easy to see that for $P=(3,1,1)$ we have $Q(P)=(4,1)$, Also by [Bas2, Lemma 2.3], up to conjugation by an element of the centralizer $\mathcal{C}_B$, any element $A\in\mathcal{N}_B$ satisfies

$$B=\left(\begin{array}{ccc|cc} 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&0&0\\ \hline 0&0&0&0&0\\ 0&0&0&0&0 \end{array}\right),\qquad A=\left(\begin{array}{ccc|cc} 0&a&b&f&g\\ 0&0&a&0&0\\ 0&0&0&0&0\\ \hline 0&0&e&0&c\\ 0&0&d&0&0 \end{array}\right).$$

We send $x\to A$, $y\to B$, and let the ideal $I=\operatorname{Ker}(R\to K[A,B])$. We now assume that $acdf\neq 0$ so that A is general enough to have Jordan block partition $Q(P)$. Let $\beta=1/(cdf)$, and let

$$g_2=y^2-\beta x^3,\qquad g_1=y-a\beta x^2,\qquad g_0=1.$$

Then I has a normal basis in the x direction (Definition 2.12): we have

$$\mathcal{A}=\mathcal{A}_{A,B}=K[A,B]\cong R/I,\quad I=\big(g_2,xg_1,x^4g_0\big)$$

with $k_0=4$, $k_1=1$ in (2.12), and the Hilbert function $H(\mathcal{A})=(1,2,1,1)$. The multiplication action $A=m_x$ of x on the classes $\langle 1,x,x^2,x^3;g_1\rangle$ in $\mathcal{A}$ has Jordan blocks given by the partition $(4,1)$ having diagonal lengths $H(\mathcal{A})$.

We have in the y-direction $I=(x^3-\beta^{-1}y^2,xy-ay^2,y^3)$: the non-homogeneous generator $x^3-\beta^{-1}y^2$ with lead term x^3 prevents I from having a standard basis in the direction y. The action of $B=m_y$ on the classes of $\langle 1,y,\beta x^3;x-ay,y^2\rangle$ in $\mathcal{A}$ verifies that $P_B=(3,1,1)$ of diagonal lengths $(1,2,2)$, which is *not* $H(\mathcal{A})$.

Now consider the associated graded algebra $\mathcal{A}^*=R/I^*$: here $I^*=(y^2,xy,x^4)$. The standard generators in the y direction (switch y, x in the Definition 2.12) are (x^4,x^3y,x^2y,xy,y^2). The action of m_y on the K-basis $\langle 1,y;x,x^2,x^3\rangle$ of $\mathcal{A}^*$ has Jordan partition $P'=(2,1,1,1)$ of diagonal lengths $H(\mathcal{A})=(1,2,1,1)$ (Lemma 2.19). (In the x direction I^* has normal generators (y^2,xy,x^4) of partition $(4,1)$, the same partition as for I.) Also, holding a constant, we have

$$I^*=\lim_{\beta\to 0} I,$$

so $P'=(2,1,1,1)$ occurs in the closure of the orbit of B (Theorem 2.20(ii)).

Here $\dim G_H=1$: a graded ideal of Hilbert function H must satisfy

$$\exists L\in R_1\mid I=\big(xL,yL,M^4\big),$$

so $G_H\cong\mathbb{P}^1$, and $I\in G_H$ is determined by the choice of the linear form L, here $L=y$. The fiber of Z_H over a point of G_H is determined here by the choice of a, β, so has dimension two.

Theorem 2.16. *Assume A, B are commuting $n\times n$ nilpotent matrices with B in Jordan form and suppose* $\dim_K K[A,B]=n$. *Let $H=H(K[A,B])$ be the Hilbert function. Let K be an algebraically closed field of characteristic zero, or of characteristic $p>j$ the socle degree of H. Then for a generic $\lambda\in\mathbb{P}^1$, the Jordan block sizes of the action of $A+\lambda B$ both on $K[A,B]\cong$*

R/I and on the associated graded algebra $\mathrm{Gr}_M K[A,B] \cong \mathrm{Gr}_M(R/I)$*, are given by the parts of* $P(H)$*. We have* $P(H) \geqslant P$ *in the POS of* (1.4)*.*

Proof. By [Br] in the case $\mathrm{char}\, K = 0$ or [I2] when $\mathrm{char}\, K = p > j$, there is an open dense set of $\lambda \in \mathbb{A}^1$, such that the ideal I has normal basis in the direction $x' = x + \lambda y$. Replacing x in (2.12) by x', so considering the standard basis $f_0, \ldots, f_{\nu-1}$ there, and considering the action of $m_{x'} = \times x'$ on the cyclic $K[x']$ subspaces of R/I generated by $1, g_1, \ldots, g_{\nu-1}$, we see that the Jordan partition of $m_{x'}$ is just $P(m_{x'}) = (k_0, \ldots, k_{\nu-1})$. This is $P(H)$ since the basis is normal.

The standard basis for the associated graded ideal is given by the initial ideal $\mathrm{In}\, I$, satisfying

$$\mathrm{In}\, I = \big(\mathrm{In}(f_\nu), \ldots, \mathrm{In}(f_1), f_0\big),$$

where here $\mathrm{In}\, f$ denotes the lowest degree graded summand of f. So the Jordan partition for the action of m_x on R/I^* is also $P(H)$. □

We thank G. McNinch for comments and a discussion that led to the following corollary. The corollary implies the special case of his result [McN, Theorem 26] where (A, B) is assumed to have a cyclic vector, and also K is algebraically closed of suitable characteristic.

Corollary 2.17. *Assume that* A, B *and the field* K *satisfy the hypotheses of Theorem* 2.16*. Then for generic* t*,* A *and* B *are in the Jacobson radical of* $\mathcal{C}_t$*, the commutator algebra of* $A + tB$*.*

Proof. The Jordan partition P_t given by the blocks of $A + tB$ for t generic is strictly decreasing, as it has the form $P(H)$. That the partition P_t has distinct parts is equivalent to the semisimple quotient $\mathcal{C}_t/\mathfrak{J}_t$ of the commutator algebra $\mathcal{C}_t \subset \mathrm{End}\, V$ of $A + tB$ satisfying $\mathcal{C}_t/\mathfrak{J}_T$ being an étale algebra—a product of fields K, one copy for each distinct part of P_t [Bas2, Lemma 2.3]. Thus A and B, being nilpotent, are in the Jacobson radical of $\mathcal{C}_t$. □

The following example communicated to us by G. McNinch shows that the restriction on $\mathrm{char}\, K$ in Theorem 2.16 is sharp.

Example 2.18. Let d be a positive integer. Let V_1 be a d-dimensional K-vector space, let V_2 be a 2-dimensional K-vector space, and let V be the tensor product $V = V_1 \otimes V_2$. Let $A = J_d \otimes I_2$, where J_d is a Jordan block of size d in the V_1 factor and I_2 the identity. So the partition of A is (d, d). And let $B = I_d \otimes J_2$. Then A and B commute. The algebra $K[A, B]$ is isomorphic to $K[x, y]/(x^d, y^2)$, and has vector space dimension $n = 2d$. Its Hilbert function is $H = (1, 2, 2, \ldots, 2, 2, 1)$ of socle degree d, so $P(H) = (d + 1, d - 1)$. (This is the answer one expects from the rule for computing tensor products of representations of the Lie algebra $sl(2)$.) If the integer d is invertible in K, the partition of $A + tB$ is indeed $(d + 1, d - 1)$ for all t not zero. But in characteristic p dividing d, the Jordan block partition of $A + tB$ is (d, d) for all t [McN, Example 22].

We isolate a result that can be concluded simply from [IY, Definition 3.9,Theorem 3.12] or from Gröbner basis theory. We use the notation from Definition 2.12.

Lemma 2.19. *Let* $\mathcal{A}$ *be a graded Artinian algebra quotient* $\mathcal{A} = R/I$ *of* R*, so* $\mathcal{A} = \bigoplus_0^j \mathcal{A}_i$*. Then we have*:

(i) *Let $x \in \mathcal{A}_1$. Then I has a standard basis in the direction x.*
(ii) *The partition P' given by the Jordan blocks of the action of x on $\mathcal{A}$ satisfies $P' = (k_0, \ldots, k_{\nu-1})$ from* (2.12)*, and has diagonal lengths $H = H(\mathcal{A})$.*

Proof. The initial monomial ideal $E(I)$ in the x-direction certainly has a basis as in (2.12), for some sequence of integers $k_0, \ldots, k_{\nu-1}$: to show a standard basis we must show that the sequence is non-increasing. However, if $k_u > k_{u-1}$ then multiples of yf_{u-1} could be used to eliminate $y^u x^{k_u}$ from the initial ideal $E(I)$.

Then (ii) follows from (2.12), since $\mathcal{A}$ is the internal direct sum of the $k[X]$ modules generated by $1, g_1, \ldots, g_{\nu-1} \in \mathcal{A}$. □

Recall that $\mathcal{U}_B$ is the open dense subset of $\mathcal{N}_B$ for which $\dim_K K[A, B] = n$. Now using the connection between Z_H and $\mathcal{H}^H(n, K)$ we have

Theorem 2.20. *Let B be nilpotent with Jordan partition P, let $A \in \mathcal{U}_B$, and let $H = H(K[A, B])$. Suppose that K is as in Theorem* 2.16*. Then*:

(i) *For generic $\lambda \in \mathbb{P}^1$ the Jordan block sizes of the action of $A + \lambda B$ on $K[A, B]$ are given by the parts of $P(H)$.*
(ii) *The closure of the* Gl_n *orbit of B contains a nilpotent matrix having partition P' whose diagonal lengths are given by H.*

Proof. We may assume that B is in Jordan form. It follows from the assumptions and Theorem 2.16 that $C_\lambda = A + \lambda B$ for λ generic satisfies, $P(C_\lambda) = P(H)$. Since the algebra $\mathcal{A} = \mathcal{A}_{A,B} = K[A, B]$ is a deformation of the associated graded algebra $\mathcal{A}^*$, the multiplication m_y on $\mathcal{A}$ is a deformation of the action m_y on $\mathcal{A}^*$, so the orbit P' of the latter is in the closure of the orbit of P. By Lemma 2.19(ii) P' has diagonal lengths H. □

Theorem 2.21. *Let B be nilpotent of partition P, and denote by $Q(P)$ the partition giving the Jordan block decomposition for a generic element $A \in \mathcal{N}_B$. Suppose that K is algebraically closed and that* $\operatorname{char} K = 0$ *or* $\operatorname{char} K > n$*. Then $Q(P)$ has decreasing parts and is the greatest $P(H)$ that occurs for Hilbert functions of length n algebras $\mathcal{A} = K[A, B]$, with $A \in \mathcal{N}_B$:*

$$Q(P) = \sup\big\{ P(H) \;\big|\; \exists A \in \mathcal{U}_B,\ H = H\big(K[A, B]\big)\big\}.$$

Proof. By the irreducibility of $\mathcal{N}_B$ (Lemma 1.5), there is an orbit $Q(P)$ whose closure contains each other orbit occurring in $\mathcal{U}_B$. By Theorem 2.20 $Q(P)$ has the form $P(H)$ for some H. Since the closure of its orbit contains each other orbit, this $P(H) = Q(P)$ is greater than every other $P(H')$ for a sequence H' among $\{H(K[A, B]), A \in \mathcal{U}_B\}$. □

We recall the natural order (2.8) on the set $\mathcal{H}(n)$ of Hilbert functions of length m. The openness on $\mathrm{Hilb}^n(R)$ of the condition

$$\dim_K I \cap M^{u+1} > s$$

shows that

$$\overline{Z_H} \cap Z_{H'} \neq \emptyset \quad \Rightarrow \quad H \leqslant H'. \tag{2.15}$$

We denote by $\mathfrak{W}_B$ the fiber over projection on the first factor of $\mathfrak{W}$ from (2.1): thus $\mathfrak{W}_B$ is isomorphic to pairs (A, v) with v a cyclic vector for (B, A); and it is acted on by the units $\mathfrak{G} = {\mathcal{C}_B}^*$ of the commutator $\mathcal{C}_B$ of B by: $g \in \mathfrak{G} \Rightarrow g(A, v) = (gAg^{-1}, g(v))$. Recall that $B = J_P$, and that $i(Q(P))$ is its largest part.

Lemma 2.22. *We have the following*:

(i) *Let* $(A, v), (A', v') \in \mathfrak{W}_B$ *satisfy, the closure of the* $\mathfrak{G}$ *orbit of* (A, v), *contains that of* (A', v). *Then* $H(K[A, B]) \leqslant H(K[A', B])$.
(ii) *Let* $(A, v) \in \mathfrak{W}_B$ *satisfy* $P_A = Q(P)$, *and let* K *satisfy,* $\operatorname{char} K = 0$ *or* $\operatorname{char} K > n$. *Then* $H(K[A, B]) = H_{Q(P)}$, *the diagonal lengths of* $Q(P)$.

Proof. The claim (i) follows from $\pi : \mathfrak{W}_B \to H^{[n]}$ being a morphism, and (2.15). Concerning (ii), let A have partition $Q(P)$, and let $H = H(K[A, B])$. By Theorem 2.20, for generic $\lambda \in \mathbb{P}^1$ the Jordan block sizes of the action of $A + \lambda B$ on $K[A, B]$ are given by the parts of $P(H)$. Since $m_{A+\lambda B}$ specializes to m_A, we have $Q(P) \leqslant P(H)$. But $Q(P)$ is the partition of the generic element $A \in \mathcal{N}_B$, which is irreducible, so $Q(P) \geqslant P(H)$, implying equality. By Lemma 2.10(i) $H = H_{Q(P)}$. □

Note that an analogous result to Lemma 2.22(ii) would hold for any irreducible subset $\mathcal{N}$ of $\mathcal{N}_B$, satisfying $A \in \mathcal{N} \Rightarrow$ the general element of the pencil $A + tB \subset \mathcal{N}$, $t \in K$.

Theorem 2.23. *Let* B *be Jordan of partition* P *and let* $\operatorname{char} K = 0$ *or* $\operatorname{char} K > n$. *Then*

$$Q(P) = P\big(H_{\min}(P)\big), \quad \text{where } H_{\min}(P) = \min\Big\{H \;\Big|\; \exists A \in \mathcal{U}_B \;\Big|\; H\big(K[A, B]\big) = H\Big\}.$$

Proof. By Lemma 2.10(i) the bijection $H \to P_H$ from Hilbert functions to partitions with decreasing parts, is order-reversing. The assertion thus follows from Theorem 2.21. □

Note that Lemma 2.22(i) may be used in place of Lemma 2.10 in the above proof.

Remark 2.24. P. Oblak has shown a formula for the index $i(Q(P))$, which is the largest part of $Q(P)$. This was proven in [Ob1] for $\operatorname{char} K = 0$, but can be shown valid in all characteristics [Bas-I]. For a Hilbert function H, the index $i(P(H))$ is by definition one greater than the socle degree j of H. This suggests that Theorems 2.21 and 2.23 might hold for $\operatorname{char} K \geqslant i(Q(P))$.

T. Košir and P. Oblak have recently resolved the question we asked in [I4, p. 3] whether $Q(P)$ is stable (Theorem 2.27). We give a short summary in order to comment on the relation of their result to the Hilbert scheme. An insight they had was that the question about stability is closely related to the case $e = 2$ of the following classical result about height two ideals.

Lemma 2.25. *Let* K *be an infinite field and* $\mathcal{A} = R/I$, $R = K\{x, y\}$ *be an Artinian quotient.*

(i) *Then* A *satisfies* $\dim_K(0 : m) = e - 1$ *if and only if the ideal* I *has* e *generators in a minimal generating set.*

(ii) *Let I have e generators in a minimal generating set. Then the Hilbert function $H(A)$ satisfies,*

$$i \geqslant \nu(I) \quad \Rightarrow \quad h_{i-1} - h_i \leqslant e - 1.$$

In particular, if I is a complete intersection ($e = 2$) then $h_{i-1} - h_i \leqslant 1$.

Comment on proof. The result (i) is shown by F.H.S. Macaulay in [Mac2] following earlier articles [Mac1,Scott], that were incomplete. The case $e = 2$ is that an Artinian ring A is Gorenstein of codimension two if and only if A is a complete intersection (CI). The usual proof given now uses the Hilbert Burch theorem about minimal resolutions for I (see [E, Theorem 20.15ff]).

The result (ii) appears to be shown for at least $e = 2$ in [Mac2]. The general case follows when $\operatorname{char} K = 0$ or $\operatorname{char} K = p > n$ from considering normal bases [Br,I2], or in all characteristics from considering "weak normal" bases [I2, Theorem 4.3]. Underlying the numerical result when $e = 2$ is that a *graded* CI such as $C = R/(x^a, y^b)$, $a \leqslant b$ has Hilbert function

$$H(C) = (1, 2, \ldots, a, a, \ldots, a, a - 1, \ldots, 1).$$

When $\mathcal{A}$ is CI of codimension two then $\mathcal{A}^*$ has a unique filtration by graded modules whose successive quotients are shifted CI's [I3]. □

Remark 2.26. When $H(\mathcal{A})$ satisfies $h_{i-1} - h_i \leqslant 1$ for $i \geqslant \nu$, then $P(H)$ has decreasing parts that differ pairwise by at least two. For example, when $H = (1, 2, 3, 4, 3, 3, 2, 1)$, $P(H) = (8, 6, 4, 1)$.

The following is the main result of [KoOb]. Recall that $B = J_P$, the Jordan nilpotent matrix of partition P. Recall that K is algebraically closed.

Theorem 2.27 *(T. Košir and P. Oblak). Let A be generic in $\mathcal{N}_B$. Then $K[A, B]$ is Gorenstein. When K is algebraically closed and* $\operatorname{char} K = 0$ *or* $\operatorname{char} K > n$ *then $Q(P)$ is stable.*

Proof idea. Their key step is to extend V. Baranovsky's result that A generic in $\mathcal{N}_B$ implies $K[A, B]$ is cyclic [Bar, Lemma 3], to show that $K[A, B]$ is also cocyclic (Gorenstein). Since height two Gorenstein Artinian algebras are CI [Mac3], it follows that for A generic in $\mathcal{N}_B$, that $P(H)$ for $H = H(K[A, B])$ has parts that differ pairwise by at least two. They conclude using Theorem 2.21 and Theorem 1.12 that $Q(P) = P(H)$ and is stable. □

Remark 2.28. The Košir–Oblak theorem gives an alternative route to the conclusion of the first step in J. Briançon's proof of his irreducibility theorem, in which he "vertically" deforms an ideal to a complete intersection ([Br], see also [I2, p. 81ff]). Conversely, J. Briançon's proof joined with V. Baranovsky's cyclicity result appears to give, for K algebraically closed of $\operatorname{char} K = 0$ or $\operatorname{char} K = p > n$, an alternative, if indirect, approach to the cocyclicity step of the Oblak–Košir result, since

(a) the vertical deformation preserves the Jordan partition of m_x [I2, (5.2) and §5AI.3];
(b) a deformation of a complete intersection remains a CI, and $\mathcal{N}_B$ is irreducible (Lemma 1.5).

J. Briançon's proof of the irreducibility of $H^{[n]}$ requires a specific step to deform the CI $(xy, x^p + y^q)$ to an order one ideal. It would be interesting to know the order $\nu_{Q(P)}$ of $H(Q(P))$ (the diagonal lengths of $Q(P)$) in terms of P. This order of $H(Q)$ is just the largest ν such that $Q_i \geqslant \nu + 1 - i$ for each i, $1 \leqslant i \leqslant \nu$.

Question. What is the closure of $\mathcal{U}_{\nu,n}$ in $H(n, K)$? (See Corollary 2.6.)

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