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Linear normality of general linear sections and some graded Betti numbers of 3-regular projective schemes ☆

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ABSTRACT

In this paper we study graded Betti numbers of any nondegenerate 3-regular algebraic set X in a projective space $\mathbb{P}^n$. More concretely, via Generic initial ideals (Gins) method we mainly consider 'tailing' Betti numbers, whose homological index is at least $\mathrm{codim}(X, \mathbb{P}^n)$. For this purpose, we first introduce a key definition 'ND(1) property', which provides a suitable ground where one can generalize the concepts such as 'being nondegenerate' or 'of minimal degree' from the case of varieties to the case of more general closed subschemes and give a clear interpretation on the tailing Betti numbers. Next, we recall basic notions and facts on Gins theory and we analyze the generation structure of the reverse lexicographic (rlex) Gins of 3-regular ND(1) subschemes. As a result, we present exact formulae for these tailing Betti numbers, which connect them with linear normality of general linear sections of $X \cap \Lambda$ with a

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linear subspace Λ of dimension at least $\operatorname{codim}(X, \mathbb{P}^n)$. Finally, we consider some applications and related examples.

Contents

1. Introduction

Throughout this paper, we will work with a closed subscheme X of codimension e in $\mathbb{P}^n$ over an algebraically closed field k of $\mathrm{char}(k) = 0$. Let I_X be $\bigoplus_{m=0}^{\infty} H^0(\mathcal{I}_{X/\mathbb{P}^n}(m))$, the defining ideal of X in the polynomial ring $R = k[x_0, x_1, \ldots, x_n]$. We mean (co)dimension and degree of $X \subset \mathbb{P}^n$ by the definitions deducing from Hilbert polynomial of R/I_X (see Convention in Section 2).

Suppose that X is 3-regular (in the sense of Castelnuovo–Mumford), that is

$$H^i(\mathcal{I}_{X/\mathbb{P}^n}(3-i)) = 0 \text{ for all } i \geq 1.$$

The set of 3-regular subschemes is quite large. It contains many known examples such as (embedded) curves of high degree, secants of rational normal scrolls, and del Pezzo varieties. Until now, 3-regularity seems not to be well-understood. For instance, in contrast with 2-regularity, there is no classification for 3-regularity even in the category of varieties. So, it is worthwhile to investigate graded Betti numbers of 3-regular subschemes.

When X is 3-regular, we could present the Betti table of X with 3 rows (e.g. see [5, Chap. 4]) and divide the whole table into four quadrants (by *dashed* lines) as in Fig. 1.

Once X is nondegenerate (i.e. I_X has no linear forms), in the first row all the Betti numbers except $\beta_{0,0}$ are zeros. For the other entries of 1st-quadrant, it was first known by Green's $K_{p,1}$-theorem for compact complex manifolds in [10] that they are also all *zeros*; we prove this in a more general category using our method (see Theorem 3.2 and Remark 3.3). In 2nd-quadrant the part corresponding to $\beta_{1,1}, \beta_{2,1}, \cdots$ is called *linear strand* of the table and many authors have studied on this subject of linear syzygies (e.g. see [5, Chap. 8, 9] for an overview of its connection toward Green and Green–Lazarsfeld conjectures of curves or see [14] for its relevance to classification of varieties of small degree). For 3rd-quadrant, there is a nice geometric interpretation related to the existence of a *degenerate secant plane* $X \cap \Lambda$, which is a finite scheme with *length*$(X \cap \Lambda) > \dim \Lambda + 1$

	0	1	2	⋯	a	a+1	⋯	e	e+1	⋯	n−1	n	n+1
0	1	0	0	⋯	0	0	⋯	0	0	⋯	0	0	0
1	0	$\beta_{1,1}$	$\beta_{2,1}$	⋯	$\beta_{\mathtt{a},1}$	$\beta_{\mathtt{a}+1,1}$	⋯	$\beta_{\mathtt{e},1}$	$\beta_{\mathtt{e}+1,1}$	⋯	$\beta_{\mathtt{n}-1,1}$	$\beta_{\mathtt{n},1}$	0
2	0	0	0	⋯	0	$\beta_{\mathtt{a}+1,2}$	⋯	$\beta_{\mathtt{e},2}$	$\beta_{\mathtt{e}+1,2}$	⋯	$\beta_{\mathtt{n}-1,2}$	$\beta_{\mathtt{n},2}$	0

Fig. 1. A typical Betti table of any nondegenerate 3-regular projective scheme X in $\mathbb{P}^n$. Here $\beta_{i,j}$ denotes the *graded Betti number* of X $\dim_k \operatorname{Tor}_i^R(R/I_X, k)_{i+j}$, $\mathtt{e} = \operatorname{codim}(X, \mathbb{P}^n)$. $\mathtt{a} = \mathtt{a}(X)$ is often called *Green–Lazarsfeld index* of $X \subset \mathbb{P}^n$.

for a linear subspace Λ of dimension $\le e$ (see e.g. [13, Theorem 2], [6, Theorem 1.1]). Recently, the first author and S. Kwak obtained a more refined result on Betti numbers in the 3rd-quadrant in this viewpoint (see [1, Theorem 1.2 and Example 3.4]).

In this paper, to complete the picture of Betti tables of 3-regular schemes, we focus on the interpretation of the $\beta_{e,2}, \beta_{e+1,2}, \cdots, \beta_{n,2}$ (we call these *tailing Betti numbers*) in the remaining part, i.e. *fourth* quadrant.

ND(1) property. As usual, we would like to assume X to be nondegenerate. But, to generalize what we have and expect in varieties to more general schemes, in many cases it is not enough. Therefore, let us begin this study by introducing a definition:

Definition 1.1 *(*ND(1) *property).* Let k be any field as above. We say that a closed subscheme $X \subset \mathbb{P}^n_k$ satisfies ND(1) *property* if for a *general* linear section Λ of each dimension $\ge e$

$$X \cap \Lambda \text{ is nondegenerate (i.e. } H^0(\mathcal{I}_{X\cap\Lambda/\Lambda}(1)) = 0).$$

Note that by definition $X \subset \mathbb{P}^n$ itself is nondegenerate if X satisfies ND(1) property. We also remark that every general linear section of $X \cap \Lambda$ also has the same property in case that X has ND(1) property.

This definition of ND(1) property is very natural. For example, every nondegenerate integral (i.e. irreducible and reduced) variety has ND(1) property. In fact, the set of closed subschemes having ND(1) property is quite large; we do not need to assume our subscheme X to be *irreducible, equi-dimensional,* or even *reduced* necessarily (see Example 1.2). Further, this is also natural in the sense that via this property one could generalize the concepts such as 'being nondegenerate' and 'of minimal degree' (i.e. degree is no less than codimension +1) to more general reducible subschemes and give a very clear meaning of the tailing Betti numbers (see Proposition 3.1, Theorem 3.2 and Theorem 4.1).

Example 1.2 *(Subschemes with* ND(1) *property).* We list the following examples of subschemes with ND(1) property:

(a) In case of X being a nondegenerate integral variety, ND(1) property of X can be deduced from Bertini type theorem. More generally, if any algebraic set X has a nondegenerate top dimensional component, then X also satisfies ND(1) property.
(b) (Case of all the components degenerate) There are algebraic sets satisfying ND(1) whose all components are degenerate. For example, consider the following saturated

ideal

$$I = (L_1L_2L_3, L_1L_2L_4, L_1L_4L_5, L_4L_5L_6, L_1L_2L_7),$$

where L_i is a generic linear form for each $i = 1, \ldots, 7$ in $R = k[x_0, x_1, x_2, x_3]$. Then the algebraic set $X \subset \mathbb{P}^3$ defined by the ideal I is the union of 5 lines and one point such that its minimal free resolution is given by

	0	1	2	3
0	1	0	0	0
1	0	0	0	0
2	0	5	5	1

Note that all the components of X are degenerate in $\mathbb{P}^3$. But, a general hyperplane section $X \cap H$ is a set of 5-points in $\mathbb{P}^2$ and it is nondegenerate. Hence, X satisfies ND(1) property.

(c) (Non-reduced case) A closed subscheme satisfying ND(1) is not necessarily reduced. A simple example is a non-reduced scheme X in $\mathbb{P}^3$ defined by the following saturated monomial ideal

$$I_X = (x_0^3, x_0^2x_1, x_0x_1^2, x_1^3, x_0^2x_2) \subset R = k[x_0, x_1, x_2, x_3],$$

then X is a one-dimensional nondegenerate closed subscheme in $\mathbb{P}^3$ (i.e. $\operatorname{codim}(X, \mathbb{P}^3) = 2$). Since I_X is a Borel fixed monomial ideal, we can verify $\operatorname{Gin}(I_{X\cap H/H}) = (x_0^2, x_0x_1^2, x_1^3)$, which has no linear form. So does $I_{X\cap H/H}$ (i.e. $X\cap H$ is nondegenerate in H). Hence X satisfies ND(1).

We also give some examples of subschemes without ND(1) property (see Example 3.4).

Main results. Now, we present our main results on Betti numbers of ND(1) subschemes:

Theorem 1.3 *(Theorems 3.2 and 4.1). Let X be any closed subscheme of codimension e in $\mathbb{P}^n$ satisfying* ND(1) *property and let I_X be the (saturated) defining ideal of X. Then, we have*

(a) *($K_{p,1}$-theorem for subschemes with* ND(1) *property)*

$$\beta_{i,1}(R/I_X) = 0 \text{ for any } i > e.$$

(b) *(Formulae for tailing Betti numbers) Suppose that X is 3-regular. For each $i \geq e$,*

$$\beta_{i,2}(R/I_X) = \sum_{\alpha=i}^{n} \binom{\alpha}{i} \dim H^1(\mathcal{I}_{X\cap\Lambda^\alpha/\Lambda^\alpha}(1)),$$

where Λ^α is a general linear space of dimension α.

In particular, the tailing Betti numbers of 3-regular scheme X with ND(1) property depend only on $\dim H^1(\mathcal{I}_{X\cap\Lambda^\alpha/\Lambda^\alpha}(1))$, the failure of 1-normality of each general linear section $X\cap\Lambda^\alpha$ with $e\le\alpha\le n$.

Here is an immediate corollary of Theorem 1.3 as follows:

Corollary 1.4. *Let X be any closed subscheme of codimension e in $\mathbb{P}^n$ satisfying* ND(1) *property and let I_X be the (saturated) defining ideal of X. Suppose that X is 3-regular. Let $\mathbf{b}$ and $\mathbf{h}$ be two row vectors as given below:*

$$\begin{aligned}\mathbf{b} &:= [\beta_{e,2}(R/I_X), \beta_{e+1,2}(R/I_X), \cdots, \beta_{n-1,2}(R/I_X), \beta_{n,2}(R/I_X)]\\ \mathbf{h} &:= [h^1(\mathcal{I}_{X\cap\Lambda^e/\Lambda^e}(1)), h^1(\mathcal{I}_{X\cap\Lambda^{e+1}/\Lambda^{e+1}}(1)), \cdots, h^1(\mathcal{I}_{X\cap\Lambda^{n-1}/\Lambda^{n-1}}(1)), h^1(\mathcal{I}_{X/\mathbb{P}^n}(1))],\end{aligned}$$

where $h^1(\mathcal{I}_{X\cap\Lambda^{n-i}/\Lambda^{n-i}}(1))$ denotes $\dim H^1(\mathcal{I}_{X\cap\Lambda^{n-i}/\Lambda^{n-i}}(1))$. Then, we have identities

$$\mathbf{b}^T = \Xi(n,e)\cdot\mathbf{h}^T \text{ and } \mathbf{h}^T = \Xi(n,e)^{-1}\cdot\mathbf{b}^T, \tag{1.1}$$

where $\mathbf{b}^T$ (resp. $\mathbf{h}^T$) is the transpose of $\mathbf{b}^T$ (resp. of $\mathbf{h}^T$) and $\Xi(n,e)$ is the invertible $(n-e+1)\times(n-e+1)$-matrix such as

$$\Xi(n,e) = \begin{bmatrix} \binom{e}{e} & \binom{e+1}{e} & \binom{e+2}{e} & \cdots & \binom{n-1}{e} & \binom{n}{e} \\ 0 & \binom{e+1}{e+1} & \binom{e+2}{e+1} & \cdots & \binom{n-1}{e+1} & \binom{n}{e+1} \\ 0 & 0 & \binom{e+2}{e+2} & \cdots & \binom{n-1}{e+2} & \binom{n}{e+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{n-1}{n-1} & \binom{n}{n-1} \\ 0 & 0 & 0 & \cdots & 0 & \binom{n}{n} \end{bmatrix}. \tag{1.2}$$

Note that any nondegenerate connected in codimension 1 algebraic set X has also ND(1) property. Further, using these tailing Betti numbers $\beta_{e,2}(R/I_X), \cdots, \beta_{n,2}(R/I_X)$, we can determine $P_X(t)$, Hilbert polynomial of any 3-regular subscheme $X\subset\mathbb{P}^n$ from this ND(1) property as follows:

Corollary 1.5 *(Corollary 4.5). Let X be any 3-regular closed subscheme in $\mathbb{P}^n$ satisfying* ND(1) *property. Set $r=\dim X$ and $e=n-r$, the codimension. Then, $P_X(t)$, Hilbert polynomial of $X\subset\mathbb{P}^n$ can be computed as*

$$P_X(t) = \left\{ e+1+\sum_{i=e}^{n}(-1)^{i-e}\binom{i}{e}\beta_{i,2}(R/I_X) \right\}\binom{t+r-1}{r}$$
$$+\sum_{i=0}^{r-1}\left[\left\{ 1-\beta_{n-i-1,2}(R/I_X)+\sum_{j=n-i}^{n}(-1)^{j-n+i}\binom{j+1}{n-i}\beta_{j,2}(R/I_X) \right\}\binom{t+i-1}{i} \right].$$

In particular, the degree of $X \subset \mathbb{P}^n$ can be given by

$$e+1+\sum_{i=e}^{n}(-1)^{i-e}\binom{i}{e}\beta_{i,2}(R/I_X)$$

and the arithmetic genus $p_a(X)$ in case of X being a curve and the irregularity $q(X)$ in case of X being a surface can be given by $\beta_{n-1,2}(R/I_X) - \binom{n+1}{n}\beta_{n,2}(R/I_X)$.

For this purpose, in Section 2 we briefly review Generic initial ideals (Gins) theory and develop some combinatorial methods to study tailing Betti numbers using Gins with *reverse lexicographic order (rlex)*. In Section 3, we first regard some properties of ND(1) subschemes, next consider results which connect these Betti numbers with those of rlex Gins, and in the end we prove an important proposition which describes the generation structure of the rlex Gins of 3-regular subschemes virtually. Finally, in Section 4, we give a proof on our main result and consider some applications and examples (see Example 4.8). Note that, in case of arithmetically Cohen–Macaulay (ACM) or arithmetically Gorenstein (AG), there exists only one tailing Betti number $\beta_{e,2}(R/I_X)$ and the meaning is well-known. Since we consider a general situation which is not necessarily ACM nor AG, our approach uses combinatorial rlex Gins techniques instead of using duality theorems and our results can cover more general cases.

2. Generic initial ideals: a brief review

Convention. We will work with the following conventions:

- (d-normality) We say that $X \subset \mathbb{P}^n$ is d-normal if the restriction map
$$H^0(\mathcal{O}_{\mathbb{P}^n}(d)) \to H^0(\mathcal{O}_X(d))$$
is surjective (i.e. $H^1(\mathcal{I}_{X/\mathbb{P}^n}(d)) = 0$). If X is 1-normal then it is also called linearly normal. The failure of 1-normality of $X \subset \mathbb{P}^n$ can be measured by $\dim H^1(\mathcal{I}_{X/\mathbb{P}^n}(1))$.
- For a coherent sheaf $\mathcal{F}$ on a projective scheme X over k, $h^i(X, \mathcal{F})$ means $\dim_k H^i(X, \mathcal{F})$.
- (Betti numbers) For a graded R-module M, we define *graded Betti numbers $\beta^R_{i,j}(M)$ of M* by $\dim_k \operatorname{Tor}^R_i(M, k)_{i+j}$. We denote it as $\beta_{i,j}(M)$ or $\beta_{i,j}$ if it is obvious. For a homogeneous ideal $I \subset R$, note that $\operatorname{Tor}^R_i(R/I, k)_{i+j} = \operatorname{Tor}^R_{i-1}(I, k)_{i-1+j+1}$. So $\beta^R_{i,j}(R/I) = \beta^R_{i-1,j+1}(I)$.

- (Dimension and degree) When we call the *dimension of a closed subscheme* $X \subset \mathbb{P}^n$, denoted by $\dim X$, it means the degree of the Hilbert polynomial of R/I_X. The *codimension of* X is $n - \dim X$. We also define the *degree of* X, denoted by $\deg(X)$, as $(\dim X)!$ times the leading coefficient of this Hilbert polynomial.
- (Arithmetic depth) When we refer the *depth of* $X \subset \mathbb{P}^n$, denoted by $\mathrm{depth}_R(X)$ or simply $\mathrm{depth}(X)$, we mean the arithmetic depth of X, i.e. $\mathrm{depth}_R(R/I_X)$.
- (Generic initial ideal) Given a homogeneous ideal $I \subset R$ and a term order τ, there is a Zariski open subset $U \subset GL_{r+1}(k)$ such that $\mathrm{in}_\tau(g(I))$ for $g \in U$ is constant. We will call this constant $\mathrm{in}_\tau(g(I))$ the *generic initial ideal of* I and denote it by $\mathrm{Gin}_\tau(I)$.
- (Borel fixed property) The generic initial ideal $\mathrm{Gin}_\tau(I)$ of I has *Borel fixed property*, which is a nice combinatorial property. In characteristic 0, we say that a monomial ideal J has Borel-fixed property if $x_i m \in J$ for a monomial m, then $x_j m \in J$ for all $j \leq i$.
- (Term order) Unless otherwise stated, we always assume the generic initial ideal with respect to *the reverse lexicographic order* (e.g. see [11]) and denote it simply by $\mathrm{Gin}(I)$.
- (Saturation and quotient) For a Borel fixed monomial ideal $J \subset R$, we will simply denote $\bigcup_{k=0}^{\infty}(J : x_n^k)$ and $(J, x_n)/(x_n)$ by $J|_{x_n \to 1}$ and $J|_{x_n \to 0}$ respectively.
- If $K = (k_0, \ldots, k_n)$ then we denote by $\mathbf{x}^K$ the monomial $\mathbf{x}^K = x_0^{k_0} \cdots x_n^{k_n}$, and denote by $|K|$ its degree $|K| = \sum_{j=0}^{n} k_j$. For monomial $\mathbf{x}^K$, we define $\max(\mathbf{x}^K) = \max\{j : k_j > 0\}$.
- (Monomial generators set) Let J be any monomial ideal of R. Then, we have a set of minimal *monomial* generators of J. We write this set as $\mathcal{G}(J)$. For each $d \geq 0$, we also write $\mathcal{G}(J)_d$ for the subset of minimal monomial generators of degree d.
- For a Borel fixed monomial ideal J, we denote the subset $\{\mathbf{x}^K \in \mathcal{G}(J)_d \mid \max(\mathbf{x}^K) = i\}$ of $\mathcal{G}(J)_d$ by $\mathcal{M}_i(d, J)$. Thus, it holds that $\mathcal{G}(J)_d = \bigcup_{i=0}^{n} \mathcal{M}_i(d, J)$.

In general, a great deal of fundamental information about a homogeneous ideal I can be obtained from $\mathrm{Gin}(I)$. Now we recall some known facts concerning generic initial ideals from [4,11], which will be used throughout the remaining parts of the paper:

Theorem 2.1. *(See [4,11].) Let I be a homogeneous ideal and L be a general linear form in $R = k[x_0, \ldots, x_n]$. Consider the ideal $\bar{I} = (I, L)/(L)$ as a homogeneous ideal of polynomial ring $S = k[x_0, \ldots, x_{n-1}]$. Then we have*

(a) $\mathrm{Gin}(\bar{I}) = \frac{(\mathrm{Gin}(I), x_n)}{(x_n)} = \mathrm{Gin}(I)|_{x_n \to 0}$;
(b) $\mathrm{Gin}(\bar{I}^{\mathrm{sat}}) = \bigcup_{k=0}^{\infty}(\mathrm{Gin}(\bar{I}) : x_{n-1}^k) = (\mathrm{Gin}(I)|_{x_n \to 0})|_{x_{n-1} \to 1}$;
(c) *I is saturated if and only if no generator of* $\mathrm{Gin}(I)$ *involves* x_n;
(d) $\mathrm{reg}(I) = \mathrm{reg}(\mathrm{Gin}(I)) =$ *the maximal degree of generators of* $\mathrm{Gin}(I)$.

Remark 2.2. By a slight abuse of notation, we denote by $(\mathrm{Gin}(I)|_{x_n\to 0})|_{x_{n-1}\to 1}$ the saturation of a monomial ideal $\frac{(\mathrm{Gin}(I),x_n)}{(x_n)}$ in $k[x_0,\dots,x_{n-1}]$. Note that we see from the definition that it could contain some monomials divisible by x_{n-1}, even though there is no such monomial in a minimal generating set. This notation was introduced by M. Green in [12].

From the following result we can compute the graded Betti numbers of Borel fixed monomial ideals. Note that generic initial ideals have the Borel fixed property (see e.g. [4,8]).

Theorem 2.3 *(Eliahou and Kervaire). Let J be a Borel fixed monomial ideal of $R = k[x_0,\dots,x_n]$. Denote by $\mathcal{G}(J)$ the set of minimal (monomial) generators of J and by $\mathcal{G}(J)_d$ the elements of $\mathcal{G}(J)$ having degree d. Then,*

$$\beta_{i,d}(R/J) = \beta_{i-1,d+1}(J) = \sum_{T\in\mathcal{G}(J)_{d+1}} \binom{\max(T)}{i-1}.$$

Proof. See the main result in [7]. □

Now, we present a useful lemma for the remaining part, which also comes from the Borel fixed property.

Lemma 2.4. *Let $J \subset R = k[x_0,\dots,x_n]$ be a Borel fixed monomial ideal and $T \in R_d \setminus J_d$ be a monomial of degree d. Suppose that $Tx_j \in J_{d+1}$ for some $j \geq \max(T)$. Then we have*

$$Tx_i \in \mathcal{G}(J)_{d+1} \textit{ for every } i \textit{ such that } \max(T) \leq i \leq j.$$

Proof. Let us write T as $x_{k_1}x_{k_2}\cdots x_{k_d}$ where $k_1 \leq k_2 \leq \cdots \leq k_d = \max(T)$. Since $Tx_j \in J_{d+1}$, by the Borel fixed property, we see that $Tx_i \in J_{d+1}$ for each i with $\max(T) \leq i \leq j$. Suppose that Tx_i is not a minimal generator of J for some $\max(T) \leq i \leq j$. Then there is a monomial $M \in J_d$ such that $Tx_i = Mx_\ell$ for some variable $x_\ell \in R_1$. Since $T \notin J_d$, we see that $M \neq T$ (so, $x_\ell \neq x_i$). Now we could write $M = x_{k_1}x_{k_2}\cdots\widehat{x_\ell}\cdots x_{k_d}x_i$ for some $k_1 \leq \ell \leq k_d$. So, using the Borel fixed property again, we see that $T = x_{k_1}x_{k_2}\cdots x_{k_d} \in J_d$, which contradicts $T \notin J_d$. □

Remark 2.5. In Lemma 2.4, if we replace the condition $Tx_j \in J_{d+1}$ with $Tx_j \in \mathcal{G}(J)_{d+1}$, then we do not need the hypothesis $T \notin J_d$. Therefore, Lemma 2.4 can be also used to show a variant as follows:

> "Let $J \subset R = k[x_0,\dots,x_n]$ be a Borel fixed monomial ideal and $T \in R_d$ be a monomial. Suppose that $Tx_j \in \mathcal{G}(J)_{d+1}$ for some $j \geq \max(T)$. Then, we have
>
> $$Tx_i \in \mathcal{G}(J)_{d+1} \text{ for each } i \text{ with } \max(T) \leq i \leq j."$$

Let I be a homogeneous ideal of R. For any monomial term order τ there exists a flat family of ideals I_t with $I_0 = \mathrm{in}_\tau(I)$ (the initial ideal of I) and I_t canonically isomorphic to I for all $t \neq 0$ (this implies that $\mathrm{in}_\tau(I)$ has the same Hilbert function as that of I). Using this result, we have

Theorem 2.6 *(The cancellation principle). (See [11, Corollary 1.21].) Choose any monomial term order τ. For any homogeneous ideal I and any i and d, there is a complex of k-modules $V_\bullet^d$ such that*

$$V_i^d \simeq \mathrm{Tor}_i^R(\mathrm{in}_\tau(I), k)_d, \quad H_i(V_\bullet^d) \simeq \mathrm{Tor}_i^R(I, k)_d.$$

In particular, this implies that

$$\beta_{i,d}(R/I) \le \beta_{i,d}(R/\mathrm{in}_\tau(I)) \text{ for any } i, d.$$

Example 2.7. Consider a complete intersection X of type $(2, 2, 2)$ in $\mathbb{P}^4$. The following are Betti tables of I_X and $\mathrm{Gin}(I_X)$.

I_X

	0	1	2	3	...
0	1	0	0	0	...
1	0	3	0	0	...
2	0	0	3	0	...
3	0	0	0	1	...

$\Longleftrightarrow$

$\mathrm{Gin}(I_X)$

	0	1	2	3	...
0	1	0	0	0	...
1	0	3	2	0	...
2	0	2	4	2	...
3	0	1	2	1	...

The cancellation principle says that the minimal free resolution of I_X is obtained from that of $\mathrm{Gin}(I_X)$ by canceling some adjacent terms of the same shift in the free resolution. These terms are diagonally adjacent in the Betti table.

Remark 2.8. From Theorem 2.6, we see that $\beta_{i,d}(R/I) \le \beta_{i,d}(R/\mathrm{Gin}(I))$ for any i, d. We can ask when the equality holds. One case that we want to describe is the following. If we have

$$\beta_{k-1,m+1}(R/\mathrm{Gin}(I)) = \beta_{k+1,m-1}(R/\mathrm{Gin}(I)) = 0, \text{ for some } k, m$$

then $\beta_{k,m}(R/I) = \beta_{k,m}(R/\mathrm{Gin}(I))$ by the cancellation principle. For example, if $\mathrm{Gin}(I)$ has a linear resolution then I also has the same graded Betti numbers as those of $\mathrm{Gin}(I)$.

Another case is the following, which will be used in this paper.

Lemma 2.9 *(Marginal Betti number). For a homogeneous ideal $I \subset R$, suppose that I is $(d+1)$-regular and the projective dimension of R/I is at most p. Then we have*

$$\beta_{p,d}(R/I) = \beta_{p,d}(R/\mathrm{Gin}(I)).$$

	0	1	2	⋯	p-1	p	p+1	⋯
0	1	0	0	⋯	0	0	0	⋯
1	0	*	*	⋯	*	*	0	⋯
⋮	⋮	⋮	⋮	⋯	⋮	⋮	⋮	⋯
d-1	0	*	*	⋯	*	*	0	⋯
d	0	*	*	⋯	*	β	0	⋯
d+1	0	0	0	⋯	0	0	0	⋯

Fig. 2. We call β the *marginal Betti number of* R/I.

Proof. Note that the projective dimension of R/I is the same as that of $R/\mathrm{Gin}(I)$ (see e.g. [2, Corollary 2.8]). So, $\beta_{p+1,d-1}(R/\mathrm{Gin}(I)) = 0$. Since I is $(d+1)$-regular, from the result of Bayer and Stillman (Theorem 2.1(d)), $\mathrm{Gin}(I)$ is also $(d+1)$-regular. Hence we have $\beta_{p-1,d+1}(R/\mathrm{Gin}(I)) = 0$. Then, it follows from Remark 2.8 that $\beta = \beta_{p,d}(R/I) = \beta_{p,d}(R/\mathrm{Gin}(I))$. (See Fig. 2.) □

3. ND(1) property and structures of reverse lexicographic Gins of 3-regular subschemes

We begin with the following result.

Proposition 3.1 *(Basic inequality for degrees of* ND(1) *subschemes). Let X be a closed subscheme of codimension e in $\mathbb{P}^n$ satisfying* ND(1) *property. Then, the following holds*

$$\deg(X) \geq e+1.$$

Proof. Let us prove by induction on $\dim X$. First, if $\dim X = 0$ (so, $e = n$), then by ND(1) property of X, $H^0(\mathcal{I}_{X/\mathbb{P}^n}(1)) = 0$ and the sequence $0 \to H^0(\mathcal{O}_{\mathbb{P}^n}(1)) \to H^0(\mathcal{O}_X(1))$ tells us that $\deg(X) \geq n+1 = e+1$, as we wish. Now, suppose that the statement holds for any ND(1) subscheme of dimension $\leq k$ for some $k \geq 0$. Then, let us consider a ND(1) subscheme X of dimension $k+1$. For a general hyperplane section $X \cap H$ still satisfies ND(1) property and has the same codimension and degree as those of X; by induction hypothesis we have

$$\deg(X) = \deg(X \cap H) \geq e+1,$$

and the assertion is proved. □

Note that this is not true in general (see Example 3.4).

Theorem 3.2 *($K_{p,1}$-theorem for* ND(1) *subscheme). Let X be a closed subscheme of codimension e in $\mathbb{P}^n$ satisfying* ND(1) *property. Then we have*

$$\beta_{i,1}(R/\mathrm{Gin}(I_X)) = 0 \text{ for any } i > e = \mathrm{codim}(X, \mathbb{P}^n).$$

Therefore, we also have $\beta_{i,1}(R/I_X) = 0$ for any $i > e$.

Proof. The cancellation principle (Theorem 2.6) implies that

$$\beta_{i,1}(R/I_X) \le \beta_{i,1}(R/\mathrm{Gin}(I_X)) \text{ for any } i.$$

Therefore, it suffices to show that

$$\beta_{i,1}(R/\mathrm{Gin}(I_X)) = 0 \text{ for any } i > e.$$

Note that the assumption on ND(1) property of X guarantees that the linear section $X \cap \Lambda$ is nondegenerate for a general linear subspace Λ of dimension $\ge e$.

Suppose that $\beta_{i,1}(R/I_X) \neq 0$ for some $i > e$. Then, from Theorem 2.3 we have

$$\begin{aligned} 0 < \beta_{i,1}(R/\mathrm{Gin}(I_X)) &= \sum_{T \in \mathcal{G}(\mathrm{Gin}(I_X))_2} \binom{\max(T)}{i-1} \\ &\Rightarrow \exists\, T_0 \in \mathcal{G}(\mathrm{Gin}(I_X))_2 \text{ such that } \max(T_0) \ge i-1 \ge e. \end{aligned}$$

By the Borel fixed property we may assume that $T_0 = x_1 x_{i-1}$ in $\mathrm{Gin}(I_X)_2$. Consider a general linear section $X \cap \Lambda$ with $\dim \Lambda = i-1$. Then, by Theorem 2.1(a) and (b),

$$\mathrm{Gin}(I_{X\cap\Lambda/\Lambda}) = \left[\frac{(\mathrm{Gin}(I_X), x_i, x_{i+1}, \ldots, x_n)}{(x_i, x_{i+1}, \ldots, x_n)}\right]^{\mathrm{sat}} = \left[\frac{(\mathrm{Gin}(I_X), x_i, x_{i+1}, \ldots, x_n)}{(x_i, x_{i+1}, \ldots, x_n)}\right]_{x_{i-1}\to 1},$$

and thus T_0 gives $x_1 \in \mathrm{Gin}(I_{X\cap\Lambda/\Lambda})$, which implies that $I_{X\cap\Lambda/\Lambda}$ has a linear form. This contradicts that $X \cap \Lambda$ is nondegenerate. $\square$

Remark 3.3. This was first known by Green's $K_{p,1}$-theorem for compact complex manifolds [10] and later by Nagel and Pitteloud for a more general setting [16]. Here, by utilizing ND(1) assumption, the proof of Theorem 3.2 is quite simple. Further, it can be used to claim the result for more general cases than those treated in [10,16]. For example, a case such as (c) in Example 1.2, a subscheme whose ideal contains some power of a linear form, cannot be covered by previous results, while this method can do.

Example 3.4 *(Non-*ND(1) *subschemes).* If a closed subscheme $X \subset \mathbb{P}^n$ does not satisfy ND(1), then Theorem 3.2 is no longer true. For examples:

(a) (Two planes in $\mathbb{P}^4$) Let $X \subset \mathbb{P}^4$ be a closed subscheme defined by

$$I_X = (x_0, x_1) \cap (x_2, x_3) \subset R = k[x_0, x_1, x_2, x_3, x_4].$$

Since X is a union of two planes in four space meeting in a single point, we see that X and $X \cap H$ are nondegenerate for a general hyperplane H. However, because $X \cap H$ is a union of two skew lines in H, for a general 2-plane Λ the intersection $X \cap \Lambda$ is a set of two points, degenerate. Thus, X does not satisfy ND(1) property.

We see that $\deg(X) = 2$ and $2 \not> 2+1$, so the inequality in Proposition 3.1 does not hold for this non-ND(1) subscheme. The Betti table of X is given by

	0	1	2	3	4
0	1	0	0	0	0
1	0	4	4	1	0
2	0	0	0	0	0

Note that codimension of X is two but $\beta_{3,1} \neq 0$, that is, Theorem 3.2 does not hold.

(b) Consider the following non-reduced closed subscheme $X \in \mathbb{P}^3$ defined by the ideal

$$\begin{aligned} I_X &= (x_0x_2, x_1x_2, x_2^2, x_0^2x_1 - x_0x_1^2) \\ &= (x_0, x_2) \cap (x_1, x_2) \cap (x_0 - x_1, x_2) \cap (x_0, x_1, x_2)^2. \end{aligned}$$

Then X is a union of three lines in a plane $x_2 = 0$ with an embedded point at the origin. Since a general hyperplane section gives 3-collinear points in a line, we see that X does not satisfy ND(1). On the other hand, Betti table of R/I_X is given by

	0	1	2	3	4	...
0	1	0	0	0	0	...
1	0	3	3	1	0	...
2	0	1	1	0	0	...

such that $\beta_{e+1,1}(R/I_X) = 1 \neq 0$, $\beta_{e+1,2}(R/I_X) = 0$ and $\beta_{e,2}(R/I_X) = 1$. In this case, Theorem 1.3(a) is not true. But, by a simple computation, we also see that $h^1(\mathcal{I}_{X/\mathbb{P}^3}(1)) = 0$ and $h^1(\mathcal{I}_{X\cap H/H}(1)) = 1$. So, even though X is non-ND(1), the relation in Theorem 1.3(b) still holds.

Corollary 3.5. *Let X be a closed scheme of codimension e in $\mathbb{P}^n$ satisfying* ND(1) *property. Suppose that X is* 3*-regular. Then we have*

$$\beta_{i,2}(R/I_X) = \beta_{i,2}(R/\mathrm{Gin}(I_X)) \text{ for any } i \geq e.$$

Proof. For each $i \geq e$, we see from Theorem 3.2 that $\beta_{i+1,1}(R/\mathrm{Gin}(I_X)) = 0$. Since X is 3-regular we also have $\beta_{i-1,3}(R/\mathrm{Gin}(I_X)) = 0$ for any i. Then it follows from Remark 2.8 that $\beta_{i,2}(R/I_X)$ is equal to $\beta_{i,2}(R/\mathrm{Gin}(I_X))$, as we wished. □

Once X is $(d+1)$-regular, X is m-normal for all $m \geq d$ (see e.g. [15, p. 100]). Then, what can we say about $h^1(\mathcal{I}_{X/\mathbb{P}^n}(d-1))$? It is quite remarkable that the failure of $(d-1)$-normality $h^1(\mathcal{I}_{X/\mathbb{P}^n}(d-1))$ can be read off from the *generation structure* of $\mathrm{Gin}(I_X)$, as the following proposition says:

Proposition 3.6. *Let X be any closed subscheme in $\mathbb{P}^n$ and I_X be the saturated defining ideal of X. Suppose that X is $(d+1)$-regular for some $d \geq 1$. Then, we have*

(a) $h^1(\mathcal{I}_{X/\mathbb{P}^n}(d-1)) = |\mathcal{M}_{n-1}(d+1, \mathrm{Gin}(I_X))|$;
(b) *(Marginal Betti number)* $\beta_{n,d}(R/I_X) = h^1(\mathcal{I}_{X/\mathbb{P}^n}(d-1))$.

Proof. (a) Let H be the hyperplane defined by a general linear form L, $\bar{I} = \frac{(I_X, L)}{(L)}$ and

$$\mathcal{M}_{n-1}(d+1, \mathrm{Gin}(I_X)) = \{T_1 x_{n-1}, T_2 x_{n-1}, \ldots, T_r x_{n-1}\}.$$

Since X is $(d+1)$-regular, X is also d-normal, that is $H^1(\mathcal{I}_{X/\mathbb{P}^n}(d)) = 0$. Then, from the exact sequence

$$0 \to H^0(\mathcal{I}_{X/\mathbb{P}^n}(d-1)) \xrightarrow{\cdot L} H^0(\mathcal{I}_{X/\mathbb{P}^n}(d)) \to H^0(\mathcal{I}_{X\cap H/H}(d)) \to H^1(\mathcal{I}_{X/\mathbb{P}^n}(d-1)) \to 0,$$

it follows that

$$\begin{aligned} h^1(\mathcal{I}_{X/\mathbb{P}^n}(d-1)) &= \dim_k(\bar{I}^{sat}/\bar{I})_d = \dim_k(\mathrm{Gin}(\bar{I}^{sat})/\mathrm{Gin}(\bar{I}))_d \\ &= \dim_k \left[\frac{(\mathrm{Gin}(I_X)|_{x_n\to 0})|_{x_{n-1}\to 1}}{\mathrm{Gin}(I_X)|_{x_n\to 0}} \right]_d \quad \text{(by Theorem 2.1(a), (b))} \cdots (*) \end{aligned}$$

Now, we claim that $(*)$ is equal to $|\mathcal{M}_{n-1}(d+1, \mathrm{Gin}(I_X))|$.

It is clear that each element of $\mathcal{M}_{n-1}(d+1, \mathrm{Gin}(I_X))$ contributes to $(*)$ by one. For the converse, choose any degree d monomial $T \in \left[(\mathrm{Gin}(I_X)|_{x_n\to 0})|_{x_{n-1}\to 1} \setminus \mathrm{Gin}(I_X)|_{x_n\to 0}\right]$, so that $\max(T) < n-1$, $T \notin \mathrm{Gin}(I_X)_d$ and $Tx_{n-1}^k \in \mathrm{Gin}(I_X)_{d+k}$ for some $k \geq 1$. Let k_0 be the minimum of such k's. Then, we are enough to show that $k_0 = 1$, for this also implies $Tx_{n-1} \in \mathcal{M}_{n-1}(d+1, \mathrm{Gin}(I_X))$ by Lemma 2.4.

Suppose that $k_0 \geq 2$. By the result of Bayer and Stillman (Theorem 2.1(d)), $\mathrm{Gin}(I_X)$ is also $(d+1)$-regular and we could write $Tx_{n-1}^{k_0}$ as Mx_ℓ for some $M \in \mathrm{Gin}(I_X)_{d+k_0-1}$ and some $x_\ell \in R_1$. Due to minimality of k_0, note that $x_\ell \neq x_{n-1}$ (i.e. $\ell < n-1$). Then, $(T/x_\ell)\cdot x_{n-1}^{k_0} \in \mathrm{Gin}(I_X)$ and by the Borel fixed property $T \cdot x_{n-1}^{k_0-1} = (T/x_\ell)\cdot x_\ell x_{n-1}^{k_0-1}$ also belong to $\mathrm{Gin}(I_X)$, which is again a contradiction to minimality of k_0.

(b) Since I_X is a saturated ideal, $\mathrm{depth}(R/I_X) \geq 1$, which implies that the projective dimension of R/I_X is at most n. Hence, we see that

$$\begin{aligned} \beta_{n,d}(R/I_X) &= \beta_{n,d}(R/\mathrm{Gin}(I_X)) \quad \text{(by Lemma 2.9)} \\ &= \sum_{T \in \mathcal{G}(\mathrm{Gin}(I_X))_{d+1}} \binom{\max(T)}{n-1} \quad \text{(by Theorem 2.3)} \\ &= \#\{\mathbf{x}^K \in \mathcal{G}(\mathrm{Gin}(I_X))_{d+1} \mid \max(\mathbf{x}^K) = n-1\} \\ &= |\mathcal{M}_{n-1}(d+1, \mathrm{Gin}(I_X))| = h^1(\mathcal{I}_{X/\mathbb{P}^n}(d-1)) \quad \text{(by (a))}, \end{aligned}$$

as we wished. □

Remark 3.7.

(a) With the same notation as in Proposition 3.6, the argument used in the proof of Proposition 3.6(a) actually shows that

$$\big(\mathrm{Gin}(\bar{I}^{sat})/\mathrm{Gin}(\bar{I})\big)_d \text{ is exactly the same as the } k\text{-vector space } \langle T_1, T_2, \ldots, T_r\rangle,$$

where $\mathcal{M}_{n-1}(d+1, \mathrm{Gin}(I_X)) = \{T_1x_{n-1}, T_2x_{n-1}, \ldots, T_rx_{n-1}\}$.

(b) Proposition 3.6(b) can be also shown using *local duality* (see e.g. [5, Appendix A] for the statement of local duality). Here, the proof is given by a completely different method using combinatorial properties of Gins.

Example 3.8 *(Failure of quadratic normality of a rational curve).* Consider a smooth rational curve $C \subset \mathbb{P}^3$ of degree 5 defined by the map:

$$[s, t] \to [s^5, s^4t, st^4, t^5].$$

Then, using `Macaulay 2` [9] we know that I_C and $\mathrm{Gin}(I_C)$ are minimally generated as:

- $I_C = (x_1x_2 - x_0x_3, x_2^4 - x_1x_3^3, x_0x_2^3 - x_1^2x_3^2, x_0^2x_2^2 - x_1^3x_3, x_1^4 - x_0^3x_2)$
- $\mathrm{Gin}(I_C) = (x_0^2, x_0x_1^3, x_1^4, x_0x_1^2x_2, x_1^3x_2)$

and the Betti table of R/I_C is given by

	0	1	2	3	4
0	1	0	0	0	0
1	0	1	0	0	0
2	0	0	0	0	0
3	0	4	6	2	0

.

Note that $\mathcal{M}_2(4, \mathrm{Gin}(I_C)) = \{x_1^3x_2, x_0x_1^2x_2\}$. Since the maximal degree of generators of $\mathrm{Gin}(I_C)$ is 4, we know that I_C is 4-regular (Theorem 2.1(d)), as we see in the table above. Then, by Proposition 3.6 we compute the failure of 2-normality of $C \subset \mathbb{P}^3$

$$h^1(\mathcal{J}_{C/\mathbb{P}^3}(2)) = |\mathcal{M}_2(4, \mathrm{Gin}(I_C))| = 2,$$

which also coincides with the marginal Betti number $\beta_{3,3}(R/I_C)$.

The following proposition is a crucial part of obtaining formulae in the main Theorem 4.1. It describes a peculiar aspect of the generation structure of the reverse lexicographic Gins of 3-regular ND(1)-subschemes: a set of generators of $\mathrm{Gin}(I_{X\cap H/H})$ can be obtained by taking the minimal generators of $\mathrm{Gin}(I_X)$ (which have degrees 2, 3), and setting $x_{n-1} = 1$. We prove this by exploiting the Borel fixed property of Gins and our key definition, ND(1) property.

Proposition 3.9. *Let X be a closed subscheme of codimension e in $\mathbb{P}^n$ satisfying* ND(1) *property. If H is a general hyperplane in $\mathbb{P}^n$ and*

$$\mathcal{M}_{n-1}(3, \mathrm{Gin}(I_X)) = \{T_1x_{n-1}, T_2x_{n-1}, \ldots, T_rx_{n-1}\},$$

then we have (here, $\bigsqcup$ means "disjoint union"):

(a) *For any* $T_k x_{n-1} \in \mathcal{M}_{n-1}(3, \mathrm{Gin}(I_X))$, $\max(T_k) \le e-1$*;*
(b) *For each* $e-1 \le i \le n-1$, $T_k x_i \in \mathcal{M}_i(3, \mathrm{Gin}(I_X))$*.*

Now, further suppose that X is 3-regular. Then, as a set of monomials we have the followings:

(c) *(degree 2 level)* $\mathcal{G}(\mathrm{Gin}(I_{X\cap H/H}))_2 = \mathcal{G}(\mathrm{Gin}(I_X))_2 \bigsqcup \{T_1, T_2, \dots, T_r\}$*;*
(d) *(degree 3 level)* $\mathcal{G}(\mathrm{Gin}(I_{X\cap H/H}))_3 \subset \mathcal{G}(\mathrm{Gin}(I_X))_3$*;*
(e) *(degree 3 level, continued) Moreover, for every* $e-1 \le i \le n-1$*, we have*

$$\mathcal{M}_i(3, \mathrm{Gin}(I_X)) = \mathcal{M}_i(3, \mathrm{Gin}(I_{X\cap H/H})) \bigsqcup \{T_1 x_i, T_2 x_i, \dots, T_r x_i\}.$$

In particular, we obtain

$$|\mathcal{M}_i(3, \mathrm{Gin}(I_X))| = |\mathcal{M}_i(3, \mathrm{Gin}(I_{X\cap H/H}))| + h^1(\mathcal{I}_{X/\mathbb{P}^n}(1)).$$

Proof. (a): Suppose that $\max(T_k) = j \ge e$ for some j. Then by the Borel fixed property, we may assume that $\mathrm{Gin}(I_X)$ contains a monomial $x_1 x_j^2$. For a general linear space Λ of dimension j, it follows from Theorem 2.1(a) and (b),

$$\mathrm{Gin}(I_{X\cap\Lambda/\Lambda}) = \left[\frac{(\mathrm{Gin}(I_X), x_{j+1}, \dots, x_n)}{(x_{j+1}, \dots, x_n)}\right]^{\mathrm{sat}} = \left[\frac{(\mathrm{Gin}(I_X), x_{j+1}, \dots, x_n)}{(x_{j+1}, \dots, x_n)}\right]_{x_j \to 1},$$

and thus $x_1 \in \mathrm{Gin}(I_{X\cap\Lambda/\Lambda})$, which implies that $I_{X\cap\Lambda/\Lambda}$ has a linear form. This contradicts that X satisfies ND(1) property.

(b): From Remark 2.5, we can say $T_k x_i \in \mathcal{G}(\mathrm{Gin}(I_X))_3$ (i.e. $T_k x_i \in \mathcal{M}_i(3, \mathrm{Gin}(I_X))$) for each $i \ge e-1$.

(c)–(e): A set of minimal generators of $\mathrm{Gin}(I_{X\cap H/H})$ can be obtained by taking the minimal generators of $\mathrm{Gin}(I_X)$ and setting $x_{n-1} = 1$. Note that we can only consider the generators having degree ≤ 3 from the assumption that X is 3-regular. Moreover, none of these generators can have degree 1 by the ND(1) property, so the ones of degree 2 in $\mathrm{Gin}(I_{X\cap H/H})$ are automatically minimal. It is then clear that the new minimal generators of degree 2 are $\{T_1, T_2, \dots, T_r\}$. Now some of the generators of $\mathrm{Gin}(I_X)$ (that do not involve x_{n-1}) will no longer be minimal in $\mathrm{Gin}(I_{X\cap H/H})$, and this happens precisely when they are divisible by one of $T_1, T_2, \dots, T_r$. We can then use part (b) to obtain the conclusions. □

Remark 3.10. Note that if the regularity of X is at least 4, it could be

$$h^1(\mathcal{I}_X(1)) > |\mathcal{M}_{n-1}(3, \mathrm{Gin}(I_X))|,$$

and we do not expect similar results. For example, if we consider the smooth rational curve $C \subset \mathbb{P}^3$ of degree 5 given by Example 3.8 then one can verify that $h^1(\mathcal{I}_C(1)) = 2$ but $|\mathcal{M}_2(3, \mathrm{Gin}(I_X))| = 0$.

4. Failure of 1-normality of general linear sections and tailing Betti numbers

Now we are ready to prove our main result.

Theorem 4.1 *(Formulae for tailing Betti numbers). Let X be a closed subscheme of codimension e in $\mathbb{P}^n$ satisfying* ND(1) *property. Suppose that X is 3-regular. Then, for any $i \geq e$ we have*

$$\beta_{i,2}(R/I_X) = \sum_{\alpha=i}^{n} \binom{\alpha}{i} h^1(\mathcal{I}_{X\cap\Lambda^\alpha/\Lambda^\alpha}(1)), \tag{4.1}$$

where Λ^α is a general linear space of dimension α.

Proof. Let $R = k[x_0, \cdots, x_n]$ and $S = R/H$ for a general hyperplane H. We will give a proof by induction on $\dim(X) \geq 0$. Suppose that $\dim(X) = 0$. In this case, we only have to consider $i = n$. Hence we have a tautological identity

$$\sum_{\alpha=i}^{n} \binom{\alpha}{i} h^1(\mathcal{I}_{X\cap\Lambda^\alpha/\Lambda^\alpha}(1)) = h^1(\mathcal{I}_{X/\mathbb{P}^n}(1))$$

and the claim follows from Proposition 3.6(b) which says that $\beta_{n,2}(R/I_X) = h^1(\mathcal{I}_{X/\mathbb{P}^n}(1))$.

Now assume that $\dim(X) \geq 1$. Note that $X \cap H$ is also 3-regular, has the same codimension, and satisfies ND(1) property in the projective space $H \cong \mathbb{P}^{n-1}$. Then, for any $i \geq e$ we have

$$\begin{aligned}
&\beta_{i,2}(R/I_X)\\
&\quad = \beta_{i,2}(R/\mathrm{Gin}(I_X)) \quad \text{(by Corollary 3.5)}\\
&\quad = \sum_{T\in\mathcal{G}(\mathrm{Gin}(I_X))_3} \binom{\max(T)}{i-1} \quad \text{(by Theorem 2.3)}\\
&\quad = \sum_{k=i-1}^{n-1} \binom{k}{i-1} |\mathcal{M}_k(3, \mathrm{Gin}(I_X))| \quad (\because \max(T) \leq n-1 \text{ for all } T \in \mathcal{G}(\mathrm{Gin}(I_X))_3)\\
&\quad = \sum_{k=i-1}^{n-1} \binom{k}{i-1} \Big(|\mathcal{M}_k(3, \mathrm{Gin}(I_{X\cap H,H}))| + h^1(\mathcal{I}_{X/\mathbb{P}^n}(1))\Big) \quad \text{(by Proposition 3.9(e))}\\
&\quad = \beta_{i,2}(S/\mathrm{Gin}(I_{X\cap H/H})) + h^1(\mathcal{I}_{X/\mathbb{P}^n}(1)) \sum_{k=i-1}^{n-1} \binom{k}{i-1}
\end{aligned}$$

$$= \beta_{i,2}(S/I_{X\cap H/H}) + \binom{n}{i} h^1(\mathcal{I}_{X/\mathbb{P}^n}(1)) \quad (\because\ X \cap H \text{ is also 3-regular with ND(1)})$$

$$= \sum_{\alpha=i}^{n-1} \binom{\alpha}{i} h^1(\mathcal{I}_{X\cap\Lambda^\alpha/\Lambda^\alpha}(1)) + \binom{n}{i} h^1(\mathcal{I}_{X/\mathbb{P}^n}(1)) \quad \text{(by inductive hypothesis)}$$

$$= \sum_{\alpha=i}^{n} \binom{\alpha}{i} h^1(\mathcal{I}_{X\cap\Lambda^\alpha/\Lambda^\alpha}(1)),$$

as we wished. $\square$

Before considering some applications of Theorem 4.1, we would like to mention that formula (4.1) can be restated in a more explicit way as follows:

Corollary 4.2. *Let X be any closed subscheme of codimension e in $\mathbb{P}^n$ satisfying* ND(1) *property. Suppose that X is* 3*-regular. Let* $\mathbf{b}$ *and* $\mathbf{h}$ *be two row vectors as given below:*

$$\mathbf{b} := [\beta_{e,2}(R/I_X), \beta_{e+1,2}(R/I_X), \cdots, \beta_{n-1,2}(R/I_X), \beta_{n,2}(R/I_X)]$$
$$\mathbf{h} := [h^1(\mathcal{I}_{X\cap\Lambda^e/\Lambda^e}(1)), h^1(\mathcal{I}_{X\cap\Lambda^{e+1}/\Lambda^{e+1}}(1)), \cdots, h^1(\mathcal{I}_{X\cap\Lambda^{n-1}/\Lambda^{n-1}}(1)), h^1(\mathcal{I}_{X/\mathbb{P}^n}(1))].$$

Then, we have

$$\mathbf{b}^T = \Xi(n,e)\cdot \mathbf{h}^T,$$

where $\mathbf{b}^T$ *(resp.* $\mathbf{h}^T$*) is the transpose of* $\mathbf{b}^T$ *(resp. of* $\mathbf{h}^T$*) and* $\Xi(n,e)$ *is the invertible* $(n-e+1)\times(n-e+1)$*-matrix such as*

$$\Xi(n,e) = \begin{bmatrix} \binom{e}{e} & \binom{e+1}{e} & \binom{e+2}{e} & \cdots & \binom{n-1}{e} & \binom{n}{e} \\ 0 & \binom{e+1}{e+1} & \binom{e+2}{e+1} & \cdots & \binom{n-1}{e+1} & \binom{n}{e+1} \\ 0 & 0 & \binom{e+2}{e+2} & \cdots & \binom{n-1}{e+2} & \binom{n}{e+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{n-1}{n-1} & \binom{n}{n-1} \\ 0 & 0 & 0 & \cdots & 0 & \binom{n}{n} \end{bmatrix}. \tag{4.2}$$

Now that $\det \Xi(n,e) = 1$, there exists the inverse matrix $\Xi(n,e)^{-1}$. In fact, the inverse $\Xi(n,e)^{-1}$ is given as the form

$$\Xi(n,e)^{-1} = \begin{bmatrix} \binom{e}{e} & -\binom{e+1}{e} & \binom{e+2}{e} & \cdots & (-1)^{n-e-1}\binom{n-1}{e} & (-1)^{n-e}\binom{n}{e} \\ 0 & \binom{e+1}{e+1} & -\binom{e+2}{e+1} & \cdots & (-1)^{n-e-2}\binom{n-1}{e+1} & (-1)^{n-e-1}\binom{n}{e+1} \\ 0 & 0 & \binom{e+2}{e+2} & \cdots & (-1)^{n-e-3}\binom{n-1}{e+2} & (-1)^{n-e-2}\binom{n}{e+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{n-1}{n-1} & -\binom{n}{n-1} \\ 0 & 0 & 0 & \cdots & 0 & \binom{n}{n} \end{bmatrix}. \tag{4.3}$$

Therefore, $\mathbf{h}^T$ uniquely determines $\mathbf{b}^T$ and vice versa. We have exact formulae for the entries of $\mathbf{h}^T$ as follows:

$$h^1(\mathcal{J}_{X\cap\Lambda^\alpha/\Lambda^\alpha}(1)) = \sum_{i=\alpha}^{n}(-1)^{i-\alpha}\binom{i}{\alpha}\beta_{i,2}(R/I_X), \tag{4.4}$$

where $e \le \alpha \le n$ and Λ^α is a general linear space of dimension α. In particular, we could see that for some $\ell \ge e$

$$\beta_{i,2}(R/I_X) = 0 \text{ for every } n \ge i \ge \ell \iff h^1(\mathcal{J}_{X\cap\Lambda^i/\Lambda^i}(1)) = 0 \text{ for every } n \ge i \ge \ell.$$

Keeping in mind this correspondence, we consider the following corollaries:

Corollary 4.3. *Let X be any* 3*-regular closed subscheme of codimension e in $\mathbb{P}^n$ satisfying* ND(1) *property. Let ℓ_0 be the projective dimension of R-module R/I_X. Then, we have:*

(a) *(Rigidity on 2-regularity) The vanishing of $\beta_{e,2}(R/I_X)$ implies that X is* 2*-regular;*
(b) *(Lower bounds for tailing Betti numbers) Suppose that X is a connected in codimension 1 algebraic set. Unless X is* 2*-regular, then for any $e \le i \le \ell_0$*

$$\beta_{i,2}(R/I_X) \ge \binom{\ell_0+1}{i+1}.$$

Proof. (a) First, note that for any $\alpha \ge e$ and $i \ge 0$ we have

$$H^i(\mathcal{J}_{X\cap\Lambda^\alpha/\Lambda^\alpha}(2-i)) \to H^{i+1}(\mathcal{J}_{X\cap\Lambda^{\alpha+1}/\Lambda^{\alpha+1}}(1-i)) \to H^{i+1}(\mathcal{J}_{X\cap\Lambda^{\alpha+1}/\Lambda^{\alpha+1}}(2-i)) = 0,$$

for $X \cap \Lambda^{\alpha+1}$ is also 3-regular. Thus,

$$\begin{aligned} h^i(\mathcal{I}_{X\cap\Lambda^\alpha/\Lambda^\alpha}(2-i)) &\geq h^{i+1}(\mathcal{I}_{X\cap\Lambda^{\alpha+1}/\Lambda^{\alpha+1}}(2-(i+1))) \\ &\geq \cdots \\ &\geq h^{i+n-\alpha}(\mathcal{I}_{X/\mathbb{P}^n}(2-(i+n-\alpha))). \end{aligned} \tag{4.5}$$

By formula (4.1) and inequality (4.5), we know that

$$\beta_{e,2}(R/I_X) \geq \sum_{\alpha=e}^{n} \binom{\alpha}{e} h^{1+n-\alpha}(\mathcal{I}_{X/\mathbb{P}^n}(2-(1+n-\alpha))),$$

a positive combination of cohomologies of some twists of the ideal sheaf $\mathcal{I}_X$. Hence, the vanishing $\beta_{e,2}(R/I_X) = 0$ forces that

$$H^i(\mathcal{I}_{X/\mathbb{P}^n}(2-i)) = 0 \text{ for all } i = 1, \cdots, \dim X + 1,$$

which implies 2-regularity of X.

(b) We will first show $h^1(\mathcal{I}_{X\cap\Lambda^\ell/\Lambda^\ell}(1)) > 0$ for every ℓ with $e \leq \ell \leq \ell_0$ by backward induction on $\ell = \dim(\Lambda^\ell)$. Suppose that $h^1(\mathcal{I}_{X\cap\Lambda^\ell/\Lambda^\ell}(1)) > 0$ with $i < \ell \leq \ell_0$. By the assumption that X is a connected in codimension 1 algebraic set, we see that

$$h^0(\mathcal{O}_{X\cap\Lambda^{i+1}/\Lambda^{i+1}}) = 1$$

for every i with $e \leq i < \ell_0$. From the following exact sequence

$$0 = H^0(\mathcal{I}_{X\cap\Lambda^{i+1}/\Lambda^{i+1}}) \to H^0(\mathcal{O}_{\Lambda^{i+1}}) \to H^0(\mathcal{O}_{X\cap\Lambda^{i+1}/\Lambda^{i+1}}) \to H^1(\mathcal{I}_{X\cap\Lambda^{i+1}/\Lambda^{i+1}}) \to 0,$$

we have $h^1(\mathcal{I}_{X\cap\Lambda^{i+1}/\Lambda^{i+1}}) = 0$. If we now apply the induction hypothesis to

$$0 = H^1(\mathcal{I}_{X\cap\Lambda^{i+1}/\Lambda^{i+1}}) \to H^1(\mathcal{I}_{X\cap\Lambda^{i+1}/\Lambda^{i+1}}(1)) \to H^1(\mathcal{I}_{X\cap\Lambda^{i}/\Lambda^{i}}(1))$$

then we have that $h^1(\mathcal{I}_{X\cap\Lambda^i/\Lambda^i}(1)) \geq h^1(\mathcal{I}_{X\cap\Lambda^{i+1}/\Lambda^{i+1}}(1)) > 0$.

Consequently, it follows from formula (4.1) that for each $e \leq i \leq \ell_0$,

$$\beta_{i,2}(R/I_X) = \sum_{\alpha=i}^{\ell_0} \binom{\alpha}{i} h^1(\mathcal{I}_{X\cap\Lambda^\alpha/\Lambda^\alpha}(1)) \geq \sum_{\alpha=i}^{\ell_0} \binom{\alpha}{i} = \binom{\ell_0+1}{i+1},$$

as we wished. □

Remark 4.4.

(a) Let X be a 3-regular closed subscheme in $\mathbb{P}^n$. Then Corollary 4.3(a) shows that if X satisfies the ND(1) property and $\beta_{e,2}(R/I_X) = 0$ then there is some sort of *rigidity* on the shape of the resolution; in this case X is 2-regular and arithmetically Cohen–Macaulay (aCM). This means the following Betti diagrams are equivalent;

Property ND(1) and $\beta_{e,2}(R/I_X) = 0$

0	1	2	...	e − 1	e	e + 1	e + 2	...
0	1	0	...	0	0	0	0	...
1	0	∗	...	∗	∗	0	0	...
2	0	∗	...	∗	0	∗	∗	...
3	0	0	...	0	0	0	0	...

$\Longleftrightarrow$

X is 2-regular and aCM

0	1	2	...	e − 1	e	e + 1	e + 2	...
0	1	0	...	0	0	0	0	...
1	0	∗	...	∗	∗	0	0	...
2	0	0	...	0	0	0	0	...
3	0	0	...	0	0	0	0	...

(b) Let C be a rational normal curve and Z be a set of general 4 points in $\mathbb{P}^3$ (see [1, Example 3.11]). Using Macaulay 2, we can compute the Betti table of $X = C \cup Z$ as follows:

0	1	2	3	4
0	1	0	0	0
1	0	0	0	0
2	0	6	6	0
3	0	0	0	1

This example shows that the condition $\beta_{e,d}(R/I_X) = 0$ does not imply d-regularity in general, even though X is a $(d + 1)$-regular closed subscheme satisfying property ND(1).

(c) The lower bound in Corollary 4.3(b) is sharp: see Example 4.8(c).

(d) In Corollary 4.3(b), the condition "connected in codimension 1" is essential and cannot be weakened. For example, let $X \subset \mathbb{P}^5$ be an algebraic set of codimension 2 defined by the ideal

$$I_X = (L_1L_2, L_1L_3, L_3L_4L_5, L_3L_4L_6, L_3L_4L_7),$$

where L_i is a generic linear form for each $i = 1, \dots, 7$. X is a union of three general 3-planes and a line, which is not connected in codimension 1. Since I_X is a generic distraction of the stable monomial ideal $J = (x_0^2, x_0x_1, x_1^3, x_1^2x_2, x_1^2x_3)$, we have that $\mathrm{Gin}(I_X) = J$ (see [3]). In this case, X satisfies ND(1) property. However, we see from the Betti table of X given by

	0	1	2	3	4
0	1	.	.	.	.
1	.	2	1	.	.
2	.	3	6	4	1

that the inequality (b) in Corollary 4.3 does not hold as $\beta_{3,2} = 4 \not\geq \binom{4+1}{3+1}$.

The tailing Betti numbers determine Hilbert polynomial of any 3-regular ND(1)-closed subscheme $X \subset \mathbb{P}^n$ precisely, so that we can read off some intrinsic invariants of X such as *arithmetic genus* in case of X being a curve or *irregularity* in case of X being a surface from these Betti numbers.

Let us recall some basic facts. Say $r = \dim X$. It is well-known that there exist unique integers $\chi_r(X, \mathcal{O}_X(1)), \cdots, \chi_0(X, \mathcal{O}_X(1))$, which determine Hilbert polynomial of $X \subset \mathbb{P}^n$

$$P_X(t) = \sum_{j=0}^{r} \chi_j(X, \mathcal{O}_X(1)) \binom{t+j-1}{j} \quad \text{or briefly,} \quad \sum_{j=0}^{r} \chi_j(X) \binom{t+j-1}{j} \tag{4.6}$$

and that $\chi_j(X, \mathcal{O}_X(1)) = \chi_{j-1}(X \cap H, \mathcal{O}_{X\cap H}(1))$ for each $1 \leq j \leq r$, because we have the relation $\chi(\mathcal{O}_X(t)) - \chi(\mathcal{O}_X(t-1)) = \chi(\mathcal{O}_{X\cap H}(t))$ from the exact sequence

$$0 \to \mathcal{O}_X(-1) \xrightarrow{\cdot H} \mathcal{O}_X \to \mathcal{O}_{X\cap H} \to 0 \quad (H\text{: a general hyperplane of } \mathbb{P}^n).$$

Thus, the determination of $P_X(t)$ is equivalent to decide all the $\chi_r(X), \cdots, \chi_0(X)$ in (4.6).

Corollary 4.5. *Let X be any 3-regular closed subscheme in $\mathbb{P}^n$ satisfying* ND(1) *property. Set $r = \dim X$ and $e = n - r$, the codimension. Then, the tailing Betti numbers $\beta_{e,2}(R/I_X), \cdots, \beta_{n,2}(R/I_X)$ completely determine $P_X(t)$, Hilbert polynomial of $X \subset \mathbb{P}^n$ as follows:*

$$\begin{aligned} P_X(t) = &\left\{ e+1+\sum_{i=e}^{n}(-1)^{i-e}\binom{i}{e}\beta_{i,2}(R/I_X)\right\}\binom{t+r-1}{r} \\ &+\sum_{i=0}^{r-1}\left[\left\{1-\beta_{n-i-1,2}(R/I_X)+\sum_{j=n-i}^{n}(-1)^{j-n+i}\binom{j+1}{n-i}\beta_{j,2}(R/I_X)\right\}\binom{t+i-1}{i}\right]. \end{aligned} \tag{4.7}$$

In particular, the degree of $X \subset \mathbb{P}^n$ can be given by

$$e+1+\sum_{i=e}^{n}(-1)^{i-e}\binom{i}{e}\beta_{i,2}(R/I_X) \tag{4.8}$$

and the arithmetic genus $p_a(X) = (-1)^r(\chi(\mathcal{O}_X) - 1)$ can be computed by

$$(-1)^r \left\{(n+1)\beta_{n,2}(R/I_X) - \beta_{n-1,2}(R/I_X)\right\}. \tag{4.9}$$

Proof. First of all, we recall *the formula* (4.4): for each α ($e \leq \alpha \leq n$)

$$h^1(\mathcal{I}_{X\cap\Lambda^\alpha/\Lambda^\alpha}(1)) = \sum_{j=\alpha}^{n}(-1)^{j-\alpha}\binom{j}{\alpha}\beta_{j,2}(R/I_X),$$

where Λ^α is a general linear space of dimension α. To get the Hilbert polynomial $P_X(t)$, it is enough to determine all the coefficients $\chi_r(X), \cdots, \chi_0(X)$ in (4.6) from $\beta_{e,2}(R/I_X), \cdots, \beta_{n,2}(R/I_X)$.

For the top coefficient $\chi_r(X)$, we see that

$$\begin{aligned}\chi_r(X) &= \chi_0(X\cap\Lambda^e) = P_{X\cap\Lambda^e}(0) = \chi(\mathcal{O}_{X\cap\Lambda^e})\\ &= h^0(\mathcal{O}_{X\cap\Lambda^e}) = h^0(\mathcal{O}_{X\cap\Lambda^e}(1)) = h^0(\mathcal{O}_{\Lambda^e}(1)) + h^1(\mathcal{J}_{X\cap\Lambda^e}(1))\\ &= e+1+\sum_{i=e}^{n}(-1)^{i-e}\binom{i}{e}\beta_{i,2}(R/I_X) \quad \text{by (4.4)}.\end{aligned}$$

Next, note that $X\cap\Lambda^{n-i}$ is also d-regular for all $d\geq 3$ for any $0\leq i<r$. Thus, we have

$$\begin{aligned}0 &\to H^1(\mathcal{J}_{X\cap\Lambda^{n-i}}) \to H^1(\mathcal{J}_{X\cap\Lambda^{n-i}}(1)) \to H^1(\mathcal{J}_{X\cap\Lambda^{n-i-1}}(1)) \to H^2(\mathcal{J}_{X\cap\Lambda^{n-i}})\\ &\to H^2(\mathcal{J}_{X\cap\Lambda^{n-i}}(1)) = 0,\end{aligned}$$

for each $0\leq i<r$. We also recall that $h^0(\mathcal{O}_{X\cap\Lambda^{n-i}}) = h^1(\mathcal{J}_{X\cap\Lambda^{n-i}})+1$ from the standard exact sequence. Using these facts, we calculate $\chi_i(X)$ as

$$\begin{aligned}\chi_i(X) &= \chi_0(X\cap\Lambda^{n-i}) = \chi(\mathcal{O}_{X\cap\Lambda^{n-i}}) = h^0(\mathcal{O}_{X\cap\Lambda^{n-i}}) - h^1(\mathcal{O}_{X\cap\Lambda^{n-i}})\\ &= h^0(\mathcal{O}_{X\cap\Lambda^{n-i}}) - h^2(\mathcal{J}_{X\cap\Lambda^{n-i}})\\ &= h^0(\mathcal{O}_{X\cap\Lambda^{n-i}}) - \left\{h^1(\mathcal{J}_{X\cap\Lambda^{n-i-1}}(1)) - h^1(\mathcal{J}_{X\cap\Lambda^{n-i}}(1)) + h^1(\mathcal{J}_{X\cap\Lambda^{n-i}})\right\}\\ &= 1-\left\{\sum_{j=n-i-1}^{n}(-1)^{j-n+i+1}\binom{j}{n-i-1}\beta_{j,2}(R/I_X)\right.\\ &\qquad \left.-\sum_{j=n-i}^{n}(-1)^{j-n+i}\binom{j}{n-i}\beta_{j,2}(R/I_X)\right\} \text{ by (4.4)}\\ &= 1-\beta_{n-i-1,2}(R/I_X) + \sum_{j=n-i}^{n}(-1)^{j-n+i}\binom{j+1}{n-i}\beta_{j,2}(R/I_X). \qquad \square\end{aligned}$$

In addition, we could express or bound the cohomologies $H^i(\mathcal{J}_{X/\mathbb{P}^n}(2-i))$ for $1\leq i\leq \dim(X)+1$, which is a measure on the *deviation* of X from 2-regularity, with the tailing Betti numbers. For instance, for a small $i\leq 3$, we can get simple formulae or bounds as below:

Corollary 4.6. *Let X be any nondegenerate 3-regular connected in codimension 1 algebraic set in $\mathbb{P}^n$. Then, we know that*

(a) $h^1(\mathcal{J}_{X/\mathbb{P}^n}(1)) = \beta_{n,2}(R/I_X)$;
(b) $h^2(\mathcal{J}_{X/\mathbb{P}^n}) = \beta_{n-1,2}(R/I_X) - (n+1)\beta_{n,2}(R/I_X)$ *if* $\dim X\geq 1$;
(c) $h^3(\mathcal{J}_{X/\mathbb{P}^n}(-1)) \geq \beta_{n-2,2}(R/I_X) - (n+1)\beta_{n-1,2}(R/I_X) + \binom{n+2}{2}\beta_{n,2}(R/I_X)$ *and* $h^3(\mathcal{J}_{X/\mathbb{P}^n}(-1)) \leq \beta_{n-2,2}(R/I_X) - n\beta_{n-1,2}(R/I_X) + \binom{n+1}{2}\beta_{n,2}(R/I_X)$ *if* $\dim X\geq 2$.

Proof. First, we recall that any nondegenerate connected in codimension 1 algebraic set has ND(1) property. So, (a) comes from part (b) of Proposition 3.6. For (b) and (c), by the assumption that X is connected in codimension 1, we see that

$$h^0(\mathcal{O}_{X\cap\Lambda^{i+1}/\Lambda^{i+1}}) = 1 \quad (\text{i.e. } h^1(\mathcal{I}_{X\cap\Lambda^{i+1}/\Lambda^{i+1}}) = 0)$$

for every i with $e \le i < n$. From the proof of Corollary 4.5 we also know that

$$\begin{aligned} h^2(\mathcal{I}_{X/\mathbb{P}^n}) &= h^1(\mathcal{I}_{X\cap\Lambda^{n-1}/\Lambda^{n-1}}(1)) - h^1(\mathcal{I}_{X/\mathbb{P}^n}(1)), \\ h^2(\mathcal{I}_{X\cap\Lambda^{n-1}/\Lambda^{n-1}}) &= h^1(\mathcal{I}_{X\cap\Lambda^{n-2}/\Lambda^{n-2}}(1)) - h^1(\mathcal{I}_{X\cap\Lambda^{n-1}/\Lambda^{n-1}}(1)). \end{aligned}$$

So, (b) is straightforward by formula (4.4). For $h^3(\mathcal{I}_{X/\mathbb{P}^n}(-1))$, we have a similar sequence as

$$\begin{aligned} H^2(\mathcal{I}_{X/\mathbb{P}^n}(-1)) \to H^2(\mathcal{I}_{X/\mathbb{P}^n}) \to H^2(\mathcal{I}_{X\cap\Lambda^{n-1}/\Lambda^{n-1}}) \to H^3(\mathcal{I}_{X/\mathbb{P}^n}(-1)) \\ \to H^3(\mathcal{I}_{X/\mathbb{P}^n}) = 0, \end{aligned}$$

since X is 3-regular. Thus, we see that

$$h^2(\mathcal{I}_{X\cap\Lambda^{n-1}/\Lambda^{n-1}}) - h^2(\mathcal{I}_{X/\mathbb{P}^n}) \le h^3(\mathcal{I}_{X/\mathbb{P}^n}(-1)) \le h^2(\mathcal{I}_{X\cap\Lambda^{n-1}/\Lambda^{n-1}}),$$

so that lower and upper bounds in (c) can be obtained by formula (4.4). □

Remark 4.7. Both lower and upper bounds for $h^3(\mathcal{I}_{X/\mathbb{P}^n}(-1))$ in Corollary 4.6(c) are sharp. See Example 4.8(b) for an optimal case of the lower bound and Example 4.8(c) for the upper bound.

Example 4.8 *(with* `Macaulay 2`*).* We check our results with some examples;

(a) Let $\widetilde{C}$ be a rational normal curve in $\mathbb{P}^{13}$ and $S^m(\widetilde{C})$ be the mth higher secant variety of dimension $\min\{2m-1, 13\}$. Now choose four points q_1, q_2, q_3, q_4 such that

$$q_1 \in \mathbb{P}^{13} \setminus S^6(\widetilde{C}), \quad q_2, q_3 \in S^6(\widetilde{C}) \setminus S^5(\widetilde{C}), \quad q_4 \in S^5(\widetilde{C}) \setminus S^4(\widetilde{C})$$

If we let $\Sigma = \langle q_1, q_2, q_3, q_4\rangle$, then we know that $C = \pi_\Sigma(\widetilde{C}) \subset \mathbb{P}^9$ is a smooth rational curve of degree 13 ($e = 8$) and, using `Macaulay 2`, we can compute its 3-regular Betti table

	0	1	2	3	4	5	6	7	8	9
0	1	.	.	.	.	.	.	.	.	.
1	.	28	103	161	134	50	6	.	.	.
2	.	3	39	190	414	518	385	168	40	4

,

where we denote the tailing Betti numbers as **boldface**. Now, we verify our theorems by this example. Since π_Σ is an isomorphic projection from $\mathbb{P}^{13}$, we get that 1-normality $h^1(\mathcal{I}_{C/\mathbb{P}^9}(1)) = 4$ and $h^1(\mathcal{I}_{C\cap H/H}(1)) = h^1(\mathcal{O}_{C\cap H}(1)) - h^1(\mathcal{O}_H(1)) = 13 - 9 = 4$. Then, by Corollary 4.2 we can see that

$$\begin{bmatrix}\beta_{8,2}\\ \beta_{9,2}\end{bmatrix} = \Xi(9,8)\cdot\begin{bmatrix}h^1(\mathcal{I}_{C\cap H/H}(1))\\ h^1(\mathcal{I}_{C/\mathbb{P}^9}(1))\end{bmatrix} = \begin{bmatrix}1 & 9\\ 0 & 1\end{bmatrix}\begin{bmatrix}4\\ 4\end{bmatrix} = \begin{bmatrix}40\\ 4\end{bmatrix},$$

which coincides with tailing Betti numbers in the table above. Moreover, $\deg(C)$ can also be read off from $8 + 1 + \binom{8}{8}\beta_{8,2} - \binom{9}{8}\beta_{9,2} = 13$ and the (arithmetic) genus of C by $\beta_{8,2} - \binom{9+1}{9}\beta_{9,2} = 0$ as Corollary 4.5 says.

(b) Let S be a smooth surface in $\mathbb{P}^8$ ($e = 6$), which is Segre embedding of $C_1 \times C_2$ where C_1 is a randomly chosen plane conic (i.e. smooth rational) curve and C_2 is a random plane cubic (i.e. elliptic) curve using `Macaulay 2`. The surface S is 3-regular and its Betti table is given by

	0	1	2	3	4	5	6	7	8
0	1	.	.	.	.	.	.	.	.
1	.	15	40	45	24	5	.	.	.
2	.	7	36	75	80	45	12	1	0

,

which shows that S is linearly normal, but not ACM surface. By Degree formula in Corollary 4.5, we can check $\deg(S) = 6 + 1 + \binom{6}{6}\cdot\mathbf{12} - \binom{7}{6}\cdot\mathbf{1} + \binom{8}{6}\cdot\mathbf{0} = 12$. The irregularity $q(S)$ is also given using tailing Betti numbers as $\mathbf{1} - (8+1)\cdot\mathbf{0} = 1$. In general, Hilbert polynomial can be given as

$$P_S(t) = 12\binom{t+1}{2} - 3\binom{t}{1},$$

by Corollary 4.5. Now, let us check the cohomology values $h^i(\mathcal{I}_{S/\mathbb{P}^8}(2-i))$. We already know that $h^1(\mathcal{I}_{S/\mathbb{P}^8}(1)) = 0$, $h^2(\mathcal{I}_{S/\mathbb{P}^8}) = h^1(\mathcal{O}_{S/\mathbb{P}^8}) = 1$. What is $h^3(\mathcal{I}_{S/\mathbb{P}^8}(-1))$? By lower and upper bounds in Corollary 4.6(c), we can bound it as

$$3 = \mathbf{12} - (8+1)\cdot\mathbf{1} + \binom{8+2}{2}\cdot\mathbf{0} \le h^3(\mathcal{I}_{S/\mathbb{P}^8}(-1)) \le \mathbf{12} - 8\cdot\mathbf{1} + \binom{8+1}{2}\cdot\mathbf{0} = 4.$$

But, by Serre's duality and Künneth's formula, we know that

$$\begin{aligned} h^3(\mathcal{I}_{S/\mathbb{P}^8}(-1)) &= h^2(\mathcal{O}_{S/\mathbb{P}^8}(-1)) = h^0(\omega_S(1)) = h^0(\omega_{C_1}(1))\cdot h^0(\omega_{C_2}(1))\\ &= h^0(\mathcal{O}_{C_1/\mathbb{P}^2}(2-3+1))\cdot h^0(\mathcal{O}_{C_2/\mathbb{P}^2}(3-3+1)) = 1\cdot 3 = 3, \end{aligned}$$

and this shows that the lower bound in Corollary 4.6(c) is sharp.

(c) Let $X \subset \mathbb{P}^{10}$ be a smooth Segre 5-fold ($e = 5$), which is a generic projection of the variety $\mathbb{P}^2 \times \mathbb{P}^3 \subset \mathbb{P}^{11}$ into $\mathbb{P}^{10}$. The degree of $X \subset \mathbb{P}^{10}$ is $\binom{3+2}{2} = 10$ and X has a 3-regular Betti table of the projective dimension $\ell_0 = 10$ as follows:

	0	1	2	3	4	5	6	7	8	9	10
0	1	0	0	0	0	0	0	0	0	0	0
1	0	6	0	0	0	0	0	0	0	0	0
2	0	20	140	331	471	465	330	165	55	11	1

.

First, we would like to mention that all the tailing Betti numbers $\beta_{i,2}$ except $\beta_{5,2}$ coincide with the lower bound $\binom{\ell_0+1}{i+1}$ in Corollary 4.3(b) in this case; that is, this lower bound is optimal. Second, by Corollary 4.2, we calculate 1-normalities of general linear sections

$$\begin{bmatrix} h^1(\mathcal{I}_{X\cap\Lambda^5/\Lambda^5}(1)) \\ h^1(\mathcal{I}_{X\cap\Lambda^6/\Lambda^6}(1)) \\ h^1(\mathcal{I}_{X\cap\Lambda^7/\Lambda^7}(1)) \\ h^1(\mathcal{I}_{X\cap\Lambda^8/\Lambda^8}(1)) \\ h^1(\mathcal{I}_{X\cap\Lambda^9/\Lambda^9}(1)) \\ h^1(\mathcal{I}_{X/\mathbb{P}^{10}}(1)) \end{bmatrix} = \Xi(10,5)^{-1} \cdot \begin{bmatrix} \beta_{5,2} \\ \beta_{6,2} \\ \beta_{7,2} \\ \beta_{8,2} \\ \beta_{9,2} \\ \beta_{10,2} \end{bmatrix}$$
$$= \begin{bmatrix} 1 & -6 & 21 & -56 & 126 & -252 \\ 0 & 1 & -7 & 28 & -84 & 210 \\ 0 & 0 & 1 & -8 & 36 & -120 \\ 0 & 0 & 0 & 1 & -9 & 45 \\ 0 & 0 & 0 & 0 & 1 & -10 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 465 \\ 330 \\ 165 \\ 55 \\ 11 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

and from these sectional 1-normalities we can compute many invariants of X; $\deg(X)$ is recovered by $h^0(\mathcal{O}_{X\cap\Lambda^5}(1)) = h^1(\mathcal{I}_{X\cap\Lambda^5/\Lambda^5}(1)) + h^0(\mathcal{O}_{\Lambda^5}(1)) = 4 + 6 = 10$ and the *sectional genus* of X, $g_s(X)$ can be obtained by $h^1(\mathcal{O}_{X\cap\Lambda^6}) = h^2(\mathcal{I}_{X\cap\Lambda^6/\Lambda^6}) = h^1(\mathcal{I}_{X\cap\Lambda^5/\Lambda^5}(1)) - h^1(\mathcal{I}_{X\cap\Lambda^6/\Lambda^6}(1)) = 4 - 1 = 3$. More generally, by Corollary 4.5 the Hilbert polynomial of X can be computed as

$$P_X(t) = 10\binom{t+4}{5} - 2\binom{t+3}{4} + \binom{t+2}{3} + \binom{t+1}{2} + \binom{t}{1} + 1,$$

which tells us $\chi(\mathcal{O}_X) = 1$ so that the irregularity $h^1(\mathcal{O}_X) = 0$. Finally, using Corollary 4.6 we check the cohomology values $h^1(\mathcal{I}_{X/\mathbb{P}^{10}}(1)) = \mathbf{1}$, $h^2(\mathcal{I}_{X/\mathbb{P}^{10}}) = \mathbf{11} - (10+1)\cdot\mathbf{1} = 0$, and $h^3(\mathcal{I}_{X/\mathbb{P}^{10}}(-1)) \leq \mathbf{55} - 10\cdot\mathbf{11} + \binom{10+1}{2}\cdot\mathbf{1} = 0$. Thus, $h^3(\mathcal{I}_{X/\mathbb{P}^{10}}(-1)) = 0$, which shows that the upper bound in Corollary 4.6(c) is also sharp.

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