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Weakly c-permutable subgroups of finite groups ☆

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ABSTRACT

A subgroup H of a group G is called weakly c-permutable in G if there exists a subgroup T of G such that $G = HT$ and $H \cap T$ is completely c-permutable in G. In this paper, we obtain some results about the weakly c-permutable subgroups and use them to determine the structures of some groups. In particular, we give some new characterizations of supersolvability and p-nilpotency of a group (and, more general, a group belonging to a given formation of finite groups) by using the weakly c-permutability of some primary subgroups. As application, we generalize a series of known results.

1. Introduction

In [10,11], Guo, Shum and Skiba introduced the concept of conditionally permutable subgroup and completely c-permutable subgroup: Let H, K be subgroups of a group G. H is said to be conditionally permutable (or in brevity, c-permutable) with T if there exists some $x \in G$ such that $HT^x = T^xH$. H is said to be completely c-permutable with T if there exists some $x \in \langle H, T\rangle$ such that $HT^x = T^xH$. If H is c-permutable (completely c-permutable) with all subgroups of G, then H is said to be c-permutable (completely c-permutable, respectively) in G. The new idea has been used to prove a series of elegant

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results on the structure of groups (see [8–12]). As a development of the above research, we now introduce the following new concept of weakly c-permutable subgroups:

Definition 1.1. Let H be a subgroup of a group G. H is said to be weakly c-permutable in G if there exists a subgroup T of G such that $G = HT$ and $H \cap T$ is completely c-permutable in G.

Obviously, all permutable subgroups and all completely c-permutable subgroups are weakly c-permutable subgroups. However the converse is not true. For example, in the symmetric group $S_5 = C_5A_4$ of degree 5, C_5 is not completely c-permutable in S_5 because C_5 cannot permute with any subgroup of S_5 with order 3, but C_5 is weakly c-permutable in S_5 since $C_5 \cap A_5 = 1$.

Recently, Wang introduced the concept of c-normal subgroup [27] and Ballester-Bolinches, Guo and Wang introduced the notion of c-supplemented subgroup [3] (also see [28]): a subgroup H of a group G is said to be a c-supplemented (c-normal) if there exists a subgroup (normal subgroup) T of G such that $G = HT$ and $H \cap T \leqslant H_G$, where H_G is the largest normal subgroup of G contained in H. Note that the condition $H \cap T \leqslant H_G$ in the concepts is actually equivalent to the condition $H \cap T = H_G$ (see [33, Lemma 2.2(1)]). We also see that many interesting results have been obtained by using the c-normal subgroups and the c-supplemented subgroups (see [17–32]).

It is easy to know that all normal subgroups, c-normal subgroups and c-supplemented subgroups are all also weakly c-permutable in G. But the following examples show that the converse is not true.

Example 1. Let $G = [C_5]C_4$, where C_5 is a cyclic group of order 5 and C_4 is an automorphism group of C_5. Then it is easy to see that the subgroup C_2 of C_4 of order 2 is weakly c-permutable in G, but it is not c-supplemented in G.

Example 2. Let $G = \langle x, y \mid x^{16} = y^4 = 1,\ x^y = x^3\rangle$. Then $\Phi(G) = \langle x^2, y^2\rangle = \langle x^2\rangle \times \langle y^2\rangle$. It is easy to see that $H = \langle y^2\rangle$ is permutable in G and consequently H is weakly c-permutable in G, but H is not c-supplemented (see [31]).

The analysis above shows that the set of all weakly c-permutable subgroups is wider than the set of all permutable subgroups, the set of all completely c-permutable subgroups, the set of all c-normal subgroups and than the set of all c-supplemented subgroups. In [24], Skiba introduced the notion of weakly s-supplemented subgroup and the notion of weakly s-permutable subgroup: a subgroup H is said to be weakly s-supplemented (weakly s-permutable) in G if G has a subgroup (a subnormal subgroup) T such that $HT = G$ and $T \cap H \leqslant H_{sG}$. We also note that our weakly c-permutable subgroup is different from the weakly s-supplemented subgroup and so is different from the weakly s-permutable subgroup. For example, let $G = [C_5]C_4$, where C_5 is a cyclic group of order 5 and C_4 is an automorphism group of C_5. Then the subgroup C_2 of C_4 of order 2 is weakly c-permutable in G. But C_2 is not weakly s-supplemented in G. In fact, if C_2 is weakly s-supplemented in G, then G has a subgroup T such that $G = C_2T$ and $C_2 \cap T \leqslant (C_2)_{sG}$. Obviously, either $(C_2)_{sG} = 1$ or $(C_2)_{sG} = C_2$. If $(C_2)_{sG} = 1$, then C_2 has a complement T in G. This implies that C_2 has also a complement in C_4, which is impossible since C_4 is a cyclic group. If $(C_2)_{sG} = C_2$, then C_2 is s-permutable and consequently $C_2 \leqslant O_2(G)$ (see [24, Lemma 2.5(6) and Lemma 2.6(3)]), which contradicts $O_2(G) = 1$. Thus, C_2 is not weakly s-supplemented in G and therefore it is not also weakly s-permutable in G. In connection with this, naturally there is a question: *Whether can we characterize the structure of finite groups by using the weak c-permutability of subgroups*? The purpose of this paper contributes to this. Our main results are as follows:

Theorem 1.1. *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and G a group. Then $G \in \mathfrak{F}$ if and only if there exists a soluble normal subgroup H such that $G/H \in \mathfrak{F}$ and all maximal subgroups of all Sylow subgroups of $F(H)$ are weakly c-permutable in G.*

Theorem 1.2. *Let $\mathfrak{F}$ be an S-closed saturated formation containing $\mathfrak{N}$ and G a group. Then $G \in \mathfrak{F}$ if and only if there exists a normal subgroup N of G such that $G/N \in \mathfrak{F}$ and every cyclic subgroup of $G^{\mathfrak{F}}$ with order 4 is weakly c-permutable in G and every minimal subgroup of $G^{\mathfrak{F}}$ is contained in $Z_{\mathfrak{F}}(G)$.*

Theorem 1.3. *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and G a group. Then $G \in \mathfrak{F}$ if and only if there exists a soluble normal subgroup H such that $G/H \in \mathfrak{F}$ and all cyclic subgroups of any non-cyclic Sylow subgroup of $F(H)$ with prime order or 4 (if the Sylow 2-subgroup of $F(H)$ is a non-abelian 2-group) are weakly c-permutable in G.*

Theorem 1.4. *Let G be a finite group and p a prime divisor of $|G|$ such that $(|G|, p^2 - 1) = 1$. Then G is p-nilpotent if and only if there exists a normal subgroup N in G such that G/N is p-nilpotent and every subgroup of N of order p^2 is weakly c-permutable in G.*

As applications, we generalize a series of known results (see Corollaries 5.1–5.24).

Throughout this paper, all groups are assumed to be finite groups. The reader is referred to the monographs of [5,7] or [15] for the notations and terminologies not mentioned in this paper.

2. Preliminaries

For the sake of convenience, we list here some notions and basic results which are needed in this paper.

We denote $M < \cdot G$ to indicate that M is a maximal subgroup of a group G. For a class $\mathfrak{F}$ of groups, a chief factor H/K of a group G is called $\mathfrak{F}$-central if $[H/K](G/C_G(H/K)) \in \mathfrak{F}$ (see [7, Definition 2.4.3]). The symbol $Z_{\mathfrak{F}}(G)$ denotes the $\mathfrak{F}$-hypercenter of a group G, that is, the product of all such normal subgroups H of G whose G-chief factors are $\mathfrak{F}$-central. A subgroup H of G is said to be $\mathfrak{F}$-hypercentral in G if $H \leqslant Z_{\mathfrak{F}}(G)$.

Recall that (see [5] or [7]) a class $\mathfrak{F}$ of groups is said to be a formation if it is closed under homomorphic image and every group G has a smallest normal subgroup (called $\mathfrak{F}$-residual and denoted by $G^{\mathfrak{F}}$) with quotient is in $\mathfrak{F}$. A formation $\mathfrak{F}$ is said to be saturated if it contains every group G with $G/\Phi(G) \in \mathfrak{F}$. A formation $\mathfrak{F}$ is said to be S-closed if every subgroup of a group G belongs to $\mathfrak{F}$ whenever $G \in \mathfrak{F}$.

We use $\mathfrak{N}$, $\mathfrak{U}$ and $\mathfrak{S}$ to denote the classes of all nilpotent groups, supersoluble groups and soluble groups, respectively. It is well known that the classes $\mathfrak{N}$, $\mathfrak{U}$ and $\mathfrak{S}$ are all S-closed saturated formations.

Lemma 2.1. *(See [10,11].) Let G be a group, $K \lhd G$ and $H \leqslant G$. Then:*

(1) *If $K \leqslant T \leqslant G$ and H is completely c-permutable with T in G, then HK/K is completely c-permutable with T/K in G/K. In particular, if H is completely c-permutable in G, then HK/K is completely c-permutable in G/K;*
(2) *If $K \leqslant H$ and H/K is completely c-permutable in G/K, then H is completely c-permutable in G;*
(3) *If $T \leqslant M \leqslant G$, $H \leqslant M$ and H is completely c-permutable with T in G, then H is completely c-permutable with T in M;*
(4) *If $T \leqslant G$ and H is completely c-permutable with T in G, then H^x is completely c-permutable with T^x in G for every $x \in G$.*

Lemma 2.2. *Let G be a group. Then:*

(1) *If H is weakly c-permutable in G and $H \leqslant M \leqslant G$, then H is weakly c-permutable in M;*
(2) *Let $N \trianglelefteq G$ and $N \leqslant H$. Then H is weakly c-permutable in G if and only if H/N is weakly c-permutable in G/N;*
(3) *Let π be a set of primes. Let N be a normal π'-subgroup of G and H a π-subgroup of G. If H is weakly c-permutable in G, then HN/N is weakly c-permutable in G/N.*

Proof. (1) If $G = HK$ and $H \cap K$ is completely c-permutable in G, then $M = M \cap G = M \cap HK = H(M \cap K)$ and $H \cap (M \cap K) = H \cap K$ is completely c-permutable in M by Lemma 2.1(3). Hence, H is weakly c-permutable in M.

(2) Suppose that H/N is weakly c-permutable in G/N. Then there exists a subgroup K/N of G/N such that $G/N = (H/N)(K/N)$ and $H/N \cap K/N = (H \cap K)/N$ is completely c-permutable in G/N. It follows from Lemma 2.1(2) that $G = HK$ and $H \cap K$ is completely c-permutable in G.

Conversely, if H is weakly c-permutable in G, then there exists a subgroup T of G such that $G = HT$ and $H \cap T$ is completely c-permutable in G. Then $G = (H/N)(TN/N)$ and $H/N \cap TN/N = N(H \cap T)/N$. By Lemma 2.1(1), $N(H \cap T)/N$ is completely c-permutable in G/N. Thus, H/N is weakly c-permutable in G/N.

(3) If H is weakly c-permutable in G, then there exists a subgroup K of G such that $G = HK$ and $H \cap K$ is completely c-permutable in G. Since $|G|_{\pi'} = |K|_{\pi'} = |KN|_{\pi'}$, we have that $|K \cap N|_{\pi'} = |N|_{\pi'} = |N|$. Hence $N \leqslant K$. Clearly $G/N = (HN/N)(K/N)$ and $HN/N \cap K/N = (H \cap K)N/N$ is completely c-permutable in G/N by Lemma 2.1(1). Hence HN/N is weakly c-permutable in G/N. $\square$

Lemma 2.3. *(See [21, Lemma 2.5].) Let G be a finite group and p the prime divisor of $|G|$ such that $(|G|, p^2 - 1) = 1$. If G/L is p-nilpotent and $p^3 \nmid |L|$, then G is p-nilpotent.*

Lemma 2.4. *(See [7, Theorem 1.8.17].) Let N be a soluble normal subgroup of a group G. If $N \cap \Phi(G) = 1$, then the Fitting subgroup $F(N)$ of N is a direct product of some abelian minimal normal subgroups of G.*

Lemma 2.5. *(See [26, Lemma 2.8].) Let M be a maximal subgroup of G and P a normal Sylow p-subgroup of G such that $G = PM$, where p is a prime. Then:*

(1) *$P \cap M$ is a normal subgroup of G;*
(2) *If $p > 2$ and all minimal subgroups of P are normal in G, then M has index p in G.*

Lemma 2.6. *(See [24, Lemma 2.16].) Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and G a group with a normal subgroup E such that $G/E \in \mathfrak{F}$. If E is cyclic, then $G \in \mathfrak{F}$.*

Lemma 2.7. *(See [24, Lemma 2.20].) Let A be a p'-automorphisms of a p-group P, where p is an odd prime. Assume that every subgroup of P with prime order is A-invariant. Then A is cyclic.*

Lemma 2.8. *(See [13, Lemma 5].) Let $\mathfrak{F}$ be S-closed local formation and H a subgroup of G. Then $H \cap Z_{\mathfrak{F}}(G) \subseteq Z_{\mathfrak{F}}(H)$.*

3. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. It is clear that the condition is necessary. We only need to prove that it is sufficient. Assume that the assertion is false and let (G, H) be a counterexample with $|G||H|$ is minimal. Let P be an arbitrary Sylow p-subgroup of $F(H)$. Clearly $P \trianglelefteq G$. We proceed the proof by the following steps.

(1) $P \cap \Phi(G) = 1$.

If not, then $1 \neq P \cap \Phi(G) \lhd G$. Let $R = P \cap \Phi(G)$. We show that G/R satisfies the hypothesis. In fact, $(G/R)/(H/R) \cong G/H \in \mathfrak{F}$. Let $F(H/R) = T/R$. Then, obviously, $F(H)/R = F(H/R)$. Let P_1/R be a maximal subgroup of P/R. Then P_1 is a maximal subgroup of P. By hypothesis, P_1 is weakly c-permutable in G. Hence, by Lemma 2.2, P_1/R is weakly c-permutable in G/R. Let $\bar{Q}_1$ be a maximal subgroup of the Sylow q-subgroup $\bar{Q}$ of $F(H)/R$, where $q \neq p$. Then, clearly, there exists a Sylow q-subgroup Q of $F(H)$ such that $\bar{Q} = QR/R$ and $\bar{Q}_1 = Q_1R/R$ with Q_1 is a maximal subgroup of Q. By hypothesis, Q_1 is weakly c-permutable in G and so Q_1R/R is weakly c-permutable in G/R by Lemma 2.2(3). This shows that $(G/R, H/R)$ satisfies the hypothesis. The minimal choice of (G, H) implies that $G/R \in \mathfrak{F}$. Since $R \subseteq \Phi(G)$ and $\mathfrak{F}$ is a saturated formation, $G \in \mathfrak{F}$, a contradiction. Thus (1) holds.

(2) $P = R_1 \times R_2 \times \cdots \times R_m$, where R_i $(i = 1, 2, \ldots, m)$ is some minimal normal subgroup of G of prime order.

Since $P \lhd G$ and $P \cap \Phi(G) = 1$, $P = R_1 \times R_2 \times \cdots \times R_m$, where R_i $(i = 1, 2, \ldots, m)$ is an abelian minimal normal subgroup of G by Lemma 2.4. We now prove that $|R_i| = p$.

Since $R_i \nsubseteq \Phi(G)$, there exists a maximal subgroup M of G such that $G = R_i M$ and $R_i \cap M = 1$. Let M_p be a Sylow p-subgroup of M and $G_p = M_p R_i$. Then G_p is a Sylow p-subgroup of G. Let P_1 be a maximal subgroup of G_p containing M_p and $P_2 = P_1 \cap P$. Then $|P : P_2| = |P : P_1 \cap P| = |PP_1 : P_1| = |G_p : P_1| = p$ and so P_2 is a maximal subgroup of P. We also have that $P_2 M_p = (P_1 \cap P)M_p = P_1 \cap PM_p = P_1 \cap G_p = P_1$ and $P_2 \cap M_p = P \cap P_1 \cap M = P \cap M_p$. By hypothesis, P_2 is weakly c-permutable in G. Hence there exists a subgroup T of G such that $G = P_2 T$ and $P_2 \cap T$ is completely c-permutable in G. Then, for an arbitrary Sylow q-subgroup Q of G with $q \neq p$, there exists an element $\alpha \in \langle P_2 \cap T, Q \rangle$ such that $(P_2 \cap T)Q^\alpha = Q^\alpha (P_2 \cap T)$. Hence $P_2 \cap T = (P_2 \cap T)(P \cap Q^\alpha) = P \cap (P_2 \cap T)Q^\alpha \trianglelefteq (P_2 \cap T)Q^\alpha$. It follows that $Q^\alpha \leqslant N_G(P_2 \cap T)$. On the other hand, $P \cap T \trianglelefteq T$ and $P \cap T \trianglelefteq P$ since P is abelian. Hence $P \cap T \trianglelefteq PT = G$ and consequently $P_2 \cap T = P_1 \cap P \cap T \trianglelefteq P_1$. It follows that $P_2 \cap T \trianglelefteq P_1 P = G_p$. This shows that both G_p and Q are contained in $N_G(P_2 \cap T)$. The arbitrary choice of q implies that $P_2 \cap T \trianglelefteq G$ and so $P_2 \cap T \leqslant (P_2)_G$. Assume that $P_2 \cap T < (P_2)_G$ and let $N = (P_2)_G T$. Then $G = P_2 T = P_2 (P_2)_G T = P_2 N$ and $P_2 \cap N = P_2 \cap (P_2)_G T = (P_2)_G (P_2 \cap T) = (P_2)_G$. This shows that there always exists a subgroup K of G such that $G = P_2 K$ and $P_2 \cap K = (P_2)_G$.

Since P is abelian, $P_2(P \cap M) \trianglelefteq P$. Thus $P_2(P \cap M) = P$ or $P_2(P \cap M) = P_2$. If $P_2(P \cap M) = P$, then $G = PM = P_2(P \cap M)M = P_2 M$ and so $P = P \cap P_2 M = P_2(P \cap M) = P_2(P \cap P_1 \cap M) = P_2(P_2 \cap M) = P_2$, a contradiction. Hence $P_2(P \cap M) = P_2$ and so $P \cap M \leqslant P_2$. Since $P \cap M \trianglelefteq G$ by Lemma 2.5, $P \cap M \leqslant (P_2)_G = P_2 \cap K$.

Assume that $K < G$. Let K_1 be a maximal subgroup of G containing K. Then $P \cap K_1 \trianglelefteq G$ by Lemma 2.5. Hence $(P \cap K_1)M$ is a subgroup of G. Since $M < \cdot G$, $(P \cap K_1)M = G$ or $(P \cap K_1)M = M$. If $(P \cap K_1)M = G = PM$, then $P = P \cap (P \cap K_1)M = (P \cap K_1)(P \cap M) = P \cap K_1$ since $P \cap M \leqslant (P_2)_G = P_2 \cap K \leqslant P \cap K_1$. It follows that $P \leqslant K_1$ and hence $G = PK \leqslant PK_1 = K_1$, a contradiction. If $(P \cap K_1)M = M$, then $P \cap K_1 \leqslant M$ and so $P_2 \cap K \leqslant P \cap K \leqslant P \cap K_1 = P \cap K_1 \cap M \leqslant P \cap M \leqslant P_2 \cap K$. Hence $P_2 \cap K = P \cap K$. Since $G = PK = P_2 K$, $|G : P| = |PK : P| = |K : (P \cap K)| = |K : (P_2 \cap K)| = |P_2 K : P_2| = |G : P_2|$, which is impossible. Thus $K = G$. It follows that $P_2 \cap K = P_2 = P_{2G} \trianglelefteq G$. Consequently, $P_2 \cap R_i \trianglelefteq G$. But since $G_p = R_i M_p = R_i P_1$ and P_1 is a maximal subgroup of G_p containing M_p, we have $R_i \nleqslant P_2 = P_1 \cap P$. The minimal normality of R_i implies that $P_2 \cap R_i = 1$. Hence $|R_i| = |R_i : (P_2 \cap R_i)| = |R_i P_2 : P_2| = |R_i(P \cap P_1) : P_2| = |(P \cap R_i P_1) : P_2| = |P \cap G_p : P_2| = |P : P_2| = p$. Therefore R_i is a cyclic group of order p.

(3) Final contradiction.

Let $R_i \subseteq H$ and $C_0 = C_H(R_i)$. We claim that the hypothesis holds for $(G/R_i, C_0/R_i)$. Indeed, since $G/C_G(R_i) \leqslant \mathrm{Aut}(R_i)$ is abelian, $G/C_G(R_i) \in \mathfrak{F}$. Consequently, $G/C_0 = G/(H \cap C_G(R_i)) \in \mathfrak{F}$. Besides, since $R_i \leqslant Z(C_0)$ and $F(H) \leqslant C_0$, we have $F(H) = F(C_0)$. Thus $F(C_0/R_i) = F(H)/R_i$. Let P/R_i be a Sylow p-subgroup of $F(H)/R_i$, where P is a Sylow p-subgroup of $F(H)$ and P_1/R_i is a maximal subgroup of P/R_i. Then P_1 is a maximal subgroup of P. By hypothesis, P_1 is weakly c-permutable in G. Hence P_1/R_i is weakly c-permutable in G/R_i by Lemma 2.2. Now assume that QR_i/R_i is the Sylow q-subgroup of $F(H)/R_i$, where $q \neq p$ and Q is the Sylow q-subgroup of $F(H)$. Then every maximal subgroup of QR_i/R_i is of the form of Q_1R_i/R_i, where Q_1 is a maximal subgroup of Q. By hypothesis and Lemma 2.2, we see that Q_1R_i/R_i is weakly c-permutable in G/R_i. This shows that $(G/R_i, C_0/R_i)$ satisfies the condition of the theorem. The minimal choice of (G, H) implies that $G \in \mathfrak{F}$ by Lemma 2.6. The final contradiction completes the proof. $\square$

We need a preliminary to give the proof of Theorem 1.2.

Lemma 3.1. *Let G be a group. If every minimal subgroup of G is contained in $Z_\infty(G)$ and every cyclic subgroup of G with order 4 is weakly c-permutable in G, then G is nilpotent.*

Proof. Suppose that the assertion is false and let G be a counterexample of minimal order. Then by Lemma 2.2 and Lemma 2.8, we see that the hypothesis holds for every proper subgroups of G. The minimal choice of G implies that G is a minimal non-nilpotent group. Then by [7, Theorem 3.4.11], we can see that G has the following properties: (i) $G = PQ$, where P is a normal Sylow p-subgroup of G and Q a non-normal cyclic Sylow q-subgroup of G; (ii) $P/\Phi(P)$ is a chief factor of G; (iii) If P is abelian, then P is an elementary abelian subgroup; (iv) If $p > 2$, the exponent of P is a prime p; if $p = 2$, then the exponent of P is 2 or 4.

If P is abelian or $p > 2$, then the exponent of P is prime. Hence by hypothesis, $P \leqslant Z_\infty(G)$. It follows that G is nilpotent. This contradiction shows that the exponent of P is 4.

Suppose that there exists an element $x \in P \setminus \Phi(P)$ such that $|x| = 2$. Let $T = \langle x\rangle^G$. Then $T \leqslant P$ and $T\Phi(P)/\Phi(P)$ is normal in $G/\Phi(P)$. Since $P/\Phi(P)$ is a chief factor of G, $P = T$, which is impossible since the exponent of P is 4. Therefore, for all $x \in P\setminus\Phi(P)$, we have that $|x| = 4$.

Now we claim that every cyclic subgroup H of P is completely c-permutable in G. In fact, since H is weakly c-permutable in G, there exists a subgroup T of G such that $G = HT$ and $H \cap T$ is completely c-permutable in G. Let $P_1 = P \cap T$. Then $P_1 \trianglelefteq T$ and hence $P_1\Phi(P)/\Phi(P)$ is normal in $T\Phi(P)/\Phi(P)$. Since $P/\Phi(P)$ is an elementary abelian p-group, $P_1\Phi(P)/\Phi(P)$ is normal in $P/\Phi(P)$. Therefore $P_1\Phi(P)/\Phi(P)$ is normal in $G/\Phi(P)$. Since $P/\Phi(P)$ is a chief factor of G, $P_1\Phi(P)/\Phi(P) = 1$ or $P_1\Phi(P)/\Phi(P) = P/\Phi(P)$. If $P_1\Phi(P)/\Phi(P) = 1$, then $P_1 \leqslant \Phi(P)$ and so $P = P \cap HT = H(P \cap T) = HP_1 = H$. This means that H is normal in G and consequently H is completely c-permutable in G. If $P_1\Phi(P)/\Phi(P) = P/\Phi(P)$, then $G = PT = T$ and thereby $H = H \cap T$ is completely c-permutable in G. Thus, our claim holds.

Let $x \in P\setminus\Phi(P)$. Then as above we see that $|x| = 4$ and $\langle x\rangle$ is completely c-permutable in G. Hence there exists an element $\alpha \in \langle\langle x\rangle, Q\rangle$ such that $\langle x\rangle Q^\alpha = Q^\alpha\langle x\rangle$ and so $\langle x\rangle Q^\alpha$ is a subgroup of G. Then since $\langle x\rangle = \langle x\rangle(Q^\alpha \cap P) = \langle x\rangle Q^\alpha \cap P \trianglelefteq \langle x\rangle Q^\alpha$, $Q^\alpha \leqslant N_G(\langle x\rangle)$. On the other hand, since $P/\Phi(P)$ is abelian, $\langle x\rangle\Phi(P)/\Phi(P) \trianglelefteq P/\Phi(P)$. This implies that $1 \neq \langle x\rangle\Phi(P)/\Phi(P) \trianglelefteq G/\Phi(P)$. However, since $P/\Phi(P)$ is chief factor of G, $\langle x\rangle\Phi(P) = P$ and consequently $\langle x\rangle = P$, a contradiction. Thus the proof is completed. $\square$

Proof of Theorem 1.2. It is clear that the condition is necessary. We only need to prove that it is sufficient.

Suppose that the assertion is false and let G be a counterexample of minimal order. Let M be a proper subgroup of G. Since $M/N \cap M \cong MN/N \leqslant G/N \in \mathfrak{F}$ and $\mathfrak{F}$ is S-closed, we have that $M/N \cap M \in \mathfrak{F}$. Since $M/M \cap G^{\mathfrak{F}} = MG^{\mathfrak{F}}/G^{\mathfrak{F}} \leqslant G/G^{\mathfrak{F}}$, $M/M \cap G^{\mathfrak{F}} \in \mathfrak{F}$ and so $M^{\mathfrak{F}} \leqslant M \cap G^{\mathfrak{F}} \leqslant G^{\mathfrak{F}}$. Thus by hypothesis and Lemma 2.8, every minimal subgroup of $M^{\mathfrak{F}}$ is contained in $Z_{\mathfrak{F}}(G) \cap M \subseteq Z_{\mathfrak{F}}(M)$. Besides, every cyclic subgroup of $M^{\mathfrak{F}}$ with order 4 is weakly c-permutable in M by Lemma 2.2. This shows that M satisfies the hypothesis and hence G is a minimal non-$\mathfrak{F}$-group. By Lemma 3.1, $G^{\mathfrak{N}}$ is nilpotent. Hence by [7, Theorem 3.4.2], G has the following properties:

(a) $G^{\mathfrak{F}}$ is a p-group, for some prime p.
(b) $G^{\mathfrak{F}}/\Phi(G^{\mathfrak{F}})$ is a minimal normal subgroup of $G/G^{\mathfrak{F}}$.
(c) If $G^{\mathfrak{F}}$ is abelian, then $G^{\mathfrak{F}}$ is an elementary abelian p-group.
(d) If $p > 2$, then $\exp(G^{\mathfrak{F}}) = p$; if $p = 2$, then $\exp(G^{\mathfrak{F}}) = 2$ or 4.

If $G^{\mathfrak{F}}$ is abelian, then $G^{\mathfrak{F}}$ is an elementary abelian subgroup by (c). Hence, by hypothesis, we have that $G^{\mathfrak{F}} \subseteq Z_{\mathfrak{F}}(G)$. It follows that $G \in \mathfrak{F}$. This contradiction shows that $G^{\mathfrak{F}}$ is non-abelian. If $\exp(G^{\mathfrak{F}}) = p$, then $G^{\mathfrak{F}} \subseteq Z_{\mathfrak{F}}(G)$ by hypothesis and consequently $G \in \mathfrak{F}$, a contradiction again. Thus, $G^{\mathfrak{F}}$ is a non-abelian 2-group and $\exp(G^{\mathfrak{F}}) = 4$. Let x be an arbitrary element of $G^{\mathfrak{F}} \setminus \Phi(G^{\mathfrak{F}})$. Then $|x| = 4$. Indeed, suppose that there exists an element $x \in G^{\mathfrak{F}} \setminus \Phi(G^{\mathfrak{F}})$ such that $|x| = 2$. Let $T = \langle x\rangle^G$. Then $T \leqslant G^{\mathfrak{F}}$ and $T\Phi(G^{\mathfrak{F}})/\Phi(G^{\mathfrak{F}})$ is normal in $G/\Phi(G^{\mathfrak{F}})$. Since $G^{\mathfrak{F}}/\Phi(G^{\mathfrak{F}})$ is a chief factor of G, $G^{\mathfrak{F}} = T$, which contradicts the fact that $\exp(G^{\mathfrak{F}}) = 4$. Then by hypothesis, $\langle x\rangle$ is weakly c-permutable in G. Hence there exists a subgroup K of G such that $G = \langle x\rangle K$ and $\langle x\rangle \cap K$ is completely c-permutable in G. If $K = G$, then $\langle x\rangle$ is completely c-permutable in G. Let $\Phi = \Phi(G^{\mathfrak{F}})$. Obviously, $G^{\mathfrak{F}}/\Phi \nsubseteq \Phi(G/\Phi)$. Thus, there exists a maximal subgroup M/Φ of G/Φ such that $G/\Phi = (G^{\mathfrak{F}}/\Phi)(M/\Phi)$. Since $G^{\mathfrak{F}}/\Phi$ is abelian, it is easy to see that $G^{\mathfrak{F}}/\Phi \cap M/\Phi = 1$. Hence $G/\Phi = [G^{\mathfrak{F}}/\Phi](M/\Phi)$. Since $\langle x\rangle$ is completely c-permutable in G, $\langle x\rangle\Phi/\Phi$ is completely c-permutable in G/Φ by Lemma 2.1. Hence there exists an element $\alpha \in \langle\langle x\rangle\Phi/\Phi, M/\Phi\rangle$ such that $(\langle x\rangle\Phi/\Phi)(M^\alpha\Phi/\Phi) \leqslant G/\Phi$. Clearly, $G^{\mathfrak{F}}/\Phi \cap M^\alpha\Phi/\Phi = 1$. Hence $\langle x\rangle\Phi/\Phi = (\langle x\rangle\Phi/\Phi)(G^{\mathfrak{F}}/\Phi \cap M^\alpha\Phi/\Phi) = (G^{\mathfrak{F}}/\Phi) \cap ((\langle x\rangle\Phi/\Phi)(M^\alpha\Phi/\Phi) \trianglelefteq ((\langle x\rangle\Phi/\Phi)(M^\alpha\Phi/\Phi)$. It follows that $M^\alpha\Phi/\Phi \leqslant N_{(G/\Phi)}(\langle x\rangle\Phi/\Phi)$. On the other hand, since $\langle x\rangle\Phi/\Phi \trianglelefteq G^{\mathfrak{F}}/\Phi$, $\langle x\rangle\Phi/\Phi \trianglelefteq G/\Phi$. This shows that $\langle x\rangle\Phi \trianglelefteq G$. Therefore $\langle x\rangle\Phi/\Phi = G^{\mathfrak{F}}/\Phi$ or $\langle x\rangle\Phi/\Phi = 1$ since $G^{\mathfrak{F}}/\Phi$ is a chief factor of G. Obviously, $\langle x\rangle\Phi/\Phi \neq 1$ by the choice of x. Hence $\langle x\rangle\Phi/\Phi = G^{\mathfrak{F}}/\Phi$ and so $G^{\mathfrak{F}} = \langle x\rangle\Phi = \langle x\rangle$, a contradiction. Thus we may assume that $K < G$. Let $P^* = G^{\mathfrak{F}} \cap K$. Then $P^* \trianglelefteq K$ and $P^* < G^{\mathfrak{F}}$.

Assume that $P^* = 1$. Then $|G^{\mathfrak{F}}| = |G : K| = |\langle x\rangle|$ since $G = \langle x\rangle K = G^{\mathfrak{F}}K$. Consequently $\langle x\rangle = G^{\mathfrak{F}}$, which contradicts the fact that $G^{\mathfrak{F}}$ is not abelian.

Hence $P^* \neq 1$. Since $K \leqslant N_G(P^*)$ and $P^* < N_{G^{\mathfrak{F}}}(P^*)$, $|G : N_G(P^*)| = |G^{\mathfrak{F}}N_G(P^*) : N_G(P^*)| = |G^{\mathfrak{F}} : N_G(P^*) \cap G^{\mathfrak{F}}| = |G^{\mathfrak{F}} : N_{G^{\mathfrak{F}}}(P^*)| < |G^{\mathfrak{F}} : P^*| = |G^{\mathfrak{F}} : G^{\mathfrak{F}} \cap K| = |G^{\mathfrak{F}}K : K| = |\langle x\rangle K : K| = |\langle x\rangle : \langle x\rangle \cap K| \leqslant 4$. Hence $|G : N_G(P^*)| = 2$ or $|G : N_G(P^*)| = 1$.

If $|G : N_G(P^*)| = 2$, then $N_G(P^*) \trianglelefteq G$. Let $P_1 = N_G(P^*) \cap G^{\mathfrak{F}}$. Then $P_1 \trianglelefteq G$. If $P_1 \leqslant \Phi(G^{\mathfrak{F}})$, then $G^{\mathfrak{F}} = G^{\mathfrak{F}} \cap \langle x\rangle K = G^{\mathfrak{F}} \cap \langle x\rangle N_G(P^*) = \langle x\rangle(G^{\mathfrak{F}} \cap N_G(P^*)) = \langle x\rangle$, a contradiction. If $P_1 \nleqslant \Phi(G^{\mathfrak{F}})$, then $1 \neq P_1G^{\mathfrak{F}}/G^{\mathfrak{F}} \trianglelefteq G/\Phi(G^{\mathfrak{F}})$. It follows from (b) that $G^{\mathfrak{F}} = P_1\Phi(G^{\mathfrak{F}}) = P_1 = G^{\mathfrak{F}} \cap N_G(P^*)$. Thus $G^{\mathfrak{F}} \leqslant N_G(P^*)$ and thereby $G = \langle x\rangle K = G^{\mathfrak{F}}N_G(P^*) = N_G(P^*)$, which contradicts $|G : N_G(P^*)| = 2$.

If $G = N_G(P^*)$, then $P^* \trianglelefteq G$. It follows that $P^*\Phi(G^{\mathfrak{F}})/\Phi(G^{\mathfrak{F}}) \trianglelefteq G/\Phi(G^{\mathfrak{F}})$. Hence by (b), $P^*\Phi(G^{\mathfrak{F}}) = \Phi(G^{\mathfrak{F}})$ or $P^*\Phi(G^{\mathfrak{F}}) = G^{\mathfrak{F}}$. If $P^*\Phi(G^{\mathfrak{F}}) = \Phi(G^{\mathfrak{F}})$, then $P^* \leqslant \Phi(G^{\mathfrak{F}})$ and consequently $G^{\mathfrak{F}} = G^{\mathfrak{F}} \cap \langle x\rangle K = \langle x\rangle(G^{\mathfrak{F}} \cap K) = \langle x\rangle P^* = \langle x\rangle$, a contradiction. If $G^{\mathfrak{F}} = P^*\Phi(G^{\mathfrak{F}}) = P^* = G^{\mathfrak{F}} \cap K$, then $G^{\mathfrak{F}} \leqslant K$, which contradicts the fact that $K < G$.

The contradiction completes the proof. □

4. Proofs of Theorems 1.3 and 1.4

Lemma 4.1. *A group G is supersoluble if and only if there exists a normal subgroup N of G such that G/N is supersoluble and every cyclic subgroup of N with prime order or* 4 *is weakly c-permutable in G.*

Proof. It is clear that the condition is necessary. We only need to prove that it is sufficient.

Suppose that the assertion is false and let G be a counterexample of minimal order. Let H be a proper subgroup of G. Since G/N is supersoluble, $H/(H \cap N) \cong HN/N$ is also supersoluble. By Lemma 2.2, every minimal subgroup of $H \cap N$ and every cyclic subgroup of $H \cap N$ of order 4 are weakly c-permutable in H. This means that H (with respect to $H \cap N$) satisfies the hypothesis. The minimal choice of G implies that H is supersoluble. This shows that G is a minimal non-supersoluble group. Hence by [7, Theorem 3.4.2 and 3.11.8], G has a non-cyclic normal Sylow p-subgroup $P = G^{\mathfrak{U}}$ for some prime p such that $P/\Phi(P)$ is chief factor of $G/\Phi(P)$ and the exponent of P is p or 4. Since G/N is supersoluble, $P \leqslant N$.

Let $x \in P\backslash\Phi(P)$. Then $|x| = p$ or 4. By hypothesis, $\langle x\rangle$ is weakly c-permutable in G. Hence there exists a subgroup K of G such that $G = \langle x\rangle K$ and $\langle x\rangle \cap K$ is completely c-permutable in G. Assume that $K < G$. Since $P/\Phi(P)$ is abelian, $(P \cap K)\Phi(P)/\Phi(P) \trianglelefteq PK/\Phi(P) = G/\Phi(P)$. Since $P/\Phi(P)$ is a chief factor of G, $P \cap K \leqslant \Phi(P)$ or $P = (P \cap K)\Phi(P) = P \cap K$. If $P \cap K \leqslant \Phi(P)$, then $P = P \cap G = P \cap \langle x\rangle K = \langle x\rangle(P \cap K) = \langle x\rangle$, a contradiction. If $P \cap K = P$, then $P \leqslant K$ and hence $G = \langle x\rangle K = K$, a contradiction again. Hence we may assume that $K = G$. Then $\langle x\rangle$ is completely c-permutable in G. Since P is a normal Sylow p-subgroup of G, P has a complement D in G by Shur–Zassenhaus theorem. Since $\langle x\rangle$ is completely c-permutable in G, there exists an element $g \in \langle\langle x\rangle, D\rangle$ such that $\langle x\rangle D^g = D^g\langle x\rangle$. Hence $\langle x\rangle = \langle x\rangle(D^g \cap P) = \langle x\rangle D^g \cap P \trianglelefteq \langle x\rangle D^g$. Consequently, $D^g \subseteq N_G(\langle x\rangle)$. On the other hand, since $P/\Phi(P)$ is abelian, $\langle x\rangle\Phi(P)/\Phi(P) \trianglelefteq P/\Phi(P)$. This implies that $\langle x\rangle\Phi(P)/\Phi(P) \trianglelefteq G/\Phi(P)$. However, since $P/\Phi(P)$ is a chief factor of G and $x \notin \Phi(P)$, $\langle x\rangle\Phi(P) = P$ and consequently $\langle x\rangle = P$, a contradiction again. The final contradiction completes the proof. □

Lemma 4.2. *Suppose that P is a minimal normal p-subgroup of G. If every minimal subgroup of P is completely c-permutable in G, then P is a cyclic subgroup of order p.*

Proof. Let D be a Sylow p-subgroup of G. Then $P \cap Z(D) \neq 1$. Suppose that L is a subgroup of $P \cap Z(D)$ of order p. Then $L \trianglelefteq P$ and so L is completely c-permutable in G. Let Q be an arbitrary Sylow q-subgroup of G with $q \neq p$. By hypothesis, there exists an element $\alpha \in \langle L, Q\rangle$ such that

$LQ^{\alpha} = Q^{\alpha}L \leqslant G$. Therefore $L = P \cap LQ^{\alpha} \trianglelefteq LQ^{\alpha}$ and so $Q^{\alpha} \leqslant N_G(L)$. The arbitrary choice of q implies that $L \trianglelefteq G$. But, since P is a minimal normal subgroup of G, we have that $P = L$. This completes the proof. $\square$

Theorem 4.3. *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and G a group. Then $G \in \mathfrak{F}$ if and only if there exists a normal subgroup H such that $G/H \in \mathfrak{F}$ and every cyclic subgroup of any non-cyclic Sylow subgroup of H with prime order or 4 (if the Sylow 2-subgroup of H is a non-abelian 2-group) is weakly c-permutable in G.*

Proof. It is clear that the condition is necessary. We only need to prove that it is sufficient. Suppose that the assertion is false and let (G, H) be a counterexample for which $|G||H|$ is minimal. Then:

(1) If T is a normal Hall subgroup of H, then the hypothesis holds for (T, T) and for $(G/T, H/T)$.

Let P be an arbitrary non-cyclic Sylow subgroup of T. By hypothesis, every cyclic subgroup N of P with prime order or 4 is weakly c-permutable in G. Then by Lemma 2.2(1), N is weakly c-permutable in T. Thus (T, T) satisfies the hypothesis.

Obviously, $(G/T)/(H/T) \in \mathfrak{F}$. Let R^*/T be a non-cyclic Sylow r-subgroup of H/T where $r \mid |H/T|$ and R a Sylow r-subgroup of R^* such that $R^* = RT$. Then R is a non-cyclic Sylow r-subgroup of H. Assume that K/T is a cyclic subgroup of R^*/T with prime order or 4. Then, obviously, $K/T = \langle x\rangle T/T$, where $\langle x\rangle$ is a subgroup of R with prime order or 4 since T is a normal Hall subgroup of H. By hypothesis, $\langle x\rangle$ is weakly c-permutable in G. Then by Lemma 2.2(3), we see that K/T is also weakly c-permutable in G/T. Thus $(G/T, H/T)$ satisfies the hypothesis.

(2) If T is a non-identity normal Hall subgroup P of H, then $T = H$.

Since T char H, $T \trianglelefteq G$. Then by (1), the hypothesis is true for $(G/T, H/T)$. Hence $G/T \in \mathfrak{F}$. It is easy to see that the hypothesis is still true for (G, T). The minimal choice of (G, H) implies that $T = H$.

(3) If p is the smallest prime of $|H|$ and P is a Sylow p-subgroup of H, then P is not cyclic.

Indeed, if P is cyclic, then by [15, IV, Theorem 2.8], H is p-nilpotent. Hence by (2), $H = P$ is cyclic. It follows from Lemma 2.6 that $G \in \mathfrak{F}$, a contradiction.

(4) H is soluble.

Let K be an arbitrary proper subgroup of H. Then $|K| < |G|$ and K/K is supersoluble. Let $\langle x\rangle$ be a cyclic subgroup of any non-cyclic Sylow subgroup of K with prime order or 4. Then, clearly, $\langle x\rangle$ is also a cyclic subgroup of a non-cyclic Sylow subgroup of H with prime order or 4. By hypothesis, $\langle x\rangle$ is weakly c-permutable in G and so $\langle x\rangle$ is weakly c-permutable in K by Lemma 2.2(1). This implies that the hypothesis is still true for (K, K). The minimal choice of (G, H) implies that K is supersoluble (since we can consider $\mathfrak{F} = \mathfrak{U}$). Hence H is a minimal non-supersoluble group and consequently H is soluble (see [7, Theorem 3.11.8]).

(5) G is a minimal non-$\mathfrak{F}$-group.

Since $\mathfrak{F}$ is a saturated formation, $G^{\mathfrak{F}} \nsubseteq \Phi(G)$. Hence there exists a maximal subgroup M such that $G = MG^{\mathfrak{F}}$. Since $G/G^{\mathfrak{F}} \cong M/(M \cap G^{\mathfrak{F}}) \in \mathfrak{F}$ and $G/H \in \mathfrak{F}$, $M^{\mathfrak{F}} \subseteq G^{\mathfrak{F}} \subseteq H$. By Lemma 2.2, we see that $(M, G^{\mathfrak{F}})$ satisfies the hypothesis. The minimal choice of G implies that $M \in \mathfrak{F}$. This shows that G is a minimal non-$\mathfrak{F}$-group. By (4), we also see that $G^{\mathfrak{F}}$ is soluble.

(6) G has the following properties: (a) $G^{\mathfrak{F}}$ is a p-group for some prime p; (b) $G^{\mathfrak{F}}/\Phi(G^{\mathfrak{F}})$ is a chief factor of G; (c) If $p > 2$, then $\exp(G^{\mathfrak{F}}) = p$. If $p = 2$, then $\exp(G^{\mathfrak{F}}) = 2$ or 4.

It follows directly from (4), (5) and [7, Theorem 3.4.2].

(7) Final contradiction.

Let $x \in G^{\mathfrak{F}} \backslash \Phi(G^{\mathfrak{F}})$. Then by (6), $|x|$ is a prime or 4. Since $G^{\mathfrak{F}} \subseteq H$, by hypothesis, we can see that $\langle x\rangle$ is weakly c-permutable in G. Hence there exists a subgroup $T \leqslant G$ such that $G = \langle x\rangle T$ and $\langle x\rangle \cap T$ is completely c-permutable in G. Assume that $T < G$. By (6), we see that $G^{\mathfrak{F}}/\Phi(G^{\mathfrak{F}})$ is abelian and consequently $(G^{\mathfrak{F}} \cap T)\Phi(G^{\mathfrak{F}})/\Phi(G^{\mathfrak{F}}) \trianglelefteq G/\Phi(G^{\mathfrak{F}})$. Since $G^{\mathfrak{F}}/\Phi(G^{\mathfrak{F}})$ is a chief factor of G, $G^{\mathfrak{F}} \cap T \leqslant \Phi(G^{\mathfrak{F}})$ or $G^{\mathfrak{F}} = (G^{\mathfrak{F}} \cap T)\Phi(G^{\mathfrak{F}}) = G^{\mathfrak{F}} \cap T$. If $G^{\mathfrak{F}} \cap T \leqslant \Phi(G^{\mathfrak{F}})$, then $\langle x\rangle = G^{\mathfrak{F}} \trianglelefteq G$. It follows from Lemma 2.6 that $G \in \mathfrak{F}$, a contradiction. Thus we may assume that $G^{\mathfrak{F}} \cap T = G^{\mathfrak{F}}$. Then $G^{\mathfrak{F}} \leqslant T$ and hence $G = \langle x\rangle T = T$. This contradiction shows that $T = G$ and so $\langle x\rangle = \langle x\rangle \cap T$ is completely c-permutable in G. By Lemma 2.1, $\langle x\rangle\Phi(G^{\mathfrak{F}})/\Phi(G^{\mathfrak{F}})$ is also completely c-permutable in $G/\Phi(G^{\mathfrak{F}})$. Since $G^{\mathfrak{F}}/\Phi(G^{\mathfrak{F}})$ is an elementary abelian group, by Lemma 4.2, $G^{\mathfrak{F}}/\Phi(G^{\mathfrak{F}})$ is a cyclic group. Hence

$G/\Phi(G^{\mathfrak{F}}) \in \mathfrak{F}$ by Lemma 2.6. This implies that $G \in \mathfrak{F}$ since $\mathfrak{F}$ is a saturated formation. The final contradiction completes the proof. □

Proof of Theorem 1.3. It is clear that the condition is necessary. We only need to prove that it is sufficient. Suppose that the assertion is false and let (G, H) be a counterexample for which $|G||H|$ is minimal. Let p be the smallest prime divisor of $|F(H)|$ and P the Sylow p-subgroup of $F(H)$. Then $P \trianglelefteq G$. Now we proceed with our proof as follows:

(1) $F(H) \neq H$ and $C_G(F(H)) \leqslant F(H)$.

If $F(H) = H$, then $G \in \mathfrak{F}$ by Theorem 4.3, a contradiction. Obviously, $C_G(F(H)) \leqslant F(H)$ since G is soluble.

(2) Let $V/P = F(H/P)$ and Q be a Sylow q-subgroup of V, where $q \mid |V/P|$. Then $q \neq p$ and either $Q \leqslant F(H)$ or $p > q$ and $C_Q(P) = 1$.

Since V/P is nilpotent, QP/P char V/P and so $QP \trianglelefteq H$. Then, it is easy to see that $p \neq q$. By Lemma 4.1, PQ is supersoluble. If $q > p$, then $Q \trianglelefteq PQ$ and so $Q \leqslant F(H)$. Now assume that $p > q$. Then $p > 2$. Since p is the minimal prime of $|F(H)|$, $F(H)$ is a q'-group. Let R be a Sylow r-subgroup of $F(H)$ where $r \neq p$. Then $r \neq q$ and so $[R, Q] \leqslant P$. Assume that for some $x \in Q$, we have $x \in C_H(P)$. Since V/P is nilpotent, $[R, \langle x\rangle] = [R, \langle x\rangle, \langle x\rangle] = 1$ by [6, Chapter 5, Theorem 3.6]. Hence $x \in C_G(F(H))$. By (1), $C_H(F(H)) \leqslant F(H)$ and so $C_Q(P) = 1$.

(3) $p > 2$.

If $p = 2$, then by (2), we see that $F(H/P) = F(H)/P$ and $2 \nmid |F(H/P)|$. This implies that if $\langle x\rangle P/P$ is an arbitrary minimal subgroup of $F(H)/P$, then $|x| = r$, where $r \neq 2$. By Lemma 2.2(3), every minimal subgroup of $F(H/P)$ is weakly c-permutable in G/P. Hence $(G/P, H/P)$ satisfies the hypothesis. The minimal choice of (G, H) implies that $G/P \in \mathfrak{F}$. Hence by Theorem 4.3, $G \in \mathfrak{F}$, a contradiction. Thus, (3) holds.

(4) Final contradiction.

Let $V/P = F(H/P)$ and Q be a Sylow q-subgroup of V, where $q \mid |V/P|$. Then by (2), either $Q \leqslant F(H)$ or $p > q$ and $C_Q(P) = 1$. In the second case, Q is cyclic by (3) and Lemma 2.7. Hence every Sylow subgroup of $F(H/P)$ either is cyclic or is contained in $F(H)$. Moreover by (2), $p \nmid |F(H/P)|$. Let K/P be a cyclic subgroup of a non-cyclic Sylow subgroup of $F(H/P)$ with prime order. Then it is easy to see that $K/P = \langle x\rangle P/P$, where $\langle x\rangle$ is a cyclic subgroup of some non-cyclic Sylow subgroup of $F(H)$ with prime order. By hypothesis, $\langle x\rangle$ is weakly c-permutable in G. Hence $\langle x\rangle P/P$ is weakly c-permutable in G/P by Lemma 2.2(3). This shows that $(G/P, H/P)$ satisfies the hypothesis. The minimal choice of (G, H) implies that $G/P \in \mathfrak{F}$. Therefore, $G \in \mathfrak{F}$ by Theorem 4.3. The final contradiction completes the proof. □

Proof of Theorem 1.4. It is clear that the condition is necessary. We only need to prove that it is sufficient. Suppose that the assertion is false and let G be a counterexample of minimal order. Then:

(1) Every proper subgroup of G is p-nilpotent.

By Lemma 2.3, we see that $|N_p| > p^2$. Let L be a proper subgroup of G. Since $L/(L \cap N) \cong LN/N \leqslant G/N$, $L/(L \cap N)$ is p-nilpotent. If $|L \cap N|_p \leqslant p^2$, then L is p-nilpotent by Lemma 2.3. If $|L \cap N|_p > p^2$, then every subgroup of $L \cap N$ of order p^2 is weakly c-permutable in L by Lemma 2.2(1). Hence L is p-nilpotent by the choice of G. This shows that G is a minimal non-p-nilpotent group.

(2) G has the following properties: (i) $G = PQ$, where $P = G^{\mathfrak{N}}$ is a normal Sylow p-subgroup of G and Q is a non-normal cyclic Sylow q-subgroup of G; (ii) $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$; (iii) If $p > 2$, then the exponent of P is p; if $p = 2$, then the exponent of P is 2 or 4; (iv) $\Phi(P) \leqslant Z(P)$; (v) p^3 dividing the order of P; (vi) $P \leqslant N$.

By (1) and [15, Theorem IV. 5.4], G is a minimal non-nilpotent group. Hence (i)–(iv) follow directly from [7, Theorem 3.4.2]. (v) follows from Lemma 2.3. (vi) is clear since $P = G^{\mathfrak{N}}$ is the p-nilpotent residual of G and G/N is p-nilpotent.

(3) P is not cyclic.

Suppose that P is cyclic. If $\exp(P) = p$, then $|P| = p$ and so $|\mathrm{Aut}(P)| = p - 1$. If $\exp(P) = 4$, then $|P| = 4$ and so $|\mathrm{Aut}(P)| = 2$. Since $N_G(P)/C_G(P)$ is isomorphic to some subgroup of $\mathrm{Aut}(P)$ and $(|G|, p - 1) = 1$, $N_G(P)/C_G(P) = 1$. Hence, by Burnside's theorem, G is p-nilpotent, a contradiction.

(4) If H is a subgroup of P of order p^2, then H is completely c-permutable in G.

By hypothesis, H is weakly c-permutable in G. Hence there exists a subgroup T of G such that $G = HT$ and $H \cap T$ is completely c-permutable in G. If $T < G$, then T is nilpotent by (2). Since $p^3 \mid |P|$ by (2), $p \mid |T|$. Let T_p be a Sylow p-subgroup of T. Then $T_p \trianglelefteq T$ and so $T \leqslant N_G(T_p)$. Then since $|H| = p^2$, $G = N_G(T_p)$ or $|G : N_G(T_p)| = p$ or $|G : N_G(T_p)| = p^2$. Assume that $G = N_G(T_p)$. Then $T_p \trianglelefteq G$. Obviously, $p^3 \nmid |G/T_p|$ and $(G/T_p)/(G/T_p) = 1$ is p-nilpotent. By Lemma 2.3, G/T_p is p-nilpotent. Hence $P \leqslant T_p$ and so $P = T_p$. It follows that $G = T$, a contradiction. Suppose that $|G : N_G(T_p)| = p$ and let $P_1 = P \cap N_G(T_p)$. Since $|P : P_1| = |P : (P \cap N_G(T_p))| = |PN_G(T_p) : N_G(T_p)| = |PT : N_G(T_p)| = |G : N_G(T_p)| = p$, P_1 is a maximal subgroup of P and so $P_1 \trianglelefteq P$. It follows that $P_1 \trianglelefteq G = PN_G(T_p)$. If $P_1 \subseteq \Phi(P)$, then $P = P \cap HN_G(T_p) = H(P \cap N_G(T_p)) = HP_1 = H$, a contradiction. Hence we can assume that $P_1 \nsubseteq \Phi(P)$. Since $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$, $P_1\Phi(P)/\Phi(P) = P/\Phi(P)$. This implies that $P = P_1$, a contradiction. Now assume that $|G : N_G(T_p)| = p^2$. Since $p^2 = |G : N_G(T_p)| \leqslant |G : T| = |HT : T| = |H : (H \cap T)| \leqslant p^2$, we have that $H \cap T = 1$. Hence, $|P : T_p| = |HT_p : T_p| = |H : (H \cap T_p)| = p^2$, which means that T_p is a 2-maximal subgroup of P. Therefore, there exists a maximal subgroup P_2 of P such that T_p is a maximal subgroup of P_2. Then $T_p \trianglelefteq P_2$ and so $P_2 \leqslant N_G(T_p)$. Hence $|G : N_G(T_p)| = |HT : N_G(T_p)| = |PN_G(T_p) : N_G(T_p)| = |P : (P \cap N_G(T_p))| \leqslant |P : P_2| = p$, a contradiction. These contradictions show that $T = G$. Thus, $H = H \cap T$ is completely c-permutable in G.

(5) There exists a subgroup H of P such that $|H| = p^2$ which is not contained in $\Phi(P)$.

If $\Phi(P) = 1$, then it is clear. Hence we may assume that $\Phi(P) \neq 1$. If $|P| = p^3$, then clearly P has a maximal subgroup of order p^2. Since P is not cyclic by (3), P has at least two different maximal subgroups P_1 and P_2. If P_1 and P_2 are all contained in $\Phi(P)$, then $P = P_1P_2 \subseteq \Phi(P)$, a contradiction. Hence, we can assume that $|P| > p^3$. Let $x \in P \backslash \Phi(P)$ and $a \in \Phi(P)$ where $|a| = p$. Since $\Phi(P) \leqslant Z(P)$, $\langle x\rangle\langle a\rangle \leqslant G$. By (2), we see that $|x| = p$ or 4. If $|x| = 4$, we can choose $H = \langle x\rangle$. If $|x| = p$, then $|\langle x\rangle\langle a\rangle| \leqslant p^2$. If $|\langle x\rangle\langle a\rangle| = p$, then $\langle x\rangle = \langle a\rangle$, a contradiction. Hence $|\langle x\rangle\langle a\rangle| = p^2$. Therefore (5) holds.

(6) Final contradiction.

By (2), $G = [P]Q$. By (5), there exists a subgroup H of P with order p^2 such that $H \nleqslant \Phi(P)$. Then by (4), H is completely c-permutable in G. Hence there exists an element $g \in \langle H, Q\rangle$ such that $HQ^g = Q^gH$. Then $H = H(Q^g \cap P) = HQ^g \cap P \trianglelefteq HQ^g$. It follows that $Q^g \subseteq N_G(H)$. On the other hand, since $P/\Phi(P)$ is abelian, $H\Phi(P)/\Phi(P) \trianglelefteq P/\Phi(P)$. This implies that $H\Phi(P)/\Phi(P) \trianglelefteq G/\Phi(P)$. However, since $P/\Phi(P)$ is chief factor of G, we obtain that $H\Phi(P) = P$ and consequently $H = P$, a contradiction. Thus the proof is completed. □

5. Some applications

Our theorems have many corollaries. We here list such special cases of them which can be found in the literature.

Theorem 1.1 immediately implies:

Corollary 5.1. *(See Ramadan [22].) Let G be a soluble group and E a normal subgroup of G such that G/E is supersoluble. If all maximal subgroups of every Sylow subgroup of $F(E)$ are normal, then G is supersoluble.*

Corollary 5.2. *(See Wei [25, Theorem 1].) Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. Suppose that G is a group with a soluble normal subgroup H such that $G/H \in \mathfrak{F}$. If all maximal subgroups of all Sylow subgroups of $F(H)$ are c-normal in G, then $G \in \mathfrak{F}$.*

Corollary 5.3. *(See Wang, Wei and Li [26, Theorem 4.5].) Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. Suppose that G is a group with a soluble normal subgroup H such that $G/H \in \mathfrak{F}$. If all maximal subgroups of all Sylow subgroups of $F(H)$ are c-supplemented in G, then $G \in \mathfrak{F}$.*

Corollary 5.4. *(See Li and X. Guo [18, Theorem 2].) Let G be a group and E a soluble normal subgroup of G such that G/E is supersoluble. If all maximal subgroups of the Sylow subgroups of $F(E)$ are c-normal in G, then G is supersoluble.*

Corollary 5.5. *(See Li and X. Guo [19, Theorem 1.2].) Suppose that G is a soluble group with a normal subgroup H such that G/H is supersoluble. If all maximal subgroups of every Sylow subgroup of $F(H)$ are complement in G, then G is supersoluble.*

Corollary 5.6. *(See X. Guo and Shum [14, Theorem 1.6].) Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. Let H be a soluble normal subgroup of a group G such that $G/H \in \mathfrak{F}$. If all maximal subgroups of every Sylow subgroup of $F(H)$ are complemented in G, then $G \in \mathfrak{F}$.*

From Theorem 1.2 we obtain:

Corollary 5.7. *(See Lam, Shum and Guo [16].) If p is an odd prime and every minimal subgroup of G is contained in $Z_\infty(G)$, then G is p-nilpotent.*

Corollary 5.8. *(See Ballester-Bolinches and Wang [2, Theorem 3.1].) Let $\mathfrak{F}$ be a saturated formation such that $\mathfrak{N} \subseteq \mathfrak{F}$. Let G be a group such that every cyclic of $G^{\mathfrak{F}}$ with order 4 is c-normal in G. Then G belongs to $\mathfrak{F}$ if and only if $\langle x\rangle$ lies in the $\mathfrak{F}$-hypercenter $Z_{\mathfrak{F}}(G)$ of G for every minimal subgroup $\langle x\rangle$ of $G^{\mathfrak{F}}$.*

Corollary 5.9. *(See Zhong and Li [32, Theorem 2.5].) Suppose that p is a prime and $K = G^{\mathfrak{N}}$ the nilpotent residual of G. Then G is p-nilpotent if every minimal subgroup of K is contained in $Z_\infty(G)$ and every cyclic $\langle x\rangle$ of K with order 4 is c-supplemented in G.*

Corollary 5.10. *(See Wang [29, Theorem 2.4].) Let G be a finite group and $K = G^{\mathfrak{N}}$ be the nilpotent residual of G. Then G is nilpotent if and only if every minimal subgroup $\langle x\rangle$ of K lies in the hypercenter $Z_\infty(G)$ of G and every cyclic element of P with order 4 is c-normal in G.*

Corollary 5.11. *(See Wang, Li and Wang [30, Theorem 4.4].) Let $\mathfrak{F}$ be a saturated formation such that $\mathfrak{N} \subseteq \mathfrak{F}$. Let G be a group such that every element of $G^{\mathfrak{F}}$ with order 4 is c-supplemented in G. Then G belongs to $\mathfrak{F}$ if and only if every element $\langle x\rangle$ with prime order lies in the $\mathfrak{F}$-hypercenter $Z_{\mathfrak{F}}(G)$ of G.*

As immediate corollaries of Theorem 4.3, we have the following:

Corollary 5.12. *(See Buckley [4].) Let G be group of odd order. If all subgroups of G of prime order are normal in G, the G is supersoluble.*

Corollary 5.13. *(See Wang [27, Theorem 4.2].) If all cyclic subgroups of G with prime order and order 4 are c-normal in G, then G is supersoluble.*

Corollary 5.14. *(See Li and X. Guo [17, Theorem 3.4].) Let N be a normal subgroup of a group G such that G/N is supersoluble. If every minimal subgroup of N is c-normal in G and for $2 \mid |N|$ either every Sylow 2-subgroup of N is an abelian group or every cyclic subgroup of N of order 4 is c-normal in G, then G is supersoluble.*

Corollary 5.15. *(See Ballester-Bolinches and Wang [2, Theorem 3.4].) Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. If all minimal subgroups and all cyclic subgroups with order 4 of $G^{\mathfrak{F}}$ are c-normal in G, then $G \in \mathfrak{F}$.*

Corollary 5.16. *(See Ballester-Bolinches and Pedraza-Aguilera [1].) Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and G a group with normal subgroup E such that $G/E \in \mathfrak{F}$. Assume that a Sylow 2-subgroup of G is abelian. If all minimal subgroups of E are permutable in G, then $G \in \mathfrak{F}$.*

Corollary 5.17. *(See Ramadan, Mohamed and Heliel [23, Theorem 3.9].) Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and G be a group. Then $G \in \mathfrak{F}$ if and only if there exists a normal subgroup H of G such that $G/H \in \mathfrak{F}$ and the subgroups of prime order or order 4 of H are c-normal in G.*

Corollary 5.18. *(See Wang, Li and Wang [30, Theorem 4.2].) Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. Assume G is a group with normal subgroup N such that $G/N \in \mathfrak{F}$. If all minimal subgroups and cyclic subgroups with order 4 of N is c-supplemented in G, then $G \in \mathfrak{F}$.*

Corollary 5.19. *(See Ballester-Bolinches, Wang and X. Guo [3, Theorem 4.1].) Let G be a group and let H be the supersoluble residual of G. If all minimal subgroups and all cyclic subgroups with order 4 of H are c-supplemented in G, then G is supersoluble.*

Corollary 5.20. *(See Zhong and Li [32, Theorem 3.1].) Let G be a group and N a normal subgroup of a group G such that G/N is supersoluble. If every minimal subgroup of E is c-supplemented in G and if every cyclic subgroup of order 4 of N is c-normal in G, then G is supersoluble.*

From Theorem 1.3 we obtain:

Corollary 5.21. *(See Li and X. Guo [18, Theorem 3].) Let G be a group and E a soluble normal subgroup of G such that G/E is supersoluble. If all minimal subgroups and all cyclic subgroups with order 4 of $F(E)$ are c-normal in G, then G is supersoluble.*

Corollary 5.22. *(See Wei [25, Theorem 2].) Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. Suppose that G is a group with a soluble normal subgroup H such that $G/H \in \mathfrak{F}$. If all minimal subgroups and all cyclic subgroups with order 4 of $F(H)$ are c-normal in G, then $G \in \mathfrak{F}$.*

Corollary 5.23. *(See Li [20, Theorem 3].) Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and G be a group. Then $G \in \mathfrak{F}$ if and only if there is a normal soluble subgroup H in G such that $G/H \in \mathfrak{F}$ and the subgroups of prime order or order 4 of $F(H)$ are c-normal in G.*

Corollary 5.24. *(See Wang, Wei and Li [26, Theorem 4.1].) Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. Suppose that G is a group with a soluble normal subgroup H such that $G/H \in \mathfrak{F}$. If all minimal subgroups and all cyclic subgroups with order 4 of $F(H)$ are c-supplemented in G, then $G \in \mathfrak{F}$.*

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