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POWER SERIES OVER NOETHERIAN DOMAINS, NAGATA RINGS, AND KRONECKER FUNCTION RINGS

GYU WHAN CHANG

Abstract. Let D be a Noetherian domain, $*$ be a star operation on D, X be an indeterminate over D, $D[\![X]\!]$ be the power series ring over D, $c(f)$ be the ideal of D generated by the coefficients of $f \in D[\![X]\!]$, and $N_* = \{f \in D[\![X]\!] \mid c(f)^* = D\}$. Moreover, if $*$ is *e.a.b.*, then we let $\mathrm{Kr}(\!(D,*)\!) = \{\frac{f}{g} \mid f, g \in D[\![X]\!], 0 \neq g$, and $c(f) \subseteq c(g)^*\}$. In this paper, we show that N_* is a saturated multiplicative set and $\mathrm{Kr}(\!(D,*)\!)$ is a Bezout domain. We then study some ring-theoretic properties of $D[\![X]\!]_{N_*}$ and $\mathrm{Kr}(\!(D,*)\!)$. For example, we prove that every invertible ideal of $D[\![X]\!]_{N_*}$ is principal; $\dim(\mathrm{Kr}(\!(D,b)\!)) = \dim_v(D)$; and if V is a valuation overring of D, then $\widehat{V} = \{\frac{f}{g} \mid f, g \in D[\![X]\!], g \neq 0$, and $c(f)V \subseteq c(g)V\}$ is a valuation overring of $D[\![X]\!]$; and $\mathrm{Kr}(\!(D,*)\!) = \bigcap\{\widehat{V} \mid V$ is a $*$-valuation overring of $D\}$.

0. Introduction

Let D be an integral domain, $qf(D)$ be the quotient field of D, $\{X_\alpha\}$ be a nonempty set of indeterminates over D, and $D[\{X_\alpha\}]$ be the polynomial ring over D. For indeterminates $X_1, \ldots, X_k$ over D, let $D[\![X_1, \ldots, X_k]\!]$ be the power series ring over D, and let $D[\![\{X_\alpha\}]\!]_1 = \bigcup D[\![\{X_{\alpha_i}\}]\!]$, where $\{X_{\alpha_i}\}$ runs over all finite subsets of $\{X_\alpha\}$. Clearly, $D[\![\{X_\alpha\}]\!]_1$ is an integral domain such that $D[\{X_\alpha\}] \subsetneq D[\![\{X_\alpha\}]\!]_1$ and $D[\![\{X_\alpha\}]\!]_1 = D[\![\{X_\alpha\}]\!]$ when $\{X_\alpha\}$ is finite. For $f \in D[\![\{X_\alpha\}]\!]_1$, let $c_D(f)$ (simply, $c(f)$) denote the ideal of D generated by the coefficients of f.

Let $*$ be a star operation on D. (Definitions related to star operations will be reviewed in Section 1.) There are two purposes of this paper. One of them is to generalize the Nagata ring to power series rings over a Noetherian domain. This also generalizes the results of [11] that if $\{X_\alpha\}$ is infinite, then $N_{*_f} := \{f \in D[\![\{X_\alpha\}]\!]_1 \mid c(f)^{*_f} = D\}$ is a saturated multiplicative set such that $\mathrm{Max}((D[\![\{X_\alpha\}]\!]_1)_{N_{*_f}}) = \{(P[\![\{X_\alpha\}]\!]_1)_{N_{*_f}} \mid P \in *_f\text{-}\mathrm{Max}(D)\}$; every invertible ideal of $(D[\![\{X_\alpha\}]\!]_1)_{N_{*_f}}$ is principal; and D is a Krull domain if and only if $(D[\![\{X_\alpha\}]\!]_1)_{N_{*_f}}$ is a Prüfer domain. The other is to study the power series ring analog of Kronecker function rings.

Precisely, in Section 1, we review basic facts on star operations, Nagata rings, and Kronecker function rings. Let $*$ be a star operation on a Noetherian domain D, $D[\![\{X_\alpha\}]\!] = D[\![\{X_\alpha\}]\!]_1$, and $N_* = \{f \in D[\![\{X_\alpha\}]\!] \mid c(f)^* = D\}$. In Section 2, we show that (i) N_* is a saturated multiplicative set and $\mathrm{Max}(D[\![\{X_\alpha\}]\!]_{N_*}) = \{P[\![\{X_\alpha\}]\!]_{N_*} \mid P \in *\text{-}\mathrm{Max}(D)\}$; (ii) $D[\![\{X_\alpha\}]\!]_{N_*}$ is a Noetherian domain with

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$\dim(D[\![\{X_\alpha\}]\!]_{N_*}) = *_w\text{-}\dim(D)$; (iii) if $0 \neq f \in D[\![\{X_\alpha\}]\!]$, then $c(f)$ is $*$-invertible if and only if $fD[\![\{X_\alpha\}]\!]_{N_*} = c(f)D[\![\{X_\alpha\}]\!]_{N_*}$; and (iv) every invertible ideal of $D[\![\{X_\alpha\}]\!]_{N_*}$ is principal.

We in Section 3 introduce the notion of Kronecker function rings of the power series ring over a Noetherian domain. Let $*$ be an *e.a.b.* star operation on a Noetherian domain, $K = qf(D)$, X be an indeterminate over D, and

$$\mathrm{Kr}(\!(D,*)\!) = \{\tfrac{f}{g} \mid f, g \in D[\![X]\!], 0 \neq g, \text{ and } c(f) \subseteq c(g)^*\}.$$

We show that (v) $\mathrm{Kr}(\!(D,*)\!)$ is a Bezout domain such that $\mathrm{Kr}(\!(D,*)\!) \cap K = D$, $\mathrm{Kr}(\!(D,*)\!) \cap K(X) = \mathrm{Kr}(D,*)$, and $I\mathrm{Kr}(\!(D,*)\!) \cap K = I^*$ for all nonzero ideals I of D; (vi) for a valuation overring V of D, if we let

$$\widehat{V} = \{\tfrac{f}{g} \mid f, g \in D[\![X]\!], g \neq 0, \text{ and } c(f)V \subseteq c(g)V\},$$

then $\widehat{V}$ is a valuation domain such that $\widehat{V} \cap K(X) = V(X)$, $\dim(\widehat{V}) = \dim(V)$, and V is discrete if and only if $\widehat{V}$ is discrete; (vii) $\mathrm{Kr}(\!(D,*)\!) = \bigcap\{\widehat{V} \mid V$ is a $*$-valuation overring of $D\}$; and (viii) $\mathrm{Kr}(\!(D,b)\!) = D(\!(X)\!)$ if and only if $\dim(D) = 1$, if and only if $\mathrm{Kr}(\!(D,b)\!) = \mathrm{Kr}(\!(D,v)\!)$.

1. Star operations, Nagata rings and Kronecker function rings

Let D be an integral domain with quotient field K. Let $F(D)$ (resp., $f(D)$) be the set of nonzero (resp., nonzero finitely generated) fractional ideals of D; so $f(D) \subseteq F(D)$, and $f(D) = F(D)$ if and only if D is Noetherian. A *star operation* $*$ on D is a mapping $* : F(D) \to F(D)$, $I \mapsto I^*$ such that for all $I, J \in F(D)$ and $0 \neq x \in K$, (i) $(xD)^* = xD$ and $(xI)^* = xI^*$, (ii) $I \subseteq I^*$, and $I \subseteq J$ implies $I^* \subseteq J^*$, and (iii) $(I^*)^* = I^*$. Given a star operation $*$ on D, one can construct two new star operations $*_f$ and $*_w$ on D by setting $I^{*_f} = \bigcup\{J^* \mid J \subseteq I$ and $J \in f(D)\}$ and $I^{*_w} = \{x \in K \mid xJ \subseteq I$ for some $J \in f(D)$ with $J^* = D\}$ for all $I \in F(D)$. We say that $*$ is of *finite type* if $*_f = *$. Clearly, $*_f$ and $*_w$ are of finite type. Also, if $I \in f(D)$, then $I^* = I^{*_f}$, and thus every star operation on a Noetherian domain is of finite type. An $I \in F(D)$ is called a *$*$-ideal* if $I^* = I$, while a $*$-ideal is a *maximal $*$-ideal* if it is maximal among proper integral $*$-ideals. Let $*\text{-Max}(D)$ be the set of maximal $*$-ideals of D. It is well known that if $* = *_f$ or $*_w$, then $*\text{-Max}(D) \neq \emptyset$ when D is not a field; each maximal $*$-ideal is a prime ideal; $D = \bigcap_{P \in *\text{-Max}(D)} D_P$; and $*_f\text{-Max}(D) = *_w\text{-Max}(D)$ [4, Theorem 2.16]. Let $*\text{-}\dim(D) = \sup\{n \mid P_1 \subsetneq P_2 \subsetneq \cdots \subsetneq P_n$ is a chain of prime $*$-ideals of $D\}$. Hence, $d\text{-}\dim(D)$ (resp., $d\text{-Max}(D)$) is just the (Krull) dimension $\dim(D)$ (resp., $\mathrm{Max}(D)$, the set of maximal ideals) of D. Also, $*_w\text{-}\dim(D) = \sup\{\mathrm{ht}P \mid P \in *_w\text{-Max}(D)\}$ because $P^{*_w} \subsetneq D$ implies $P^{*_w} = P$ for all nonzero prime ideals P of D.

The most well known examples of star operations include the v-, t-, w-, and d-operations. For $I \in F(D)$, let $I^{-1} = \{x \in K \mid xI \subseteq D\}$; so $I^{-1} \in F(D)$. The v-operation is defined by $I^v = (I^{-1})^{-1}$, the t-operation (resp., w-operation) is given by $t = v_f$ (resp., $w = v_w$), and the d-operation is the identity function on $F(D)$, i.e., $I^d = I$ for all $I \in F(D)$; so $d_f = d_w = d$. Let $*_1$ and $*_2$ be star operations on D. We say that $*_1 \leq *_2$ if $I^{*_1} \subseteq I^{*_2}$ for all $I \in F(D)$. It is known that if $*$ is a star operation, then $*_w \leq *_f \leq *$, $d \leq * \leq v$, $*_f \leq t$, and $*_w \leq w$.

An $I \in F(D)$ is said to be $*$-invertible if $(II^{-1})^* = D$. We say that D is a *Prüfer $*$-multiplication domain* (P$*$MD) if each $I \in f(D)$ is $*_f$-invertible. Let $\text{Inv}^*(D)$ be the group of $*$-invertible fractional ideals of D under $I \times J = (IJ)^*$, and let $\text{Prin}(D)$ be its subgroup of principal fractional ideals. Then the *$*$-class group* of D is an abelian group $\text{Cl}_*(D) = \text{Inv}^*(D)/\text{Prin}(D)$. In particular, $\text{Cl}_d(D)$ is the Picard group $\text{Pic}(D)$ of D. Clearly, $\text{Pic}(D) \subseteq \text{Cl}_*(D) \subseteq \text{Cl}_t(D)$ for any star operation $*$ on D, $\text{Cl}_*(D) = \{0\}$ if and only if each $*$-invertible ideal of D is principal, and if D is a Krull domain, then $Cl_t(D)$ is the usual divisor class group. Also, a Prüfer (resp., Krull) domain D is a Bezout domain (resp., UFD) if and only if $Cl_t(D) = \{0\}$.

1.1. **Nagata rings.** Let $*$ be a star operation on D and $N_* = \{f \in D[\{X_\alpha\}] \mid c(f)^* = D\}$. The Dedekind-Mertens Lemma says that if $f, g \in D[\{X_\alpha\}]$, then there exists an integer $n = n(f,g) \geq 1$ such that $c(f)^{n+1}c(g) = c(f)^n c(fg)$. Hence, $c(f)^* = c(g)^* = D$ if and only if $c(fg)^* = D$, and thus N_* is a saturated multiplicative subset of $D[\{X_\alpha\}]$ such that $N_* = N_{*_f} = N_{*_w}$. The ring $D[\{X_\alpha\}]_{N_*}$ is called the ($*$-)*Nagata ring* of D (with respect to $*$), and hence $D[\{X_\alpha\}]_{N_d}$ is just the Nagata ring $D(\{X_\alpha\})$ of D. Recall from [19] that $\text{Max}(D[\{X_\alpha\}]_{N_*}) = \{P[\{X_\alpha\}]_{N_*} \mid P \in *_f\text{-Max}(D)\}$; a nonzero ideal I of D is $*_f$-invertible if and only if $ID[\{X_\alpha\}]_{N_*}$ is invertible; for $0 \neq f \in D[\{X_\alpha\}]$, $c(f)$ is $*_f$-invertible if and only if $c(f)D[\{X_\alpha\}]_{N_*} = fD[\{X_\alpha\}]_{N_*}$; every invertible ideal of $D[\{X_\alpha\}]_{N_*}$ is principal; and D is a P$*$MD if and only if $D[\{X_\alpha\}]_{N_*}$ is a Prüfer domain. It is obvious that if $*_1$ and $*_2$ are star operations with $(*_1)_f \leq (*_2)_f$, then $N_d \subseteq N_{*_1} \subseteq N_{*_2} \subseteq N_v$.

Lemma 1.1. *The following statements are equivalent for star operations $*_1$ and $*_2$ on D.*

(1) $N_{*_1} = N_{*_2}$.
(2) $D[\{X_\alpha\}]_{N_{*_1}} = D[\{X_\alpha\}]_{N_{*_2}}$.
(3) $(*_1)_f$-*Max*$(D) = (*_2)_f$-*Max*(D).
(4) $(*_1)_w = (*_2)_w$.

Proof. (1) $\Leftrightarrow$ (2) This follows from the fact that N_{*_i} is saturated for $i = 1, 2$.

(1) $\Rightarrow$ (3) Let $M \in (*_1)_f\text{-Max}(D)$. If $M^{(*_2)_f} = D$, then there is a nonzero finitely generated ideal $I \subseteq M$ such that $I^{*_2} = D$. So if we choose $f \in D[\{X_\alpha\}]$ with $c(f) = I$, then $c(f)^{*_2} = D$. Hence, $f \in N_{*_1} = N_{*_2}$, and thus $D = c(f)^{*_1} \subseteq M^{*_1} = M$, a contradiction. Thus, $M^{(*_2)_f} \subsetneq D$. A similar argument shows that $(M^{(*_2)_f})^{(*_1)_f} \subsetneq D$, and therefore, $M^{(*_2)_f} = M$ and $M \in (*_2)_f\text{-Max}(D)$. Thus, $(*_1)_f\text{-Max}(D) \subseteq (*_2)_f\text{-Max}(D)$, and by a symmetric argument, we have equality.

(3) $\Rightarrow$ (4) If $I \in F(D)$, then $I^{(*_1)_w} = \bigcap_{P \in (*_1)_f\text{-Max}(D)} ID_P = \bigcap_{P \in (*_2)_f\text{-Max}(D)} ID_P = I^{(*_2)_w}$ [4, Corollary 2.10]. Thus, $(*_1)_w = (*_2)_w$.

(4) $\Rightarrow$ (1) Since $*_f\text{-Max}(D) = *_w\text{-Max}(D)$ for any star operation $*$ on D, $c(f)^{*_w} = D$ if and only if $c(f)^{*_f} = D$ for all $0 \neq f \in D[\{X_\alpha\}]$. Hence, $c(f)^{*_1} = D$ if and only if $c(f)^{*_2} = D$, and thus, $N_{*_1} = N_{*_2}$. $\square$

Corollary 1.2. *If each maximal ideal of D is a t-ideal (e.g., $dim(D) = 1$), then $D[\{X_\alpha\}]_{N_*} = D(\{X_\alpha\})$ for every star operation $*$ on D.*

Proof. This follows directly from Lemma 1.1 because $\text{Max}(D) = t\text{-Max}(D)$ (and hence $d = *_w = w$) by assumption. $\square$

It is shown that if D is a Noetherian domain with finitely many star operations, then $\dim(D) = 1$ [16, Theorem 2.1], and hence each maximal ideal of D is a t-ideal. Thus, $D[\{X_\alpha\}]_{N_*} = D(\{X_\alpha\})$ for any star operation $*$ on D.

1.2. **Kronecker function rings.** A star operation $*$ on D is said to be *endlich arithmetisch brauchbar* (*e.a.b.*) if, for $A, B, C \in f(D)$, $(AB)^* \subseteq (AC)^*$ implies $B^* \subseteq C^*$. Clearly, $*$ is *e.a.b.* if and only if $*_f$ is *e.a.b.*. We know that if D admits an *e.a.b.* star operation, then D is integrally closed [15, Corollary 32.8]. Conversely, if D is integrally closed, then the map $b : F(D) \to F(D)$ defined by $I^b = \bigcap\{IV \mid V$ is a valuation overring of $D\}$ is an *e.a.b.* star operation on D.

Lemma 1.3. [15, Theorem 32.12] *Let $*$ be an e.a.b. star operation of finite type on an integrally closed domain D. Then there exists a set $\{V_\alpha\}$ of valuation overrings of D such that $I^* = \bigcap_\alpha IV_\alpha$ for all $I \in F(D)$.*

If $*$ is an *e.a.b.* star operation on D, then

$$\mathrm{Kr}(D, *) = \{\tfrac{f}{g} \mid f, g \in D[\{X_\alpha\}], g \neq 0, \text{ and } c(f) \subseteq c(g)^*\},$$

called the *Kronecker function ring* of D (with respect to $*$), is a Bezout domain such that $D[\{X_\alpha\}]_{N_*} \subseteq \mathrm{Kr}(D, *)$, $f\mathrm{Kr}(D, *) = c(f)\mathrm{Kr}(D, *)$ for $f \in D[\{X_\alpha\}]$, and D is a P$*$MD if and only if $D[\{X_\alpha\}]_{N_*} = \mathrm{Kr}(D, *)$.

Lemma 1.4. [15, Remark 32.7] *Let $*_1$ and $*_2$ be e.a.b. star operations on D. Then $Kr(D, *_1) = Kr(D, *_2)$ if and only if $(*_1)_f = (*_2)_f$.*

We recall that if D is an integrally closed Noetherian domain, then D is a Krull domain, and hence each maximal t-ideal of D is of height-one, i.e., t-Max$(D) = X^1(D)$, where $X^1(D)$ is the set of height-one prime ideals of D.

Let $eS(D)$ (resp., $wS(D)$) be the set of *e.a.b.* star operations of finite type (resp., star operations $*$ with $*_w = *$) on D. Then, by Lemmas 1.1 and 1.4, the cardinality of the set of Nagata rings (resp., Kronecker function rings) of D is equal to $|wS(D)|$ (resp., $|eS(D)|$). We next study how many Nagata rings and Kronecker functions rings there are when D is an integrally closed Noetherian domain.

Proposition 1.5. *Let D be an integrally closed Noetherian domain and let $X(D)$ be the set of height-two prime ideals of D. Then $2^{|X(D)|} \leq |wS(D)| \leq |eS(D)|$, and hence if $dim(D) \geq 3$, then $eS(D)$ and $wS(D)$ are both uncountable.*

Proof. Let Λ be a set of height-two prime ideals of D, i.e., $\Lambda \subseteq X(D)$, and let $\Delta = \{P \in X^1(D) \mid P \nsubseteq Q$ for all $Q \in \Lambda\}$. Then $D = \bigcap_{P \in \Lambda \cup \Delta} D_P$, and so if we let $I^{*_\Lambda} = \bigcap_{P \in \Lambda \cup \Delta} ID_P$ for all $I \in F(D)$, then $*_\Lambda$ is a star operation on D [3, Theorem 1] and $*_\Lambda$-Max$(D) = \Lambda \cup \Delta$ (*Proof.* If $P \in \Lambda \cup \Delta$, then, clearly, $P^{*_\Lambda} = P$. Also, if M is a prime ideal of D with $P \subsetneq M$, then $M^{*_\Lambda} = D$. Hence, P is a maximal $*_\Lambda$-ideal. Conversely, if P_0 is a maximal $*_\Lambda$-ideal, then $P_0 \subseteq P$ for some $P \in \Lambda \cup \Delta$, and since P is a maximal $*_\Lambda$-ideal by the previous sentence, we have $P_0 = P$.).

Hence, if Λ_1 and Λ_2 are two distinct sets of height-two prime ideals of D, then $(*_{\Lambda_1})_w \neq (*_{\Lambda_2})_w$ by Lemma 1.1. Thus, $2^{|X(D)|} \leq |wS(D)|$. Next, recall from [8, Lemma 3.1] that if $*$ is a star operation of finite type, then we can construct an *e.a.b.* star operation $*_c$ such that $*_w \leq *_c$ and $*$-Max$(D) = *_c$-Max(D). This means that $|wS(D)| \leq |eS(D)|$. Finally, note that if $\dim D \geq 3$, then $|X(D)| = \infty$ [20, Theorem 144], and thus $eS(D)$ and $wS(D)$ are both uncountable. $\square$

For more on Kronecker function rings and Nagata rings, see Fontana-Loper's interesting survey article [13] or [15, Section 32].

2. Nagata Rings

Let D be an integral domain, $qf(D) = K$, $*$ be a star operation on D, $\{X_\alpha\}$ be a nonempty set of indeterminates over D,

$$D[\![\{X_\alpha\}]\!] = D[\![\{X_\alpha\}]\!]_1 \text{ and } N_* = \{f \in D[\![\{X_\alpha\}]\!] \mid c(f)^* = D\}.$$

Let I be a nonzero ideal of D. Then $I[\![\{X_\alpha\}]\!] = \{f \in D[\![\{X_\alpha\}]\!] \mid c(f) \subseteq I\}$, and hence $I[\![\{X_\alpha\}]\!]$ is an ideal of $D[\![\{X_\alpha\}]\!]$ such that $ID[\![\{X_\alpha\}]\!] \subseteq I[\![\{X_\alpha\}]\!]$; equality holds when I is finitely generated; and I is a prime ideal if and only if $I[\![\{X_\alpha\}]\!]$ is a prime ideal.

Lemma 2.1. *Let D be a Noetherian domain.*

(1) $N_* = D[\![\{X_\alpha\}]\!] \setminus \bigcup_{P \in *\text{-}Max(D)} P[\![\{X_\alpha\}]\!]$. *Hence, N_* is a saturated multiplicative subset of $D[\![\{X_\alpha\}]\!]$.*

(2) $Max(D[\![\{X_\alpha\}]\!]_{N_*}) = \{P[\![\{X_\alpha\}]\!]_{N_*} \mid P \in *\text{-}Max(D)\}$.

Proof. (1) $f \in N_* \Leftrightarrow c(f)^* = D$, $\Leftrightarrow c(f) \nsubseteq P$ for all $P \in *\text{-Max}(D)$, $\Leftrightarrow f \notin P[\![\{X_\alpha\}]\!]$ for all $P \in *\text{-Max}(D)$, $\Leftrightarrow f \in D[\![\{X_\alpha\}]\!] \setminus \bigcup_{P \in *\text{-Max}(D)} P[\![\{X_\alpha\}]\!]$.

(2) Let A be an ideal of $D[\![\{X_\alpha\}]\!]$ such that $A \nsubseteq P[\![\{X_\alpha\}]\!]$ for all $P \in *\text{-Max}(D)$. By [15, Proposition 4.8], it suffices to show that there is an $f \in A$ such that $f \notin P[\![\{X_\alpha\}]\!]$ for all $P \in *\text{-Max}(D)$; equivalently, $c(f)^* = D$. Note that $*_f = *$; so we can choose some $f_1, \dots, f_n \in A$ such that $(c(f_1) + \cdots + c(f_n))^* = D$. Since D is Noetherian, we can choose a polynomial $g_i \in D[\{X_\alpha\}]$ such that $c(g_i) = c(f_i)$ and $f_i - g_i$ does not have a monomial of degree $\leq \deg(g_i)$. Let $X \in \{X_\alpha\}$ and

$$f = f_1 + f_2 X^{\mu(g_1)+1} + f_3 X^{\mu(g_1)+\mu(g_2)+2} + \cdots + f_n X^{\mu(g_1)+\cdots+\mu(g_{n-1})+n-1},$$

where $\mu(g_i) = \deg(g_i)$ for $i = 1, \dots, n-1$. Then $f \in A$ and $c(f)^* = D$. □

As in the polynomial ring case, we call $D[\![\{X_\alpha\}]\!]_{N_*}$ the *($*$-)Nagata ring* of D with respect to $*$. In particular, we denote by $D(\!(\{X_\alpha\})\!)$ the Nagata ring $D[\![\{X_\alpha\}]\!]_{N_*}$ when $* = d$. The next corollary with Proposition 1.5 shows that a Noetherian domain D has at least $2^{|X(D)|}$ Nagata rings, where $X(D)$ is the set of height-two prime ideals of D.

Corollary 2.2. *Let $*_1$, $*_2$, and $*$ be star operations on a Noetherian domain D.*

(1) $N_d \subseteq N_* \subseteq N_v$.

(2) $D[\![\{X_\alpha\}]\!]_{N_{*_1}} = D[\![\{X_\alpha\}]\!]_{N_{*_2}}$ *if and only if* $N_{*_1} = N_{*_2}$, *if and only if* $*_1$-$Max(D) = *_2$-$Max(D)$, *if and only if* $(*_1)_w = (*_2)_w$.

(3) *If each maximal ideal of D is a t-ideal, then* $D[\![\{X_\alpha\}]\!]_{N_*} = D(\!(\{X_\alpha\})\!)$.

Proof. By Lemma 2.1, N_* is a saturated multiplicative set of $D[\![\{X_\alpha\}]\!]$. Hence, the result can be proved by an argument similar to the proof of Lemma 1.1. □

We next give the Noetherian property of $D[\![\{X_\alpha\}]\!]_{N_*}$ that is already known when $|\{X_\alpha\}| = \infty$ and $* = d$ [11, Theorem 2.6].

Proposition 2.3. *If D is a Noetherian domain, then $D[\![\{X_\alpha\}]\!]_{N_*}$ is a Noetherian domain and dim$(D[\![\{X_\alpha\}]\!]_{N_*}) = *_w$-dim$(D)$. In particular, $D(\!(\{X_\alpha\})\!)$ is a Noetherian domain with dim$(D(\!(\{X_\alpha\})\!)) =$ dim(D).*

Proof. Let $P \in *$-Max(D). If $|\{X_\alpha\}| = \infty$, then $\mathrm{ht}(P[\![\{X_\alpha\}]\!]) = \mathrm{ht}P < \infty$ and every prime ideal of $D[\![\{X_\alpha\}]\!]$ contained in $P[\![\{X_\alpha\}]\!]$ is finitely generated [11, Lemma 2.4]. Also, if $|\{X_\alpha\}| < \infty$, then $D[\![\{X_\alpha\}]\!]$ is Noetherian [7, Theorem 7] and $\mathrm{ht}(P[\![\{X_\alpha\}]\!]) = \mathrm{ht}P < \infty$. Thus, $D[\![\{X_\alpha\}]\!]_{N_*}$ is Noetherian and $\dim(D[\![\{X_\alpha\}]\!]_{N_*}) = *_w$-$\dim(D)$ by Lemma 2.1(2). $\square$

The next result is the power series ring analog of [19, Theorem 2.12] that if $0 \neq f \in D[X]$, then $c(f)$ is $*_f$-locally principal, i.e., $c(f)D_P$ is principal for every maximal $*_f$-ideal P of D if and only if $fD[X]_{N_*} = c(f)D[X]_{N_*}$, if and only if $c(f)D[X]_{N_*}$ is principal.

Proposition 2.4. *Let D be a Noetherian domain. Then the following statements are equivalent for all $0 \neq f \in D[\![\{X_\alpha\}]\!]$.*

(1) *$c(f)$ is $*$-invertible.*
(2) *$c(f)$ is $*$-locally principal.*
(3) *$fD[\![\{X_\alpha\}]\!]_{N_*} = c(f)D[\![\{X_\alpha\}]\!]_{N_*}$.*
(4) *$c(f)D[\![\{X_\alpha\}]\!]_{N_*}$ is principal.*
(5) *$c(f)D[\![\{X_\alpha\}]\!]_{N_*}$ is locally principal.*

Proof. (1) $\Leftrightarrow$ (2) This follows from [19, Proposition 2.6] because D is Noetherian (hence $* = *_f$ and $c(f)$ is finitely generated).

(2) $\Rightarrow$ (3) Let $P \in *$-Max(D) and $c(f)D_P = aD_P$ for some $a \in D$. Since $c(f)$ is finitely generated, there is an $s \in D \setminus P$ such that $s \cdot c(\frac{f}{a}) = \frac{s}{a} \cdot c(f) \subseteq D$. Also, by [15, Propostion 7.4], we may assume that a is a coefficient of f. Hence, $\frac{f}{a} \in D[\![\{X_\alpha\}]\!]_{P[\![\{X_\alpha\}]\!]} \setminus P[\![\{X_\alpha\}]\!]_{P[\![\{X_\alpha\}]\!]}$, and thus

$$\begin{aligned}(fD[\![\{X_\alpha\}]\!]_{N_*})_{P[\![\{X_\alpha\}]\!]_{N_*}} &= fD[\![\{X_\alpha\}]\!]_{P[\![\{X_\alpha\}]\!]}\\ &= aD[\![\{X_\alpha\}]\!]_{P[\![\{X_\alpha\}]\!]}\\ &= c(f)D[\![\{X_\alpha\}]\!]_{P[\![\{X_\alpha\}]\!]}\\ &= (c(f)D[\![\{X_\alpha\}]\!]_{N_*})_{P[\![\{X_\alpha\}]\!]_{N_*}}.\end{aligned}$$

Thus, by Lemma 2.1 and [15, Theorem 4.10],

$$\begin{aligned}fD[\![\{X_\alpha\}]\!]_{N_*} &= \bigcap_{P\in *\text{-Max}(D)} fD[\![\{X_\alpha\}]\!]_{P[\![\{X_\alpha\}]\!]}\\ &= \bigcap_{P\in *\text{-Max}(D)} c(f)D[\![\{X_\alpha\}]\!]_{P[\![\{X_\alpha\}]\!]}\\ &= c(f)D[\![\{X_\alpha\}]\!]_{N_*}.\end{aligned}$$

(3) $\Rightarrow$ (4) $\Rightarrow$ (5) Clear.

(5) $\Rightarrow$ (2) Let $P \in *$-Max(D). Then $PD[\![\{X_\alpha\}]\!]_{N_*}$ is a maximal ideal of $D[\![\{X_\alpha\}]\!]_{N_*}$ by Lemma 2.1, and hence $c(f)D[\![\{X_\alpha\}]\!]_{P[\![\{X_\alpha\}]\!]} = aD[\![\{X_\alpha\}]\!]_{P[\![\{X_\alpha\}]\!]}$ for some $a \in c(f)$ [15, Proposition 7.4]. Thus, $c(f)D_P = c(f)D[\![\{X_\alpha\}]\!]_{P[\![\{X_\alpha\}]\!]} \cap K = aD[\![\{X_\alpha\}]\!]_{P[\![\{X_\alpha\}]\!]} \cap K = aD_P$. $\square$

Let $*$ be a star operation on D. Then $\mathrm{Pic}(D[X]_{N_*}) = \{0\}$ [19, Theorem 2.14]. This was generalized to power series rings as follows: *Let* $\{X_\alpha\}$ *be an infinite set of indeterminates over* D*, and let* $N_{*_f} = \{f \in D[\![\{X_\alpha\}]\!] \mid c(f)^{*_f} = D\}$*. Then* $Pic(D[\![\{X_\alpha\}]\!]_{N_{*_f}}) = \{0\}$*, i.e., every invertible ideal of* $D[\![\{X_\alpha\}]\!]_{N_{*_f}}$ *is principal* [11, Proposition 3.6]. However, we don't know if this result holds when $\{X_\alpha\}$ is finite. We next in Theorem 2.7 show that if D is Noetherian, then $\mathrm{Pic}(D[\![\{X_\alpha\}]\!]_{N_*}) = \{0\}$ for any set $\{X_\alpha\}$ of indeterminates. To do this, we first need two lemmas.

Lemma 2.5. *Let* y *be an indeterminate over* $D[\![\{X_\alpha\}]\!]$ *and* $N = \{f \in D[\![\{X_\alpha\}]\!][y] \mid c_D(f)^* = D\}$*. If* $|\{X_\alpha\}| < \infty$*, then*

(1) N *is a multiplicative subset of* $D[\![\{X_\alpha\}]\!][y]$ *and*
(2) $(D[\![\{X_\alpha\}]\!]_{N_*})(y) = (D[\![\{X_\alpha\}]\!][y])_N$*.*

Proof. (1) Note that $D[\![\{X_\alpha\}]\!][\![y]\!] = D[\![\{X_\alpha\}\cup\{y\}]\!]$ [15, Exercise 11 on page 11] and $D[\![\{X_\alpha\}]\!][y] \subseteq D[\![\{X_\alpha\}]\!][\![y]\!]$. Hence if we set $N' = \{f \in D[\![\{X_\alpha\}]\!][\![y]\!] \mid c_D(f)^* = D\}$, then N' is a multiplicative set by Lemma 2.1 and $N = N' \cap D[\![\{X_\alpha\}]\!][y]$. Thus, N is a multiplicative set.

(2) ($\subseteq$) Let $u \in (D[\![\{X_\alpha\}]\!]_{N_*})(y)$. Then

$$\begin{aligned} u &= \left(\frac{g_0 + g_1 y + \cdots + g_m y^m}{h}\right)/\left(\frac{f_0 + f_1 y + \cdots + f_n y^n}{h'}\right) \\ &= h'(g_0 + g_1 y + \cdots + g_m y^m)/h(f_0 + f_1 y + \cdots + f_n y^n), \end{aligned}$$

where $h, h' \in N_*$ and $g_i, f_j \in D[\![\{X_\alpha\}]\!]$ such that $(f_0, f_1, \ldots, f_n)D[\![\{X_\alpha\}]\!]_{N_*} = D[\![\{X_\alpha\}]\!]_{N_*}$. Note that $(c(f_0) + c(f_1) + \cdots + c(f_n))^* = D$ by Lemma 2.1(2) and $c_D(f_0 + f_1 y + \cdots + f_n y^n) = c(f_0) + c(f_1) + \cdots + c(f_n)$; so $f_0 + f_1 y + \cdots + f_n y^n \in N$. Thus, by (1), $h(f_0 + f_1 y + \cdots + f_n y^n) \in N$, and hence $u \in (D[\![\{X_\alpha\}]\!][y])_N$.

($\supseteq$) Let $h = h_0 + h_1 y + \cdots + h_k y^k \in N$, where $h_i \in D[\![\{X_\alpha\}]\!]$. Clearly, $c_D(h) = c(h_0) + c(h_1) + \cdots + c(h_k)$, and hence $(c(h_0) + c(h_1) + \cdots + c(h_k))^* = D$. Thus, $h \in (D[\![\{X_\alpha\}]\!]_{N_*})[y] \setminus \bigcup_{P \in *\text{-Max}(D)} (P[\![\{X_\alpha\}]\!]_{N_*})[y]$, and since $\mathrm{Max}(D[\![\{X_\alpha\}]\!]_{N_*}) = \{P[\![\{X_\alpha\}]\!]_{N_*} \mid P \in *\text{-Max}(D)\}$ by Lemma 2.1(2), $\frac{1}{h} \in (D[\![\{X_\alpha\}]\!]_{N_*})(y)$. Therefore $D[\![\{X_\alpha\}]\!][y]_N \subseteq (D[\![\{X_\alpha\}]\!]_{N_*})(y)$. □

Lemma 2.6. *Let* V *be a valuation domain,* $\{X_\alpha\}$ *be a nonempty set of indeterminates over* V*, and* $0 \neq f, g \in V[\![\{X_\alpha\}]\!]$*. If* $c_V(f)$ *and* $c_V(g)$ *are finitely generated, then* $c_V(fg) = c_V(f)c_V(g)$*.*

Proof. Let $c(h) = c_V(h)$ for $h = f, g$. Then $c(f) = aV$ and $c(g) = bV$ for some $0 \neq a, b \in V$ because $c(f)$ and $c(g)$ are finitely generated. Hence $c(\frac{f}{a}) = V$ and $c(\frac{g}{b}) = V$; equivalently, $\frac{f}{a}, \frac{g}{b} \in V[\![\{X_\alpha\}]\!] \setminus M[\![\{X_\alpha\}]\!]$, where M is the maximal ideal of V. Note that $V[\![\{X_\alpha\}]\!] \setminus M[\![\{X_\alpha\}]\!]$ is a multiplicative subset of $V[\![\{X_\alpha\}]\!]$; so $\frac{f}{a} \cdot \frac{g}{b} \in V[\![\{X_\alpha\}]\!] \setminus M[\![\{X_\alpha\}]\!]$, and thus $c(\frac{f}{a} \cdot \frac{g}{b}) = V$. Therefore $c(fg) = ab \cdot c(\frac{f}{a} \cdot \frac{g}{b}) = (aV)(bV) = c(f)c(g)$. □

Theorem 2.7. *Let* D *be a Noetherian domain. Then* $Pic(D[\![\{X_\alpha\}]\!]_{N_*}) = \{0\}$*, i.e., every invertible ideal of* $D[\![\{X_\alpha\}]\!]_{N_*}$ *is principal.*

Proof. By [11, Proposition 3.6], we may assume that $|\{X_\alpha\}| < \infty$. Let $A = (f_0, f_1, \ldots, f_n)D[\![\{X_\alpha\}]\!]_{N_*}$ be an invertible ideal of $D[\![\{X_\alpha\}]\!]_{N_*}$, where $f_i \in D[\![\{X_\alpha\}]\!]$.

Let y be an indeterminate over $D[\![\{X_\alpha\}]\!]$ and $g = f_0 + f_1 y + \cdots + f_n y^n$. Then, by [2, Theorem 1], $A(y) = g(D[\![\{X_\alpha\}]\!]_{N_*})(y)$.

Note that if we set $N = \{f \in D[\![\{X_\alpha\}]\!][y] \mid c_D(f)^* = D\}$, then $(D[\![\{X_\alpha\}]\!]_{N_*})(y) = D[\![\{X_\alpha\}]\!][y]_N$ by Lemma 2.5. Also, note that $c(f_0)$ is finitely generated; hence there is an integer $k_1 \geq 1$ such that $c(f_0 + f_1 X^{k_1}) = c(f_0) + c(f_1)$, where $X \in \{X_\alpha\}$. Repeating this process, there are positive integers $k_2, \ldots, k_n$ such that if we let $f = f_0 + f_1 X^{k_1} + \cdots + f_n X^{k_n}$, then $c(f) = c(f_0) + c(f_1) + \cdots + c(f_n)$ and $fD[\![\{X_\alpha\}]\!][y]_N \subseteq A(y) = gD[\![\{X_\alpha\}]\!][y]_N$.

Let $h \in N$ and $h_1 \in D[\![\{X_\alpha\}]\!][y]$ be such that $f = g \cdot \frac{h_1}{h}$ or $fh = gh_1$. We claim that $c_D(h_1)^* = D$. If not, there is a $P \in *\text{-Max}(D)$ with $c_D(h_1) \subseteq P$. Let (V, M) be a valuation overring of D with $M \cap D = P$ [15, Theorem 19.6]; so $c_V(h_1) = c_D(h_1)V \subseteq M$. However, note that $c_V(h) = V$ and $c_V(f) = c_V(g)$; hence $c_V(f) = c_V(f)c_V(h) = c_V(fh) = c_V(gh_1) = c_V(g)c_V(h_1) = c_V(f)c_V(h_1)$ by Lemma 2.6, and since $c_V(f)$ is finitely generated, we have $c_V(h_1) = V$, a contradiction. Thus $c_D(h_1)^* = D$ and so $f(D[\![\{X_\alpha\}]\!]_{N_*})(y) = A(y)$. Therefore $fD[\![\{X_\alpha\}]\!]_{N_*} = f(D[\![\{X_\alpha\}]\!]_{N_*})(y) \cap qf(D[\![\{X_\alpha\}]\!]_{N_*}) = A(y) \cap qf(D[\![\{X_\alpha\}]\!]_{N_*}) = A$ [15, Proposition 33.1(4)]. $\square$

Let $*$ be a star operation on D. A *$*$-quasi-Prüfer domain* D is an integral domain in which if Q is a prime ideal in $D[\{X_\alpha\}]$ and $Q \subseteq P[\{X_\alpha\}]$, for some prime $*$-ideal P of D, then $Q = (Q \cap D)D[\{X_\alpha\}]$. As in [6], we say that an ideal A of $D[\![\{X_\alpha\}]\!]_{N_*}$ is *formally extended from* D if $A = I[\![\{X_\alpha\}]\!]_{N_*}$ for some ideal I of D. Finally, D is said to be of *finite t-character* if each nonzero nonunit of D is contained in only finitely many maximal t-ideals of D. It is known that integral domains on which $t = v$ (e.g., Noetherian domains and Krull domains) are of finite t-character [17, Theorem 1.3].

Proposition 2.8. *The following statements are equivalent for a Noetherian domain D.*

(1) *D is a $*$-quasi-Prüfer domain.*
(2) *Every prime $*$-ideal of D is a maximal $*$-ideal, i.e., $*$-dim$(D) = 1$.*
(3) *Every prime ideal of $D[\![\{X_\alpha\}]\!]_{N_*}$ is formally extended from D.*

Proof. (1) $\Rightarrow$ (2) It is known that D is a $*$-quasi-Prüfer domain if and only if D is t-quasi-Prüfer and $*_w = w$ [9, Theorem 2.16]. Also, a Notherian domain is a t-quasi-Prüfer domain if and only if t-dim$(D) = 1$ [18, Theorem 3.7]. Thus, $*$-dim$(D) = 1$.

(2) $\Rightarrow$ (3) If M is a nonzero prime ideal of $D[\![\{X_\alpha\}]\!]_{N_*}$, then M is maximal by Proposition 2.3, and thus $M = P[\![\{X_\alpha\}]\!]_{N_*}$ for some $P \in *\text{-Max}(D)$ by Lemma 2.1.

(3) $\Rightarrow$ (1) Let P be a maximal $*$-ideal of D. If $P_t = D$, then we can choose $a, b \in P$ so that $(a, b)_v = D$ because Noetherian domains are of finite t-character. Let Q be a prime ideal of $D[\![\{X_\alpha\}]\!]$ such that $Q \subseteq P[\![\{X_\alpha\}]\!]$ and Q is minimal over $a + bX$ where $X \in \{X_\alpha\}$. Clearly, Q is a t-ideal, and since $Q = P_1[\![\{X_\alpha\}]\!]$ for some prime ideal P_1 of D by assumption, $Q = Q_t = (P_1[\![\{X_\alpha\}]\!])_t \supseteq ((a, b)D[\![\{X_\alpha\}]\!])_v = (a, b)_v[\![\{X_\alpha\}]\!] = D[\![\{X_\alpha\}]\!]$, a contradiction (see [12, Proposition 2.1] for the fourth equality). Hence, $P_t \subsetneq D$, and so $P_t = P$ because $* \leq t$. Thus, each maximal $*$-ideal is a t-ideal, and therefore $*_w = w$.

Next, if ht$P = n$, then there are $a_1, \dots, a_n \in P$ such that P is minimal over $(a_1, \dots, a_n)$ [20, Theorem 153]. Let $f = a_1X + \cdots + a_nX^n$ and let Q_0 be a minimal prime ideal of $fD[\![\{X_\alpha\}]\!]$ such that $Q_0 \subseteq P[\![\{X_\alpha\}]\!]$. Then $Q_0 = (Q_0 \cap D)[\![\{X_\alpha\}]\!]$ by assumption, and hence $(a_1, \dots, a_n) \subseteq Q_0 \cap D \subseteq P$. Since P is minimal over $(a_1, \dots, a_n)$, we have $Q_0 = P[\![\{X_\alpha\}]\!]$. Also, since $D[\![\{X_\alpha\}]\!]$ is Noetherian, $1 \leq n = \text{ht}P \leq \text{ht}(P[\![\{X_\alpha\}]\!]) = \text{ht}Q_0 \leq 1$. Hence, ht$P = 1$, and therefore, t-dim$(D) = 1$. Thus, D is a $*$-quasi-Prüfer domain (see the proof of (1) $\Rightarrow$ (2)). $\square$

It is well known that an integrally closed Noetherian domain D is a Krull domain [15, Theorem 43.4], and hence t-dim$(D) = 1$.

Corollary 2.9. *Let D be an integrally closed Noetherian domain.*

(1) *Every prime ideal of $D[\![\{X_\alpha\}]\!]_{N_*}$ is formally extended from D if and only if $* = v$ on D.*
(2) *$D[\![\{X_\alpha\}]\!]_{N_v}$ is a principal ideal domain (PID).*

Proof. (1) ($\Rightarrow$) The proof of (1) $\Rightarrow$ (2) of Proposition 2.8 shows that $*_w = w$. Therefore, $* = v$ because $*_w \leq * \leq v$ and $w = t = v$ on a Krull domain. ($\Leftarrow$) Since D is a Krull domain, $*$-dim$(D) = t$-dim$(D) = 1$. Thus, the result follows from Proposition 2.8.

(2) Note that $D[\![\{X_\alpha\}]\!]$ is a Krull domain [14, Theorem 2.1], and so integrally closed. Hence, $D[\![\{X_\alpha\}]\!]_{N_v}$ is a one-dimensional integrally closed Noetherian domain by Proposition 2.3, and so a Dedekind domain [15, Theorem 37.8]. Thus, by Theorem 2.7, $D[\![\{X_\alpha\}]\!]_{N_v}$ is a PID. $\square$

3. Kronecker Function Rings

Let D be an integral domain with quotient field K, X be an indeterminate over D, and $D[\![X]\!]$ be the power series ring over D. Let $*$ be an *e.a.b.* star operation on D; in this case, D is integrally closed, and so if D is Noetherian, then D is a Krull domain and t-Max$(D) = X^1(D)$.

Let D be an integrally closed Noetherian domain. In this section, we use an *e.a.b.* star operation $*$ on D to generalize the Kronecker function ring Kr$(D, *)$ to the power series ring Kr$(\!(D, *)\!)$. It is noteworthy that the results of this section can be extended to $D[\![\{X_\alpha\}]\!]_1$ and their proofs are the same as in the case of $D[\![X]\!]$. Also, if R is an overring of D, then $R[\![X]\!] \subseteq qf(D[\![X]\!])$ if and only if $D = R$ [22, Theorem 3.4].

Lemma 3.1. *Let $*$ be an e.a.b. star operation on a Noetherian domain D.*

(1) *$c_D(fg)^* = (c_D(f)c_D(g))^*$ for all $0 \neq f, g \in D[\![X]\!]$.*
(2) *For $A, B, C \in F(D)$, if $(AB)^* \subseteq (AC)^*$, then $B^* \subseteq C^*$.*
(3) *The v-operation on D is an e.a.b. star operation.*

Proof. (1) Clearly, $*$ is of finite type, and hence there exists a family $\{V_\alpha\}$ of valuation overrings of D such that $I^* = \bigcap_\alpha IV_\alpha$ for all $I \in F(D)$ [15, Theorem 32.12]. Note that $c_{V_\alpha}(h) = c_D(h)V_\alpha$ for all $h \in D[\![X]\!]$; so $c_{V_\alpha}(f)$ and $c_{V_\alpha}(g)$ are finitely generated. Hence $c_{V_\alpha}(fg) = c_{V_\alpha}(f)c_{V_\alpha}(g)$ for all α by Lemma 2.6, and therefore $c_D(fg)^* = (c_D(f)c_D(g))^*$.

(2) This follows because A, B, C are finitely generated.

(3) Since D admits an *e.a.b.* star operation, D is integrally closed, and thus D is a Krull domain. Hence, every nonzero fractional ideal of D is v-invertible. Thus, v is an *e.a.b.* star operation. □

In [5, Theorem 2.13], the authors introduced the power series ring analog of the Kronecker function ring for an integral domain D satisfying $c(fg)^v = (c(f)c(g))^v$ for all $0 \neq f, g \in D[\![X]\!]$ as follows:

$$D^{\hat{v}} = \{\tfrac{f}{g} \mid f, g \in D[\![X]\!] \text{ with } 0 \neq g \text{ and } f = 0 \text{ or } c(f)^v \subseteq c(g)^v\}.$$

They then showed that $D^{\hat{v}}$ is a completely integrally closed Bezout domain.

Theorem 3.2. *Let D be an integrally closed Noetherian domain, $*$ be an e.a.b. star operation on D, and*

$$\mathrm{Kr}(\!(D,*)\!) = \{\frac{f}{g} \mid f, g \in D[\![X]\!], 0 \neq g, \text{ and } c_D(f) \subseteq c_D(g)^*\}.$$

(1) $\mathrm{Kr}(\!(D,*)\!)$ *is a Bezout domain with* $D[\![X]\!]_{N_*} \subseteq \mathrm{Kr}(\!(D,*)\!) \subseteq qf(D[\![X]\!])$.
(2) $\mathrm{Kr}(\!(D,*)\!) \cap K = D$ *and* $\mathrm{Kr}(\!(D,*)\!) \cap K(X) = \mathrm{Kr}(D,*)$.
(3) $f\mathrm{Kr}(\!(D,*)\!) \cap K = c(f)^*$ *and* $f\mathrm{Kr}(\!(D,*)\!) = c(f)^*\mathrm{Kr}(\!(D,*)\!) = c(f)\mathrm{Kr}(\!(D,*)\!)$ *for all* $0 \neq f \in D[\![X]\!]$.
(4) *If I is a nonzero ideal of D, then* $I\mathrm{Kr}(\!(D,*)\!) \cap K = I^*$.
(5) $\mathrm{Kr}(\!(D,b)\!) \subseteq \mathrm{Kr}(\!(D,*)\!) \subseteq \mathrm{Kr}(\!(D,v)\!)$.

Proof. (1) Claim 1. $\mathrm{Kr}(\!(D,*)\!)$ is well-defined. (*Proof.* Let $0 \neq f, g, h, k \in D[\![X]\!]$ be such that $c(f) \subseteq c(g)^*$ and $\frac{f}{g} = \frac{k}{h}$. Then $fh = gk$, and hence $(c(g)c(k))^* = c(gk)^* = c(fh)^* = (c(f)c(h))^* \subseteq (c(g)c(h))^*$. Thus, $c(k)^* \subseteq c(h)^*$.)

Claim 2. $\mathrm{Kr}(\!(D,*)\!)$ is an integral domain. (*Proof.* Assume that $\frac{f}{g}$ and $\frac{k}{h}$ are nonzero elements of $\mathrm{Kr}(\!(D,*)\!)$. Then $c(fh+gk)^* \subseteq (c(fh)+c(gk))^* = (c(fh)^* + c(gk)^*)^* = ((c(f)c(h))^* + (c(g)c(k))^*)^* \subseteq (c(g)c(h))^* = c(gh)^*$. Hence $\frac{f}{g} + \frac{k}{g} = \frac{fh+gk}{gh} \in \mathrm{Kr}(\!(D,*)\!)$. Also, $c(fk)^* = (c(f)c(k))^* \subseteq (c(g)c(h))^* = c(gh)^*$, and hence $\frac{f}{g} \cdot \frac{k}{h} = \frac{fk}{gh} \in \mathrm{Kr}(\!(D,*)\!)$.)

Claim 3. $\mathrm{Kr}(\!(D,*)\!)$ is a Bezout domain. (*Proof.* It suffices to show that $(f,g)\mathrm{Kr}(\!(D,*)\!)$ is principal for all $0 \neq f, g \in D[\![X]\!]$. Since D is Noetherian, $c(f)$ is finitely generated, and so there is an integer $n \geq 1$ such that if we let $h = f + gX^n$, then $c(h) = c(f) + c(g)$. Clearly, $h \in (f,g)\mathrm{Kr}(\!(D,*)\!)$. Also, $\frac{f}{h}, \frac{g}{h} \in \mathrm{Kr}(\!(D,*)\!)$, and hence $(f,g)\mathrm{Kr}(\!(D,*)\!) \subseteq h\mathrm{Kr}(\!(D,*)\!)$.)

Claim 4. $D[\![X]\!]_{N_*} \subseteq \mathrm{Kr}(\!(D,*)\!) \subseteq qf(D[\![X]\!])$. (*Proof.* This is clear.)

(2) Clearly, $D \subseteq \mathrm{Kr}(\!(D,*)\!) \cap K$. For the reverse containment, let $0 \neq u = \frac{f}{g} \in \mathrm{Kr}(\!(D,*)\!) \cap K$, where $f, g \in D[\![X]\!]$ with $c(f)^* \subseteq c(g)^*$. Then $ug = f$, and so $uc(g)^* = c(f)^* \subseteq c(g)^*$. Hence, $u \in uD \subseteq D$, and thus $\mathrm{Kr}(\!(D,*)\!) \cap K \subseteq D$. Also, it is obvious that $\mathrm{Kr}(\!(D,*)\!) \cap K(X) = \mathrm{Kr}(D,*)$.

(3) If $\alpha \in c(f)^*$, then $\frac{\alpha}{f} \in \mathrm{Kr}(\!(D,*)\!)$, and hence $\alpha \in f\mathrm{Kr}(\!(D,*)\!)$. Thus, $c(f)^* \subseteq f\mathrm{Kr}(\!(D,*)\!) \cap K$. Conversely, assume $0 \neq u \in f\mathrm{Kr}(\!(D,*)\!) \cap K$. Then there are some $0 \neq g, h \in D[\![X]\!]$ such that $c(h)^* \subseteq c(g)^*$ and $u = f \cdot \frac{h}{g}$. Hence, $ug = fh$, and so $uc(g)^* = (c(f)c(h))^* \subseteq (c(f)c(g))^*$. Thus, $u \in uD \subseteq c(f)^*$. Thus, $f\mathrm{Kr}(\!(D,*)\!) \cap K = c(f)^*$. This also proves the second result because $c(f)$ is finitely generated.

(4) This is an immediate consequence of (3) because we can choose a power series $f \in D[\![X]\!]$ such that $I = c(f)$.

(5) This follows because $b \leq * \leq v$. □

Corollary 3.3. *If $*_1$ and $*_2$ are e.a.b. star operations on a Noetherian domain D such that $Kr(\!(D, *_1)\!) = Kr(\!(D, *_2)\!)$, then $*_1 = *_2$.*

Proof. Let I be a nonzero fractional ideal of D. If $I \subseteq D$, then there is an $f \in D[X]$ such that $c(f) = I$, and hence $I^{*_1} = c(f)^{*_1} = f\mathrm{Kr}(\!(D, *_1)\!) \cap K = f\mathrm{Kr}(\!(D, *_2)\!) \cap K = c(f)^{*_2} = I^{*_2}$ by Theorem 3.2. Next, if $I \nsubseteq D$, then there is a $0 \neq d \in D$ with $dI \subseteq D$. Hence, $dI^{*_1} = (dI)^{*_1} = (dI)^{*_2} = dI^{*_2}$, and thus $I^{*_1} = I^{*_2}$. Therefore, $*_1 = *_2$. □

Remark 3.4. Let D be an integrally closed Noetherian domain. Then D is a Krull domain, and hence the v-operation is an *e.a.b.* star operation. Note that b-Max$(D) = \mathrm{Max}(D)$ and t-dim$(D) = 1$. Hence, $b = v$ if and only if $\dim(D) = 1$, and in this case, $\mathrm{Kr}(\!(D, b)\!) = \mathrm{Kr}(\!(D, v)\!)$ is a PID (see Corollaries 2.9 and 3.13). Thus, if $\dim(D) \geq 2$, then D has at least two distinct *e.a.b.* star operations b and v, and by Corollary 3.3, $\mathrm{Kr}(\!(D, b)\!) \subsetneq \mathrm{Kr}(\!(D, v)\!)$. More precisely, by Proposition 1.5 and Corollary 3.3, there are at least $2^{|X(D)|}$ Kronecker function rings of D, where $X(D)$ is the set of height-two prime ideals of D.

Let v be a valuation on K, associated with V, and let X be an indeterminate over V. For each $f = a_0 + a_1X + \cdots a_nX^n \in V[X]$, let $v^*(f) = inf\{v(a_i)\}$. Then v^* is a valuation on $K(X)$, called the *trivial extension of v to $K(X)$*, and its valuation ring is $V(X)$; so $V(X) = \{\frac{f}{g} \mid f, g \in D[X], g \neq 0$, and $c(f)V \subseteq c(g)V\}$. Clearly, the value group of v^* is equal to that of v.

Let M be the maximal ideal of V. Then $V(X) = V[X]_{M[X]}$ and $V(\!(X)\!) = V[\![X]\!]_{M[\![X]\!]}$. Hence $V(X)$ is a valuation domain, while $V(\!(X)\!)$ is a valuation domain if and only if V is a rank one DVR [1, Theorems 1 and 2].

Proposition 3.5. *Let V be a valuation overring of a Noetherian domain D, v be the valuation of V, and $\widehat{V} = \{\frac{f}{g} \mid f, g \in D[\![X]\!], g \neq 0$, and $c(f)V \subseteq c(g)V\}$.*

1. *$\widehat{V}$ is a valuation domain such that $\widehat{V} \cap K(X) = V(X)$.*
2. *If M is the maximal ideal of V, then $\widehat{M} = \{\frac{f}{g} \mid f, g \in D[\![X]\!], g \neq 0$, and $c(f)V \subsetneq c(g)V\}$ is the maximal ideal of $\widehat{V}$.*
3. *If $f \in D[\![X]\!]$, then $c(f)V = aV$ for some coefficient a of f.*
4. *For $f \in D[\![X]\!]$, let $v^*(f) = v(a)$, where $a \in D$ with $c(f)V = aV$ by (2). Then v^* is a valuation on $qf(D[\![X]\!])$, associated with $\widehat{V}$, and its value group is the same as that of v.*
5. *$dim(\widehat{V}) = dim(V)$.*
6. *V is discrete if and only if $\widehat{V}$ is discrete.*

Proof. (1) Let $0 \neq f, g, h, k \in D[\![X]\!]$ be such that $c_V(f) \subseteq c_V(g)$ and $\frac{f}{g} = \frac{k}{h}$. Then $fh = gk$, and hence, by Lemma 2.6, $c_V(g)c_V(k) = c_V(gk) = c_V(fh) = c_V(f)c_V(h) \subseteq c_V(g)c_(h)$. Note that $c_V(g)$ is finitely generated, and so principal. Hence, $c_V(h) \subseteq c_V(k)$. This shows that $\widehat{V}$ is well-defined and $\widehat{V} \cap K(X) = V(X)$.

Next, to show that $\widehat{V}$ is a valuation domain, let $0 \neq f, g, h_1, h_2 \in D[\![X]\!]$ be such that $\frac{f}{h_1}, \frac{g}{h_2} \in \widehat{V}$. Then, by Lemma 2.6, $c(fg)V = c_V(fg) = c_V(f)c_V(g) \subseteq c_V(h_1)c_V(h_2) = c_V(h_1h_2) = c(h_1h_2)V$, and thus $\frac{f}{h_1} \cdot \frac{g}{h_2} \in \widehat{V}$. Again, by Lemma 2.6, $c(fh_2 + gh_1)V = c_V(fh_2 + gh_1) \subseteq c_V(fh_2) + c_V(gh_1) = c_V(f)c_V(h_2) + c_V(g)c_V(h_1) \subseteq c_V(h_1)c_V(h_2) = c_V(h_1h_2)$, and hence $\frac{f}{h_1} + \frac{g}{h_2} \in \widehat{V}$. Thus, $\widehat{V}$ is an integral domain because $\widehat{V} \subseteq qf(D[\![X]\!])$. Also, since V is a valuation domain, either $c(f)V \subseteq c(g)V$ or $c(g)V \subseteq c(f)V$ for all $0 \neq f, g \in D[\![X]\!]$. Hence $\frac{f}{g} \in \widehat{V}$ or $\frac{g}{f} \in \widehat{V}$, and therefore $\widehat{V}$ is a valuation domain.

(2) Let $0 \neq f, g \in D[\![X]\!]$. Then $\frac{f}{g}$ is a unit of $\widehat{V}$ if and only if $c(f)V = c(g)V$, and since $\widehat{V}$ is quasi-local by (1), $\widehat{M}$ is the maximal ideal of $\widehat{V}$.

(3) This follows because $c(f)$ is finitely generated.

(4) Let $0 \neq f, g \in D[\![X]\!]$, and let $c(f)V = aV$ and $c(g)V = bV$. Note that $c(fg)V = c_V(fg) = c_V(f)c_V(g) = (aV)(bV) = abV$ by Lemma 2.6; hence $v^*(fg) = v(ab) = v(a) + v(b) = v^*(f) + v^*(g)$. Note also that $c(f + g) = (a_i + b_i)V$ for some $a_i \in c(f)$ and $b_i \in c(g)$ by (3). Hence $v^*(f + g) = v(a_i + b_i) \geq \min\{v(a_i), v(b_i)\} \geq \min\{v(a), v(b)\} = \min\{v^*(f), v^*(g)\}$. Thus, v^* is a valuation on $qf(D[\![X]\!])$ whose value group is the same as that of v. Next, $\frac{f}{g} \in \widehat{V} \Leftrightarrow aV = c(f)V \subseteq c(g)V = bV$, $\Leftrightarrow v(\frac{a}{b}) \geq 0$, $\Leftrightarrow v^*(g) = v(b) \leq v(a) = v^*(f)$, $\Leftrightarrow v^*(\frac{f}{g}) \geq 0$. Therefore, $\widehat{V}$ is the valuation ring of v^*.

(5) It is known that if G is the value group of v, then the rank of G (as a totally ordered abelian group) is equal to $\dim(V)$ [21, Theorem 5.17]. Thus, by (4), $\dim(\widehat{V}) = \dim(V)$.

(6) This follows from [15, Exercise 22 on p. 205], because the value group of v^* is the same as that of v by (4). $\square$

A valuation overring V of D is called a *$*$-valuation overring* of D if $I^* \subseteq IV$ for all $I \in f(D)$. It is known that if $*$ is *e.a.b.*, then $\mathrm{Kr}(D, *) = \bigcap\{V(X) \mid V$ is a $*$-valuation overring of $D\}$ [13, Proposition 13].

Lemma 3.6. *Let $*$ be an e.a.b. star operation on a Noetherian domain D. Then V is a $*$-valuation overring of D if and only if $\widehat{V}$ is an overring of $Kr(\!(D, *)\!)$.*

Proof. Assume that V is a $*$-valuation overring of D, and let $0 \neq f, g \in D[\![X]\!]$ with $\frac{f}{g} \in \mathrm{Kr}(\!(D, *)\!)$. Then $c(f)V = c(f)^*V \subseteq c(g)^*V = c(g)V$ because $c(f)$ and $c(g)$ are finitely generated, and hence $\frac{f}{g} \in \widehat{V}$. Thus $\mathrm{Kr}(\!(D, *)\!) \subseteq \widehat{V}$. For the reverse implication, it suffices to show that if $0 \neq h \in D[X]$, then $c(h)^* \subseteq c(h)V$. Recall that $c(h)^* = h\mathrm{Kr}(\!(D, *)\!) \cap K$ by Theorem 3.2(3); so if we let $h_1 \in D[X]$ with $c(h_1) = c(h)^*$ (note that $c(h)^*$ is finitely generated), then $c(h_1)^* = c(h)^*$, and thus $\frac{h_1}{h} \in \mathrm{Kr}(\!(D, *)\!) \subseteq \widehat{V}$. Thus $c(h)^* \subseteq c(h)^*V = c(h_1)V \subseteq c(h)V$. $\square$

We say that D has *valuative dimension* n, denoted by $\dim_v(D) = n$, if each valuation overring of D has dimension at most n and if there is a valuation overring V of D with $\dim(V) = n$. It is known that if D is an n-dimensional Noetherian domain, then $\dim_v(D) = n$ [15, Corollary 30.10]. Let v be a valuation on K, where K is the quotient field of a Noetherian domain D, associated with V, and let v^* be the valuation on $qf(D[\![X]\!])$, associated with $\widehat{V}$. Then, as in the polynomial ring case, we will say that v^* is the *trivial extension* of v to $qf(D[\![X]\!])$ (equivalently, $\widehat{V}$ is called the trivial extension of V to $qf(D[\![X]\!])$).

Corollary 3.7. *Let $*$ be an e.a.b. star operation on a Noetherian domain D, W be a valuation overring of* $\mathrm{Kr}(\!(D,*)\!)$*, and $V = W \cap K$.*

(1) *W is the trivial extension of V to $qf(D[\![X]\!])$, i.e., $W = \widehat{V}$.*
(2) *V is a $*$-valuation overring of D.*
(3) *$dim(\mathrm{Kr}(\!(D,b)\!)) = dim_v(D)$.*

Proof. (1) Let v^* be the valuation on $qf(D[\![X]\!])$ associated with W, and let v be the restriction of v^* to K. Let $0 \neq f \in D[\![X]\!]$. Then $c(f)V = aV$ for some $a \in D$ by Proposition 3.5(3). Also, since $c(f)\mathrm{Kr}(\!(D,*)\!) = f\mathrm{Kr}(\!(D,*)\!)$ by Theorem 3.2(3), $aW = c(f)W = fW$ because $\mathrm{Kr}(\!(D,*)\!) \subseteq W$. Thus, $v^*(f) = v^*(a) = v(a)$. So if $0 \neq f, g \in D[\![X]\!]$, then $\frac{f}{g} \in W \Leftrightarrow v^*(f) \geq v^*(g) \Leftrightarrow c(f)V \subseteq c(g)V \Leftrightarrow \frac{f}{g} \in \widehat{V}$. Thus, $W = \widehat{V}$.

(2) This follows from (1) and Lemma 3.6.

(3) Since $\mathrm{Kr}(\!(D,b)\!)$ is a Bezout domain by Theorem 3.2(1), the result follows directly from (1), Lemma 3.6 and Proposition 3.5(5). □

Corollary 3.8. *Let $*$ be an e.a.b. star operation on a Noetherian domain D. Then* $\mathrm{Kr}(\!(D,*)\!) = \bigcap\{\widehat{V} \mid V$ *is a $*$-valuation overring of D*$\}$.

Proof. This follows from Lemma 3.6 and Corollary 3.7 because $\mathrm{Kr}(\!(D,*)\!)$ is integrally closed and so the intersection of valuation overrings [15, Theorem 19.8]. □

It is known that if P is a height-one prime ideal of an integrally closed Noetherian domain D, then D_P is a rank-one DVR, and hence $ID_P = (ID_P)^v = I^vD_P$ for all $I \in F(D)$; so D_P is a v-valuation overring of D.

Corollary 3.9. *If D is an integrally closed Noetherian domain, then* $\mathrm{Kr}(\!(D,v)\!)$ *is a PID and* $\mathrm{Kr}(\!(D,v)\!) = D[\![X]\!]_{N_v}$.

Proof. By Corollary 2.9, it suffices to show that $\mathrm{Kr}(\!(D,v)\!) = D[\![X]\!]_{N_v}$. Note that $D[\![X]\!]_{N_v} = \bigcap_{P \in X^1(D)} D[\![X]\!]_{P[\![X]\!]}$, because $\mathrm{Max}(D[\![X]\!]_{N_v}) = \{P[\![X]\!]_{N_v} \mid P \in X^1(D)\}$ by Lemma 2.1. Let $P \in X^1(D)$. Then D_P and $D[\![X]\!]_{P[\![X]\!]}$ are both rank-one DVRs. Also, obviously, $D[\![X]\!]_{P[\![X]\!]} \subseteq \widehat{D_P}$ because $c(f)D_P = D_P$ for all $f \in D[\![X]\!] \setminus P[\![X]\!]$, and thus $D[\![X]\!]_{P[\![X]\!]} = \widehat{D_P}$. So, by Theorem 3.2 and Corollary 3.8, $\mathrm{Kr}(\!(D,v)\!) \subseteq \bigcap\{\widehat{D_P} \mid P \in X^1(D)\} = D[\![X]\!]_{N_v} \subseteq \mathrm{Kr}(\!(D,v)\!)$. Thus, $\mathrm{Kr}(\!(D,v)\!) = D[\![X]\!]_{N_v}$. □

Corollary 3.10. *Let $*$ be an e.a.b. star operation on a Noetherian domain D. If T is an overring of* $\mathrm{Kr}(\!(D,*)\!)$*, then $T = \bigcap_\alpha \widehat{V_\alpha}$ for some set $\{V_\alpha\}$ of $*$-valuation overrings of D containing $T \cap K$.*

Proof. Clearly, T is integrally closed because T is an overring of a Bezout domain. Hence, if we let $\{W_\alpha\}$ be the set of valuation overrings of T, then $T = \bigcap_\alpha W_\alpha$. Put $V_\alpha = W_\alpha \cap K$. Then V_α is a $*$-valuation overring of D containing $T \cap K$ such that $W_\alpha = \widehat{V_\alpha}$ by Corollary 3.7. Thus, $T = \bigcap_\alpha \widehat{V_\alpha}$. □

It is known that $\{V_\alpha\}$ is a set of valuation overrings of D, then $\bigcap V_\alpha(X)$ is a Bezout domain (cf. [15, Theorem 32.11]).

Corollary 3.11. *Let $\{V_\alpha\}$ be a set of valuation overrings of a Noetherian domain D, and let $T = \bigcap \widehat{V_\alpha}$. Then T is a Bezout domain.*

Proof. Let $R = \bigcap_\alpha V_\alpha$, and let $\widehat{R} = \{\frac{f}{g} \mid f, g \in D[\![X]\!], g \neq 0$, and $c(f)V_\alpha \subseteq c(g)V_\alpha$ for all $\alpha\}$. Then $\widehat{R}$ is a Bezout domain by an argument similar to the proof of Theorem 3.2(1). Clearly, $T = \widehat{R}$, and thus T is a Bezout domain. □

Corollary 3.12. *Let R be an integrally closed overring of a Noetherian domain D, $\{V_\alpha\}$ be the set of valuation overrings of R, and*

$$\widehat{R} = \{\tfrac{f}{g} \mid f, g \in D[\![X]\!], g \neq 0, \text{ and } c(f)V_\alpha \subseteq c(g)V_\alpha \text{ for all } \alpha\}.$$

Then $\widehat{R}$ is a Bezout domain such that $\widehat{R} \cap K = R$ and $D(\!(X)\!) \subseteq \widehat{R} \subseteq qf(D[\![X]\!])$.

Proof. Clearly, $\widehat{R} = \bigcap \widehat{V_\alpha}$ and $D(\!(X)\!) \subseteq \widehat{R} \subseteq qf(D[\![X]\!])$. Hence, $\widehat{R}$ is a Bezout domain by Corollary 3.11 and $\widehat{R} \cap K = (\bigcap \widehat{V_\alpha}) \cap K = \bigcap(\widehat{V_\alpha} \cap K(X) \cap K) = \bigcap(V_\alpha(X) \cap K) = \bigcap V_\alpha = R$ by Proposition 3.5(1). □

Corollary 3.13. *The following statements are equivalent for an integrally closed Noetherian domain D.*

(1) *D is a Dedekind domain.*
(2) $\mathrm{Kr}(\!(D, b)\!) = D(\!(X)\!)$.
(3) *$D(\!(X)\!)$ is a Prüfer domain.*
(4) *$dim(D) = 1$.*
(5) $\mathrm{Kr}(\!(D, b)\!) = \mathrm{Kr}(\!(D, v)\!)$.

Proof. (1) ⇒ (2) This follows from Corollary 3.9, because a Dedekind domain is a Krull domain on which $d = t = v$.

(2) ⇒ (3) This is an immediate consequence of Theorem 3.2(1).

(3) ⇒ (1) [10, Corollary 7].

(1) ⇔ (4) Recall that an integrally closed Noetherian domain is a Krull domain; hence the result follows from the fact that a Krull domain is a Dedekind domain if and only if it is of dimension one.

(1) ⇒ (5) This follows because $b = d = v$ on a Dedekind domain.

(5) ⇒ (4) Let M be a maximal ideal of D. If $\mathrm{ht}M \geq 2$, then $M_t = D$, and so there is a nonzero subideal I of M such that $I_v = D$. Since I is finitely generated, there is a polynomial $f \in D[X]$ such that $c(f) = I$. Clearly, $f \in N_v$, and so $\frac{1}{f} \in \mathrm{Kr}(\!(D, b)\!)$ by (5) and Corollary 3.9. Hence, $c(f) = c(f)^b = D$, a contradiction. Thus, $\mathrm{ht}M = 1$, and so D is one-dimensional. □

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