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Stanley depth of squarefree monomial ideals

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ABSTRACT

In this paper, we answer a question posed by Y.H. Shen. We prove that if I is an m-generated squarefree monomial ideal in the polynomial ring $S = K[x_1, \dots, x_n]$ with K a field, then $\operatorname{sdepth} I \geqslant n - \lfloor m/2 \rfloor$. The proof is inductive and uses the correspondence between a Stanley decomposition of a monomial ideal and a partition of a particular poset into intervals established by Herzog, Vladoiu and Zheng.

1. Introduction

In [1], Stanley introduced the idea of what is now called the Stanley depth of a $\mathbb{Z}^n$-graded module over a commutative ring and conjectured that the Stanley depth is always at least the module's depth. While some special cases of the conjecture have been resolved, it still remains largely open. Herzog, Vladoiu and Zheng considered the Stanley depth of monomial ideals in [2] and showed that the Stanley depth of a monomial ideal can be computed by partitioning a finite poset associated to the ideal into intervals. Biró, Howard, Keller, Trotter and Young used this correspondence in [3] to show that for K a field, the Stanley depth of the maximal ideal $(x_1, \dots, x_n) \subseteq K[x_1, \dots, x_n]$ is exactly $\lceil n/2 \rceil$. Shen subsequently showed in [4] that the result of Biro et al. can be extended to show that the Stanley depth of a complete intersection monomial ideal minimally-generated by m monomials is $n - \lfloor m/2 \rfloor$. Shen's proof relies on a theorem of Cimpoeaş [5] which states that the Stanley depth of a complete intersection monomial ideal is equal to that of its radical, which allows for a focus on squarefree ideals.

In addition to finding the exact value of the Stanley depth of an m-generated complete intersection monomial ideal, Shen proved two results regarding squarefree monomial ideals that are not complete

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intersection monomial ideals. In particular, [4] shows that if $I \subseteq K[x_1,\dots,x_n]$ is a 3-generated squarefree monomial ideal, then $\operatorname{sdepth} I \geqslant n-1$, and if I is a 4-generated squarefree monomial ideal, then $\operatorname{sdepth} I \geqslant n-2$. An example is also given to show that the inequality may be strict. The following theorem, which is the main result of this paper, provides an affirmative answer to Shen's Question 4.3.

Theorem 1.1. *Let K be a field and $I \subseteq K[x_1,\dots,x_n]$ be an m-generated squarefree monomial ideal. Then* $\operatorname{sdepth} I \geqslant n - \lfloor m/2 \rfloor$.

Shen's proof for the $m = 3$ case of Theorem 1.1 involves four steps spanning multiple pages. The proof relies upon finding an appropriate interval partition of the poset associated to I but passes a number of times between the language of ideals and the combinatorial question of partitioning. Our proof of Theorem 1.1 is considerably shorter and proceeds by inductively constructing an interval partition with the desired property. This suggests that the combinatorial approach provides a powerful tool for resolving questions involving Stanley depth.

Before restating Theorem 1.1 in terms of interval partitions and giving a proof in Section 3, we establish our notation and the needed background in Section 2.

2. Notation and background

For a positive integer n, let $[n] = \{1,2,\dots,n\}$ and let $\mathbf{2}^n$ denote the Boolean algebra consisting of all subsets of $[n]$. For $x \leqslant y$ in a poset $\mathbf{P}$, we let $[x,y] = \{z\colon\ x \leqslant z \leqslant y\}$ and call $[x,y]$ an *interval* in $\mathbf{P}$. An *antichain* in a poset is a set of pairwise incomparable elements of the poset. For $\mathbf{2}^n$, an antichain is simply a set of subsets of $[n]$ such that no subset is contained in any other. If $\mathbf{P}$ is a poset and $x \in \mathbf{P}$, we let $U[x] = \{y \in \mathbf{P}\colon\ y \geqslant x\}$ and call this the up-set of x. An element x of $\mathbf{P}$ is minimal if there is no y with $y \leqslant x$ in $\mathbf{P}$.

Throughout this paper, we will let K denote a field and $S = K[x_1,\dots,x_n]$. If M is a finitely generated $\mathbb{Z}^n$-graded S-module, $u \in M$ is a homogeneous element, and $Z \subseteq \{x_1,\dots,x_n\}$, then we call the K-subspace $uK[Z]$ of M generated by the elements uv where v is a monomial in $K[Z]$ a *Stanley space* of dimension $|Z|$ if $uK[Z]$ is a free $K[Z]$-module. We call a presentation of the $\mathbb{Z}^n$-graded K-vector space M as a finite direct sum of Stanley spaces $\mathcal{D}: M = \bigoplus_{i=1}^m u_i K[Z_i]$ in the category of $\mathbb{Z}^n$-graded K-vector spaces a *Stanley decomposition of M*. The *Stanley depth of D*, $\operatorname{sdepth}\mathcal{D}$, is $\min\{|Z_i|\colon\ 1 \leqslant i \leqslant m\}$ and the *Stanley depth of M* is then

$$\operatorname{sdepth} M = \max_{\mathcal{D}} \operatorname{sdepth}\mathcal{D},$$

where the maximum is taken over all Stanley decompositions $\mathcal{D}$ of M.

In [2], Herzog et al. introduced a powerful connection between the Stanley depth of a monomial ideal and a combinatorial partitioning problem for partially ordered sets. For $c \in \mathbb{N}^n$, let x^c denote the monomial $x_1^{c(1)} x_2^{c(2)} \cdots x_n^{c(n)}$. Let $I = (x^{a_1},\dots,x^{a_r}) \subseteq S$ be a monomial ideal. Let $g \in \mathbb{N}^n$ be such that $g \geqslant a_i$ for all i. The characteristic poset of I with respect to g, denoted $\mathbf{P}_I^g$, is defined as the induced subposet of $\mathbb{N}^n$ with ground set

$$\bigl\{c \in \mathbb{N}^n \bigm| c \leqslant g \text{ and there is } i \text{ such that } c \geqslant a_i\bigr\}.$$

(The definition in [2] includes the more general case of the module I/J where $J \subseteq I$ are monomial ideals, but this definition is sufficient for our purposes.)

Let $\mathcal{C}$ be a partition of $\mathbf{P}_I^g$ into intervals. For $J = [x,y] \in \mathcal{C}$, define

$$Z_J = \bigl\{i \in [n] \bigm| y(i) = g(i)\bigr\}.$$

Define the Stanley depth of a partition $\mathcal{C}$ to be $\operatorname{sdepth}\mathcal{C} = \min_{J \in \mathcal{C}} |Z_J|$ and the Stanley depth of the poset $\mathbf{P}_I^g$ to be $\operatorname{sdepth}\mathbf{P}_I^g = \max_{\mathcal{C}} \operatorname{sdepth}\mathcal{C}$, where the maximum is taken over all partitions $\mathcal{C}$ of $\mathbf{P}_I^g$ into intervals. Herzog et al. showed in [2] that $\operatorname{sdepth} I = \operatorname{sdepth}\mathbf{P}_I^g$.

If I is squarefree, then we may take $g = (1, \dots, 1)$ and work inside $\{0,1\}^n$, which is isomorphic to $\mathbf{2}^n$. A monomial v in S then can be identified with the subset of $[n]$ whose elements correspond to the subscripts of the variables appearing in v. If $I = (v_1, \dots, v_m)$ is squarefree and $A_i \subseteq [n]$ corresponds to v_i, then the definition of $\mathbf{P}_I^g$ clearly simplifies to $\mathbf{P}_I^g = \bigcup_{i=1}^m U[A_i]$ as a subposet of $\mathbf{2}^n$. For an interval $J = [X, Y]$, we then have that $|Z_J|$ corresponds to $|Y|$.

3. A combinatorial theorem and proof

With the definitions of the previous section, we are now ready to state and prove the combinatorial version of Theorem 1.1. They are clearly equivalent once we note that if I is minimally generated by $v_1, \dots, v_m$, then the sets corresponding to the v_i must form an antichain.

Theorem 3.1. *Let $\mathcal{A}$ be an antichain of size m in $\mathbf{2}^n$ and let $\mathbf{P} = \bigcup_{A \in \mathcal{A}} U[A]$. Then there exists a partition $\mathcal{C}$ of $\mathbf{P}$ into intervals such that for each interval $[X, Y] \in \mathcal{C}$, $|Y| \geqslant n - \lfloor m/2 \rfloor$.*

Proof. Let k be the number of elements of $[n]$ that appear in at least two sets in $\mathcal{A}$. Our proof will be by induction on k. The base case is $k = 0$, which follows from Theorem 2.4 of [4] since the sets in $\mathcal{A}$ being disjoint corresponds to a complete intersection monomial ideal.

We assume that the result holds (for all n and m) if at most $k \geqslant 0$ elements of $[n]$ occur in more than one set in an antichain. Let $\mathcal{A}$ be an antichain in $\mathbf{2}^n$ in which exactly $k+1$ elements of $[n]$ occur in more than one set. Fix an element $x \in [n]$ that appears in at least two elements of $\mathcal{A}$. Let $\mathcal{A}_0 = \{A \in \mathcal{A}\colon\ x \notin A\}$ and $\mathcal{A}_1 = \{A \in \mathcal{A}\colon\ x \in A\}$. Since $x \notin A$ for all $A \in \mathcal{A}_0$, we can define $\mathbf{P}_0 = \bigcup_{A \in \mathcal{A}_0} U[A]$, taking up-sets inside the lattice of subsets of $[n] - \{x\}$ so that we may view $\mathbf{P}_0$ as a subposet of $\mathbf{2}^{n-1}$. We also define

$$\mathbf{P}_1 = \left(\bigcup_{A \in \mathcal{A}_1} U[A] \right) \cup \left(\bigcup_{A \in \mathcal{A}_0} U\big[A \cup \{x\}\big] \right)$$

as a subposet of $\mathbf{2}^n$ and note that it is isomorphic to the poset

$$\left(\bigcup_{A \in \mathcal{A}_1} U\big[A - \{x\}\big] \right) \cup \left(\bigcup_{A \in \mathcal{A}_0} U[A] \right),$$

which is a subposet of the lattice of subsets of $[n] - \{x\}$ and thus isomorphic to a subposet of $\mathbf{2}^{n-1}$. We will use $\mathbf{P}_1$ to refer to whichever of these representations is more convenient. Although these representations do not give, strictly speaking, a representation of $\mathbf{P}_1$ as the union of the up-sets of elements of an antichain, we note that $\mathbf{P}_1$ can be represented in this way using the minimal elements $\mathcal{M}$ of $\mathcal{A}_1 \cup \{A \cup \{x\}\colon\ A \in \mathcal{A}_0\}$. We note that $\mathbf{P}$ is the disjoint union of $\mathbf{P}_0$ and $\mathbf{P}_1$ when they are considered as subposets of $\mathbf{2}^n$.

Note that since x does not occur in any of the sets of $\mathbf{P}_0$, at most k of the elements of $[n] - \{x\}$ occur in more than one set of $\mathcal{A}_0$, so by induction we may partition $\mathbf{P}_0$ into a collection $\mathcal{C}_0$ of intervals such that for each $[X, Y] \in \mathcal{C}_0$,

$$|Y| \geqslant (n-1) - \left\lfloor \frac{|\mathcal{A}_0|}{2} \right\rfloor \geqslant (n-1) - \left\lfloor \frac{m-2}{2} \right\rfloor = n - \left\lfloor \frac{m}{2} \right\rfloor.$$

Similarly, x does not appear in any of the sets of the second representation of $\mathbf{P}_1$, so by induction we have a partition $\mathcal{C}_1$ of $\mathbf{P}_1$ into intervals so that for each $[X, Y] \in \mathcal{C}_1$,

$$|Y| \geqslant (n-1) - \left\lfloor \frac{|\mathcal{M}|}{2} \right\rfloor \geqslant (n-1) - \left\lfloor \frac{m}{2} \right\rfloor.$$

Since our goal is to think of $\mathbf{P}$ as being the disjoint union of $\mathbf{P}_0$ and $\mathbf{P}_1$ inside $\mathbf{2}^n$, we now lift the intervals of $\mathcal{C}_1$ and form the partition

$$\mathcal{C}_1' = \big\{\big[X \cup \{x\}, Y \cup \{x\}\big]: [X, Y] \in \mathcal{C}_1\big\},$$

which is a partition of the first representation of $\mathbf{P}_1$. Now notice that the upper bounds of the intervals have increased in size by one, so that for each $[X, Y] \in \mathcal{C}_1'$ we have $|Y| \geqslant n - \lfloor m/2 \rfloor$.

The partition $\mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_1'$ is the desired partition of $\mathbf{P}$ with $|Y| \geqslant n - \lfloor m/2 \rfloor$ for each $[X, Y] \in \mathcal{C}$. $\square$

4. Conclusion

It is worth noting that the bound of Theorem 3.1 is only nontrivial for a fraction of the possible values of m, since $\mathbf{2}^n$ contains an antichain of size $\binom{n}{\lfloor n/2 \rfloor}$. It may be possible to formulate a stronger bound involving the number of elements of $[n]$ that appear in more than one set in the antichain $\mathcal{A}$. In fact, we can use $\mathcal{A}$ to form a hypergraph $([n], \mathcal{A})$, and perhaps studying this hypergraph will provide additional insight. Furthermore, the techniques developed thus far for interval partitioning seem to just scratch the surface. An investigation of other classes of monomial ideals that have characteristic posets that have been studied in other contexts may prove fruitful to both algebra and combinatorics.

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