Journal of Algebra 368 (2012) 182–198

Contents lists available at SciVerse ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

On some minimal supervarieties of exponential growth

Onofrio Mario Di Vincenzo [a,1], Ernesto Spinelli [b,*]

[a] Dipartimento di Matematica e Informatica, Università degli Studi della Basilicata, Via dell'Ateneo Lucano, 10, 85100 Potenza, Italy
[b] Dipartimento di Matematica "G. Castelnuovo", Università degli Studi di Roma "La Sapienza", Piazzale Aldo Moro 5, 00185 Roma, Italy

ARTICLE INFO

Article history:
Received 9 December 2011
Available online 15 July 2012
Communicated by Nicolás Andruskiewitsch

MSC:
16W55
16R50
16R10

Keywords:
Superalgebras
Polynomial identities
Superexponent

ABSTRACT

In the present paper we investigate minimal supervarieties of given superexponent over fields of characteristic zero. We show that any minimal supervariety of finite basic rank is generated by one of the minimal superalgebras, introduced by Giambruno and Zaicev in 2003. Furthermore it is proved that any minimal superalgebra, whose graded simple components of the semisimple part are simple, generates a minimal supervariety. Finally we state that the same conclusion holds when the semisimple part of a minimal superalgebra has exactly two arbitrary graded simple components.

1. Introduction

Let F be a field of characteristic zero and let $F\langle X\rangle$ be the free associative algebra on a countable set X over F. If $\mathcal{V}$ is a variety of associative algebras over F, let us denote by $\mathrm{Id}(\mathcal{V})$ the T-ideal of $F\langle X\rangle$ associated to $\mathcal{V}$. Recall that $\mathrm{Id}(\mathcal{V})$ is a two-sided ideal invariant under all endomorphisms of $F\langle X\rangle$ and consists of the polynomial identities satisfied by all the algebras of $\mathcal{V}$. An algebra A generates $\mathcal{V}$ if the set $\mathrm{Id}(A)$ of polynomial identities it satisfies coincides with $\mathrm{Id}(\mathcal{V})$. In this case, we write $\mathcal{V} = \mathrm{var}(A)$. A variety $\mathcal{V}$ is called non-trivial if $\mathrm{Id}(\mathcal{V})$ is non-zero and $\mathcal{V}$ is proper if it is non-trivial and contains a non-zero algebra.

A fundamental tool for the investigation of varieties is the concept of *codimension sequence*, introduced by Regev in the seminal paper [23]. For every variety $\mathcal{V}$ and for every $n \geqslant 1$ the *n-th codimension* $c_n(\mathcal{V})$ of $\mathcal{V}$ is the dimension of the vector space $\frac{P_n}{P_n \cap \mathrm{Id}(\mathcal{V})}$, where P_n is the space of

* Corresponding author.
E-mail addresses: onofrio.divincenzo@unibas.it (O.M. Di Vincenzo), spinelli@mat.uniroma1.it (E. Spinelli).
1 The author was supported by Università della Basilicata.

0021-8693/$ – see front matter
http://dx.doi.org/10.1016/j.jalgebra.2012.07.005

multilinear polynomials of degree n in a fixed set of n variables. Since F has characteristic zero and, hence, $\mathrm{Id}(\mathcal{V})$ is completely determined by multilinear polynomials it contains, $\{c_n(\mathcal{V})\}_{n\geqslant 1}$ gives us, in some sense, the growth of the identities of given degree of the variety $\mathcal{V}$. The starting point for understanding the asymptotic behaviour of this sequence is a result of [23] stating that, when $\mathcal{V}$ is proper, $\{c_n(\mathcal{V})\}_{n\geqslant 1}$ is exponentially bounded, that is there exists a constant $a > 0$ such that $c_n(\mathcal{V}) \leqslant a^n$ for all n. Later Kemer in [19] characterized the varieties whose codimension sequence is polynomially bounded (see also [15]). From his work it turns out that there exists no variety with intermediate growth of the codimensions between polynomial and exponential. A fundamental step in this setting is a remarkable result due to Giambruno and Zaicev ([13] and [14]) establishing that for any proper variety $\mathcal{V}$ the limit

$$\exp(\mathcal{V}) := \lim_{m\to+\infty} \sqrt[m]{c_m(\mathcal{V})}$$

exists and is a non-negative integer, which is called the *exponent* of the variety $\mathcal{V}$. If $\mathcal{V} = \mathrm{var}(A)$, we write $\exp(A) := \exp(\mathcal{V})$, the exponent of the algebra A. The most important feature of the exponent is that it provides an integral scale allowing us to measure the growth of any variety. By virtue of this, the theory has developed towards a classification of varieties according to the asymptotic behaviour of their codimension sequences. A deep result in this direction is provided by Giambruno and Zaicev in [17] with the characterization of *minimal varieties* of given exponent $d \geqslant 2$, namely those varieties $\mathcal{V}$ such that $\exp(\mathcal{V}) = d$ and $\exp(\mathcal{U}) < d$ for all proper subvarieties $\mathcal{U}$ of $\mathcal{V}$. In particular, they proved that a variety is minimal of exponent d if, and only if, it is generated by the Grassmann envelope of a so-called *minimal superalgebra*.

More generally, superalgebras play a basic role in the theory of varieties. In fact, according to a celebrated result of Kemer (see [20]), any T-ideal of polynomial identities of a non-trivial variety coincides with the T-ideal of polynomial identities satisfied by the Grassmann envelope of a suitable finite-dimensional superalgebra. Nevertheless, their graded polynomial identities have been the object of a good deal of independent attention.

Let $F\langle Y \cup Z\rangle$ be the free associative algebra on the disjoint countable sets of variables $Y := \{y_1, y_2, \ldots\}$ and $Z := \{z_1, z_2, \ldots\}$. It has a natural superalgebra structure if we require that the variables from Y have degree 0 and those from Z have degree 1. The superalgebra $F\langle Y \cup Z\rangle$ is called the *free associative superalgebra* on Y and Z over F. An element $f(y_1, \ldots, y_n, z_1, \ldots, z_m)$ of $F\langle Y \cup Z\rangle$ is said to be a *$\mathbb{Z}_2$-graded polynomial identity* or *superidentity* for an F-superalgebra $A = A^{(0)} \oplus A^{(1)}$ if $f(a_1, \ldots, a_n, b_1, \ldots, b_m) = 0$ for all $a_1, \ldots, a_n \in A^{(0)}$ and $b_1, \ldots, b_m \in A^{(1)}$. Let $T_2(A)$ be the set of all the $\mathbb{Z}_2$-graded identities satisfied by A, which is easily seen to be a *T_2-ideal* of $F\langle Y \cup Z\rangle$, namely an ideal of $F\langle Y \cup Z\rangle$ invariant under all the endomorphisms of $F\langle Y \cup Z\rangle$ preserving the grading. A *variety of associative superalgebras* or *supervariety* $\mathcal{V}^{sup}$ is the class of all associative F-superalgebras whose T_2-ideals of graded identities contain the T_2-ideal of $F\langle Y \cup Z\rangle$ associated to $\mathcal{V}^{sup}$, denoted by $T_2(\mathcal{V}^{sup})$. The supervariety $\mathcal{V}^{sup}$ is generated by the superalgebra A if $T_2(\mathcal{V}^{sup}) = T_2(A)$ and in this case we write $\mathcal{V}^{sup} = \mathrm{supvar}(A)$. By following the same procedure of the ordinary case, let us consider the *n-th $\mathbb{Z}_2$-graded codimension* (or *n-th supercodimension*) $c_n^{sup}(A)$ of a superalgebra A defined as $\dim_F \frac{P_n^{sup}}{P_n^{sup}\cap T_2(A)}$, where P_n^{sup} denotes the space of multilinear polynomials of degree n of $F\langle Y \cup Z\rangle$ in $y_1, z_1, \ldots, y_n, z_n$. Since F has characteristic zero, the spaces $P_n^{sup} \cap T_2(A)$ determine $T_2(A)$. In [12] it was proved that $\{c_n^{sup}(A)\}_{n\geqslant 1}$ is exponentially bounded if, and only if, A satisfies an ordinary polynomial identity. In [3] the authors captured the exponential growth of this sequence in the case in which A is finitely generated and PI by proving that the $\lim_{m\to+\infty} \sqrt[m]{c_m^{sup}(A)}$ exists and is a non-negative integer, called the *$\mathbb{Z}_2$-graded exponent* or *superexponent* of A and denoted by $\exp_2(A)$. If $\mathcal{V}^{sup} = \mathrm{supvar}(A)$, set $\exp_2(\mathcal{V}^{sup}) := \exp_2(A)$, the superexponent of the supervariety $\mathcal{V}^{sup}$. Recently, the existence of the graded exponent has been established for any PI algebra graded by a finite abelian group (see [7,1] and [10]).

As in the ordinary case, in the superalgebras setting a considerable amount of work has been devoted to the investigation of the asymptotic behaviour of supercodimensions. In [11] the authors provided a list of five superalgebras characterizing the supervarieties of *polynomial growth*, namely those supervarieties with polynomially bounded supercodimension sequence. More precisely, they showed that a supervariety $\mathcal{V}^{sup}$ has polynomial growth if, and only if, none of the superalgebras of the list belongs to $\mathcal{V}^{sup}$. As a consequence, the supervarieties generated by those superalgebras are the only supervarieties of *almost polynomial growth*, namely, in terms of superexponent, those varieties of superalgebras of superexponent $d \geqslant 2$ such that any proper subvariety has superexponent at most 1. Following this direction, La Mattina in [22] has completely classified all subvarieties of the varieties of superalgebras of almost polynomial growth by determining, among the other things, all their minimal subvarieties of polynomial growth.

In this framework it becomes of particular interest to investigate minimal varieties of PI superalgebras of given superexponent, which are the natural generalization of supervarieties of almost polynomial growth.

Definition 1.1. A variety $\mathcal{V}^{sup}$ of PI associative superalgebras is said to be *minimal of superexponent d* if $\exp_2(\mathcal{V}^{sup}) = d$ and $\exp_2(\mathcal{U}^{sup}) < d$ for all proper subvarieties $\mathcal{U}^{sup}$ of $\mathcal{V}^{sup}$.

The present paper originates from the attempt to characterize supervarieties of *finite basic rank* (that is generated by a finitely generated and PI superalgebra) which are minimal with respect to their superexponent. In the first part we prove that any such supervariety is generated by a minimal superalgebra. In the second one we show that any minimal superalgebra in which all the graded simple components of the semisimple part are simple generates a minimal supervariety. Finally we prove that the same conclusion holds when the semisimple part of a minimal superalgebra has exactly two arbitrary graded simple components.

The corresponding problem for finite-dimensional algebras with involution was studied and solved in a series of papers ([6,8] and [9]) in which special block triangular matrix algebras with involution were constructed.

Before proceeding, a simple consideration is in order. As mentioned above, varieties of superalgebras of superexponent 1, namely varieties of polynomial growth, have been characterized in [11]. In particular, this result states that a $\mathbb{Z}_2$-graded algebra has polynomially bounded supercodimension sequence if, and only if, it satisfies a certain set of $\mathbb{Z}_2$-graded polynomial identities. It is easy to see that any graded identity of this set is a consequence of the polynomials $[y_1, y_2]$ and z_1, which are the generators of the T_2-ideal of $\mathbb{Z}_2$-graded polynomial identities satisfied by F (with its natural grading $(F, 0)$). Obviously, the supervariety generated by F is minimal of superexponent 1. By collecting all these deductions one has that $\mathrm{supvar}(F)$ is the unique minimal supervariety of polynomial growth. For this reason, in the sequel we shall deal with minimal supervarieties of exponential growth.

2. Preliminaries and minimal superalgebras

Throughout the rest of the paper, unless otherwise stated, let F denote a field of characteristic zero and all the algebras are assumed to be associative and to have the same ground field F.

Recall that an algebra A is a *superalgebra* (or $\mathbb{Z}_2$-graded algebra) if it has a vector space decomposition $A = A^{(0)} \oplus A^{(1)}$ such that $A^{(0)}A^{(0)} + A^{(1)}A^{(1)} \subseteq A^{(0)}$ and $A^{(0)}A^{(1)} + A^{(1)}A^{(0)} \subseteq A^{(1)}$. The subspaces $A^{(0)}$ and $A^{(1)}$ are said to be the *even* and the *odd* component of A, respectively. Accordingly, the elements of $A^{(0)}$ are called *even* (or homogeneous of degree 0) and those of $A^{(1)}$ are called *odd* (or homogeneous of degree 1). An element w of A is *homogeneous* if it is homogeneous of degree 0 or 1 (and we write $|w| = 0$ and $|w| = 1$, respectively), whereas a subspace $V \subseteq A$ is *homogeneous* if $V = (V \cap A^{(0)}) \oplus (V \cap A^{(1)})$. The superalgebra A is called *graded simple* (or $\mathbb{Z}_2$*-simple*) if the multiplication is non-trivial and it has no non-trivial homogeneous ideals.

Assume that A is a finite-dimensional superalgebra and $J = J(A)$ is its Jacobson radical. Then J is homogeneous and set $J^{(i)} := J \cap A^{(i)}$ for $i = 0, 1$. Moreover, by the generalization of the Wedderburn–Malcev Theorem we can write $A = A_{ss} + J$, where A_{ss} is a maximal semisimple subalgebra of A having an induced $\mathbb{Z}_2$-grading. Also A_{ss} can be written as the direct sum of graded simple superalgebras whose structure is well known, at least when the ground field is algebraically closed (see [24]). In fact, they are one of the following types:

(a) $M_{k,l} := \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, where $k \geqslant l \geqslant 0$, $k \neq 0$, $A \in M_k$, $D \in M_l$, $B \in M_{k\times l}$ and $C \in M_{l\times k}$, endowed with the grading $M_{k,l}^{(0)} := \left(\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right)$ and $M_{k,l}^{(1)} := \left(\begin{smallmatrix} 0 & B \\ C & 0 \end{smallmatrix}\right)$;
(b) $M_m(F \oplus tF)$, where $t^2 = 1$ with grading (M_m, tM_m),

where, for any pair of positive integers m and s, the symbol $M_{m\times s}$ means the F-vector space of all rectangular matrices with m rows and s columns, and $M_m := M_{m\times m}$. When in (a) $l = 0$, $M_{k,0}$ is nothing but the matrix algebra M_k with the trivial grading $(M_k, 0)$. In the sequel we refer to superalgebras of type (a) and (b) as *canonical* simple superalgebras.

Giambruno and Zaicev in [17] introduced the definition of *minimal* superalgebra in the following manner.

Definition 2.1. Let F be an algebraically closed field. An F-superalgebra A is called *minimal* if it is finite-dimensional and $A = A_{ss} + J$ where

(i) $A_{ss} = A_1 \oplus \cdots \oplus A_n$ with $A_1, \ldots, A_n$ canonical simple superalgebras;
(ii) there exist homogeneous elements $w_{12}, \ldots, w_{n-1,n} \in J^{(0)} \cup J^{(1)}$ and minimal graded idempotents $e_1 \in A_1, \ldots, e_n \in A_n$ such that

$$e_i w_{i,i+1} = w_{i,i+1} e_{i+1} = w_{i,i+1}, \quad i = 1, \ldots, n-1$$

and

$$w_{12} w_{23} \cdots w_{n-1,n} \neq 0;$$

(iii) $w_{12}, \ldots, w_{n-1,n}$ generate J as two-sided ideal of A.

We observe that, when $n = 1$, A is nothing but a canonical simple superalgebra.

In Lemma 3.5 of [17] it was shown that the minimal superalgebra A has the following vector space decomposition:

$$A = \bigoplus_{1\leqslant i\leqslant j\leqslant n} A_{ij}, \tag{1}$$

where $A_{11} := A_1, \ldots, A_{nn} := A_n$ and, for all $i < j$,

$$A_{ij} := A_i w_{i,i+1} A_{i+1} \cdots A_{j-1} w_{j-1,j} A_j.$$

Moreover $J = \bigoplus_{i<j} A_{ij}$ and $A_{ij} A_{kl} = \delta_{jk} A_{il}$, where δ_{jk} is the Kronecker delta.

As stressed in Chapter 8 of [18], the order of the simple components $A_1, \ldots, A_n$ of the semisimple part A_{ss} of a minimal superalgebra A is important. For this reason in the sequel we shall tacitly agree that if $A = A_{ss} + J$ is a minimal superalgebra with semisimple part $A_{ss} = A_1 \oplus \cdots \oplus A_n$, then $A_1 J A_2 J \cdots J A_n \neq 0$. According to the main result of [3], $\exp_2(A) = \dim_F A_{ss}$.

3. Generators of minimal supervarieties of finite basic rank

The aim of this section is to show that any minimal supervariety of finite basic rank is generated by a suitable minimal superalgebra. The first important feature of these superalgebras (which will be very useful for our aims) is the following result.

Lemma 3.1. *Let A be a finite-dimensional superalgebra over an algebraically closed field. Then there exists a minimal superalgebra $B \subseteq A$ such that* $\exp_2(A) = \exp_2(B)$.

Proof. See Lemma 8.1.4 of [18]. □

We are in position to state the first of our main results.

Proposition 3.2. *Let $\mathcal{V}^{sup}$ be a supervariety of finite basic rank. If $\mathcal{V}^{sup}$ is minimal of superexponent $d \geqslant 2$, then $\mathcal{V}^{sup} = \operatorname{supvar}(B)$, where B is a suitable minimal superalgebra.*

Proof. Since $\mathcal{V}^{sup}$ is of finite basic rank, according to Kemer's result (see Theorem 2.2 of [20]) there exists a finite-dimensional superalgebra A over F such that $\mathcal{V}^{sup} = \operatorname{supvar}(A)$ and $\exp_2(A) = d$. Let $\overline{F}$ be the algebraic closure of F and $\overline{A} := A \otimes_F \overline{F}$. Then $\overline{A}$ has an induced grading given by $\overline{A}^{(0)} = A^{(0)} \otimes_F \overline{F}$ and $\overline{A}^{(1)} = A^{(1)} \otimes_F \overline{F}$. It follows that the n-th supercodimension $c_n^{sup}(A)$ of A over F coincides with the n-th supercodimension $c_n^{sup}(\overline{A})$ of $\overline{A}$ over $\overline{F}$. Hence $\exp_2(\overline{A}) = d$ (over $\overline{F}$). By Lemma 3.1, $\operatorname{supvar}(\overline{A})$ contains a minimal superalgebra B of superexponent d. Since $\overline{A}$ as an F-superalgebra belongs to $\mathcal{V}^{sup}$, it follows that B belongs to $\mathcal{V}^{sup}$. By the minimality of $\mathcal{V}^{sup}$ we get that $\mathcal{V}^{sup} = \operatorname{supvar}(B)$, and this concludes the proof. □

Also by taking into account the characterization of minimal varieties of associative algebras of given exponent (see [16] and [17]), it is natural to conjecture that a supervariety of finite basic rank is minimal if, and only if, it is generated by a suitable minimal superalgebra. Hence the open question to tackle is whether the supervariety generated by a minimal superalgebra is minimal with respect to its superexponent.

Let us discuss the obvious strategy to use for this end. Let A be a minimal superalgebra and let $\mathcal{V}^{sup} := \operatorname{supvar}(A)$. Let $\mathcal{U}^{sup} \subseteq \mathcal{V}^{sup}$ such that $\exp_2(\mathcal{V}^{sup}) = \exp_2(\mathcal{U}^{sup})$. We aim to show that $\mathcal{U}^{sup} = \mathcal{V}^{sup}$. Since $\mathcal{V}^{sup}$ satisfies some Capelli identities, $\mathcal{U}^{sup}$ has finite basic rank (see Theorem 11.4.3 of [18]). By Kemer's result, $\mathcal{U}^{sup}$ is generated by a finite-dimensional superalgebra B'. On the other hand, according to Lemma 3.1, there exists a minimal superalgebra B such that $T_2(B') \subseteq T_2(B)$ and $\exp_2(B') = \exp_2(B)$. Consequently, $T_2(A) \subseteq T_2(B)$ and $\exp_2(A) = \exp_2(B)$. We have to prove that $T_2(A) = T_2(B)$. Thus we reduce our problem to comparing T_2-ideals of identities of minimal superalgebras with the same superexponent.

The following result will be crucial for our aims.

Lemma 3.3. *Let $A = A_{ss} + J(A)$ and $B = B_{ss} + J(B)$ be minimal superalgebras such that $T_2(A) \subseteq T_2(B)$ and $\exp_2(A) = \exp_2(B)$. Then $A_{ss} = B_{ss}$.*

Before proving the lemma, the following considerations are in order. Let us denote by G the infinite-dimensional Grassmann algebra over F with its natural $\mathbb{Z}_2$-grading $G = G^{(0)} \oplus G^{(1)}$. If $A = A^{(0)} \oplus A^{(1)}$ is a superalgebra, we use $G(A)$ for the Grassmann envelope of A, namely $G(A) = (A^{(0)} \otimes G^{(0)}) \oplus (A^{(1)} \otimes G^{(1)})$. A celebrated theorem of Kemer says that every T-ideal of $F\langle X\rangle$ is the T-ideal of the identities of $G(A)$, where A is a suitable finite-dimensional superalgebra. Furthermore, a T-ideal I of $F\langle X\rangle$ is *verbally prime* if $I_1 I_2 \subseteq I$ for any T-ideals I_1, I_2 of $F\langle X\rangle$ implies that either $I_1 \subseteq I$ or $I_2 \subseteq I$. According to a well-known result of Kemer (see [20]), a proper T-ideal I is verbally prime if, and only if, $I = \operatorname{Id}(G(A))$ for some canonical simple superalgebra A.

The key role played by minimal superalgebras in PI theory and the connection with verbally prime T-ideals is in the classification of minimal varieties of algebras of given exponent, which is the core of [17].

Theorem 3.4. *(See 7.5 of [17].) Let $\mathcal{V}$ be a variety of algebras such that $\exp(\mathcal{V}) \geqslant 2$. The following statements are equivalent:*

(i) *$\mathcal{V}$ is a minimal variety of exponent d;*
(ii) *$\operatorname{Id}(\mathcal{V})$ is a product of verbally prime T-ideals;*
(iii) *$\mathcal{V} = \operatorname{var}(G(A))$, for some minimal superalgebra $A = A_{ss} + J$ such that $\dim_F A_{ss} = d$.*

In particular, according to Corollary 4.4 of [17], if $\mathcal{V} = \mathrm{var}(G(A))$ is a minimal variety of given exponent and $A = (A_1 \oplus \cdots \oplus A_n) + J$ is the Wedderburn–Malcev decomposition of A, then

$$\mathrm{Id}(\mathcal{V}) = \mathrm{Id}\bigl(G(A_1)\bigr) \cdots \mathrm{Id}\bigl(G(A_n)\bigr). \tag{2}$$

Proof of Lemma 3.3. From the assumption that $T_2(A) \subseteq T_2(B)$ it follows that $T_2(G(A)) \subseteq T_2(G(B))$ as well. Consequently, $\mathrm{Id}(G(A)) \subseteq \mathrm{Id}(G(B))$. Moreover, by virtue of the main result of [14],

$$\exp\bigl(G(A)\bigr) = \exp_2(A) = \exp_2(B) = \exp\bigl(G(B)\bigr).$$

Since $G(A)$ is minimal with respect to its exponent, one has that

$$\mathrm{Id}\bigl(G(A)\bigr) = \mathrm{Id}\bigl(G(B)\bigr).$$

Let $A = (A_1 \oplus \cdots \oplus A_n) + J(A)$ and $B = (B_1 \oplus \cdots \oplus B_m) + J(B)$ be the Wedderburn–Malcev decompositions of the superalgebras A and B, respectively. The relation (2) yields

$$\mathrm{Id}\bigl(G(A)\bigr) = \mathrm{Id}\bigl(G(A_1)\bigr) \cdots \mathrm{Id}\bigl(G(A_n)\bigr)$$

and

$$\mathrm{Id}\bigl(G(B)\bigr) = \mathrm{Id}\bigl(G(B_1)\bigr) \cdots \mathrm{Id}\bigl(G(B_m)\bigr).$$

By invoking Lemma 8.5.4 of [18], the above equalities give us $m = n$ and $\mathrm{Id}(G(A_i)) = \mathrm{Id}(G(B_i))$ for all $1 \leqslant i \leqslant n$. By using the same arguments of the proof of Lemma 8.5.1 of [18], we conclude that A_i is isomorphic (as a superalgebra) to B_i for all $1 \leqslant i \leqslant n$. Since A_i and B_i are canonical simple superalgebras, this is equivalent to saying that $A_i = B_i$ for all $1 \leqslant i \leqslant n$, and this concludes the proof. □

A first immediate consequence of Lemma 3.3 is the following

Corollary 3.5. *Any canonical simple superalgebra of dimension $d > 1$ generates a minimal supervariety of superexponent d.*

4. Minimal superalgebras with simple $\mathbb{Z}_2$-simple components

Let us consider the complete matrix algebra M_n for some integer $n \geqslant 2$. A $\mathbb{Z}_2$-grading on M_n is called *elementary* if there exists an n-tuple $(g_1, \ldots, g_n) \in \mathbb{Z}_2^n$ such that the matrix units E_{ij} of M_n are homogeneous and $E_{ij} \in M_n^{(k)}$ if, and only if, $k = g_j - g_i$. In an equivalent manner, we can say that it is defined a map $|\,|: \{1, \ldots, n\} \to \mathbb{Z}_2$ inducing a grading on M_n by setting the degree of E_{ij} equal to $|j| - |i|$.

It is clear that canonical simple superalgebras of the form $M_{k,l}$ have the structure of superalgebras with elementary grading given by the $(k+l)$-tuple $(\underbrace{0, \ldots, 0}_{k\ \text{times}}, \underbrace{1, \ldots, 1}_{l\ \text{times}})$.

Now we show that, if A is a minimal superalgebra in which all the graded simple components of the semisimple part are simple (as algebras), then it can be realized as a block triangular matrix algebra with an elementary grading. In the sequel, for a finite sequence of positive integers $(d_1, \ldots, d_n)$, let us denote by $UT(d_1, \ldots, d_n)$ the algebra of upper block triangular matrices of size $d_1, \ldots, d_n$.

Proposition 4.1. *Let $A = A_{ss} + J$ be a minimal superalgebra. If $A_{ss} = A_1 \oplus \cdots \oplus A_n$, where $A_j = M_{k_j,l_j}$ for all j, then A is isomorphic (as a superalgebra) to $UT(k_1 + l_1, \ldots, k_n + l_n)$ equipped with a suitable elementary grading.*

Proof. For all $1 \leqslant i \leqslant n$, let $|\,|_i$ be the map defining the grading on the component $A_i = M_{k_i,l_i}$. Set $\eta_0 := 0$ and, for all $1 \leqslant i \leqslant n$, $\alpha_i := k_i + l_i$ and $\eta_i := \sum_{k=1}^{i} \alpha_k$, we want to construct a map

$$|\,|_A : \{1, \dots, \eta_n\} \to \mathbb{Z}_2$$

inducing a grading on $UT(\alpha_1, \dots, \alpha_n)$ so that A and $S := (UT(\alpha_1, \dots, \alpha_n), |\,|_A)$ are isomorphic as superalgebras.

For all $1 \leqslant i \leqslant n$, let $\mathrm{Bl}_i := \{\eta_{i-1} + 1, \dots, \eta_i\}$. Then we have a partition of the set $\{1, \dots, \eta_n\}$ into *n blocks*, $\mathrm{Bl}_1, \dots, \mathrm{Bl}_n$. At this point, let us define the map $|\,|_A$ block-by-block, according to the degrees $|w_{i,i+1}|$ of the homogeneous radical elements $w_{i,i+1}$ appearing in Definition 2.1, in the following manner. Set

$$|j|_A := |j|_1 \quad \forall j \in \mathrm{Bl}_1.$$

Then, inductively, for all $2 \leqslant i \leqslant n$ let us define

$$|j|_A := |j - \eta_{i-1}|_i \quad \forall j \in \mathrm{Bl}_i$$

if

- either $|t|_A = |t - \eta_{i-2}|_{i-1}$ for all $t \in \mathrm{Bl}_{i-1}$ and $|w_{i-1,i}| = 0$
- or $|t|_A = |t - \eta_{i-2}|_{i-1} + 1$ for all $t \in \mathrm{Bl}_{i-1}$ and $|w_{i-1,i}| = 1$.

Otherwise, set

$$|j|_A := |j - \eta_{i-1}|_i + 1 \quad \forall j \in \mathrm{Bl}_i.$$

If the E_{ij}'s are the usual matrix units of S, set, for all $1 \leqslant i \leqslant j \leqslant n$ and $1 \leqslant r \leqslant \alpha_i$, $1 \leqslant s \leqslant \alpha_j$, $E_{rs}^{(ij)} := E_{\eta_{i-1}+r, \eta_{j-1}+s}$ and observe that, for all $1 \leqslant i \leqslant n$, $|E_{rs}^{(ii)}|_A = |s|_i - |r|_i$. Hence there exists a graded isomorphism ϕ_i between the i-th diagonal block M_{α_i} of S (endowed with the grading induced by that on S) and the $\mathbb{Z}_2$-simple component A_i of A. We can assume that in this isomorphism $E_{11}^{(ii)} \mapsto e_i$, where the e_i's are the minimal graded idempotents of Definition 2.1.

Now, let us consider the subspaces A_{ij} of A appearing in the decomposition (1). By the definition $A_{i,i+1} = A_i w_{i,i+1} A_{i+1}$, whereas, for the indices $j > i + 1$, setting $w_{ij} := w_{i,i+1} w_{i+1,i+2} \cdots w_{j-1,j}$, we get that $A_{ij} = A_i w_{ij} A_j$. In fact, we notice that any minimal graded idempotent of the superalgebra $M_{l,m}$ is a minimal idempotent of the algebra M_{l+m}. Thus, for all $1 \leqslant k \leqslant n - 1$, e_k is a minimal idempotent of A_k and hence one has that

$$w_{k-1,k} A_k w_{k,k+1} = w_{k-1,k} e_k A_k e_k w_{k,k+1}$$

is the vector subspace generated by $\{w_{k-1,k}, w_{k,k+1}\}$.

At this stage, observing that the matrix unit $E_{rs}^{(ij)}$ of S coincides with $E_{r1}^{(ii)} E_{11}^{(ij)} E_{1s}^{(jj)}$, take the linear map $\phi : S \to A$ such that

$$E_{rs}^{(ij)} \mapsto \begin{cases} \phi_i(E_{r1}^{(ii)}) \cdot w_{ij} \cdot \phi_j(E_{1s}^{(jj)}) & \text{if } i < j; \\ \phi_i(E_{rs}^{(ij)}) & \text{if } i = j. \end{cases}$$

Straightforward computations show that ϕ is a graded isomorphism from S to A, and this concludes the proof. $\square$

In this way we reduce the investigation of minimal superalgebras in which all the graded simple components of the semisimple part are simple to that of upper block triangular matrix algebras equipped with elementary gradings. In particular, we are interested in studying possible different elementary $\mathbb{Z}_2$-gradings on the same block triangular matrix algebra.

Throughout the sequel, if $|\,|_A$ and $|\,|_B$ are maps defining an elementary $\mathbb{Z}_2$-grading on $UT(\alpha_1,\dots,\alpha_n)$, set $\eta_0 := 0$, for all $1 \leqslant i \leqslant n$ let us define $\eta_i := \sum_{k=1}^{i} \alpha_k$, $\mathrm{Bl}_i := \{\eta_{i-1}+1,\dots,\eta_i\}$,

$$p_i := \bigl|\bigl\{j \bigm| j \in \mathrm{Bl}_i,\ |j|_A = 0\bigr\}\bigr| \quad \text{and} \quad q_i := \bigl|\bigl\{j \bigm| j \in \mathrm{Bl}_i,\ |j|_A = 1\bigr\}\bigr|.$$

Furthermore, let r_i and s_i be the integers corresponding to p_i and q_i with respect to the grading induced by $|\,|_B$. It immediately follows that, for all $1 \leqslant i \leqslant n$,

$$p_i + q_i = r_i + s_i. \tag{3}$$

We prove now an easy (but useful) isomorphism criterion.

Lemma 4.2. *Let $A := (UT(\alpha_1,\dots,\alpha_n), |\,|_A)$ and $B := (UT(\alpha_1,\dots,\alpha_n), |\,|_B)$. If either $(p_1,\dots,p_n) = (r_1,\dots,r_n)$ or $(p_1,\dots,p_n) = (s_1,\dots,s_n)$, then A and B are isomorphic as superalgebras.*

Proof. We notice that, if $(g_1,\dots,g_m) \in \mathbb{Z}_2^m$ defines an elementary grading on M_m, then, for all $\sigma \in S_m$ the element $(g_{\sigma(1)},\dots,g_{\sigma(m)})$ of $\mathbb{Z}_2^m$ defines an elementary grading on M_m isomorphic to that induced by $(g_1,\dots,g_m)$.

Now, according to (3), if $(p_1,\dots,p_n) = (r_1,\dots,r_n)$, then $(q_1,\dots,q_n) = (s_1,\dots,s_n)$. On the other hand, if $(p_1,\dots,p_n) = (s_1,\dots,s_n)$ it must be $(q_1,\dots,q_n) = (r_1,\dots,r_n)$. Therefore, in the case in which $(p_1,\dots,p_n) = (r_1,\dots,r_n)$, it is easy to see that the grading on B is obtained by a permutation σ of the elements of the η_n-tuple inducing the grading on A such that $\sigma(\mathrm{Bl}_i) = \mathrm{Bl}_i$. Thus the result follows from the above observation.

Hence assume that $(p_1,\dots,p_n) = (s_1,\dots,s_n)$. The grading induced by the map $|\,|_{A'}$ defined by $|i|_{A'} := |i|_A + 1$ for all $1 \leqslant i \leqslant \eta_n$ on $UT(\alpha_1,\dots,\alpha_n)$ is the same as that induced by $|\,|_A$. If $(p'_1,\dots,p'_n)$ is the n-tuple corresponding to $(p_1,\dots,p_n)$ with respect to $|\,|_{A'}$, one has that $(p'_1,\dots,p'_n) = (q_1,\dots,q_n) = (r_1,\dots,r_n)$ and, as above, the expected conclusion holds. $\square$

In the next lemma we show that the equality among the sequence of p_j's and that of r_j's or s_j's holds as soon as the graded identities satisfied by A are satisfied also by B. This is crucial in order to state the main result of the section.

Lemma 4.3. *Let $A := (UT(\alpha_1,\dots,\alpha_n), |\,|_A)$ and $B := (UT(\alpha_1,\dots,\alpha_n), |\,|_B)$. If $T_2(A) \subseteq T_2(B)$, then either $(p_1,\dots,p_n) = (r_1,\dots,r_n)$ or $(p_1,\dots,p_n) = (s_1,\dots,s_n)$. Consequently, A and B are isomorphic as superalgebras.*

For its proof we need to introduce some terminology and preliminary results. Let us recall that, for any positive integer n, the *n-th Capelli polynomial* $\mathrm{Cap}_n(x_1,\dots,x_n,x_{n+1},\dots,x_{2n+1})$ is the element of the free associative algebra $F\langle X\rangle$ defined as

$$\sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) x_{n+1} x_{\sigma(1)} x_{n+2} x_{\sigma(2)} x_{n+3} \cdots x_{2n} x_{\sigma(n)} x_{2n+1},$$

whereas, when $n = 0$, it is nothing but the $(n+1)$-st variable x_{n+1}. The *Standard polynomial* in n variables $\mathrm{St}_n(x_1,\dots,x_n)$ is the element of $F\langle X\rangle$ defined as

$$\sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)}.$$

They play a prominent role in PI theory and we shall use them as a main tool throughout the rest of the section.

We quote now some results dealing with evaluations of Capelli and Standard polynomials in matrix algebras without reporting their proofs, which are straightforward computations or consequences of the Amitsur–Levitzki Theorem.

Lemma 4.4. *The Standard polynomial* $\mathrm{St}_t(x_1,\dots,x_t)$ *is a polynomial identity for the upper block triangular matrix algebra* $UT(d_1,\dots,d_n)$ *if, and only if,* $t \geqslant 2(d_1+\dots+d_n)$.

Lemma 4.5. *The Capelli polynomial* $\mathrm{Cap}_t(x_1,\dots,x_{2t+1})$ *is a polynomial identity for the upper block triangular matrix algebra* $UT(d_1,\dots,d_n)$ *if, and only if,* $t \geqslant n+\sum_{i=1}^n d_i^2$. *In particular, set* $k := n-1+\sum_{i=1}^n d_i^2$, *for any* $1 \leqslant l \leqslant d_1$ *and* $1+\sum_{i=1}^{n-1} d_i \leqslant m \leqslant \sum_{i=1}^n d_i$ *there exists an evaluation of* $\mathrm{Cap}_k(x_1,\dots,x_{2k+1})$ *in* $UT(d_1,\dots,d_n)$ *at matrix units equal to* E_{lm}.

Lemma 4.6. *Let* $k>l$ *be integers. The polynomial* $\mathrm{St}_{2k-1}(y_1,\dots,y_{2k-1})$ *is not a graded polynomial identity for the simple superalgebra* $M_{k,l}$ *and all its non-zero graded evaluations in* $M_{k,l}$ *are linear combinations of the matrix units* E_{ij} *for* $1 \leqslant i,j \leqslant k$. *In particular, for any* $1 \leqslant i,j \leqslant k$ *there exists a graded evaluation of* $\mathrm{St}_{2k-1}(y_1,\dots,y_{2k-1})$ *in* $M_{k,l}$ *equal to* E_{ij}.

We are in position to prove Lemma 4.3. We recall that the notations are those introduced before Lemma 4.2.

Proof of Lemma 4.3. Our proof will be by induction on n. Before proceeding, we notice that, in any case, we can assume without loss of generality that $p_1 \geqslant q_1$, $r_1 \geqslant s_1$, $|j|_A = 0$ for all $1 \leqslant j \leqslant p_1$ and $|k|_B = 0$ for all $1 \leqslant k \leqslant r_1$.

Let $n=1$ and assume, if possible, that $p_1 > r_1$. By virtue of (3), $q_1 < s_1$. An easy application of Lemma 4.6 shows that the element $\mathrm{St}_{2q_1}(y_1,\dots,y_{2q_1}) \cdot z_1 \cdot \mathrm{St}_{2q_1}(y_{2q_1+1},\dots,y_{4q_1})$ of $F\langle Y \cup Z\rangle$ is in $T_2(A)$, but not in $T_2(B)$. This contradicts the original assumption that $T_2(A) \subseteq T_2(B)$. Consequently, $p_1 = r_1$, and we are done.

Thus, suppose that $n \geqslant 2$ and consider the superalgebras

$$A' := \bigl(UT(\alpha_1,\dots,\alpha_{n-1}), |\,|_{A'}\bigr)$$

and

$$B' := \bigl(UT(\alpha_1,\dots,\alpha_{n-1}), |\,|_{B'}\bigr),$$

where $|\,|_{A'}$ and $|\,|_{B'}$ are the restrictions of the maps $|\,|_A$ and $|\,|_B$ to the set $\{1,\dots,\eta_{n-1}\}$, respectively. Let us prove that $T_2(A') \subseteq T_2(B')$. Assume, if possible, that the inclusion does not hold. Hence there exists $f \in T_2(A') \setminus T_2(B')$. Let us denote by A_n the n-th diagonal block of A with the grading induced by the restriction of $|\,|_A$ to Bl_n and by $\overline{B}$ the subalgebra $UT(\alpha_{n-1},\alpha_n)$ equipped with the grading induced by the restriction of $|\,|_B$ to the set $\mathrm{Bl}_{n-1} \cup \mathrm{Bl}_n$. Observe that the set $T_2(A_n) \setminus T_2(\overline{B})$ is non-empty (otherwise $\exp_2(A_n) \geqslant \exp_2(\overline{B})$, which is false since $\exp_2(\overline{B}) > \exp_2(A_n)$). Let us pick one element $g \in T_2(A_n) \setminus T_2(\overline{B})$ whose variables are pairwise different from those involved in f. Set $u := y+z$, where y is an even variable and z an odd variable of $F\langle Y \cup Z\rangle$ involved neither in f nor in g. Then fug is not a $\mathbb{Z}_2$-graded identity for the superalgebra B, but $fug \in T_2(A') \cdot T_2(A_n) \subseteq T_2(A)$, which contradicts the fact that $T_2(A) \subseteq T_2(B)$.

Therefore we can apply the induction assumption to the superalgebras A' and B' and, by combining it with (3), conclude that either

$$(p_1,\dots,p_{n-1}) = (r_1,\dots,r_{n-1}) \quad \text{and} \quad (q_1,\dots,q_{n-1}) = (s_1,\dots,s_{n-1}) \tag{4}$$

or

$$(p_1,\dots,p_{n-1}) = (s_1,\dots,s_{n-1}) \quad \text{and} \quad (q_1,\dots,q_{n-1}) = (r_1,\dots,r_{n-1}). \tag{5}$$

In the same manner, let us consider the superalgebras

$$A'' := \big(UT(\alpha_2, \ldots, \alpha_n), |\,|_{A''}\big)$$

and

$$B'' := \big(UT(\alpha_2, \ldots, \alpha_n), |\,|_{B''}\big),$$

where $|\,|_{A''}$ and $|\,|_{B''}$ are the restrictions of the maps $|\,|_A$ and $|\,|_B$ to the set $\{\eta_1 + 1, \ldots, \eta_n\}$, respectively. As above, $T_2(A'') \subseteq T_2(B'')$. In this case from the induction assumptions it follows that either

$$(p_2, \ldots, p_n) = (r_2, \ldots, r_n) \quad \text{and} \quad (q_2, \ldots, q_n) = (s_2, \ldots, s_n) \tag{6}$$

or

$$(p_2, \ldots, p_n) = (s_2, \ldots, s_n) \quad \text{and} \quad (q_2, \ldots, q_n) = (r_2, \ldots, r_n). \tag{7}$$

If the equalities (4) and (6) occur, one has that $(p_1, \ldots, p_n) = (r_1, \ldots, r_n)$. The same is true also in the case in which (5) and (6) hold. In fact, in such an event, $p_1 = s_1 \leqslant r_1 = q_1$. Consequently, $p_1 = q_1 = r_1 = s_1$, and we are done. The proof is concluded also when (5) and (7) are satisfied since these equalities yield $(p_1, \ldots, p_n) = (s_1, \ldots, s_n)$.

Thus, assume that the relations (4) and (7) hold simultaneously. This implies that, when $n > 2$,

$$p_i = q_i = r_i = s_i \quad \forall 2 \leqslant i \leqslant n-1.$$

If $p_1 = q_1$, it must be also $r_1 = p_1 = q_1 = s_1$ and we have nothing to show, since $(p_1, \ldots, p_n) = (s_1, \ldots, s_n)$. Therefore, assume that $p_1 > q_1$. We can also suppose that $p_n \neq q_n$, otherwise their equality implies that $(p_1, \ldots, p_n) = (r_1, \ldots, r_n)$. Furthermore, we can assume that $|j|_A = 0$ and $|j|_B = 1$ for all $\eta_{n-1} + 1 \leqslant j \leqslant \eta_{n-1} + p_n$.

At this stage, we distinguish two cases: when $p_n < q_n$ and when $p_n > q_n$.

Now, the homogeneous components of degree 0 of A and B are isomorphic to a direct sum of upper block triangular matrix algebras, namely

$$A^{(0)} \cong UT(p_1, \ldots, p_n) \oplus UT(q_1, \ldots, q_n)$$

and

$$B^{(0)} \cong UT(r_1, \ldots, r_n) \oplus UT(s_1, \ldots, s_n).$$

Set $t := q_n + \sum_{i=1}^{n-1} p_i$, and let us consider the element of $F\langle Y \cup Z\rangle$

$$f := \mathrm{St}_{2t-1}(y_1, \ldots, y_{2t-1}).$$

Under the assumption that $p_n < q_n$, by Lemma 4.4 we get that $f \in T_2(A)$. But $\mathrm{St}_{2t-1}(x_1, \ldots, x_{2t-1})$ is not a polynomial identity for $UT(r_1, \ldots, r_n)$, since $q_n = r_n$. Therefore f is not in $T_2(B)$, which is a contradiction.

Finally, suppose, if possible, that $p_n > q_n$. Let us consider the polynomials of $F\langle Y \cup Z\rangle$

$$f_1 := y_1^{(1)} \mathrm{St}_{2p_1-1}\big(y_2^{(1)}, \ldots, y_{2p_1}^{(1)}\big) y_{2p_1+1}^{(1)}$$

and

$$f_n := y_1^{(n)} \mathrm{St}_{2p_n-1}\big(y_2^{(n)}, \ldots, y_{2p_n}^{(n)}\big) y_{2p_n+1}^{(n)},$$

where $y_1^{(1)}, \dots, y_{2p_1+1}^{(1)}, y_1^{(n)}, \dots, y_{2p_n+1}^{(n)}$ are pairwise distinct variables from Y. If A_1 is the first diagonal block of A with the grading induced by the restriction of $|\,|_A$ to Bl_1, from Lemma 4.6 it follows that $f_1 \notin T_2(A_1)$ and $f_n \notin T_2(A_n)$. Denoted by B_1 and B_n the first and the n-th diagonal block of B with the grading induced by the restriction of $|\,|_B$ to Bl_1 and Bl_n, respectively, since $A_1 \cong B_1$ and $A_n \cong B_n$, $f_1 \notin T_2(B_1)$ and $f_n \notin T_2(B_n)$ as well.

Set $p := n - 1 + \sum_{i=1}^{n} \alpha_i^2$. According to Lemma 4.5, for any $1 \leqslant l \leqslant \eta_1$ and $\eta_{n-1} + 1 \leqslant m \leqslant \eta_n$ there exists an evaluation of $\mathrm{Cap}_p(x_1, \dots, x_{2p+1})$ in $UT(\alpha_1, \dots, \alpha_n)$ equal to E_{lm}. In particular, we can choose to evaluate the variables $x_1, \dots, x_{2p+1}$ at matrix units. Consequently, for any $1 \leqslant l \leqslant \eta_1$ and $\eta_{n-1} + 1 \leqslant m \leqslant \eta_n$ there exist suitable homogeneous variables $v_1, \dots, v_{2p+1}$ in $Y \cup Z$ different from those involved in f_1 and f_n such that the polynomial $\mathrm{Cap}_p(v_1, \dots, v_{2p+1})$ has a graded evaluation $\mathrm{Cap}_p(\bar{v}_1, \dots, \bar{v}_{2p+1}) = E_{lm}$ in the superalgebra B. Indeed, to construct such a polynomial let us consider one of the evaluations $\bar{x}_1, \dots, \bar{x}_{2p+1}$ at matrix units of the variables $x_1, \dots, x_{2p+1}$ in $UT(\alpha_1, \dots, \alpha_n)$ so that $\mathrm{Cap}_p(\bar{x}_1, \dots, \bar{x}_{2p+1}) = E_{lm}$. Hence, for all $1 \leqslant k \leqslant 2p + 1$, we choose to take v_k from Y (Z, respectively) if the degree of $\bar{x}_k$ with respect to the grading $|\,|_B$ is 0 (1, respectively). As an element of the free superalgebra $F\langle Y \cup Z\rangle$, the so constructed polynomial $\mathrm{Cap}_p(v_1, \dots, v_{2p+1})$ has degree equal to the degree of E_{lm} in B, namely $|m|_B - |l|_B$.

At this stage, let us fix $1 \leqslant i \leqslant p_1$ and $\eta_{n-1} + 1 \leqslant j \leqslant \eta_{n-1} + p_n$ and pick one element of $F\langle Y \cup Z\rangle$ of the form $\mathrm{Cap}_p(v_1, \dots, v_{2p+1})$ constructed as above having a graded evaluation in B equal to E_{ij}. Let us call this polynomial g. Since $|j|_B = 1$ and $|i|_B = 0$, g has total degree 1. Furthermore, by applying Lemma 4.6 it is easy to see that there exists in B a graded evaluation of f_1 equal to E_{ii} and one of f_n which is equal to E_{jj} (it is sufficient to evaluate f_1 in B_1 and f_n in B_n, respectively). Therefore we can conclude that there exists a non-zero graded evaluation of the polynomial $h := f_1 g f_n$ in B.

Now let us look at the graded evaluations of h in the superalgebra A. Since h is multilinear, we consider only graded evaluations of h at matrix units. First of all, we observe that any non-zero graded evaluation $\nu : F\langle Y \cup Z\rangle \to A$ of g in A is such that $\nu(g) \in J(A)^{n-1}$. Hence, it is a linear combination of the matrices E_{lj}, where $1 \leqslant l \leqslant \eta_1$ and $\eta_{n-1} + 1 \leqslant j \leqslant \eta_n$. As $J(A)^n = 0$, any possible non-zero graded evaluation of h in A must be such that f_1 must be evaluated in A_1 and f_n in A_n. But, by invoking Lemma 4.6, these evaluations of the f_i's are in the square of length p_i of $A_i^{(0)}$. Since g has total degree 1, this forces to be h in $T_2(A)$. Thus h is in $T_2(A) \setminus T_2(B)$, which contradicts the original assumption that $T_2(A) \subseteq T_2(B)$. Therefore it must be $p_n = q_n$, and this concludes the proof. $\square$

Remark. The above lemma shows that one can distinguish the elementary $\mathbb{Z}_2$-gradings on $UT(\alpha_1, \dots, \alpha_n)$ by their superidentities. This result is in the same spirit of those of [21] and [2], having the aim to decide if graded algebras in some important class are determined, up to graded isomorphism, by their graded identities.

At this point it is easy to obtain the main statement of the section, whose proof uses exactly the same arguments of the discussion before Lemma 3.3.

Theorem 4.7. *Let $A = A_{ss} + J$ be a minimal superalgebra. If $A_{ss} = A_1 \oplus \dots \oplus A_n$, where $A_j = M_{k_j, l_j}$ for all j, then* $\mathrm{supvar}(A)$ *is minimal of superexponent* $\dim_F A_{ss}$.

Proof. Set $\mathcal{V}^{sup} := \mathrm{supvar}(A)$ and let us consider a subvariety $\mathcal{U}^{sup} \subseteq \mathcal{V}^{sup}$ such that $\exp_2(\mathcal{V}^{sup}) = \exp_2(\mathcal{U}^{sup})$. Since $\mathcal{V}^{sup}$ satisfies some Capelli identities, $\mathcal{U}^{sup}$ has finite basic rank and, by Kemer's result, it is generated by a finite-dimensional superalgebra B'. According to Lemma 3.1, there exists a minimal superalgebra B such that $T_2(B') \subseteq T_2(B)$ and $\exp_2(B') = \exp_2(B)$. Therefore $T_2(A) \subseteq T_2(B)$ and $\exp_2(A) = \exp_2(B)$ as well.

By virtue of Lemma 3.3 we know that $B_{ss} = A_1 \oplus \dots \oplus A_n$. Moreover, by invoking Proposition 4.1, both A and B can be realized as the same block triangular matrix algebra, $UT(k_1 + l_1, \dots, k_n + l_n)$, equipped with a suitable elementary grading constructed as in the proof of the mentioned result. At this stage, we can apply Lemma 4.3 and conclude that A is isomorphic to B as a superalgebra. Consequently $T_2(A) = T_2(B)$, and the proof is done. $\square$

Giambruno and Zaicev proved that a variety of associative algebras of finite basic rank is minimal if, and only if, it is generated by an upper block triangular matrix algebra $UT(d_1,\dots,d_n)$ and that the T-ideal of identities of $UT(d_1,\dots,d_n)$ is factorable, namely it coincides with $\mathrm{Id}(M_{d_1})\cdots\mathrm{Id}(M_{d_n})$ (Theorems 1 and 2 of [16]). Working with superalgebras, the situation appears more involved. In fact, as we have shown above, minimal superalgebras in which all the graded simple components of the semisimple part are simple generate minimal supervarieties, but in general they do not generate the same supervariety even if they have the same graded components $A_1,\dots,A_n$ (for instance, the case in which $A_1=\cdots=A_n=F$ has been discussed in [4]). As a consequence, in general we cannot hope that the T_2-ideal of superidentities of these minimal superalgebras coincides with the product of the T_2-ideals of superidentities of the graded simple components $A_1,\dots,A_n$. Nevertheless, in any case the T-ideal of identities of their Grassmann envelope is equal to $\mathrm{Id}(G(A_1))\cdots\mathrm{Id}(G(A_n))$ (Corollary 4.4 of [17]).

5. Minimal superalgebras with two $\mathbb{Z}_2$-simple components

In this section we shall discuss in detail the special case in which the semisimple part of a minimal superalgebra has exactly two $\mathbb{Z}_2$-simple components, A_1 and A_2.

First of all, let us describe the structure of such a superalgebra when at least one of A_1 and A_2 is a canonical simple superalgebra of type (b) by using the language of ϕ-actions. In fact, it is well known that any superalgebra A can be viewed as an algebra with ϕ-action, where ϕ is an automorphism of A of order at most 2. Indeed, the homomorphism ϕ of $A=A^{(0)}\oplus A^{(1)}$ defined by $\phi(a_0):=a_0$ and $\phi(a_1):=-a_1$ for any $a_0\in A^{(0)}$ and $a_1\in A^{(1)}$ is an automorphism of A of order at most 2. Conversely, if A is an algebra with an automorphism ϕ of order at most 2, then, by setting $A^{(0)}:=\{a \mid a\in A,\ \phi(a)=a\}$ and $A^{(1)}:=\{a \mid a\in A,\ \phi(a)=-a\}$, A is a superalgebra with grading $(A^{(0)},A^{(1)})$.

Case I. Assume that A_1 and A_2 are both non-simple, $A_1=M_m(F\oplus cF)$ and $A_2=M_n(F\oplus dF)$, where $c^2=d^2=1$, and let us call $A=A_{ss}+J$ the minimal superalgebra having $A_{ss}=A_1\oplus A_2$. According to Definition 2.1, there exist homogeneous idempotents e_1,e_2 (of degree zero) and a homogeneous element $w:=w_{12}$ such that

$$e_1we_2=e_1w=we_2=w.$$

By regarding A as a ϕ-algebra, for $i=1,2$, we can write $A_i=I_i\oplus\phi(I_i)$ where I_i is a minimal two-sided ideal of A_i and $e_i=\rho_i+\phi(\rho_i)$ with ρ_i a non-homogeneous minimal idempotent. Hence

$$\begin{aligned} w=e_1we_2&=\big(\rho_1+\phi(\rho_1)\big)w\big(\rho_2+\phi(\rho_2)\big)\\ &=\rho_1w\rho_2+\rho_1w\phi(\rho_2)+\phi(\rho_1)w\rho_2+\phi(\rho_1)w\phi(\rho_2). \end{aligned}$$

As $w\neq 0$, either $\rho_1w\rho_2+\phi(\rho_1)w\phi(\rho_2)\neq 0$ or $\rho_1w\phi(\rho_2)+\phi(\rho_1)w\rho_2\neq 0$. For a suitable choice of ϵ_1 and ϵ_2, we can write the above non-zero element as $v:=\epsilon_1w\epsilon_2+\phi(\epsilon_1)w\phi(\epsilon_2)$, with, possibly, $u:=\phi(\epsilon_1)w\epsilon_2+\epsilon_1w\phi(\epsilon_2)\neq 0$. Furthermore

$$\phi(v)=\begin{cases} v & \text{if } |w|=0;\\ -v & \text{if } |w|=1.\end{cases}$$

Thus v is homogeneous and $(\epsilon_1+\phi(\epsilon_1))v=v=v(\epsilon_2+\phi(\epsilon_2))$. Set $v_1:=\epsilon_1w\epsilon_2$ and $v_2:=\phi(\epsilon_1)w\phi(\epsilon_2)$, one has that

$$\phi(v_i)=\begin{cases} v_{i+1} & \text{if } |w|=0;\\ -v_{i+1} & \text{if } |w|=1,\end{cases}$$

where the subindex $i+1$ of v is obviously intended modulo 2. We write u as u_1+u_2, with $u_1 := \phi(\epsilon_1)w\epsilon_2$ and $u_2 := \epsilon_1 w\phi(\epsilon_2)$ and argue in the same manner for it. Now, for the subspace A_{12} appearing in the decomposition (1) of A we get

$$A_{12} = A_1 w A_2 = A_1(v+u)A_2 = A_1 v A_2 \oplus A_1 u A_2.$$

Moreover,

$$A_1 v A_2 = A_1(v_1+v_2)A_2 = I_1 v_1 I_2 \oplus \phi(I_1)v_2\phi(I_2)$$

and

$$A_1 u A_2 = A_1(u_1+u_2)A_2 = \phi(I_1)u_1 I_2 \oplus I_1 u_2 \phi(I_2).$$

Since $I_1 \cong M_m$ and $I_2 \cong M_n$, the modules appearing in the decomposition of A_{12}, if non-zero, are isomorphic to $M_{m\times n}$. Therefore A can be represented as a block triangular matrix algebra of the following form:

$$\begin{pmatrix} B_1 & 0 & J_1 & J_2 \\ 0 & B_2 & J_3 & J_4 \\ 0 & 0 & C_1 & 0 \\ 0 & 0 & 0 & C_2 \end{pmatrix}, \quad B_i \in M_m,\ C_i \in M_n,\ J_j \in M_{m\times n}, \tag{8}$$

with $J_2 = J_3 = 0$ if $u = 0$. The ϕ-action on A depends on the degree of the radical element w. In particular, ϕ maps the element $\left(\begin{smallmatrix} a & 0 & x & y \\ 0 & b & z & h \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \end{smallmatrix}\right)$ of A into

$$\begin{pmatrix} b & 0 & h & z \\ 0 & a & y & x \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \alpha \end{pmatrix} \quad \text{if } |w| = 0, \qquad \begin{pmatrix} b & 0 & -h & -z \\ 0 & a & -y & -x \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \alpha \end{pmatrix} \quad \text{if } |w| = 1.$$

Consequently, if w is even, the grading on A is given by

$$\left(\begin{pmatrix} B & 0 & X & Y \\ 0 & B & Y & X \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & C \end{pmatrix}, \begin{pmatrix} B & 0 & X & Y \\ 0 & -B & -Y & -X \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & -C \end{pmatrix}\right),$$

otherwise it is given by

$$\left(\begin{pmatrix} B & 0 & X & Y \\ 0 & B & -Y & -X \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & C \end{pmatrix}, \begin{pmatrix} B & 0 & X & Y \\ 0 & -B & Y & X \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & -C \end{pmatrix}\right).$$

We notice that the above gradings are equivalent, that is the so constructed superalgebras are isomorphic.

Case II. Assume that A_1 is non-simple and A_2 is simple, $A_1 = M_m(F \oplus cF)$ and $A_2 = M_{k,l}$, where $c^2 = 1$ and $k \geqslant l$, and let us again call A the minimal superalgebra having semisimple part equal to $A_1 \oplus A_2$. As before, there exist non-trivial homogeneous idempotents e_1, e_2 and a homogeneous element $w := w_{12}$ such that $e_1 w e_2 = e_1 w = w e_2 = w \neq 0$. Let us write $A_1 = I_1 \oplus \phi(I_1)$, where I_1 is a minimal two-sided ideal of A_1. Thus $e_1 = \rho_1 + \phi(\rho_1)$, with ρ_1 a non-homogeneous minimal idempotent, and

$$w = e_1 w e_2 = \big(\rho_1 + \phi(\rho_1)\big) w e_2 = \rho_1 w e_2 + \phi(\rho_1) w e_2 \neq 0.$$

Hence $\rho_1 w e_2 \neq 0$ and $\phi(\rho_1) w e_2 = \pm\phi(\phi(\rho_1) w e_2) \neq 0$. Set $v_1 := \rho_1 w e_2$ and $v_2 := \phi(\rho_1) w e_2$, one has that

$$\phi(v_i) = \begin{cases} v_{i+1} & \text{if } |w| = 0; \\ -v_{i+1} & \text{if } |w| = 1. \end{cases}$$

Now, the subspace A_{12} of A coincides with $I_1 w A_2 \oplus \phi(I_1) w A_2$. But $I_1 w A_2 = I_1 v_1 A_2$ and $\phi(I_1) w A_2 = \phi(I_1) v_2 A_2$ and, since $I_1 \cong M_m$, these modules are isomorphic to $M_{m\times(k+l)}$. Hence the superalgebra A can be represented as a block triangular matrix algebra of the following form:

$$\begin{pmatrix} B_1 & 0 & J_1 & J_2 \\ 0 & B_2 & J_3 & J_4 \\ 0 & 0 & C & D \\ 0 & 0 & E & H \end{pmatrix},$$

with $B_i \in M_m$, $J_1, J_3 \in M_{m\times k}$, $J_2, J_4 \in M_{m\times l}$, $C \in M_k$, $H \in M_l$, $D \in M_{k\times l}$, $E \in M_{l\times k}$. If w is even, the grading on A is given by

$$\left(\begin{pmatrix} B & 0 & X & Y \\ 0 & B & X & -Y \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & H \end{pmatrix}, \begin{pmatrix} B & 0 & X & Y \\ 0 & -B & -X & Y \\ 0 & 0 & 0 & D \\ 0 & 0 & E & 0 \end{pmatrix}\right),$$

otherwise one has

$$\left(\begin{pmatrix} B & 0 & X & Y \\ 0 & B & -X & Y \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & H \end{pmatrix}, \begin{pmatrix} B & 0 & X & Y \\ 0 & -B & X & -Y \\ 0 & 0 & 0 & D \\ 0 & 0 & E & 0 \end{pmatrix}\right).$$

Also in this case the above gradings are equivalent.

When $A_1 = M_{k,l}$ and $A_2 = M_m(F \oplus cF)$, the superalgebra A can be represented as

$$\begin{pmatrix} C & D & J_1 & J_2 \\ E & H & J_3 & J_4 \\ 0 & 0 & B_1 & 0 \\ 0 & 0 & 0 & B_2 \end{pmatrix},$$

with $C \in M_k$, $H \in M_l$, $D \in M_{k\times l}$, $E \in M_{l\times k}$, $B_i \in M_m$, $J_1, J_2 \in M_{k\times m}$ and $J_3, J_4 \in M_{l\times m}$, with (unique) grading given by

$$\left(\begin{pmatrix} C & 0 & X & X \\ 0 & H & Y & -Y \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & B \end{pmatrix}, \begin{pmatrix} 0 & D & X & -X \\ E & 0 & Y & Y \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & -B \end{pmatrix}\right).$$

It is interesting to notice that the above superalgebras can be obtained as homogeneous subalgebras of matrix algebras endowed with an elementary grading. More precisely we get

Proposition 5.1. *Let $A = A_{ss} + J$ be a minimal superalgebra such that $A_{ss} = A_1 \oplus A_2$, where A_1, A_2 are canonical simple superalgebras with at least one of them of type* (b)*. Then A is isomorphic to one of the following superalgebras*:

(i) $\begin{pmatrix} B_1 & B_2 & J_1 & J_2 \\ B_2 & B_1 & J_3 & J_4 \\ 0 & 0 & C_1 & C_2 \\ 0 & 0 & C_2 & C_1 \end{pmatrix}$, *where $B_i \in M_m$, $C_i \in M_n$, $J_j \in M_{m\times n}$, with grading induced by the $(2m+2n)$-tuple $(\underbrace{0,\dots,0}_{m\text{ times}}, \underbrace{1,\dots,1}_{m\text{ times}}, \underbrace{0,\dots,0}_{n\text{ times}}, \underbrace{1,\dots,1}_{n\text{ times}})$, if $A_1 = M_m(F \oplus cF)$ and $A_2 = M_n(F \oplus dF)$*;

(ii) $\begin{pmatrix} B_1 & B_2 & J_1 & J_2 \\ B_2 & B_1 & J_2 & J_1 \\ 0 & 0 & C_1 & C_2 \\ 0 & 0 & C_2 & C_1 \end{pmatrix}$, *where $B_i \in M_m$, $C_i \in M_n$, $J_i \in M_{m\times n}$, with grading induced by the $(2m+2n)$-tuple $(\underbrace{0,\dots,0}_{m\text{ times}}, \underbrace{1,\dots,1}_{m\text{ times}}, \underbrace{0,\dots,0}_{n\text{ times}}, \underbrace{1,\dots,1}_{n\text{ times}})$, if $A_1 = M_m(F \oplus cF)$ and $A_2 = M_n(F \oplus dF)$*;

(iii) $\begin{pmatrix} B_1 & B_2 & J_1 & J_2 \\ B_2 & B_1 & J_3 & J_4 \\ 0 & 0 & C & D \\ 0 & 0 & E & H \end{pmatrix}$, *where $B_i \in M_m$, $J_1, J_3 \in M_{m\times k}$, $J_2, J_4 \in M_{m\times l}$, $C \in M_k$, $H \in M_l$, $D \in M_{k\times l}$, $E \in M_{l\times k}$, with grading induced by the $(2m+k+l)$-tuple $(\underbrace{0,\dots,0}_{m\text{ times}}, \underbrace{1,\dots,1}_{m\text{ times}}, \underbrace{0,\dots,0}_{k\text{ times}}, \underbrace{1,\dots,1}_{l\text{ times}})$, if $A_1 = M_m(F \oplus cF)$ and $A_2 = M_{k,l}$*;

(iv) $\begin{pmatrix} C & D & J_1 & J_2 \\ E & H & J_3 & J_4 \\ 0 & 0 & B_1 & B_2 \\ 0 & 0 & B_2 & B_1 \end{pmatrix}$, *where $C \in M_k$, $H \in M_l$, $D \in M_{k\times l}$, $E \in M_{l\times k}$, $B_i \in M_m$, $J_1, J_2 \in M_{k\times m}$ and $J_3, J_4 \in M_{l\times m}$, with grading induced by the $(k+l+2m)$-tuple $(\underbrace{0,\dots,0}_{k\text{ times}}, \underbrace{1,\dots,1}_{l\text{ times}}, \underbrace{0,\dots,0}_{m\text{ times}}, \underbrace{1,\dots,1}_{m\text{ times}})$, if $A_1 = M_{k,l}$ and $A_2 = M_m(F \oplus cF)$.*

Proof. (i) Let us consider the elements $W := \begin{pmatrix} a & 0 & x & y \\ 0 & a & y & x \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \end{pmatrix}$ of $A^{(0)}$ and $T := \begin{pmatrix} a & 0 & x & y \\ 0 & -a & -y & -x \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & -b \end{pmatrix}$ of $A^{(1)}$. Then the linear map such that

$$W \mapsto \begin{pmatrix} a & 0 & x+y & 0 \\ 0 & a & 0 & x-y \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \end{pmatrix}, \qquad T \mapsto \begin{pmatrix} 0 & a & 0 & x-y \\ a & 0 & x+y & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & b & 0 \end{pmatrix}$$

is an isomorphism of superalgebras.

(ii) Analogously to (1) by taking $y = 0$.

(iii) Let us pick $R := \begin{pmatrix} a & 0 & x & y \\ 0 & a & x & -y \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \end{pmatrix} \in A^{(0)}$ and $S := \begin{pmatrix} a & 0 & x & y \\ 0 & -a & -x & y \\ 0 & 0 & 0 & b \\ 0 & 0 & c & 0 \end{pmatrix} \in A^{(1)}$. Then the linear map such that

$$R \mapsto \begin{pmatrix} a & 0 & x & 0 \\ 0 & a & 0 & y \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \end{pmatrix}, \qquad S \mapsto \begin{pmatrix} 0 & a & 0 & y \\ a & 0 & x & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & c & 0 \end{pmatrix}$$

is the required isomorphism.

(iv) Let us pick $U := \begin{pmatrix} a & 0 & x & x \\ 0 & b & y & -y \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & c \end{pmatrix} \in A^{(0)}$ and $V := \begin{pmatrix} 0 & a & x & -x \\ b & 0 & y & y \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & -c \end{pmatrix} \in A^{(1)}$. In this case the required isomorphism is given by the linear map such that

$$U \mapsto \begin{pmatrix} a & 0 & x & 0 \\ 0 & b & 0 & y \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & c \end{pmatrix}, \qquad V \mapsto \begin{pmatrix} 0 & a & 0 & x \\ b & 0 & y & 0 \\ 0 & 0 & 0 & c \\ 0 & 0 & c & 0 \end{pmatrix}. \qquad \square$$

We give now sufficient conditions so that the T_2-ideal of superidentities of minimal superalgebras with two graded simple components is factorable. To this end we premise a simple observation.

Remark 5.2. Let us consider the canonical simple superalgebra $A := M_m(F \oplus cF)$ for some $m \geqslant 1$ and a polynomial $f(y_1, \dots, y_l, z_1, \dots, z_n)$ of $F\langle Y \cup Z\rangle$. Then $f(y_1, \dots, y_l, z_1, \dots, z_n) \in T_2(A)$ if, and only if, the polynomial $f(x_1, \dots, x_l, x_{l+1}, \dots, x_{l+n})$ of the free algebra $F\langle X\rangle$ is in $\mathrm{Id}(M_m)$.

Theorem 5.3. *Let $A = A_{ss} + J$ be a minimal superalgebra such that $A_{ss} = A_1 \oplus A_2$, where A_1, A_2 are canonical simple superalgebras. Then $T_2(A) = T_2(A_1) \cdot T_2(A_2)$ if one of the following conditions is satisfied*:

(i) *at least one of A_1 and A_2 is of type* (b);
(ii) *A_1 and A_2 are both simple $\mathbb{Z}_2$-simple and there exists $1 \leqslant i \leqslant 2$ such that $A_i = M_{k_i,k_i}$.*

Proof. If A_1 and A_2 are as in (ii), Proposition 4.1 yields that A is isomorphic to an upper block triangular matrix algebra equipped with a suitable elementary grading. At this point, the result directly follows by combining Theorems 5.4 and 4.5 of [5].

Therefore assume that at least one of A_1 and A_2 is non-simple as an algebra. Proposition 5.1 states that A is isomorphic to a homogeneous subalgebra R of a suitable matrix algebra M_α endowed with an elementary grading. If A is as in the cases (i), (iii) or (iv) of Proposition 5.1, then R is a $\mathbb{Z}_2$-graded block triangular matrix algebra of the form $\begin{pmatrix} R_1 & U \\ 0 & R_2 \end{pmatrix}$. Clearly R_1 is isomorphic to A_1 and R_2 to A_2, whereas U is the bimodule of all rectangular matrix algebras of suitable size (and depending on that of the matrices of A_1 and A_2). Furthermore, if A_i is the non-simple graded simple component of A_{ss}, then the superalgebra R_i is $\mathbb{Z}_2$-regular (see Proposition 5.2 of [5] and its proof). By applying Theorem 4.5 of [5] we conclude that $T_2(A)$ is factorable.

Finally suppose that A is as in Proposition 5.1(ii). Hence A can be represented as in (8) with $J_2 = J_3 = 0$. Now, let us consider a polynomial $f(y_1, \dots, y_r, z_1, \dots, z_s) \in T_2(A)$. If $A_1 = M_m(F \oplus cF)$ and $A_2 = M_n(F \oplus dF)$, the polynomial $f(x_1, \dots, x_r, x_{r+1}, \dots, x_{r+s})$ of $F\langle X\rangle$ is an identity for $UT(m, n)$. According to Theorem 2 of [16], $f(x_1, \dots, x_{r+s}) \in \mathrm{Id}(M_m) \cdot \mathrm{Id}(M_n)$. From Remark 5.2 we get that $f(y_1, \dots, y_r, z_1, \dots, z_s) \in T_2(A_1) \cdot T_2(A_2)$, and the expected conclusion holds. $\square$

An easy consequence of Theorem 5.3 is the following result.

Theorem 5.4. *Let $A = A_{ss} + J$ be a minimal superalgebra such that $A_{ss} = A_1 \oplus A_2$, where A_1, A_2 are canonical simple superalgebras. Then the supervariety generated by A is minimal of superexponent* $\dim_F(A_1 \oplus A_2)$.

Proof. When A_1 and A_2 are both simple (as algebras) the result is a special case of Theorem 4.7.

Thus assume that at least one of A_1 and A_2 is a canonical simple superalgebra of type (b). By using exactly the same arguments of the proof of Theorem 4.7, we reduce to considering a minimal superalgebra B with semisimple part $B_{ss} = A_{ss}$ such that $T_2(A) \subseteq T_2(B)$. At this point, it is sufficient to show that $T_2(A) = T_2(B)$, but this directly follows from Theorem 5.3(i). $\square$

References

[1] E. Aljadeff, A. Giambruno, D. La Mattina, Graded polynomial identities and exponential growth, J. Reine Angew. Math. 650 (2011) 83–100.

[2] E. Aljadeff, D. Haile, Simple G-graded algebras and their polynomial identities, Trans. Amer. Math. Soc., in press.

[3] F. Benanti, A. Giambruno, M. Pipitone, Polynomial identities on superalgebras and exponential growth, J. Algebra 269 (2003) 422–438.

[4] O.M. Di Vincenzo, P. Koshlukov, A. Valenti, Gradings on the algebra of upper triangular matrices and their graded identities, J. Algebra 275 (2004) 550–566.

[5] O.M. Di Vincenzo, R. La Scala, Block-triangular matrix algebras and factorable ideals of graded polynomial identities, J. Algebra 279 (2004) 260–279.

[6] O.M. Di Vincenzo, R. La Scala, Minimal algebras with respect to their $*$-exponent, J. Algebra 317 (2007) 642–657.

[7] O.M. Di Vincenzo, V. Nardozza, On the existence of the graded exponent for finite dimensional $\mathbb{Z}_p$-graded algebras, Canad. Math. Bull. 55 (2012) 271–284.

[8] O.M. Di Vincenzo, E. Spinelli, On the $*$-minimality of algebras with involution, J. Algebra 323 (2010) 121–131.

[9] O.M. Di Vincenzo, E. Spinelli, A characterization of $*$-minimal algebras with involution, Israel J. Math. 186 (2011) 381–400.

[10] A. Giambruno, D. La Mattina, Graded polynomial identities and codimensions: Computing the exponential growth, Adv. Math. 225 (2010) 859–881.

[11] A. Giambruno, S. Mishchenko, M. Zaicev, Polynomial identities on superalgebras and almost polynomial growth, Comm. Algebra 29 (2001) 3787–3800.

[12] A. Giambruno, A. Regev, Wreath products and P.I. algebras, J. Pure Appl. Algebra 35 (1985) 133–149.

[13] A. Giambruno, M. Zaicev, On codimension growth of finitely generated associative algebras, Adv. Math. 140 (1998) 145–155.

[14] A. Giambruno, M. Zaicev, Exponential codimension growth of PI algebras: An exact estimate, Adv. Math. 142 (1999) 221–243.

[15] A. Giambruno, M. Zaicev, A characterization of algebras with polynomial growth of the codimensions, Proc. Amer. Math. Soc. 129 (2001) 59–67.

[16] A. Giambruno, M. Zaicev, Minimal varieties of algebras of exponential growth, Adv. Math. 174 (2003) 310–323.

[17] A. Giambruno, M. Zaicev, Codimension growth and minimal superalgebras, Trans. Amer. Math. Soc. 355 (2003) 293–308.

[18] A. Giambruno, M. Zaicev, Polynomial Identities and Asymptotic Methods, Math. Surveys Monogr., vol. 122, Amer. Math. Soc., Providence, RI, 2005.

[19] A.R. Kemer, T-ideals with power growth of the codimensions are Specht, Sibirsk. Mat. Zh. 19 (1978) 54–69 (in Russian), English translation: Sib. Math. J. 19 (1978) 37–48.

[20] A.R. Kemer, Ideals of Identities of Associative Algebras, Transl. Math. Monogr., vol. 87, Amer. Math. Soc., Providence, RI, 1991.

[21] P. Koshlukov, M. Zaicev, Identities and isomorphisms of graded simple algebras, Linear Algebra Appl. 432 (2010) 3141–3148.

[22] D. La Mattina, Varieties of superalgebras of almost polynomial growth, J. Algebra 336 (2011) 209–226.

[23] A. Regev, Existence of identities in $A \otimes B$, Israel J. Math. 11 (1972) 131–152.

[24] C.T.C. Wall, Graded Brauer groups, J. Reine Angew. Math. 213 (1963) 187–199.