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Rings and modules characterized by opposites of injectivity

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ABSTRACT

In a recent paper, Aydoğdu and López-Permouth have defined a module M to be N -subinjective if every homomorphism $N \rightarrow M$ extends to some $E(N) \rightarrow M$, where $E(N)$ is the injective hull of N . Clearly, every module is subinjective relative to any injective module. Their work raises the following question: What is the structure of a ring over which every module is injective or subinjective relative only to the smallest possible family of modules, namely injectives? We show, using a dual opposite injectivity condition, that such a ring R is isomorphic to the direct product of a semisimple Artinian ring and an indecomposable ring which is (i) a hereditary Artinian serial ring with $J^2 = 0$; or (ii) a QF-ring isomorphic to a matrix ring over a local ring. Each case is viable and, conversely, (i) is sufficient for the said property, and a partial converse is proved for a ring satisfying (ii). Using the above mentioned classification, it is also shown that such rings coincide with the fully saturated rings of Trlifaj except, possibly, when von Neumann regularity is assumed. Furthermore, rings and abelian groups which satisfy these opposite injectivity conditions are characterized.

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1. Introduction and preliminaries

There is an extensive literature on the injectivity of modules and rings, a considerable amount of which involves notions derived from relative injectivity, i.e. that of a module with respect to a fixed module, inspired by Baer's criterion. Introduced in a similar vein, the kind of injectivity (namely, subinjectivity) and the opposite of injectivity induced by it (namely, indigence), as discussed by Aydoğdu and López-Permouth in [3], seem to offer a new perspective on the topic and to be abundant with interesting questions: Let R be an associative ring with identity and M and N be unital right R -modules. M is said to be N -subinjective if every homomorphism $N \rightarrow M$ can be extended to some homomorphism $E(N) \rightarrow M$, where $E(N)$ is the injective hull of N . The class of all modules N such that M is N -subinjective is called the subinjectivity domain of M , and is denoted by $\text{In}^{-1}(M)$. If N is injective, then M is vacuously N -subinjective. So, the smallest possible subinjectivity domain is the class of injective modules. A module with such a subinjectivity domain is defined in [3] to be indigent. The existence of indigent modules for an arbitrary ring is unknown, but an affirmative answer is known for some rings such as \mathbb{Z} and Artinian serial rings (see [3]).

In this paper, we address some questions raised by and studied in [3]. The first question considered here is the following: What is the structure of a ring over which every right module is indigent or injective? In order to approach this problem, we use the following notion, which is a sort of dual to the notion of indigence: We call a module M a test for injectivity by subinjectivity (t.i.b.s.) if the only modules which are M -subinjective are the injective ones. Such modules exist over any ring (Proposition 1). An easy observation shows that the rings in question are precisely those whose modules are injective or t.i.b.s. (Proposition 2). We then prove that such a ring is isomorphic to the direct product of a semisimple Artinian ring and an indecomposable ring T such that (i) T is a hereditary Artinian serial ring with $J^2 = 0$; or, (ii) T is a QF-ring isomorphic to a matrix ring over a local ring (Theorem 3). An example of each case exists. Conversely, the condition (i) is a sufficient one for an indecomposable ring (Proposition 10), and we have a partial converse for the case (ii), which yields a characterization of QF-rings that are isomorphic to a matrix ring over a local ring (Theorem 14).

Next, we mention the connection between rings whose right modules are injective or indigent and fully saturated rings of Trlifaj [14]. Using Theorem 3 (see the preceding paragraph), it is shown that the two types of rings coincide when the underlying ring is not semisimple or a von Neumann regular ring (Theorem 16). It follows as a consequence that nonsemisimple rings of the former type is in fact a subclass of the latter (Corollary 17).

We show that, for a ring R , R_R is a t.i.b.s. if and only if R is a right hereditary and right Noetherian ring (Theorem 19). Hence, a commutative domain is Dedekind if and only if it is a t.i.b.s. as a module (Corollary 20).

Abelian groups that are t.i.b.s. (respectively, indigent) are completely characterized in Theorem 26 (respectively, Theorem 27).

Aydoğdu and López-Permouth also investigate when semisimple modules over a ring are indigent, and they use the indigence of the direct sum, say S , of representatives of simples. We extend their results proving that a commutative domain is Dedekind if and only if S is indigent, and a right perfect ring is right Harada if and only if S is indigent.

As a by-product of our results the answer to another question raised in [3] is obtained: There do exist poor modules (i.e. modules injective relative only to semisimples, see [1] and [7]) which are not indigent, in particular over a PCI-domain which is not a division ring (Remark 9).

For a module M , $\text{soc}(M)$, $Z(M)$ and $\text{rad}(M)$ denote the socle, the singular submodule and the Jacobson radical of M , respectively. For any ring in our discussion, J will stand for the Jacobson radical of that ring. The composition length of a module M will be denoted by $\text{cl}(M)$. For two modules M and N over a ring R , $\text{Ext}(M, N)$ will stand for $\text{Ext}_R^1(M, N)$. A ring R is called a right SI-ring if every singular right R -module is injective. Such rings were completely characterized by Goodearl in [10]. For all other basic or background material, we refer the reader to [11].

2. Rings whose modules are injective or indigent

Definition 1. A module N is said to be a test for injectivity by subinjectivity (t.i.b.s.) if the only N -subinjective modules are injective modules.

Proposition 1. *Every ring has a t.i.b.s.*

Proof. Let R be a ring and $N = \bigoplus I$, where I ranges among (proper) essential right ideals of R , and assume that X is an N -subinjective module. Let A be a right ideal of R , and $f : A \rightarrow X$ be any homomorphism. We may assume, without loss of generality, that A is essential in R_R . Then, the copy of A in N that is a direct summand of N is essential in an injective submodule, say Q , of $E(N)$. So, there is an embedding $\phi : R_R \rightarrow Q$ fixing A . Since X is N -subinjective, $f(\phi^{-1})|_A$ (here, A is the copy in N) extends to some $h : E(N) \rightarrow X$. Thus, $h\phi$ is the desired extension of f to $R \rightarrow X$. \square

If a ring R has an injective module which is also a t.i.b.s., then R is clearly semisimple Artinian. Thus, it is natural to ask the following question:

Question. What is the structure of a ring over which every right module is injective or a t.i.b.s.?

It turns out that such rings coincide with rings whose modules are injective or indigent. The following proposition states this obvious fact without proof. We will use this proposition freely in the sequel.

Proposition 2. *The following conditions are equivalent for a ring R :*

- (i) *Every right R -module is injective or a t.i.b.s.;*
- (ii) *Every right R -module is injective or indigent;*
- (iii) *If $A, B \in \text{Mod-}R$ and A is B -subinjective, then A or B is injective.*

In this case, the class of indigent modules and that of t.i.b.s. modules coincide.

For convenience, we will define the following condition for a ring R :

(P): R satisfies the equivalent conditions of [Proposition 2](#).

The main theorem of this section classifies rings satisfying (P).

Theorem 3. *Let R be a ring satisfying the condition (P). Then $R \cong S \times T$, where S is a semisimple Artinian ring and T is an indecomposable ring satisfying one of the following conditions:*

- (i) *T is a hereditary Artinian serial ring with $J^2 = 0$;*
- (ii) *T is a QF-ring isomorphic to $M_n(\Gamma)$ for some local ring Γ .*

Before proving the theorem, we will first give some lemmas.

Lemma 4. *If R is a ring satisfying (P), then R is a right SI-ring or $R \cong S \times T$, where S is a semisimple Artinian ring and T is a ring such that $Z(T_T)$ is essential in T_T .*

Proof. If R is not right SI, then there is a noninjective singular (right R -)module C , which belongs to the subinjectivity domain of any nonsingular module, implying that nonsingular modules are (semisimple and) injective [[3](#), [Proposition 4.11](#)]. Let D be a complement in R_R of $Z_2(R_R)$. Then, by what is said above, D is semisimple and injective, so that $R_R = D \oplus Z_2(R_R)$. This is clearly a ring direct sum. Put $S = D$ and $T = Z_2(R_R)$. Then S is a semisimple Artinian ring and $Z(T_T)$ is essential in T_T . \square

Lemma 5. *If R is a ring satisfying (P), then either R is a right semiartinian ring or it is a right Noetherian right V -ring.*

Proof. Assume that R is not right semiartinian. Then, there exists a nonzero module A with $\text{soc}(A) = 0$. Then every submodule of A is S -subinjective for any semisimple

module S . Since $\text{soc}(A) = 0$, A must have a noninjective submodule, say B . By the assumption of (P) and the above argument, all semisimple modules are injective, meaning that R is right Noetherian and right V . \square

Lemma 6. *Let R be a right Artinian ring with homogeneous right socle and $Z(R_R)$ essential in R_R . If every simple right R -module is injective or indigent, then R is a QF-ring isomorphic to a matrix ring over a local ring.*

Proof. In this situation, $R = \bigoplus_{i=1}^n e_i R$, where e_i form a complete set of orthogonal primitive idempotents, and no $e_i R$ is simple. Now we will see that each $e_i R$ is either injective or small in its injective hull: Fix $i \in \{1, \dots, n\}$ and assume that $e_i R$ is not injective. Pick any $k \in \{1, \dots, n\}$ and let J stand for the Jacobson radical of R . If $\frac{e_k R}{e_k J}$ does not embed in $\text{soc}(e_i R)$, then $\text{Hom}(e_k R, \text{soc}(e_i R)) = 0$, whence $\text{soc}(e_i R)$ is $e_k R$ -subinjective. This implies, by our assumption, that $e_k R$ is injective. In particular, $\frac{e_i R}{e_i J}$ must embed in $\text{soc}(e_i R)$. Now let E be the injective hull of $e_i R$ and assume that $e_i R + K = E$ for some $K \subseteq E$. Put $V = \frac{e_i R}{e_i J}$. We will show that, in this situation, V is $e_i R$ -subinjective: Let $f : e_i R \rightarrow V$ be any nonzero homomorphism. Then $\text{Ker}(f) = e_i J$. Note that $K \cap e_i R \subseteq e_i J$, so that the zero map $K \rightarrow V$ and f agree on $K \cap e_i R$. Thus, f extends to the map $g : E \rightarrow V$ via $g(e_i r + k) = f(e_i r)$ ($r \in R, k \in K$), showing that V is $e_i R$ -subinjective. But since V is not injective, $e_i R$ must be injective by assumption, contradicting our hypothesis. Therefore, we cannot have such a module as K above, and hence $e_i R$ is small in its injective hull. Also, by these arguments, if $e_t R$ is not injective, then $\frac{e_t R}{e_t J} \cong \frac{e_i R}{e_i J}$, implying that $e_t R \cong e_i R$.

Now let k be such that $e_k R$ is injective, pick any proper submodule X of $e_k R$, and let $e_i R$ be as chosen in the preceding paragraph. Now, $\frac{e_k R}{e_k J} \cong \frac{e_i R}{e_i J} (= V)$ would imply $e_i R \cong e_k R$, a contradiction. So, we clearly have $\text{Hom}(\frac{e_k R}{X}, V) = 0$, so that V is $\frac{e_k R}{X}$ -subinjective. Recall that V embeds in $\text{soc}(e_i R)$, so it is not injective. This implies, by our hypothesis, that $\frac{e_k R}{X}$ is injective, hence uniform. In particular, $e_k R$ is uniserial because all of its nonzero factors are uniform. Now, let $Y = \text{soc}(e_k R)$ and assume that $\text{soc}(\frac{e_k R}{Y}) \cong V$. As $\text{soc}(R_R)$ is homogeneous by assumption, and V embeds in $\text{soc}(e_i R)$, we have $E(V) \cong e_k R$. Then, $\frac{e_k R}{Y}$, being uniform, would embed in $e_k R$, which would yield a contradiction because $\frac{e_k R}{Y}$ is nonzero injective. So, we must have $\text{soc}(\frac{e_k R}{Y}) \not\cong V$. Then V is $\text{soc}(\frac{e_k R}{Y})$ -subinjective as well as not injective, implying that $\text{soc}(\frac{e_k R}{Y})$ is injective, whence equal to $\frac{e_k R}{Y}$. Thus, $\text{cl}(e_k R) = 2$. This argument shows that every injective $e_k R$ has composition length = 2 (under the assumption of the existence of a noninjective principle indecomposable, namely $e_i R$).

Next, we will see that the $E (= E(e_i R))$ defined above is isomorphic to one of the $e_t R$: First, $E = E_1 \oplus \dots \oplus E_m$ for some indecomposable injective modules E_k (they are finitely many since $e_i R$ is essential in E). Since R is right Artinian, for each $k = 1, \dots, m$, $\text{rad}(E_k)$ is a small submodule of E_k . E_k is a sum of homomorphic images of the modules $e_t R$. However, since any map $\phi : e_t R \rightarrow E_k$ extends to some $\phi' : E(e_t R) \rightarrow E_k$, we have $\phi(e_t R) = \phi'(e_t R) \subseteq \phi'(\text{rad}(E(e_t R))) \subseteq \text{rad}(E_k)$ whenever $e_t R$ is small in its injective

hull. Since $\text{rad}(E_k)$ is small in E_k , this means that E_k is the sum of the images of some injective $e_t R$. This implies, in particular, that there exists at least one injective $e_t R$. Recall from the previous paragraph that, whenever $e_t R$ is injective, then all factors of it are injective. Thus, E_k is isomorphic to a factor of some injective $e_t R$. Now let, for each l , $\pi_l : E_1 \oplus \dots \oplus E_m \rightarrow E_l$ be the obvious projection. Since $e_i R$ is not singular, there exists some l such that $\pi_l(e_i R)$ is not singular; this is because $e_i R \subseteq \bigoplus_{t=1}^m \pi_t(e_i R)$. In this case E_l must be isomorphic to some injective $e_k R$. Since $\text{cl}(E_l) = 2$ by the preceding paragraph and $\text{soc}(E_l)$ is singular, we must have $\pi_l(e_i R) = E_l \cong e_k R$. But then, by projectivity, we obtain $e_i R \cong e_k R$, contradicting our choice of $e_i R$ at the beginning. This shows that we cannot have a noninjective $e_i R$. Thus, R is clearly a QF-ring. And since R has homogeneous socle, all $e_t R$ are pairwise isomorphic, yielding that $R \cong M_n(\Gamma)$, where $\Gamma \cong \text{End}_R(e_t R)$ is local. Clearly R has a unique simple right module. \square

Remark 7. For the purposes of Theorem 3 only, one does not need the full strength of Lemma 6 above: By an easy argument, [3, Proposition 2.9] implies that a ring R with (P) is either right hereditary or right self-injective. However, we need Lemma 6 to also prove Theorem 14.

A right PCI-ring R is one whose cyclic right modules, except possibly R itself, are all injective (see [9]). Such a ring is right Noetherian by a result of Damiano [5].

Lemma 8. *A right PCI-domain satisfying (P) is a division ring.*

Proof. Let R be a right PCI-domain with property (P). So, it is right Noetherian. Assume, contrarily, that R is not a division ring. Then R is not right Artinian. Hence, $E(R_R)$ is not Noetherian. Let A be a nonzero proper right ideal of R . Since R is a right PCI domain, $\frac{R}{A}$ is injective, hence it splits in $\frac{E(R_R)}{A}$. So, there exists a submodule G of $E(R_R)$ such that $R + G = E(R_R)$ and $G \cap R = A$. In particular, $G \neq E(R_R)$. Furthermore, G is not finitely generated, because otherwise so would be $E(R_R)$, contradicting the above observation. This implies that $\text{Hom}(G, R_R) = 0$ (since R is a right PCI-domain), so that R_R is G -subinjective, which is a contradiction since neither G nor R_R is injective. The conclusion now follows. \square

Remark 9. In [3, Section 5], Aydoğdu and López-Permouth raise the question whether there are any poor modules which are not indigent. Lemma 8 and [7, Proposition 5] imply an affirmative answer to this question: There do exist such modules, in particular over right PCI-domains.

Now we can prove our theorem.

Proof of Theorem 3. Assume that R satisfies the condition (P). Since any direct sum of injective modules is sub-injective relative to any cyclic module, by our assumption,

either all cyclics are injective or any direct sum of injectives is injective. The former implies, by the well-known Osofsky-Theorem (see [13]), that R is semisimple Artinian. So, in any case, R is right Noetherian.

Take any two semisimple modules A and B which are orthogonal (i.e. they do not have nonzero isomorphic submodules). Since $\text{Hom}(A, B) = 0$, either A or B must be injective. So, there is at most one homogeneous component of $\text{soc}(R_R)$ which is not injective. Since R is right Noetherian, the sum of the injective homogeneous components, call S , is injective. Thus, we have a ring direct sum $R = S \oplus T$, where S is semisimple Artinian and T has homogeneous right socle. Again by Noetherianity, we have a ring direct sum $T = R_1 \oplus \dots \oplus R_k$, where R_k are indecomposable rings satisfying (P). Also, for $i \neq j$, all right ideals of R_i are vacuously subinjective relative to all right ideals of R_j as T -modules. So, either all right ideals of R_i are injective or those of R_j are injective, whence either R_i or R_j is semisimple Artinian. This means that at most one of R_i can be nonsemisimple (in fact, it implies that k is at most 2, as T has homogeneous right socle). Adjoining now all the semisimple parts of the above decomposition to S , we can assume in the rest of the proof, without loss of generality, that T is ring-indecomposable and nonsemisimple.

Case 1: Assume T is not right SI. Then, by Lemma 4, there is a ring direct sum $T = T_1 \oplus T_2$, where T_1 is semisimple Artinian and $Z(T_2 T_2)$ is an essential right ideal of T_2 . However, since T is now an indecomposable and nonsemisimple ring by the preceding paragraph, $T = T_2$, so that $Z(T_T)$ is essential in T_T .

Assume that T is not right semiartinian. Then, T is a right Noetherian and right V-ring by Lemma 5. However, such a ring decomposes into simple rings by Faith–Ornstein Theorem (see [8, 3.20]). Since T is indecomposable, T must then be simple, contradicting that $Z(T_T)$ is essential in T_T . So this case is not possible.

So, now, T is right semiartinian. Then, T is right Artinian because of Noetherianity. Thus, T is a ring satisfying the hypotheses of Lemma 6. Thus, Lemma 6 yields the case (ii).

Case 2: Now assume that T is a right SI-ring. As in the preceding paragraph, by Lemma 5 and right Noetherianity of T , T is either right Artinian or right Noetherian and right V. By [10, Theorem 3.11], there is a ring direct sum $T = T' \oplus T_1 \oplus \dots \oplus T_l$, where $\frac{T'}{\text{soc}(T')}$ is semisimple and T_i are Morita equivalent to right SI-domains (equivalently, right PCI-domains). Since T is now ring-indecomposable and nonsemisimple, all except one of the rings in that decomposition must be zero. Assume T is right Noetherian and right V. Then, T' is semisimple Artinian, hence zero. This implies that T is Morita equivalent to a right PCI-domain. But since (P) is Morita invariant (follows routinely from Proposition 2 and [3, Lemma 2.2]), T must then be semisimple Artinian by Lemma 8, contradicting our assumption above. So, this situation is not possible.

Then now, T is right Artinian, so that $T_1 \oplus \dots \oplus T_l$ is semisimple, hence zero. Then, $\frac{T}{\text{soc}(T)}$ is semisimple. In this case $T_T = \bigoplus_{i=1}^n e_i T$, where $\{e_i : i = 1, \dots, n\}$ is a complete set of primitive orthogonal idempotents. Let $i \in \{1, \dots, n\}$. Assume that $e_i T$ is not simple. Then $\frac{e_i T}{\text{soc}(e_i T)}$ is local, semisimple (and injective, because it is now singular), whence it is simple. Thus, $e_i J(T) = \text{soc}(e_i T)$. Also, since, by nonsingularity of T_T ,

$\text{Hom}(e_i T, \text{soc}(e_i T)) = 0$, $\text{soc}(e_i T)$ is vacuously $e_i T$ -subinjective but not injective, whence $e_i T$ must be injective by assumption. So, if $e_i T$ is not simple, then by the above arguments, $e_i T$ is injective with $\text{soc}(e_i T) = e_i J(T)$, implying that $\text{cl}(e_i T) = 2$. So T is a direct sum of right ideals which are simple or injective with composition length 2. Then, by [6, 13.5] and since right SI-rings are right hereditary we obtain (i). \square

We will now consider the converse of Theorem 3:

Proposition 10. *A hereditary Artinian serial ring R which is indecomposable (or has homogeneous right socle) with $J^2 = 0$ satisfies (P).*

Proof. In any case, R has homogeneous right socle, and it is right SI by [10, Theorem 3.11]. Assume that A and B are right R -modules such that A is B -subinjective. Also let B be noninjective. Since R is Artinian, $B = B' \oplus D$ for some injective module B' and a module $0 \neq D$ which does not contain any nonzero injective submodules. By the SI condition and our assumption, $Z(D) = 0$. Now, $E(D) = \bigoplus_{j \in I} E_j$, where E_j are indecomposable injective modules.

Let $R_R = \bigoplus_{j=1}^n e_j R$, where e_j are a complete set of orthogonal primitive idempotents. By our assumptions, any nonsimple $e_j R$ is injective. Let $e_i R$ be injective. Then, since R has homogeneous socle, $e_i R$ is the injective hull of the only nonsingular simple right R -module.

Now take any $t \in I$ and let $\pi_t : \bigoplus_{j \in I} E_j \rightarrow E_t$ be the obvious projection. By the preceding paragraph, $E_t \cong e_t R$, so that it is projective. As D does not contain a nonzero injective submodule, $\pi_t(D) \neq E_t$, whence $\pi_t(D) = \text{soc}(E_t)$. By projectivity of the latter (as this ring is hereditary), it turns out that D has a direct summand isomorphic to $S = \text{soc}(e_i R)$. Since R is right SI, $A = A' \oplus C$, where A' is singular injective and C is nonsingular. Assume that $C \neq 0$. Now, by nonsingularity of C , $\text{soc}(C) = \bigoplus_{\alpha \in \Omega} S_\alpha$, where $S_\alpha \cong S$ for all $\alpha \in \Omega$. C is S -subinjective. Thus, for each α , the isomorphism $S \rightarrow S_\alpha$ extends to some monomorphism $f_\alpha : E(S) \rightarrow C$. Clearly, $C = \bigoplus_{\alpha \in \Omega} f_\alpha(E(S))$, implying that C is injective. Therefore A is injective, proving that R has the property (P). \square

Example 11. $R = \begin{pmatrix} D & D \\ 0 & D \end{pmatrix}$ (D a division ring) satisfies (P) by Proposition 10.

Example 12. Any Artinian chain ring satisfies (P) and is as in Theorem 3, by [3, Theorem 5.1].

We do not know precisely which QF-rings (isomorphic to a matrix ring over a local ring) satisfy the condition (P). As we will see in the next section, this question is equivalent to a long standing open question raised in the work of Trlifaj, namely that of determining those fully saturated rings that are QF. The next proposition is related to this question and is a partial converse for the case (ii) of Theorem 3; another one will

be given by [Theorem 14](#). Before we proceed, let us note that a QF ring R is isomorphic to a matrix ring over a local ring if and only if it has homogeneous socle.

Proposition 13. *Let R be a serial QF ring with homogeneous socle. Then R satisfies (P).*

Proof. Here, $R \cong (eR)^k$, for some primitive idempotent e of R . Let A and B be two right R -modules such that A is B -subinjective. Since R is Artinian serial, every module is a direct sum of cyclic uniserial modules. Assume, contrarily, that neither A nor B is injective. Then each of these modules has a noninjective cyclic uniserial direct summand. So, we can assume without loss of generality that A and B are both cyclic uniserial. It follows from our assumptions that for each $n \in \mathbb{N}$, there exists, up to isomorphism, at most one cyclic uniserial module of length n , and it embeds in eR .

Now, if $\text{cl}(B) \leq \text{cl}(A)$, then B embeds in A via some monomorphism, say, f . By assumption, f extends to some monomorphism $g : E(B) \rightarrow A$, contradicting our assumption that A is not injective. Else, if $\text{cl}(B) > \text{cl}(A)$, then, by uniqueness (up to isomorphism) of a module of the same length as A , there is an epimorphism $t : B \rightarrow A$. However, t cannot be extended to an element of $\text{Hom}(E(B), A)$, since $B \subseteq \text{rad}(E(B))$. Thus, either case leads to a contradiction, yielding the conclusion. \square

[Lemma 6](#) inspires the following characterization of QF-rings which are isomorphic to a matrix ring over a local ring. Notice that it also provides a partial answer to the sufficiency of the condition [Theorem 3 \(ii\)](#) for the condition (P).

Theorem 14. *The following conditions are equivalent for a ring R :*

- (i) R is a nonsemisimple QF-ring that is isomorphic to a matrix ring over a local ring;
- (ii) R is right Artinian with homogeneous right socle containing $Z(R_R)$ essentially, and every simple right module is indigent or injective;
- (iii) The left-hand version of (ii).

In this case, there is a unique simple right module and it is indigent.

Proof. $(ii) \Rightarrow (i)$ is by [Lemma 6](#), and $(i) \Leftrightarrow (iii)$ will follow by symmetry and $(i) \Leftrightarrow (ii)$. So, we only prove $(i) \Rightarrow (ii)$, and it suffices to see the last part only:

In this case, $R_R \cong (eR)^n$ for some primitive idempotent e of R , and there is a unique simple right R -module, say V . We claim that V is indigent. Let V be A -subinjective for some right module A . Since R is QF, we can split a maximal injective submodule off A and assume, without loss of generality, that A is a nonzero module that contains no nonzero injective submodules. Since R is Artinian, $\text{rad}(A)$ is small in A , so that A has a simple image, which, by assumption, must be isomorphic to V . So, let $f : A \rightarrow V$ be an epimorphism. Pick a cyclic submodule C of A such that $f(C) = V$. Since V is A -subinjective, f extends to some $g : E(A) \rightarrow V$. Clearly, $C + \text{Ker}(g) = E(A)$. Let E' be

an injective submodule of $E(A)$ essentially containing C . Then $C + (\text{Ker}(g) \cap E') = E'$. Now, since R is Artinian and E' essentially contains a cyclic module (namely C), and since each indecomposable injective is the hull of the same simple, $E' = E_1 \oplus \dots \oplus E_t$ for some $E_k \cong eR$. Let $\pi_i : E_1 \oplus \dots \oplus E_t \rightarrow E_i$ be the obvious projection, for each $i \in I$. Since A (hence C) does not contain a nonzero injective submodule and E_i are projective, $\pi_i(C) \subseteq \text{rad}(E_i)$. Thus, $C \subseteq \bigoplus_{i=1}^t \pi_i(C) \subseteq \text{rad}(E')$. However, this implies that $\text{Ker}(g)$ contains E' , a contradiction. The conclusion now follows. \square

3. Fully saturated rings and the condition (P)

In [14], Trlifaj defined what he called a Whitehead test module for injectivity (i-test module) as follows: A module M is an i-test module if N is injective whenever $\text{Ext}(M, N) = 0$. He also considered nonsemisimple rings whose non-projective modules are i-test, named such a ring fully saturated, and showed that they fall into three classes. First, let us point out that the notions of i-test and t.i.b.s. are related: It can easily be seen that if M is a t.i.b.s. then $\frac{E(M)}{M}$ is an i-test module. Trlifaj showed in [14, Theorem 6.15], using homological and set theoretic techniques, that a fully saturated ring is the direct sum of a semisimple ring and an indecomposable ring which is isomorphic to a full matrix ring over a local QF-ring, or is Morita equivalent to a 2×2 upper triangular matrix ring over a division ring, or is a simple von Neumann regular ring with all right ideals countably generated and with a unique simple right module.

In this section we will show, using Theorem 3, that a ring that is not von Neumann regular is fully saturated if and only if it satisfies (P). As a consequence, it follows that a nonsemisimple ring with (P) is fully saturated.

Lemma 15. *Let R be a nonsemisimple QF ring. Then R is fully saturated if and only if R satisfies (P). In this case, the classes of non-projective modules, noninjective modules, t.i.b.s. modules and i-test modules coincide.*

Proof. Assume R is fully saturated. Let A and B be two modules such that A is B -subinjective. Assume that B is not injective. We claim that A is injective: Consider the exact sequence

$$0 \rightarrow B \rightarrow E(B) \rightarrow \frac{E(B)}{B} \rightarrow 0.$$

Then, we have the following long exact sequence

$$\begin{aligned} 0 \rightarrow \text{Hom}\left(\frac{E(B)}{B}, A\right) &\rightarrow \text{Hom}(E(B), A) \rightarrow \text{Hom}(B, A) \\ &\rightarrow \text{Ext}\left(\frac{E(B)}{B}, A\right) \rightarrow \text{Ext}(E(B), A) \rightarrow \dots \end{aligned}$$

Now, by the QF assumption, $E(B)$ is projective, so that $\text{Ext}(E(B), A) = 0$. Furthermore, since A is B -subinjective, $\text{Hom}(E(B), A) \rightarrow \text{Hom}(B, A)$ is an epimorphism, implying that $\text{Ext}(\frac{E(B)}{B}, A) = 0$. Since B is not injective, $\frac{E(B)}{B}$ is not projective, whence, by assumption of fully saturated, A must be injective. This shows that R satisfies the condition (P).

Conversely, assume that R satisfies (P), and let B be a non-projective module. Let $\text{Ext}(B, N) = 0$ for some module N . Consider the exact sequence

$$0 \rightarrow A \rightarrow P \rightarrow B \rightarrow 0,$$

where P is projective (hence injective). In a similar way to the above argument, one can obtain the exactness of

$$0 \rightarrow \text{Hom}(B, N) \rightarrow \text{Hom}(P, N) \rightarrow \text{Hom}(A, N) \rightarrow 0,$$

implying, by [3, Lemma 2.2] and since P is injective, that N is A -subinjective. But since P is injective–projective, and B is noninjective, A cannot be injective. Thus, by assumption, N is injective. Therefore R is fully saturated. \square

Theorem 16. *Let R be a ring which is not von Neumann regular. R is fully saturated if and only if R satisfies (P).*

Proof. Let R be a fully saturated ring which is not regular. Note that the direct product of a semisimple Artinian ring and a ring satisfying (P) satisfies (P). So, by [14, Theorem 6.15], we can assume, without loss of generality, that R is either QF or Morita equivalent to a 2×2 upper triangular matrix ring over a division ring. In the former case, R satisfies (P) by Lemma 15. In the latter one, a 2×2 upper triangular matrix ring over a division ring has (P) by Example 11. Since (P) is a Morita invariant property, R satisfies (P) as well.

Conversely, let R be a nonregular ring satisfying (P). By Theorem 3, $R \cong S \times T$, where S is semisimple Artinian and T is an indecomposable ring which is either QF or hereditary Artinian serial with $J^2 = 0$. In the former case, R is QF as well, so that it is fully saturated by Lemma 15. So, assume the latter case and let M be a nonprojective right T -module, and N be a right T -module such that $\text{Ext}(M, N) = 0$. Note that $T_T = \bigoplus_{i=1}^n e_i T$, where $\{e_i : i = 1, \dots, n\}$ is a complete set of primitive orthogonal idempotents and each $e_i T$ is either injective with composition length 2, or simple; and furthermore, T_T has homogeneous socle. We will assume T is not semisimple Artinian, so that there exists some $e = e_i$ such that eT has composition length 2. In this case, both M and N are direct sums of cyclic uniserial modules, and each cyclic uniserial module is isomorphic to one of eT , $\text{soc}(eT)$, and $\frac{eT}{\text{soc}(eT)}$, where all except $\text{soc}(eT)$ are injective modules. Since M is not projective, it has a cyclic uniserial direct summand, say A , which is isomorphic to $\frac{eT}{\text{soc}(eT)}$. We want to show that N is injective. Assume, contrarily, that it is not. Then N has a cyclic uniserial direct summand, say B , which is

isomorphic to $\text{soc}(eT)$. Then, we have $\text{Ext}(A, B) = 0$, whence $\text{Ext}(\frac{eT}{\text{soc}(eT)}, \text{soc}(eT)) = 0$. However, the exact sequence

$$0 \rightarrow \text{soc}(eT) \rightarrow eT \rightarrow \frac{eT}{\text{soc}(eT)} \rightarrow 0$$

is not split, yielding a contradiction. Thus, N must be injective. Therefore T is fully saturated, and hence so is R . This completes the proof. \square

Corollary 17. *A nonsemisimple ring R satisfying (P) is fully saturated.*

Proof. If R is nonsemisimple and regular with property (P), then R is right Noetherian by the first paragraph of the proof of Theorem 3, whence it is semisimple Artinian by regularity. So, in the von Neumann regularity case, R is fully saturated. The conclusion follows by Theorem 16. \square

Remark 18. Notice that in order to obtain the result (P) \Rightarrow fully saturated for a non-semisimple ring, we have used Theorem 3. We do not know a direct (homological) proof of this fact that is independent from Theorem 3 and its techniques. Such a proof would help make a route to reach the conclusion of Theorem 3 through Trlifaj's result [14, Theorem 6.15]. However, that route would be an arduous as well as a more indirect one, considering the homological and set theoretical machinery involved in [14], whereas our methods are based on basic ring and module theory.

To the best of our knowledge, the existence of a nonsemisimple fully saturated simple von Neumann regular ring is not known. If such an example does not exist, then, by Theorem 16, fully saturated rings and rings with property (P) will coincide.

Our construction of a t.i.b.s. (Proposition 1) and Trlifaj's construction of an i-test [14, Proposition 1.2] are the same module (in fact, one could obtain the existence of a t.i.b.s. from the existence of an i-test module), showing that every ring has a module which is both a t.i.b.s. and an i-test. However, in general, they do not coincide, as there may exist projective modules that are t.i.b.s. over nonsemisimple Artinian rings (see Theorem 19).

Finally, note that a nonsemisimple serial QF ring with homogeneous socle is fully saturated, by Proposition 13 and Lemma 15. Such a ring is necessarily indecomposable.

4. When the ring is a t.i.b.s.

Theorem 19. *The following are equivalent for a ring R :*

- (i) R_R is a t.i.b.s.;
- (ii) R is right hereditary and right Noetherian.

Proof. Any direct sum of injectives is R_R -subinjective. So, the condition (i) implies that any such sum is injective, hence R is right Noetherian. Furthermore, R is right hereditary by [3, Proposition 2.9].

Conversely, let R be right hereditary and right Noetherian. Assume that A is a module which is R_R -subinjective. Then, for any $a \in A$, the map $R_R \rightarrow A$ determined by left multiplication by a extends to some $h_a : E(R_R) \rightarrow A$. By hereditary assumption, $h_a(E(R_R))$ is an injective module containing aR . Thus, $h_a(E(R_R))$ contains an essential (injective) extension, call E_a , of aR . By Zorn's lemma, one can choose a maximal independent family Λ of cyclic submodules of A . Then, $\bigoplus_{a \in \Lambda} aR$ is essential in A . By Noetherianity, $\bigoplus_{a \in \Lambda} E_a = A$, proving that A is injective, and hence R_R is a t.i.b.s. \square

Corollary 20. *A commutative domain R is Dedekind if and only if it is a t.i.b.s.*

Theorem 19 inspires the question whether every nonzero projective module over a right hereditary right Noetherian ring is a t.i.b.s. However, this is not correct as the following example shows:

Example 21. Let

$$R = \begin{pmatrix} F & F & F \\ 0 & F & F \\ 0 & 0 & F \end{pmatrix},$$

where F is a field. Then, for

$$I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & F & F \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & F \end{pmatrix},$$

S is I -subinjective as $\text{Hom}(I, S) = 0$, whereas S is not injective, showing that I is not a t.i.b.s.

Since, over a right hereditary ring, every projective right module is isomorphic to a direct sum of right ideals, we have the following result, which follows from **Theorem 19**.

Corollary 22. *The following are equivalent for a ring R :*

- (i) *Every nonzero projective right R -module is t.i.b.s;*
- (ii) *Every nonzero right ideal of R is a t.i.b.s.*

Proposition 23. *The following are equivalent for a ring R :*

- (i) *R is semisimple Artinian;*
- (ii) *R is a ring as described in **Corollary 22** and $\text{soc}(R_R) \neq 0$.*

Proof. (i) \Rightarrow (ii) is immediate.

(ii) \Rightarrow (i) In this case, R is right hereditary and right Noetherian by Theorem 19. Let S be a simple right ideal of R_R . If S is injective, then, by assumption, R is semisimple Artinian, proving the claim. Else, assume that S is not injective. Since S is projective, $R_R = I_1 \oplus S_1$ for some right ideals I_1 and $S_1 \cong S$. If I_1 is injective, then we are done. So, assume that I_1 is not injective. Then, $I_1 \neq 0$, whence it is a t.i.b.s. Again by assumption, S is not I_1 -subinjective, so that $\text{Hom}(I_1, S) \neq 0$, and thus $I_1 = I_2 \oplus S_2$, where $S_2 \cong S$. Since R is right Noetherian, this process will stop when I_n is injective and thus R is semisimple Artinian. \square

The condition $\text{soc}(R_R) \neq 0$ is not redundant as the next example shows:

Example 24. Every nonzero ideal I of \mathbb{Z} is a t.i.b.s. by Theorem 19 and the fact that $I \cong \mathbb{Z}$.

In the same vein as the above results, it is natural to ask what happens if all nonzero cyclic modules are t.i.b.s. We close this section with the following result.

Proposition 25. *Every nonzero cyclic right R -module is a t.i.b.s. if and only if R is semisimple Artinian.*

Proof. It is enough to verify one direction only. Assume that every nonzero cyclic right R -module is a t.i.b.s. By Theorem 19, R is then right hereditary and right Noetherian. If a cyclic right module is injective, the conclusion immediately follows. So, we assume, without loss of generality, that no cyclic is injective. Then, there is a unique simple right R -module, say S , upto isomorphism. Then, $\text{Hom}(S, R_R) \neq 0$, so that R contains a copy of S . Since R is right hereditary, this implies that S is projective. However, since by our assumption R is not semisimple Artinian, R contains an essential right ideal I , so that $\frac{R}{I}$ is a simple singular right R -module. This contradicts the uniqueness of S upto isomorphism. Thus, the assumption that no cyclic is injective is false, whence R is semisimple Artinian. \square

5. When abelian groups are indigent or t.i.b.s.

Theorem 26. *An abelian group G is a t.i.b.s. if and only if G has a direct summand isomorphic to \mathbb{Z} .*

Proof. Suppose G is a t.i.b.s. Then $\text{Hom}(G, \mathbb{Z}) \neq 0$. Let $f : G \rightarrow \mathbb{Z}$ be a nonzero homomorphism. Then $\frac{G}{\text{Ker}(f)} \cong n\mathbb{Z}$ is projective. So that $G = \text{Ker}(f) \oplus G'$ with $G' \cong \mathbb{Z}$. Conversely, if $G = A \oplus A'$ with $A' \cong \mathbb{Z}$, then G is a t.i.b.s. since \mathbb{Z} is a t.i.b.s. by Theorem 19. \square

For an abelian group G , let $T(G)$ and $T_p(G)$ denote the torsion and the p -torsion parts of G , respectively. Then $T(G) = \bigoplus_p T_p(G)$, where p ranges over the prime integers. Every abelian group G can be written in the form $G = D \oplus B$, where D is divisible (= injective), and B contains no nonzero divisible subgroups (it is easy to see that D is unique, and B is unique upto isomorphism). Note that a group G is divisible if and only if $pG = G$ for each prime p . If $D = 0$, then G is called reduced.

It is shown in [3] that the group $\bigoplus_p \frac{\mathbb{Z}}{p\mathbb{Z}}$ (p ranges over all primes) is indigent. The next result characterizes indigent abelian groups.

Theorem 27. *The following are equivalent for an abelian group G .*

- (i) G is indigent.
- (ii) $T_p(G) \neq pT_p(G)$ for each prime p .
- (iii) The reduced part of $T(G)$ contains a submodule isomorphic to $\bigoplus_p \frac{\mathbb{Z}}{p\mathbb{Z}}$, where p ranges over all primes.

Proof. (ii) \Rightarrow (iii) is clear.

(i) \Rightarrow (ii) Suppose $pT_p(G) = T_p(G)$ for some prime p . On the other hand, for a prime $q \neq p$, we always have $qT_p(G) = T_p(G)$. Hence $T_p(G)$ is divisible, and so injective. Now it is straightforward to check that G is $\frac{\mathbb{Z}}{p\mathbb{Z}}$ -subinjective, obtaining a contradiction.

(iii) \Rightarrow (i) Suppose G is N -subinjective for some abelian group N . We will show that N is injective, equivalently, that $qN = N$ for every prime q . Assume, contrarily, that $pN \neq N$ for some prime p . Since $\frac{N}{pN}$ is nonzero semisimple, N has a factor isomorphic to $\frac{\mathbb{Z}}{p\mathbb{Z}}$.

Now $G = D \oplus B$, where D is divisible and B is reduced. Then $T(G) = T(D) \oplus T(B)$, where $T(D)$ is clearly divisible and $T(B)$ is reduced. So, by assumption, $T(B)$ contains a copy of $\frac{\mathbb{Z}}{p\mathbb{Z}}$. Then, there is a nonzero map $f : N \rightarrow T(B)$, which, by assumption of N -subinjectivity, extends to some $g : E(N) \rightarrow G$. Thus, $\text{Im}(g)$ is divisible. Let $\pi : D \oplus B \rightarrow B$ be the obvious projection. If $\text{Im}(g)$ were not contained in D , $\pi(\text{Im}(g))$ would be a nonzero divisible module contained in B , a contradiction. But then, $\text{Im}(f) \subseteq \text{Im}(g) \cap B = 0$, again a contradiction. Now the conclusion follows. \square

Corollary 28. *An abelian group G is indigent if and only if its torsion part is indigent.*

6. Rings with semisimple indigent modules

In [3], Aydoğdu and López-Permouth investigate when a ring has a semisimple indigent module. They show that this is the case when R is the ring of integers [3, Corollary 4.5], or when R is Artinian serial with $J(R)^2 = 0$. In either case, they prove that the direct sum S of a complete set of representatives of simple right modules is indigent. Here, we extend those results:

Proposition 29. *Let R be a commutative domain and S be the direct sum of a complete set of representatives of simple right R -modules. R is Dedekind if and only if S is indigent.*

Proof. If R is a Dedekind domain, a module M is injective if and only if $\text{rad}(M) = M$ by [2, Lemma 4.4]. Then S is indigent by the proof of [3, Proposition 4.4].

Conversely, assume that S is indigent and take any divisible module D . Also assume, without loss of generality that R is not a division ring. Let A be any maximal submodule of D . Then, by commutativity of D , $\frac{D}{A}$ is annihilated by some maximal ideal I of R . However, since I is nonzero by assumption, $DI = D$ by divisibility of D , and hence $(\frac{D}{A})I = \frac{D}{A}$, contradicting our choice of A . Thus $\text{rad}(D) = D$. Then, S is clearly D -subinjective, so that D is injective. Thus, divisible modules are injective, proving that D is Dedekind. \square

A ring is called a right Harada ring if it is a perfect ring over which every nonsmall right module contains a nonzero injective submodule (see [4]). This is equivalent to the condition that any injective right module is supplemented and supplement submodules are injective (see [4, Theorem 3.1.12] and [12, Proposition 4.8]). An Artinian serial ring with $J^2 = 0$ is a two-sided Harada ring.

Lemma 30. *Let R be right perfect and A be a right R -module. If every simple right R -module is A -subinjective, then $\text{rad}(A) = A \cap \text{rad}(E(A))$.*

Proof. Let $x \in A \cap \text{rad}(E(A))$ and assume, contrarily, that $x \notin \text{rad}(A)$. Then there is a maximal submodule B of A with $x \notin B$. Since $\frac{A}{B}$ is A -subinjective by assumption, the obvious projection $f : A \rightarrow \frac{A}{B}$ with $\text{Ker}(f) = B$ extends to some $g : E(A) \rightarrow \frac{A}{B}$. Thus, $A + \text{Ker}(g) = E(A)$. Since $\text{Ker}(g) \cap A = B$ and $xR + B = A$, we have $xR + \text{Ker}(g) = E(A)$, contradicting the choice of x . This yields the conclusion. \square

Lemma 31. *The following are equivalent for a module A over a right perfect ring R :*

- (i) *Every simple right R -module is A -subinjective;*
- (ii) *A is a supplement in $E(A)$.*

Proof. Assume (i). By perfectness assumption, $\frac{E(A)}{\text{rad}(E(A))}$ is semisimple, implying by Lemma 30 that $A + B = E(A)$ for some submodule B of $E(A)$ with $A \cap B \subseteq \text{rad}(E(A)) \cap A = \text{rad}(A)$. Since R is right perfect, $\text{rad}(A)$ is small in A , showing that A is a supplement in $E(A)$.

Conversely, assume (ii). Let V be any simple right R -module. Let $f : A \rightarrow V$ be a nonzero map. By assumption, $A + B = E(A)$ for some submodule B of $E(A)$ such that $A \cap B$ is small in A . Thus, $\text{Ker}(f)$ contains $A \cap B$. Define $g : A + B \rightarrow V$ with $g(a + b) = f(a)$ for $a \in A$ and $b \in B$. Then g is clearly well-defined and it extends f . \square

Proposition 32. *Let R be a right perfect ring and S be the direct sum of a complete set of representatives of simple right R -modules. R is a right Harada ring if and only if S is indigent.*

Proof. Assume that R is right perfect and S is indigent. Next, let E be injective. Then E is supplemented. Let A be a supplement in E . Clearly, A is then a supplement in $E(A)$ as well. Note that S is finitely generated here. Then, by Lemma 31 and [3, Proposition 2.4], S is A -subinjective, implying, by assumption, that A is injective. Hence, R is right Harada.

Conversely, let R be a right Harada ring. Assume that S is A -subinjective and put $E = E(A)$. Again by Lemma 31, A is a supplement in E . So, by assumption, A is injective, showing that S is indigent. \square

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