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Koszul–Young flattenings and symmetric border rank of the determinant

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ABSTRACT

We present new lower bounds for the symmetric border rank of the $n \times n$ determinant for all n. Further lower bounds are given for the 3×3 permanent.

1. Introduction

The determinant polynomial is ubiquitous, its properties have been extensively studied. However basic questions regarding its complexity are still not understood. Lower bounds for the (symmetric) border rank of a polynomial provide a measurement of its complexity and, as such, have become an area of growing interest. In this paper we use techniques developed in [12] to explore this question. We prove a new lower bound for the symmetric border rank of the $n \times n$ determinant.

Definition 1.1. Let V be a vector space and let S^dV denote homogeneous degree d polynomials on V^*. Given $P \in S^dV$, define its symmetric rank $R_s(P)$ by

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$$R_s(P) = \min \left\{ r \in \mathbb{N} : P = \sum_{i=1}^{r} (v_i)^d, v_i \in V \right\}.$$

Symmetric rank is not semi-continuous under taking limits or Zariski closure, so we introduce symmetric border rank.

Definition 1.2. Let $P \in S^d V$. Define the symmetric border rank of P, $\underline{R}_s(P)$ to be

$$\underline{R}_s(P) = \min \left\{ r \in \mathbb{N} : P \in \overline{\{T : R_s(T) = r\}} \right\}$$

where the overline denotes Zariski closure.

Theorem 1.3. *For $n \geq 5$, the following are lower bounds on the symmetric border rank of the determinant, $\underline{R}_s(\det_n)$.*

For n even:

$$\underline{R}_s(\det_n) \geq \left(1 + \tfrac{8(-8+6n^2+n^3)}{(-1+n)(2+n)(4+n)^2(-2+n^2)}\right) \binom{n}{\frac{n}{2}}^2.$$

For n odd:

$$\underline{R}_s(\det_n) \geq \left(1 + \tfrac{16(9+8n+n^2)}{(3+n)(5+n)^2(-2+n^2)}\right) \binom{n}{\frac{n-1}{2}}^2.$$

Remark 1.4. Previously known lower bounds were

$$\underline{R}_s(\det_n) \geq \tbinom{n}{\frac{n}{2}}^2$$

for n even, and

$$\underline{R}_s(\det_n) \geq \tbinom{n}{\frac{n-1}{2}}^2$$

for n odd.

Remark 1.5. Asymptotically, our bound is

$$\underline{R}_s(\det_n) \gtrsim \tfrac{2^{2n+1}}{\pi \cdot n} + \tfrac{2^{2n+1}}{\pi \cdot n^4}$$

whereas the previous lower bounds are approximately $\underline{R}_s(\det_n) \gtrsim \frac{2^{2n+1}}{\pi \cdot n}$.

Theorem 1.6. $\underline{R}_s(\det_4) \geq 38$*.*

Remark 1.7. The previous bound was $\underline{R}_s(\det_4) \geq 36$.

Using a Macaulay2 [8] package developed by Steven Sam [14], we also show

Theorem 1.8.

$$\underline{R}_s(\det_3) \geq 14$$

and

$$\underline{R}_s(\mathrm{perm}_3) \geq 14.$$

Remark 1.9. The previous bounds were

$$\underline{R}_s(\det_3) \geq 9$$

and

$$\underline{R}_s(\mathrm{perm}_3) \geq 9.$$

Definition 1.10. Let $P \in S^d V$. We define the Chow rank of P, $\mathrm{rank}_{Chow}(P)$, as

$$\mathrm{rank}_{Chow}(P) = \min\{k : P = \sum_{i=1}^{k} \ell_{i1} \dots \ell_{id} \mid \ell_{ij} \in V\}.$$

In [9] it is shown that $\mathrm{rank}_{Chow}(\mathrm{perm}_3) = 4$. Prior to this it was known that $\mathrm{rank}_{Chow}(\mathrm{perm}_3) \leq 4$ [13,7]. Given $\mathrm{rank}_{Chow}(\mathrm{perm}_3) = 4$, results from [2] and [1] proving $\underline{R}_s(x_1 \cdots x_d) \leq 2^{d-1}$ show $\underline{R}_s(\mathrm{perm}_3) \leq 16$. In summary:

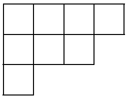
Corollary 1.11. $14 \leq \underline{R}_s(\mathrm{perm}_3) \leq 16$.

We may compare these lower bounds with known bounds on other ranks. $R_s(\det_n) \geq \binom{n}{\lfloor \frac{n}{2} \rfloor}^2 + n^2 - (\lfloor \frac{n}{2} \rfloor + 1)^2$ shown in [11], and for cactus rank, $\mathrm{krank}(\det_n) \geq \binom{2n}{n} - \binom{2n-2}{n-1}$ shown in [4] and before this it was known that $\mathrm{krank}(\det_n) \geq \frac{1}{2}\binom{2n}{n}$ [15]. Known upper bounds for symmetric rank of $\det_n$ are $R_s(\det_n) \leq \left(\frac{5}{6}\right)^{\lfloor n/3 \rfloor} 2^{n-1} n!$ [3] which also serve as upper bounds for symmetric border rank since $\underline{R}_s(T) \leq R_s(T)$ for any symmetric tensor.

2. Background

Throughout this paper Young flattenings, a tool developed and used by Landsberg and Ottaviani [12], will be used extensively. The irreducible polynomial representations of the general linear group, $GL(V)$, are parametrized by partitions π, where π has at most $\dim V$ parts, see, e.g. [5,6]. It is helpful to record these partitions visually by Young diagrams, which are left aligned diagrams consisting of boxes such that the ith row of the diagram has π_i many boxes.

Example 2.1. The Young diagram corresponding to the partition $(4, 3, 1)$ of 8 is



Proposition 2.2 *(Pieri Formula). (See e.g. [6,10].) Let $S_\pi V$ be an irreducible representation of $GL(V)$. Then as a $GL(V)$-module*

$$S_\pi V \otimes S_{(d)} V = \bigoplus_{\substack{\mu \\ \ell(\mu) \leq \dim V}} S_\mu V$$

where the partitions μ are obtained by adding d boxes to π so that no two boxes are added to the same column.

Definition 2.3. Let partitions λ and μ be such that $S_\mu V \subset S_\lambda V \otimes S_{(d)} V$. Given $P \in S_{(d)} V$ we obtain a linear map $\mathcal{F}_{\lambda,\mu}(P) : S_\lambda V \to S_\mu V$ called a **Young Flattening** via projecting the Pieri product $S_\lambda V \otimes P$ to $S_\mu V$.

Proposition 2.4. *(See Proposition 4.1 of [12].) Let $[x^d] \in v_d(\mathbb{P}V)$ and assume that rank$(\mathcal{F}_{\lambda,\mu}(x^d)) = t$. If $\underline{R}_s(P) \leq r$, then rank$(\mathcal{F}_{\lambda,\mu}(P)) \leq rt$.*

3. A preliminary result

A preliminary result will be presented to make the method clear and prove the bound in the case $n = 4$. By Proposition 2.4, to find a high lower bound for $\underline{R}_s(\det_n)$, we need to define a flattening such that $\mathrm{rank}(\mathcal{F}_{\lambda,\mu}(\det_n))$ is big and $\mathrm{rank}(\mathcal{F}_{\lambda,\mu}(x^n))$ is small. Given n dimensional vector spaces A and B, and $\alpha \in S^d(A \otimes B)^*$, we will write $\alpha \lrcorner \det_n$ to denote the tensor contraction of α and $\det_n$.

Remark 3.1. If α is a minor of the determinant in the dual space $(A \otimes B)^*$, then $\alpha \lrcorner \det_n$ is a minor on the complementary indices in the primal space.

For a tensor $\beta \in S^{n-d}(A \otimes B)$, let $\widehat{\beta} \in (A \otimes B) \otimes S^{n-d-1}(A \otimes B)$ be the image of β under partial polarization. Let $X^i_j := a_i \otimes b_j$ and for $I, J \subset [n]$ with $|I| = |J| = n - d$, let Δ^I_J denote the $(n-d) \times (n-d)$ minor on the indices in I and J.

Remark 3.2. $\widehat{\Delta}^I_J = \sum_{\substack{i \in I \\ j \in J}} (-1)^{i+j} X^i_j \otimes \Delta^{I \setminus \{i\}}_{J \setminus \{j\}}$.

Remark 3.3. The "standard" flattening of the determinant is $\det_{d,n-d} : S^d(A \otimes B)^* \longrightarrow S^{n-d}(A \otimes B)$ defined by $\alpha \mapsto \alpha \lrcorner \det_n$. Then $\mathrm{Im}(\det_{d,n-d})$ is spanned by the $(n-d) \times (n-d)$ minors of the determinant.

Define the Young flattening

$$\det{}^{\wedge 1}_{d,n-d} : \bigwedge^{n-d} A \otimes \bigwedge^{n-d} B \otimes (A \otimes B) \longrightarrow \bigwedge^{n-d-1} A \otimes \bigwedge^{n-d-1} B \otimes \bigwedge^{2} (A \otimes B)$$

$$\Delta^I_J \otimes v \mapsto \sum_{\substack{i \in I \\ j \in J}} (-1)^{i+j} X^i_j \wedge v \otimes \Delta^{[n-d] \smallsetminus \{i\}}_{[n-d] \smallsetminus \{j\}}$$

and extend linearly.

Lemma 3.4. $\mathrm{Im}(\det^{\wedge 1}_{d,n-d})$ *is contained in*

$$S_{2,1^{n-d-1}} A \otimes S_{1^{n-d+1}} B \oplus S_{1^{n-d+1}} A \otimes S_{2,1^{n-d-1}} B$$
$$\oplus S_{2,1^{n-d-1}} A \otimes S_{2,1^{n-d-1}} B. \tag{$*$}$$

Proof. Decomposing $\bigwedge^{n-d} A \otimes \bigwedge^{n-d} B \otimes A \otimes B$ as a $GL_n \times GL_n$-module we get

$$S_{2,1^{n-d-1}} A \otimes S_{1^{n-d+1}} B \oplus S_{1^{n-d+1}} A \otimes S_{2,1^{n-d-1}} B$$
$$\oplus S_{2,1^{n-d-1}} A \otimes S_{2,1^{n-d-1}} B \oplus S_{1^{n-d+1}} A \otimes S_{1^{n-d+1}}$$

and $\bigwedge^{n-d-1} A \otimes \bigwedge^{n-d-1} B \otimes \bigwedge^{2} (A \otimes B)$ as $GL_n \times GL_n$-module decomposes as

$$S_{1^{n-d+1}} A \otimes S_{3,1^{n-d-2}} B \oplus S_{2,1^{n-d-1}} A \otimes S_{3,1^{n-d-2}} B$$
$$\oplus S_{2,2,1^{n-d-3}} A \otimes S_{3,1^{n-d-2}} B \oplus S_{1^{n-d+1}} A \otimes S_{2,1^{n-d-1}} B$$
$$\oplus (S_{2,1^{n-d-1}} A \otimes S_{2,1^{n-d-1}} B)^{\oplus 2} \oplus S_{2,2,1^{n-d-3}} A \otimes S_{2,1^{n-d-1}} B$$
$$\oplus S_{3,1^{n-d-2}} A \otimes S_{1^{n-d+1}} B \oplus S_{3,1^{n-d-2}} A \otimes S_{2,1^{n-d-1}} B$$
$$\oplus S_{3,1^{n-d-2}} A \otimes S_{2,2,1^{n-d-3}} B \oplus S_{2,1^{n-d-1}} A \otimes S_{1^{n-d+1}} B$$
$$\oplus S_{2,1^{n-d-1}} A \otimes S_{2,2,1^{n-d-3}} B$$

The irreducible modules in Lemma 3.4 are the only irreducible modules appearing in both decompositions. By Schur's lemma, we conclude that the module $(*)$ must contain $\mathrm{Im}(\det^{\wedge 1}_{d,n-d})$. □

It must now be verified for each irreducible module in $(*)$, that $\det^{\wedge 1}_{d,n-d}$ is not the zero map on the module. Since each irreducible module appears with multiplicity 1, then for a given irreducible module with highest weight π, finding any highest weight vector $v \in \bigwedge^{n-d} A \otimes \bigwedge^{n-d} B \otimes (A \otimes B)$ of weight π such that $\det^{\wedge 1}_{d,n-d}(v) \neq 0$ proves $\det^{\wedge 1}_{d,n-d}$ is nonzero on the entire module.

Lemma 3.5. $\det^{\wedge 1}_{d,n-d}$ *is an isomorphism on the irreducible module* $S_{2,1^{n-d-1}} A \otimes S_{2,1^{n-d-1}} B$.

Proof. Consider $a_1 \wedge \ldots \wedge a_{n-d} \otimes a_1 \otimes b_1 \wedge \ldots \wedge b_{n-d} \otimes b_1$, a highest weight vector of the irreducible module $S_{2,1^{n-d-1}}A \otimes S_{2,1^{n-d-1}}B$. Its projection into $(A \otimes B) \otimes \bigwedge^{n-d} A \otimes \bigwedge^{n-d} B$ is a nonzero multiple of

$$X_1^1 \otimes \Delta_{[n-d]}^{[n-d]}.$$

Then

$$\begin{aligned}&\det{}_{d,n-d}^{\wedge 1}(X_1^1 \otimes \Delta_{[n-d]}^{[n-d]})\\ &= \sum_{\substack{i \in [n-d] \\ j \in [n-d]}} (-1)^{i+j} X_1^1 \wedge X_j^i \otimes \Delta_{[n-d] \setminus \{j\}}^{[n-d] \setminus \{i\}}.\end{aligned}$$

Note that the term $X_1^1 \wedge X_2^1 \otimes \Delta_{[n-d]\setminus\{2\}}^{[n-d]\setminus\{1\}}$ will not cancel in the sum. $\square$

Lemma 3.6. $\det_{d,n-d}^{\wedge 1}$ *is an isomorphism on the irreducible modules* $S_{2,1^{n-d-1}}A \otimes S_{1^{n-d+1}}B$ *and by symmetry* $S_{1^{n-d+1}}A \otimes S_{2,1^{n-d-1}}B$ *is not in the kernel.*

Proof. Consider $a_1 \wedge \ldots \wedge a_{n-d} \otimes a_1 \otimes b_1 \wedge \ldots \wedge b_{n-d+1}$, a highest weight vector of the irreducible module $S_{2,1^{n-d-1}}A \otimes S_{1^{n-d+1}}B$. Its projection into $(A \otimes B) \otimes \bigwedge^{n-d} A \otimes \bigwedge^{n-d} B$ is a nonzero multiple of

$$\sum_{j \in [n-d+1]} (-1)^j X_j^1 \otimes \Delta_{[n-d+1]\setminus\{j\}}^{[n-d]}.$$

Then

$$\begin{aligned}&\det{}_{d,n-d}^{\wedge 1}\left(\textstyle\sum_{j \in [n-d+1]} (-1)^j X_j^1 \otimes \Delta_{[n-d+1]\setminus\{j\}}^{[n-d]}\right)\\ &= \sum_{j \in [n-d+1]} \sum_{\substack{i \in [n-d] \\ k \in [n-d+1]\setminus\{j\}}} (-1)^j (-1)^{i+\tilde{k}} X_j^1 \wedge X_k^i \otimes \Delta_{[n-d+1]\setminus\{j,k\}}^{[n-d]\setminus\{i\}}\end{aligned}$$

where

$$\tilde{k} := \begin{cases} k, & k < j \\ k-1, & j < k. \end{cases}$$

Note that $X_1^1 \wedge X_2^1 \otimes \Delta_{[n-d+1]\setminus\{1,2\}}^{[n-d]\setminus\{1\}}$ does not cancel in the sum. $\square$

Finding a value of d with respect to n that maximizes the rank of $\det_{d,n-d}^{\wedge 1}$ and dividing by the rank of $[x^n]_{d,n-d}^{\wedge 1}$ we demonstrate the following theorem.

Theorem 3.7. *For $n \geq 3$, the following are lower bounds on the symmetric border rank of the determinant, $\underline{R}_s(\det_n)$.*

For n even:

$$\underline{R}_s(\det_n) \geq \left(1 + \frac{4}{(-1+n)(2+n)^2}\right)\binom{n}{\frac{n}{2}}^2$$

For n odd:

$$\underline{R}_s(\det_n) \geq \left(1 + \frac{8}{(-1+n)(3+n)^2}\right)\binom{n}{\frac{n-1}{2}}^2.$$

4. Proof of main theorem

To prove the main theorem, we use the map

$$\det{}^{\wedge 2}_{d,n-d} : \bigwedge^{n-d} A \otimes \bigwedge^{n-d} B \otimes \bigwedge^2 (A \otimes B) \longrightarrow \bigwedge^{n-d-1} A \otimes \bigwedge^{n-d-1} B \otimes \bigwedge^3 (A \otimes B)$$

defined by

$$\Delta^I_J \otimes v \wedge w \mapsto \sum_{\substack{i \in I \\ j \in J}} (-1)^{i+j} X^i_j \wedge v \wedge w \otimes \Delta^{[n-d] \smallsetminus \{i\}}_{[n-d] \smallsetminus \{j\}}$$

and extended linearly. It remains to find the rank of $\det^{\wedge 2}_{d,n-d}$.

Lemma 4.1. $\mathrm{Im}(\det^{\wedge 2}_{d,n-d})$ *is contained in*

$$\begin{aligned} & S_{3,1^{n-d-1}} A \otimes S_{1^{n-d+2}} B \oplus S_{1^{n-d+2}} A \otimes S_{3,1^{n-d-1}} B \oplus S_{3,1^{n-d-1}} A \otimes S_{2,1^{n-d}} B \\ \oplus & S_{2,1^{n-d}} A \otimes S_{3,1^{n-d-1}} B \oplus S_{3,1^{n-d-1}} A \otimes S_{2,2,1^{n-d-2}} B \\ \oplus & S_{2,2,1^{n-d-2}} A \otimes S_{3,1^{n-d-1}} B \oplus S_{2,1^{n-d+1}} A \otimes S_{2,1^{n-d+1}} B \\ \oplus & S_{2,1^{n-d+1}} A \otimes S_{2,2,1^{n-d-1}} B \oplus S_{2,2,1^{n-d-1}} A \otimes S_{2,1^{n-d+1}} B \end{aligned}$$

Proof. Decomposing $\bigwedge^{n-d} A \otimes \bigwedge^{n-d} B \otimes \bigwedge^2 (A \otimes B)$ and $\bigwedge^{n-d-1} A \otimes \bigwedge^{n-d-1} B \otimes \bigwedge^3 (A \otimes B)$ as $GL_n \times GL_n$-modules, one sees that only the irreducibles listed in the lemma appear in both decompositions and that the minimum multiplicity each appears with is 1. By Schur's Lemma, no other irreducible may be in the image. $\square$

The above lemma gives us an idea as to the largest lower bound that this particular flattening could achieve. However, we are not guaranteed that this is the image. To proceed, for each irreducible module in the lemma we must find a highest weight vector and compute $\det^{\wedge 2}_{d,n-d}$ on this vector. Note since each module appears with multiplicity 1, finding a single highest weight vector of the correct highest weight on which the flattening is nonzero is sufficient.

Lemma 4.2. $\det^{\wedge 2}_{d,n-d}$ *is an isomorphism on the irreducible module* $S_{3,1^{n-d-1}}A \otimes S_{1^{n-d+2}}B$ *and by symmetry on* $S_{1^{n-d+2}}A \otimes S_{3,1^{n-d-1}}B$.

Proof. Consider $a_1 \wedge \ldots \wedge a_{n-d} \otimes a_1 \otimes a_1 \otimes b_1 \wedge \ldots \wedge b_{n-d+2}$, a highest weight vector of the irreducible module $S_{3,1^{n-d-1}}A \otimes S_{1^{n-d+2}}B$. Its projection into $\bigwedge^2(A \otimes B) \otimes \bigwedge^{n-d} A \otimes \bigwedge^{n-d} B$ is a multiple of

$$\sum_{1 \le i < j \le n-d+2} (-1)^{i+j} X_i^1 \wedge X_j^1 \otimes \Delta^{[n-d]}_{[n-d+2] \smallsetminus \{i,j\}}.$$

Then

$$\det^{\wedge 2}_{d,n-d}\left(\sum_{1 \le i < j \le n-d+2} (-1)^{i+j} X_i^1 \wedge X_j^1 \otimes \Delta^{[n-d]}_{[n-d+2] \smallsetminus \{i,j\}}\right)$$
$$= \sum_{1 \le i < j \le n-d+2} \left(\sum_{h=1}^{n-d} \sum_{k \in [n-d+2] \smallsetminus \{i,j\}} (-1)^{\tilde{k}+h} (-1)^{i+j} X_i^1 \wedge X_j^1 \wedge X_k^h \otimes \Delta^{[n-d] \smallsetminus \{h\}}_{[n-d+2] \smallsetminus \{i,j,k\}}\right)$$

where

$$\tilde{k} := \begin{cases} k, & k < i < j \\ k-1, & i < k < j \\ k-2, & i < j < k. \end{cases}$$

Then note that the term $X_1^1 \wedge X_2^1 \wedge X_3^1 \otimes \Delta^{[n-d] \smallsetminus \{1\}}_{[n-d+2] \smallsetminus \{1,2,3\}}$ does not cancel in the sum. $\square$

Lemma 4.3. $\det^{\wedge 2}_{d,n-d}$ *is an isomorphism on the irreducible module* $S_{3,1^{n-d-1}}A \otimes S_{2,1^{n-d}}B$ *and by symmetry on* $S_{2,1^{n-d}}A \otimes S_{3,1^{n-d-1}}B$.

Proof. Consider $a_1 \wedge \ldots \wedge a_{n-d} \otimes a_1 \otimes a_1 \otimes b_1 \wedge \ldots \wedge b_{n-d+1} \otimes b_1$, a highest weight vector of the irreducible module $S_{3,1^{n-d-1}}A \otimes S_{2,1^{n-d}}B$. Its projection to $\bigwedge^2(A \otimes B) \otimes \bigwedge^{n-d} A \otimes \bigwedge^{n-d} B$ is a multiple of

$$\sum_{i=2}^{n-d+1} (-1)^i X_1^1 \wedge X_i^1 \otimes \Delta^{[n-d]}_{[n-d+1] \smallsetminus \{i\}}.$$

Then

$$\det^{\wedge 2}_{d,n-d}\left(\sum_{i=2}^{n-d+1} (-1)^i X_1^1 \wedge X_i^1 \otimes \Delta^{[n-d]}_{[n-d+1] \smallsetminus \{i\}}\right)$$
$$= \sum_{k=1}^{n-d} \sum_{i=2}^{n-d+1} \sum_{j \in [n-d+1] \smallsetminus \{i\}} (-1)^i (-1)^{\tilde{j}+k} X_1^1 \wedge X_i^1 \wedge X_j^k \otimes \Delta^{[n-d] \smallsetminus \{k\}}_{[n-d+1] \smallsetminus \{i,j\}}$$

where

$$\tilde{j} := \begin{cases} j, & j < i \\ j-1, & i < j. \end{cases}$$

The observation that $X_1^1 \wedge X_3^1 \wedge X_2^1 \otimes \Delta^{[n-d]\setminus\{1\}}_{[n-d+1]\setminus\{2,3\}}$ does not cancel demonstrates the lemma. □

Lemma 4.4. $\det^{\wedge 2}_{d,n-d}$ *is an isomorphism on the irreducible module* $S_{3,1^{n-d-1}}A \otimes S_{2,2,1^{n-d-2}}B$ *and by symmetry on* $S_{2,2,1^{n-d-2}}A \otimes S_{3,1^{n-d-1}}B$.

Proof. Consider $a_1 \wedge \ldots \wedge a_{n-d} \otimes a_1 \otimes a_1 \otimes b_1 \wedge \ldots \wedge b_{n-d} \otimes b_1 \wedge b_2$, a highest weight vector of the irreducible module $S_{3,1^{n-d-1}}A \otimes S_{2,2,1^{n-d-2}}B$. Its projection to $\bigwedge^2(A \otimes B) \otimes \bigwedge^{n-d} A \otimes \bigwedge^{n-d} B$ is a multiple of

$$X_1^1 \wedge X_2^1 \otimes \Delta^{[n-d]}_{[n-d]}.$$

Then

$$\det^{\wedge 2}_{d,n-d}\left(X_1^1 \wedge X_2^1 \otimes \Delta^{[n-d]}_{[n-d]}\right) = \sum_{i,j=1}^{n-d} (-1)^{j+i} X_1^1 \wedge X_2^1 \wedge X_j^i \otimes \Delta^{[n-d]\setminus\{i\}}_{[n-d]\setminus\{j\}}.$$

We may see this is not zero since the term $X_1^1 \wedge X_2^1 \wedge X_3^1 \otimes \Delta^{[n-d]\setminus\{1\}}_{[n-d]\setminus\{3\}}$ appears in the sum only once. □

Lemma 4.5. $\det^{\wedge 2}_{d,n-d}$ *is an isomorphism on the irreducible module* $S_{2,1^{n-d}}A \otimes S_{2,1^{n-d}}B$.

Proof. Consider $a_1 \wedge \ldots \wedge a_{n-d+1} \otimes a_1 \otimes b_1 \wedge \ldots \wedge b_{n-d+1} \otimes b_1$, a highest weight vector of the irreducible module $S_{2,1^{n-d}}A \otimes S_{2,1^{n-d}}B$. Its projection to $\bigwedge^2(A \otimes B) \otimes \bigwedge^{n-d} A \otimes \bigwedge^{n-d} B$ is a multiple of

$$\sum_{i=1}^{n-d+1} \sum_{j=2}^{n-d+1} (-1)^{i+j} X_1^1 \wedge X_j^i \otimes \Delta^{[n-d+1]\setminus\{i\}}_{[n-d+1]\setminus\{j\}} + \sum_{i=1}^{n-d+1} \sum_{j=2}^{n-d+1} (-1)^{i+j} X_1^i \wedge X_j^1 \otimes \Delta^{[n-d+1]\setminus\{i\}}_{[n-d+1]\setminus\{j\}}.$$

Then

$$\begin{aligned} &\det^{\wedge 2}_{d,n-d}\Bigg(\sum_{i=1}^{n-d+1} \sum_{j=2}^{n-d+1} (-1)^{i+j} X_1^1 \wedge X_j^i \otimes \Delta^{[n-d+1]\setminus\{i\}}_{[n-d+1]\setminus\{j\}} \\ &\qquad + \sum_{i=1}^{n-d+1} \sum_{j=2}^{n-d+1} (-1)^{i+j} X_1^i \wedge X_j^1 \otimes \Delta^{[n-d+1]\setminus\{i\}}_{[n-d+1]\setminus\{j\}}\Bigg) \\ &= \sum_{i=1}^{n-d+1} \sum_{j=2}^{n-d+1} \sum_{\substack{k \in [n-d+1]\setminus\{i\} \\ l \in [n-d+1]\setminus\{j\}}} (-1)^{i+j}(-1)^{\tilde{k}+\tilde{l}} X_1^1 \wedge X_j^i \wedge X_l^k \otimes \Delta^{[n-d+1]\setminus\{i,k\}}_{[n-d+1]\setminus\{j,l\}} \\ &+ \sum_{i=1}^{n-d+1} \sum_{j=2}^{n-d+1} \sum_{\substack{k \in [n-d+1]\setminus\{i\} \\ l \in [n-d+1]\setminus\{j\}}} (-1)^{i+j}(-1)^{\tilde{k}+\tilde{l}} X_1^i \wedge X_j^1 \wedge X_l^k \otimes \Delta^{[n-d+1]\setminus\{i,k\}}_{[n-d+1]\setminus\{j,l\}} \end{aligned}$$

where

$$\tilde{k} := \begin{cases} k, & k < i \\ k-1, & i < k \end{cases}$$

and

$$\tilde{l} := \begin{cases} l, & l < j \\ l-1, & j < l. \end{cases}$$

Since $X_1^1 \wedge X_2^1 \wedge X_1^2 \otimes \Delta_{[n-d+1]\smallsetminus\{2,1\}}^{[n-d+1]\smallsetminus\{1,2\}}$ does not cancel the lemma is proven. □

Lemma 4.6. $\det_{d,n-d}^{\wedge 2}$ *is an isomorphism on the irreducible module* $S_{2,2,1^{n-d-2}}A \otimes S_{2,1^{n-d}}B$ *and by symmetry on* $S_{2,1^{n-d}}A \otimes S_{2,2,1^{n-d-2}}B$.

Proof. Consider $a_1 \wedge \ldots \wedge a_{n-d} \otimes a_1 \wedge a_2 \otimes b_1 \wedge \ldots \wedge b_{n-d+1} \otimes b_1$, a highest weight vector of the irreducible module $S_{2,2,1^{n-d-2}}A \otimes S_{2,1^{n-d}}B$. Its projection to $\bigwedge^2(A \otimes B) \otimes \bigwedge^{n-d} A \otimes \bigwedge^{n-d} B$ is a multiple of

$$\sum_{i=1}^{n-d+1} (-1)^i X_1^1 \wedge X_i^2 \otimes \Delta_{[n-d+1]\smallsetminus\{i\}}^{[n-d]} + \sum_{i=1}^{n-d+1} (-1)^i X_i^1 \wedge X_1^2 \otimes \Delta_{[n-d+1]\smallsetminus\{i\}}^{[n-d]}.$$

Then

$$\det_{d,n-d}^{\wedge 2}\left(\sum_{i=1}^{n-d+1} (-1)^i X_1^1 \wedge X_i^2 \otimes \Delta_{[n-d+1]\smallsetminus\{i\}}^{[n-d]} + \sum_{i=1}^{n-d+1} (-1)^i X_i^1 \wedge X_1^2 \otimes \Delta_{[n-d+1]\smallsetminus\{i\}}^{[n-d]}\right)$$
$$= \sum_{k=1}^{n-d}\sum_{i=1}^{n-d+1}\sum_{j\in[n-d+1]\smallsetminus\{i\}} (-1)^i(-1)^{\tilde{j}+k} X_1^1 \wedge X_i^2 \wedge X_j^k \otimes \Delta_{[n-d+1]\smallsetminus\{i,j\}}^{[n-d]\smallsetminus\{k\}}$$
$$+ \sum_{k=1}^{n-d}\sum_{i=1}^{n-d+1}\sum_{j\in[n-d+1]\smallsetminus\{i\}} (-1)^i(-1)^{\tilde{j}+k} X_i^1 \wedge X_1^2 \wedge X_j^k \otimes \Delta_{[n-d+1]\smallsetminus\{i,j\}}^{[n-d]\smallsetminus\{k\}}$$

where

$$\tilde{j} := \begin{cases} j, & j < i \\ j-1, & i < j. \end{cases}$$

Observing that $X_1^1 \wedge X_1^2 \wedge X_2^1 \otimes \Delta_{[n-d+1]\smallsetminus\{1,2\}}^{[n-d]\smallsetminus\{1\}}$ does not cancel proves the lemma. □

Lemma 4.7. *The image of* $\det_{d,n-d}^{\wedge 2}$ *consists of all of the irreducible modules in the decomposition in Lemma 4.1.*

Proof. This is demonstrated by the preceding lemmas. □

Lemma 4.8. $\dim(\mathrm{Im}(\det^{\wedge 2}_{d,n-d}))$ *has a maximum at* $d = \lfloor \frac{n}{2} \rfloor$.

Proof. Begin by factoring $\dim(\mathrm{Im}(\det^{\wedge 2}_{d,n-d}))$ into the form $f(n,d)\binom{n}{d}^2$, where $f(n,d)$ is a rational function of n and d. In particular

$$\begin{aligned} f(n,d) := {} & \frac{(n+2)(n+1)(n-d)(d)(d-1)}{(n-d+2)^2(n-d+1)} + \frac{(n+2)(n+1)^2(n-d)(d)}{(n-d+2)^2} \\ & + \frac{(n+2)(n+1)^2(n-d)(n)(n-d-1)}{2(n-d+2)(n-d+1)} + \frac{(n+1)^2(n)(n-d-1)(d)}{(n-d+1)(n-d+2)} \\ & + \frac{(n+1)^2(d)^2}{(n-d+2)^2} \end{aligned}$$

Then consider

$$f(n,d)\binom{n}{d}^2 - f(n,d+1)\binom{n}{d+1}^2$$

and rewrite it as

$$\left(f(n,d) - f(n,d+1)\frac{(n-d)^2}{(d+1)^2}\right)\binom{n}{d}^2.$$

Notice that $f(n,d)-f(n,d+1)\frac{(n-d)^2}{(d+1)^2} < 0$ for $d = \lfloor \frac{n}{2} \rfloor - 1$ and $f(n,d)-f(n,d+1)\frac{(n-d)^2}{(d+1)^2} > 0$ for $d = \lfloor \frac{n}{2} \rfloor$ and conclude the lemma. □

Remark 4.9. The requirement for $n \geq 5$ in the main theorem, is so that the length of all partitions $S_{1^{n-d+2}}A$, $S_{2,1^{n-d}}A$, $S_{3,1^{n-d-1}}A$, and $S_{2,2,1^{n-d-2}}A$ (respectively B) does not exceed $\dim(A) = \dim(B) = n$. Hence, all of the irreducible modules in the decomposition in Lemma 4.1 occur when $d = \lfloor \frac{n}{2} \rfloor$.

Remark 4.10. $\mathrm{rank}([(X^i_j)^n]^{\wedge 2}_{d,n-d}) = \binom{n^2-1}{2}$. Which may be verified easily by noticing the image of contracting $\alpha \in S^d(A \otimes B)^*$ with $(X^i_j)^n$ is in the span of $(X^i_j)^{n-d}$ and $\widehat{(X^i_j)^{n-d}}$ is in the span of $(X^i_j)^{n-d-1} \otimes X^i_j$. Hence $\mathrm{Im}([(X^i_j)^n]^{\wedge 2}_{d,n-d}))$ are of the form $(X^i_j)^{n-d-1} \otimes X^i_j \wedge v \wedge w$, where v and w cannot be in the span of X^i_j.

The main theorem follows by substituting $\lfloor \frac{n}{2} \rfloor$ into $f(n,d)$ from the proof of Lemma 4.8, dividing by $\binom{n^2-1}{2}$ which is the rank from Remark 4.10, and simplifying.

5. 3 × 3 Determinant and permanent

Define the partitions $\pi_n = ((n-1)^{n+1}, (n-2)^{n+1}, \ldots, 1^{n+1})$ and $\tilde{\pi}_n = (n, \pi_n)$. For example, $\pi_3 = (2^4, 1^4)$ and let $\tilde{\pi}_3 = (3, 2^4, 1^4)$. Note that $\dim(S_{\pi_3}\mathbb{C}^9) = \dim(S_{\tilde{\pi}_3}\mathbb{C}^9) = 1050$. For a polynomial $\phi \in S^3\mathbb{C}^9$, define the Young flattening

$$\mathcal{F}_{\pi_3,\tilde{\pi}_3}(\phi) : S_{\pi_3}\mathbb{C}^9 \to S_{\tilde{\pi}_3}\mathbb{C}^9$$

by the labeled Pieri product restricted to shape $\tilde{\pi}_3$

$$T_{\pi_3} \otimes \phi = \sum c_{T_{\pi_3},\tilde{T}_{\tilde{\pi}_3}} \tilde{T}_{\tilde{\pi}_3}$$

where T_{π_3} and $\tilde{T}_{\tilde{\pi}_3}$ are semi-standard fillings of tableaux of shape π_3 and $\tilde{\pi}_3$ respectively and where $c_{T_{\pi_3},\tilde{T}_{\tilde{\pi}_3}}$ is obtained by adding boxes to π_3 such as to obtain a tableau of shape $\tilde{\pi}_3$ and for each monomial in ϕ, label the boxes with the variable names in all permutations and straighten. $c_{T_{\pi_3},\tilde{T}_{\tilde{\pi}_3}}$ is the coefficient of $\tilde{T}_{\tilde{\pi}_3}$.

Consider the polynomial $(x_{3,3})^3 \in S^3\mathbb{C}^9$, we immediately see that if T_{π_3} has any box labeled $x_{3,3}$, then $\mathcal{F}_{\pi_3,\tilde{\pi}_3}((x_{3,3})^3) = 0$. Since this is the only restriction of tableaux,

$$\dim \operatorname{Im}(\mathcal{F}_{\pi_3,\tilde{\pi}_3}((x_{3,3})^3)) = \dim S_{\pi_3}\mathbb{C}^8 = 70.$$

By Proposition 2.4, if $[x^3] \in v_3(\mathbb{P}\mathbb{C}^9)$ has $\operatorname{rank}(\mathcal{F}_{\mu,\nu}(x^3)) = p$, then for $[\phi] \in \mathbb{P}S^3\mathbb{C}^9$ with rank r, $\operatorname{rank}(\mathcal{F}_{\mu,\nu}(\phi)) \leq rp$. Thus the maximum lower bound on symmetric border rank on polynomial $\phi \in S^3\mathbb{C}^9$ this method may achieve is

$$\underline{R}_s(\phi) \geq 15$$

This being when $\dim \operatorname{Im}(\mathcal{F}_{\pi_3,\tilde{\pi}_3}(\phi)) = 1050$. Applying this flattening to $\det_3$ and perm_3 and using the Macaulay2 [8] package `PieriMaps` developed by Steven Sam [14] we get

$$\dim \operatorname{Im}(\mathcal{F}_{\pi_3,\tilde{\pi}_3}(\det_3)) = 950$$

and

$$\dim \operatorname{Im}(\mathcal{F}_{\pi_3,\tilde{\pi}_3}(\operatorname{perm}_3)) = 934.$$

These give the following lower bounds

$$\underline{R}_s(\det_3) \geq 14$$

and

$$\underline{R}_s(\operatorname{perm}_3) \geq 14.$$

This is an improvement from the classical lower bound for the determinant of 9 and the bound obtained from the Koszul–Young flattening $\det_{1,2}^{\wedge 2}$ of 12.

The following code is used to complete the above computations.

```
loadPackage"PieriMaps"
A=QQ[x_(0,0)..x_(2,2)]
time MX = pieri({3,2,2,2,2,1,1,1,1},{1,5,9},A);
rank diff(x_(0,0)^3,MX)
f = det genericMatrix(A,x_(0,0), 3,3)
rank diff(f,MX)
g =x_(0,2)*x_(1,1)*x_(2,0)+x_(0,1)*x_(1,2)*x_(2,0)+
   x_(0,2)*x_(1,0)*x_(2,1)+x_(0,0)*x_(1,2)*x_(2,1)+
   x_(0,1)*x_(1,0)*x_(2,2)+x_(0,0)*x_(1,1)*x_(2,2)
rank diff(g,MX)
```

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