Journal of Algebra 494 (2018) 40–76

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Coordinates at stable points of the space of arcs

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ARTICLE INFO

Article history:
Received 10 March 2017
Available online 5 October 2017
Communicated by Steven Dale Cutkosky

MSC:
13A02
13A18
14B05
14B25
14E15
14J17
32S05

Keywords:
Space of arcs
Divisorial valuation
Graded algebra

ABSTRACT

Let X be a variety over a field k and let X_∞ be its space of arcs. Let P_E be the stable point of X_∞ defined by a divisorial valuation ν_E on X. Assuming char $k = 0$, if X is smooth at the center of P_E, we make a study of the graded algebra associated to ν_E and define a finite set whose elements generate a localization of the graded algebra modulo étale covering. This provides an explicit description of a minimal system of generators of the local ring $\mathcal{O}_{X_\infty,P_E}$. If X is singular, we obtain generators of $P_E \,/\, P_E^2$ and conclude that embdim $\mathcal{O}_{(X_\infty)_{\mathrm{red}},P_E} =$ embdim $\widehat{\mathcal{O}_{X_\infty,P_E}} \leq \widehat{k}_E + 1$ where $\widehat{k}_E$ is the Mather discrepancy of X with respect to ν_E. This provides algebraic tools for explicit computations of the local rings $\widehat{\mathcal{O}_{X_\infty,P_E}}$.

1. Introduction

The space of arcs X_∞ of an algebraic variety X was introduced by J. Nash in the 60's [20]. He expected to detect from arc families those components of the exceptional locus of the resolutions of singularities $Y \to X$ which are invariant by birational equivalence. The space X_∞ is an intrinsic object associated to X which allows to construct invariants of

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https://doi.org/10.1016/j.jalgebra.2017.09.031
0021-8693/

the variety: J. Denef and F. Loeser [4] made a systematic construction of some invariants of X using motivic integration on X_∞. They also considered X_∞ together with its scheme structure, an idea which was further developed by S. Ishii and J. Kollar [13].

Precisely, J. Nash asked "how complete is the representation of essential components by arc families", that is, what essential valuations are determined by the irreducible components of the space of arcs X_∞^{Sing} centered at some singular point of X. Divisorial valuations whose center appears as an irreducible component of the exceptional locus of every resolution of singularities of X are called essential valuations.

The arc families considered by J. Nash correspond to certain fat points P of X_∞. These fat points are *stable points*, as defined by the author in [23] lemma 3.1 and [24] definition 3.1 (see also [25]). It is natural to expect that those geometric properties of X_∞ with respect to an arc family are reflected in the algebraic properties of the local ring $\mathcal{O}_{X_\infty,P}$. An important role is played by the following algebraic property: the ideal of definition of a stable point P of X_∞ in a neighborhood of P, with its reduced structure, is finitely generated ([24] theorem 4.1, see 2.4 below). This implies that the complete local ring $\widehat{\mathcal{O}_{X_\infty,P}}$ is a Noetherian ring ([24], corollary 4.6).

Most advances on the Nash program on arc families use our Curve Seletion Lemma ([24] corollary 4.8) which is an easy consequence of the previous property, and is valid over a perfect field of any characteristic. For instance, if $\dim \mathcal{O}_{X_\infty,P} = 1$ then P is the generic point of an irreducible component of X_∞^{Sing} ([25] corollary 5.12). This is what occurs for essential valuations in toric varieties ([13] theorem 3.16), for nonuniruled ([16] theorem 3.3) and for terminal valuations ([8] theorem 3.3). Known counterexamples involve a local ring $\widehat{\mathcal{O}_{X_\infty,P}}$ of dimension greater than or equal to 2: for the counterexample in [13] see [25] remark 5.16, and for the ones in [14] see example 4.13 below. On the other hand, essential valuations are characterized in different ways: minimal elements in the cone for toric varieties ([3] theorem 1.10 and [13] section 3), nonruled ([1] proposition 4) and some divisorial valuations with discrepancy 1 (resp. 2) over certain canonical (resp. terminal) isolated singularities ([7] lemma 5.2 and [14]).

The local rings $\widehat{\mathcal{O}_{X_\infty,P}}$ involved in these results and examples have nevertheless a simple algebraic structure: nonreduced curves and some mildly singular surfaces. But in general their structure is much more complicated. Our purpose in this article is to develop appropriate algebraic tools for computing the local rings $\widehat{\mathcal{O}_{X_\infty,P}}$, P a stable point. We construct (Corollary 4.11) a presentation of $\widehat{\mathcal{O}_{X_\infty,P}}$ by concrete generators and relations:

$$\widehat{\mathcal{O}_{X_\infty,P}} \;\cong\; \kappa(P)\left[\left[\{X_{j,r;n}\}_{(j,r;n)\in\mathcal{C}}\right]\right] \Big/ \;\widetilde{I}$$

where $P = P_{eE}$ is the generic point of the family of arcs with contact $e \geq 1$ with an exceptional divisor E and the cardinal of $\mathcal{C}$ is $e(\widehat{k}_E + 1)$. Here $\widehat{k}_E$ is the Mather discrepancy of X with respect to E (see [12], [5]). This provides a framework to recover geometric properties and invariants of a variety X. We apply our result in Corollary 4.11 to understand the geometry of the space of arcs in the examples of [14] (Example 4.13).

If X is smooth at the center P_0 of a stable point P of X_∞, then the local ring $\mathcal{O}_{X_\infty,P}$ is regular and essentially of finite type over some field. Applying the Motivic Change of Variables formula ([4], lemma 3.4), it is proved that $\dim \mathcal{O}_{X_\infty,P_{eE}} = e(k_E+1)$ where k_E is the discrepancy of X with respect to E (Proposition 2.6). Assuming furthermore that char $k=0$, we give an explicit description of a minimal set of generators of P_{eE}, i.e. a regular system of parameters of $\mathcal{O}_{X_\infty,P_{eE}}$.

If X is not smooth at P_0, let $X \to \mathbb{A}^d_k$ be a general projection. A set of generators of the image of P_{eE} in $(\mathbb{A}^d)_\infty$ provides a set of generators of P_{eE} ([25], prop. 4.5). From this it follows that the embedding dimension of the ring $\widehat{\mathcal{O}_{X_\infty,P_{eE}}}$ is bounded from above by $e(\widehat{k}_E+1)$. A further work has been done by H. Mourtada and the author [19] to prove that this actually defines a minimal system of coordinates of (X_∞, P_{eE}) and to extract some consequences about the dimension of the ring $\widehat{\mathcal{O}_{X_\infty,P_{eE}}}$.

A study of the graded algebra associated to the divisorial valuation ν_E is crucial in our study. One of the main ideas in our proof is to define some "approximate roots" $\{q_{j,r}\}_{(j,r)\in\mathcal{J}}$ in $\mathrm{gr}_{\nu_E}\mathcal{O}_{\mathbb{A}^d,P_0}$, where $X \to \mathbb{A}^d$ is a general projection (Definition 3.4). The techniques used here are similar to those used by B. Teissier ([28], [10], [27]) and are of independent interest from the point of view of valuation theory. Although the $q_{j,r}$'s do not generate $\mathrm{gr}_{\nu_E}\mathcal{O}_{\mathbb{A}^d,P_0}$ (in general $\mathrm{gr}_{\nu_E}\mathcal{O}_{\mathbb{A}^d,P_0}$ is not finitely generated for $d \geq 3$), they generate a localization of $\mathrm{gr}_{\nu_E}\mathcal{O}_{\mathbb{A}^d,P_0}$ modulo étale covering (Theorem 3.8). This is done in section 3. In section 4 we describe minimal coordinates of $(\mathbb{A}^d)_\infty$ at the image $P^{\mathbb{A}^d}_{eE}$ of P_{eE} in $(\mathbb{A}^d)_\infty$ from the $q_{j,r}$'s. From this we obtain a regular system of parameters of $\mathcal{O}_{X_\infty,P_{eE}}$ if X is smooth at P_0 (Theorem 4.8), and a system of coordinates of (X_∞, P_{eE}) for general X (Corollary 4.10).

2. Preliminaries

2.1. Let k be a perfect field. For any scheme over k, let X_∞ denote the *space of arcs of* X. It is a (not of finite type) k-scheme whose K-rational points are the K-arcs on X (i.e. the k-morphisms Spec $K[[t]] \to X$), for any field extension $k \subseteq K$. More precisely, $X_\infty := \varprojlim X_n$ where, for $n \in \mathbb{N}$, X_n is the k-scheme of finite type whose K-rational points are the K-arcs of order n on X (i.e. the k-morphisms Spec $K[[t]]/(t)^{n+1} \to X$). In fact, the projective limit is a k-scheme because the natural morphisms $X_{n'} \to X_n$, for $n' \geq n$, are affine morphisms. We denote by $j_n : X_\infty \to X_n$, $n \geq 0$, the natural projections.

Given $P \in X_\infty$, with residue field $\kappa(P)$, we denote by h_P : Spec $\kappa(P)[[t]] \to X$ the induced $\kappa(P)$-arc on X. The image in X of the closed point of Spec $\kappa(P)[[t]]$, or equivalently, the image P_0 of P by $j_0 : X_\infty \to X = X_0$ is called the *center* of P. Then, h_P induces a morphism of k-algebras $h_P^\sharp : \mathcal{O}_{X,j_0(P)} \to \kappa(P)[[t]]$; we denote by ν_P the function $\mathrm{ord}_t h_P^\sharp : \mathcal{O}_{X,j_0(P)} \to \mathbb{N} \cup \{\infty\}$.

The space of arcs of $\mathbb{A}^N_k = \mathrm{Spec}\, k[x_1,\ldots,x_N]$ is $(\mathbb{A}^N_k)_\infty = \mathrm{Spec}\, k[\underline{X}_0,\ldots,\underline{X}_n,\ldots]$ where for $n \geq 0$, $\underline{X}_n = (X_{1;n},\ldots,X_{N;n})$ is an N-uple of variables. For any $f \in k[x_1,\ldots,x_N]$, let $\sum_{n=0}^\infty F_n\, t^n$ be the Taylor expansion of $f(\sum_n \underline{X}_n\, t^n)$, hence $F_n \in$

$k[\underline{X}_0, \ldots, \underline{X}_n]$. If $X \subseteq \mathbb{A}_k^N$ is affine, and $I_X \subset k[x_1, \ldots, x_N]$ is the ideal defining X in $\mathbb{A}_k^N$, then we have $X_\infty = \text{Spec } k[\underline{X}_0, \ldots, \underline{X}_n, \ldots] \,/\, (\{F_n\}_{n \geq 0, f \in I_X})$.

2.2. For $r, m \in \mathbb{N}$, $0 \leq r \leq m$, let $A_k^{r,m} := k[[x_1, \ldots, x_r]][x_{r+1}, \ldots, x_m]$ and let $X \subseteq \text{Spec } A_k^{r,m}$ be an affine irreducible k-scheme. A point P of X_∞ is *stable* if there exist $G \in \mathcal{O}_{X_\infty} \setminus P$, such that, for $n >> 0$, the map $X_{n+1} \longrightarrow X_n$ induces a trivial fibration

$$\overline{j_{n+1}(Z(P))} \cap (X_{n+1})_G \longrightarrow \overline{j_n(Z(P))} \cap (X_n)_G$$

with fiber $\mathbb{A}_k^d$, where $d = \dim X$, $Z(P)$ is the set of zeros of P in X_∞, $\overline{j_n(Z(P))}$ is the closure of $j_n(Z(P))$ in X_n and $(X_n)_G$ is the open subset $X_n \setminus Z(G)$ of X_n. This definition is extended to any element X in $\mathcal{X}_k$, being $\mathcal{X}_k$ the subcategory of the category of k-schemes defined by all separated k-schemes which are locally of finite type over some Noetherian complete local ring R_0 with residue field k ([25] def. 3.3). Note that $\mathcal{X}_k$ contains the separated k-schemes of finite type and it also contains the k-schemes Spec $\widehat{R}$, being $\widehat{R}$ the completion of a local ring R which is a k-algebra of finite type. In [24] and [25] a theory of stable points of X_∞ is developed. One important property of these points is the following:

Proposition 2.3. *([25], prop. 3.7 (iv)) Let P be a stable point of X_∞. For $n \geq 0$, let P_n be the prime ideal $P \cap \mathcal{O}_{\overline{j_n(X_\infty)}}$, where $\overline{j_n(X_\infty)}$ is the closure of $j_n(X_\infty)$ in X_n, with the reduced structure. Then we have that $\dim \mathcal{O}_{\overline{j_n(X_\infty)}, P_n}$ is constant for $n >> 0$, and since*

$$\dim \mathcal{O}_{X_\infty, P} \leq sup_n \dim \mathcal{O}_{\overline{j_n(X_\infty)}, P_n}$$

it implies that $\dim \mathcal{O}_{X_\infty, P} < \infty$.

And the main result in the theory of stable points is:

2.4. **Finiteness property of the stable points.** *([24] th. 4.1, [25] 3.10)*

Let P be a stable point of X_∞, then the formal completion $\widehat{\mathcal{O}_{(X_\infty)_{red}, P}}$ of the local ring of $(X_\infty)_{red}$ at P is a Noetherian ring.

Moreover, if X is affine, then there exists $G \in \mathcal{O}_{X_\infty} \setminus P$ such that the ideal $P\left(\mathcal{O}_{(X_\infty)_{red}}\right)_G$ is a finitely generated ideal of $\left(\mathcal{O}_{(X_\infty)_{red}}\right)_G$. In particular $P\mathcal{O}_{(X_\infty)_{\text{red}}, P}$ is finitely generated.

Besides we have $\widehat{\mathcal{O}_{X_\infty, P}} \cong \widehat{\mathcal{O}_{(X_\infty)_{\text{red}}, P}}$ ([25] th. 3.13). Hence, from 2.4 it follows that the maximal ideal of $\widehat{\mathcal{O}_{X_\infty, P}}$ is $P\widehat{\mathcal{O}_{X_\infty, P}}$, and even more, $\widehat{P^n} = P^n \widehat{\mathcal{O}_{X_\infty, P}}$ for every $n > 0$ (see [2] chap. III, sec. 2, no. 12, corol. 2). Therefore, if P is a stable point of X_∞ then

$$\text{embdim } \widehat{\mathcal{O}_{X_\infty, P}} \;=\; \text{embdim } \mathcal{O}_{(X_\infty)_{\text{red}}, P}.$$

Though this article we will consider étale morphisms. The following holds:

Proposition 2.5. *Let $X, Z \in \mathcal{X}_k$ and let $\sigma : X \to Z$ be an étale k-morphism. Then we have*

$$X_\infty \cong Z_\infty \times_Z X$$

in particular, X_∞ is étale over Z_∞. Therefore, the morphism $\sigma_\infty : X_\infty \to Z_\infty$ induces a map

$$\{\textit{stable points of } X_\infty\} \to \{\textit{stable points of } Z_\infty\}$$

and, if Q is a stable point of X_∞ and P its image by the previous map, then $\mathcal{O}_{Z_\infty,P} \to \mathcal{O}_{X_\infty,Q}$ is étale and

$$\widehat{\mathcal{O}_{X_\infty,Q}} \cong \widehat{\mathcal{O}_{Z_\infty,P}} \otimes_{\kappa(P)} \kappa(Q). \tag{1}$$

Proof. We may suppose that $Z = \text{Spec } A$, $X = \text{Spec } B$ where $B = (A[x]/(f))_g$, $f, g \in A[x]$ and the class of $f'(x)$ in B is a unit ([22], chap. V, th. 1). Then the stability property in [4] (see also [25] (8) in 3.4) implies that

$$X_\infty \ = \ \text{Spec } \left(A_\infty[X_0] \ / \ (F_0)\right)_{G_0}$$

where $A_\infty = \mathcal{O}_{Z_\infty}$. From this it follows that $X_\infty \cong Z_\infty \times_Z X$. Moreover, for $n \geq 0$, we have

$$X_n \ = \ \text{Spec } \left(A_n[X_0] \ / \ (F_0)\right)_{G_0}$$

that is, $X_n \cong Z_n \times_Z X$. From this, the stability property [4], lemma 4.1, and the definition of stable point, it follows that, if Q is a stable point of X_∞ then its image P by σ_∞ is a stable point of Z_∞.

For the last assertion note that, if $\widehat{X} := \text{Spec } \widehat{\mathcal{O}_{X,Q_0}}$, being Q_0 the center of Q in X, then Q induces a stable point $\widehat{Q}$ in $\widehat{X}_\infty$ because $h_Q : \text{Spec } \kappa(Q)[[t]] \to X$ factorizes through $\widehat{X}$, and we have

$$\widehat{\mathcal{O}_{X_\infty,Q}} \cong \widehat{\mathcal{O}_{\widehat{X}_\infty,\widehat{Q}}}. \tag{2}$$

Analogously, $\widehat{\mathcal{O}_{Z_\infty,P}} \cong \widehat{\mathcal{O}_{\widehat{Z}_\infty,\widehat{P}}}$, where $\widehat{Z} := \text{Spec } \widehat{\mathcal{O}_{Z,P_0}}$ and $\widehat{P}$ is the stable point of $\widehat{Z}_\infty$ induced by P. Therefore, in order to prove (1) we may suppose that $Z = \text{Spec } A$, $X = \text{Spec } B$ where A and B are complete local rings and $X \to Z$ is local étale, hence $B \cong A \otimes_{\kappa(P_0)} \kappa(Q_0)$ ([22] VIII corol. to lemme 2 and [11] III exer. 10.4). Now, $X_\infty \cong Z_\infty \times_Z X$, therefore

$$B_\infty \cong A_\infty \otimes_{\kappa(P_0)} \kappa(Q_0) \qquad \text{and} \qquad (B_\infty)_Q \cong (A_\infty)_P \otimes_{\kappa(P_0)} \kappa(Q_0).$$

Therefore $(A_\infty)_P \to (B_\infty)_Q$ is étale, hence $\widehat{(A_\infty)_P} \to \widehat{(B_\infty)_Q}$ is also étale, and from [22], VIII corol. to lemme 2, it follows that $\widehat{(B_\infty)_Q} \cong \widehat{(A_\infty)_P} \otimes_{\kappa(P)} \kappa(Q)$, thus (1) holds.

The inequality in Proposition 2.3 may be strict. However, if X is nonsingular at P_0, then we will next show that equality holds.

Proposition 2.6. *Let P be a stable point of X_∞. If X is nonsingular at the center P_0 of P, then the ring $\mathcal{O}_{X_\infty,P}$ is regular and essentially of finite type over a field, and we have*

$$\dim \mathcal{O}_{X_\infty,P} = sup_n \dim \mathcal{O}_{\overline{j_n(X_\infty)},(P)_n}.$$

Proof. The first statement is prop. 4.2 in [25]. The second one also follows from the proof of [25], prop. 4.2. In fact, by Proposition 2.5 and since there exists an étale morphism from a neighborhood of P_0 to a subset of $A_k^{r,d-r}$, where $d = \dim X$, we may suppose that $X \subseteq A_k^{r,d-r}$. In this case we have

$$\mathcal{O}_{X_\infty} = \mathcal{O}_X[\underline{X}_1, \ldots, \underline{X}_n, \ldots] \quad \text{and} \quad \mathcal{O}_{X_n} = \mathcal{O}_X[\underline{X}_1, \ldots, \underline{X}_n],\ n \geq 0$$

where $\underline{X}_n = (X_{1;n}, \ldots, X_{d;n})$, $n \geq 1$. By 2.4, there exist a finite number of polynomials $G_1, \ldots, G_s, G \in \mathcal{O}_{X_\infty}$ such that $P = ((G_1, \ldots, G_s) : G^\infty)$. If $n_0 \in \mathbb{N}$ is such that $\mathcal{O}_{\overline{j_{n_0}(X_\infty)}}$ contains $G_1, \ldots, G_s, G$, then $k(\underline{X}_{n_0+1}, \ldots, \underline{X}_n, \ldots) \subset \mathcal{O}_{X_\infty,P}$. This implies that

$$\mathcal{O}_{X_\infty,P} \;\cong\; k(\underline{X}_{n_0+1}, \ldots, \underline{X}_n, \ldots) \otimes_k \mathcal{O}_{\overline{j_{n_0}(X_\infty)},P_{n_0}}$$

hence we conclude the result.

2.7. Let X be a reduced separated k-scheme of finite type and let ν be a divisorial valuation on X, i.e. ν is a divisorial valuation on an irreducible component of X. Then there exists a proper and birational morphism $\pi : Y \to X$, with Y normal, such that the center of ν on Y is a divisor E of Y. We also denote by ν_E the valuation ν. Let $\pi_\infty : Y_\infty \to X_\infty$ be the morphism on the spaces of arcs induced by π. Let $Y_\infty^{E_{\text{reg}}}$ be the inverse image of $E \cap \text{Reg}(Y)$ by the natural projection $j_0^Y : Y_\infty \to Y$, which is an irreducible subset of Y_∞, and let N_E be the closure of $\pi_\infty(Y_\infty^{E_{\text{reg}}})$. Then N_E is an irreducible subset of X_∞, let P_E be the generic point of N_E. More generally, for every $e \geq 1$, let $Y_\infty^{eE_{\text{reg}}} := \{Q \in Y_\infty \ /\ \nu_Q(I_E) = e\}$, where I_E is the ideal defining E in an open affine subset of $\text{Reg}(Y)$ (the set $Y_\infty^{eE_{\text{reg}}}$ will be also denoted by Y_∞^{eE} if Y is nonsingular). Then $Y_\infty^{eE_{\text{reg}}}$ is an irreducible subset of Y_∞, let N_{eE} be the closure of $\pi_\infty(Y_\infty^{eE_{\text{reg}}})$ and P_{eE} (also denoted by P_{eE}^X) be the generic point of N_{eE}. Note that P_{eE} only depends on e and on the divisorial valuation $\nu = \nu_E$, more precisely, if $\pi' : Y' \to X$ is another proper and birational morphism, with Y' normal, such that the center E' of ν on Y' is a divisor, then the point $P_{eE'}$ defined by e and E' coincides with P_{eE}. We have that P_{eE} is a stable point of X_∞ ([25], prop. 4.1, see also [24], prop. 3.8).

2.8. With the notation in 2.7, the image of the canonical homomorphism $d\pi : \pi^*(\wedge^d\Omega_X) \to \wedge^d\Omega_Y$ is an invertible sheaf at the generic point of E. That is, there exists a nonnegative integer $\widehat{k}_E$ such that the fibre at E of the sheaf $d\pi(\pi^*(\wedge^d\Omega_X))$ is isomorphic to the fibre at E of $\mathcal{O}_Y(-\widehat{k}_E E)$. We call $\widehat{k}_E$ the *Mather discrepancy* of X with respect to the prime divisor E. Note that $\widehat{k}_E \neq 0$ implies that π is not an isomorphism at the generic point of E, and that $\widehat{k}_E$ only depends on the divisorial valuation $\nu = \nu_E$. We have:

$$\sup_n \dim \mathcal{O}_{\overline{j_n(X_\infty)},(P_{eE})_n} = e\,(\widehat{k}_E + 1) \tag{3}$$

([4], lemma 3.4, [9], theorem 3.9). Hence by Proposition 2.3 we have

$$\dim \mathcal{O}_{X_\infty,P_{eE}} \leq e\,(\widehat{k}_E + 1).$$

Moreover, let P be a stable point of X_∞ and let P_0 be its center. If P_0 is the generic point of X then ν_P is trivial. Otherwise, ν_P is a divisorial valuation ([25], (vii) in prop. 3.7 and prop. 3.8), i.e. there exists $\pi : Y \to X$ birational and proper such that the center of ν_P on Y is a divisor E and there exists $e \in \mathbb{N}$ such that $\nu_P = e\nu_E$. There exists a stable point $P^Y \in Y_\infty$ whose image by π_∞ is P ([25], prop. 4.1). Therefore $P^Y \supseteq P^Y_{eE}$ and $P \supseteq P_{eE}$. Now, assume that X is nonsingular at P_0, and recall that in this nonsingular case we have $\widehat{k}_E = k_E$, where k_E is the *discrepancy of* X with respect to E, which is defined to be the coefficient of E in the divisor $K_{Y/X}$ with exceptional support which is linearly equivalent to $K_Y - \pi^*(K_X)$ ([6], appendix). Applying prop. 2.6 and lemma 4.3 in [4] we conclude

Corollary 2.9. *Let P be a stable point of X_∞. Suppose that X is nonsingular at the center P_0 of P, and that P_0 is not the generic point of X and $\nu_P = e\nu_E$. Then $\mathcal{O}_{X_\infty,P}$ is a regular ring of dimension*

$$\dim \mathcal{O}_{X_\infty,P} = ek_E + \dim \mathcal{O}_{Y_\infty,P^Y}.$$

In particular

$$\dim \mathcal{O}_{X_\infty,P_{eE}} = e(k_E + 1).$$

The following question is open:

Question 2.10. Let P be a stable point of X_∞ and suppose that X is analytically irreducible at P_0. If the local ring $\mathcal{O}_{X_\infty,P}$ is regular, is X nonsingular at the center P_0 of P?

3. On the graded algebra of the local ring of a smooth scheme associated to a divisorial valuation

From now on, let k be a field of characteristic 0. Through this article, we will denote by $k < y_1, \ldots, y_r >$ the henselization of the local ring $k[y_1, \ldots, y_r]_{(y_1,\ldots,y_r)}$, being $y_1, \ldots, y_r$ indeterminacies (see [22] for more details on henselization).

Let $\eta : Y \to \mathbb{A}^d_k$ be a k-morphism dominant and generically finite, where Y is a nonsingular k-scheme, let E be a divisor on Y and let P_0 be the center on $\mathbb{A}^d_k$ of the valuation defined by E. In this section we will define elements $\{q_{j,r}\}_{(j,r)\in\mathcal{J}}$ in the fraction field of $\mathcal{O}_{\mathbb{A}^d,P_0}$ (Proposition 3.3) whose initial forms generate a localization of the graded algebra $gr_{\nu_E}\mathcal{O}_{\mathbb{A}^d,P_0}$ modulo étale covering. In section 4 we will prove that they have the property of determining a basis of $P^{\mathbb{A}^d}_{eE}/(P^{\mathbb{A}^d}_{eE})^2$, being $P^{\mathbb{A}^d}_{eE}$ the image by η_∞ of the generic point of Y^{eE}_∞ (see 2.7). From this and applying Proposition 2.5, we will conclude analogous results for a smooth surface X and a divisorial valuation on X (Theorems 3.8 and 4.8).

Let us apply the description of the morphism η appearing in [25], proof of prop. 4.5 (see (4) below). First, we may suppose that Y is an affine k-scheme. In fact, we may replace Y by an open affine subset which contains the generic point ξ_E of E. Let $u \in \mathcal{O}_U$, where U is an open subset of Y that contains ξ_E, such that u defines a local equation of E. Since η is dominant and generically finite, there exist local coordinates $x_1, \ldots, x_d$ in an open subset of $\mathbb{A}^d$ that contains $\eta(\xi_E)$ such that the image of x_1 in $\mathcal{O}_{Y,\xi_E}$ is $g\ u^{m_1}$, where $m_1 > 0$ and g is a unit in $\mathcal{O}_{Y,\xi_E}$. By restricting U and adding a m_1-th root of g, we can define an étale morphism $\varphi : \widetilde{U} \to U$ such that the image of x_1 in $\mathcal{O}_{\widetilde{U}}$ is $u_1^{m_1}$ where u_1 is a local equation of the strict transform $\widetilde{E}$ of E in $\widetilde{U}$. Moreover, since char $k = 0$, and $\Omega_{\mathbb{A}^d} \otimes K(Y) \cong \Omega_Y \otimes K(Y)$, we may restrict $\widetilde{U}$ and U and define $\{u_1, \ldots, u_d\} \subset \mathcal{O}_{\widetilde{U}}$, $\{x_1, \ldots, x_d\} \subset \mathcal{O}_V$, where V is an open subset of $\mathbb{A}^d_k$, determining respective regular systems of parameters in a closed point $y_0 \in \widetilde{E}$ and in $\eta \circ \varphi(y_0)$, and such that, if we identify $x_1, \ldots, x_d$ with their images by $\eta^\sharp : \mathcal{O}_{V,\eta(y_0)} \to \mathcal{O}_{\widetilde{U},y_0}$, then

$$\begin{array}{ll} x_1 & = u_1^{m_1} \\ x_2 & = \sum_{1\leq i\leq m_2} \lambda_{2,i}\ u_1^i + u_1^{m_2}\ u_2 \\ x_3 & = \sum_{1\leq i\leq m_3} \lambda_{3,i}(u_2)\ u_1^i + u_1^{m_3}\ u_3 \\ \ldots & \ldots\ldots \\ x_\delta & = \sum_{1\leq i\leq m_\delta} \lambda_{\delta,i}(u_2,\ldots,u_{\delta-1})\ u_1^i + u_1^{m_\delta}\ u_\delta \\ x_{\delta+1} & = u_{\delta+1} \\ \ldots & \ldots\ldots \\ x_d & = u_d \end{array} \tag{4}$$

where $\delta = \mathrm{codim}_{\mathbb{A}^d}\ \overline{\eta(\xi_E)}$,

$$\begin{array}{l} m_1 \leq \mathrm{ord}_{u_1} x_j = min\{i\ /\ \lambda_{j,i} \neq 0\} \quad \text{for } 2 \leq j \leq \delta, \\ 0 < m_1 \leq m_2 \leq \ldots \leq m_\delta, \end{array}$$

$\lambda_{j,i}(u_2, \ldots, u_{j-1}) \in k[[u_2, \ldots, u_{j-1}]]$, for $2 \leq j \leq \delta$, $0 \leq i \leq m_j$, and, given $j' < j$, if $i < m_{j'}$ then $\lambda_{j,i} \in k[[u_2, \ldots, u_{j'-1}]]$. Moreover, since $x_j - u_1^{m_j} u_j$ belongs to $k[[u_1, \ldots, u_{j-1}]]$ and is integral over $k[u_1, \ldots, u_d]_{(u_1,\ldots,u_d)}$, it is also integral over $k[u_1, \ldots, u_{j-1}]_{(u_1,\ldots,u_{j-1})}$. Therefore, after a possible replacement of y_0 by another point in an open subset of $\widetilde{U} \cap \widetilde{E}$, we may suppose that, for $2 \leq j \leq \delta$ and $0 \leq i \leq m_j$, $\lambda_{j,i}(u_2, \ldots, u_{j-1})$ belongs to the henselization $k < u_2, \ldots, u_{j-1} >$ of the local ring $k[u_2, \ldots, u_{j-1}]_{(u_2,\ldots u_{j-1})}$, and, if $i < m_{j'}$, $j' < j$, then $\lambda_{j,i}$ belongs to $k < u_2, \ldots, u_{j'-1} >$.

Besides, from the expression (4) it follows that there exists an open neighborhood of y_0 in $\widetilde{E}$ whose closed points y_0' satisfy the same property, i.e. there exists a regular system of parameters of y_0' and of $\eta \circ \varphi(y_0')$ for which (4) holds. In fact, replace u_i by $u_i' = u_i + c_i \mod u_1$, for $2 \leq i \leq d$, where $(c_i)_i$ lies in an open subset of k^{d-1}. Hence, we may suppose with no loss of generality that

$$\begin{array}{l} \lambda_{j,i}(u_2, \ldots, u_{j-1}) \in k < u_2, \ldots, u_{j-1} > \text{ for } 2 \leq j \leq \delta,\ 0 \leq i \leq m_j \\ \text{if } i < m_{j'}, j' < j, \text{ then } \lambda_{j,i} \in k < u_2, \ldots, u_{j'-1} > \\ \text{if } \lambda_{j,i}(u_2, \ldots, u_{j-1}) \neq 0 \text{ then it is a unit in } k < u_2, \ldots, u_{j-1} > \\ \lambda_{j,m_j}(u_2, \ldots, u_{j-1}) \text{ is a unit, for } 2 \leq j \leq d. \end{array} \tag{5}$$

Note that $\widetilde{U}$ is nonsingular. Note also that $\wedge^d \Omega_V$ is an invertible sheaf, hence the image of $d\eta : \eta^*(\wedge^d \Omega_V) \to \wedge^d \Omega_U$ is an invertible sheaf. The order a_E in E of the corresponding divisor is equal to the order in $\widetilde{E}$ of the image of $d(\eta \circ \varphi) : (\eta \circ \varphi)^*(\wedge^d \Omega_V) \to \wedge^d \Omega_{\widetilde{U}}$. So, from now on, after a possible replacement of Y by $\widetilde{U}$ and of $\eta : Y \to \mathbb{A}^d$ by $\eta \circ \varphi : \widetilde{U} \to V$, we will suppose that (4) is a local expression of η. Besides, from (4) it follows that:

$$a_E = m_1 + \ldots + m_\delta - 1. \tag{6}$$

Lemma 3.1. *Let A be a finitely generated k-algebra and let $\theta : Y \to Spec\ A[x, y]$ be a k-morphism, where x, y are indeterminacies. Let j, $2 \leq j \leq d+1$ and suppose that there exists a multiplicative system S_{j-1} of $A[x]$ and there exist elements*

$$l_{j'} \in S_{j-1}^{-1} A[x] \quad \textit{for } 2 \leq j' \leq j-1$$

such that, if we set $v_{j'} := \theta^\sharp(l_{j'})$ for $2 \leq j' \leq j-1$, then $\{u_1, v_2, \ldots, v_{j-1}, u_j, \ldots, u_d\}$ is a regular system of parameters of $\mathcal{O}_{Y,y_0}$. Suppose that the images of x, y by $\theta^\sharp$ are given by $x \mapsto u_1^{m_1}$ and

$$y \mapsto \sum_{m_1 \leq i \leq m} \lambda_i(v_2, \ldots, v_{j-1})\ u_1^i + u_1^m\ \varrho \quad mod\ (u_1)^{m+1} \tag{7}$$

where $m \geq m_1$, $\varrho \in \mathcal{O}_{Y,y_0}$ and $\lambda_i(v_2, \ldots, v_{j-1}) \in k < v_2, \ldots, v_{j-1} >$. Set

$$\begin{array}{ll} \mathfrak{e} & := g.c.d.(\{m_1\} \cup \{i \;/\; \lambda_i \neq 0\}), \qquad \beta_0 := m_1, \qquad e_0 \;:= \beta_0 \\ \beta_{r+1} & := min\,\{i \;/\; \lambda_i \neq 0 \text{ and } g.c.d.\{\beta_0, \ldots, \beta_r, i\} < e_r\ \} \text{ and} \\ e_{r+1} & := g.c.d.\{\beta_0, \ldots, \beta_{r+1}\} \quad \text{for } 1 \leq r \leq g-1, \text{ where } g \text{ is such that } e_g = \mathfrak{e} \\ \beta_{g+1} & := m. \end{array} \tag{8}$$

Let $n_0 = 1$ *and* $n_r := \frac{e_{r-1}}{e_r}$ *for* $1 \leq r \leq g$ *and let* $\overline{\beta}_0 = \beta_0$ *and* $\overline{\beta}_r$, $1 \leq r \leq g+1$, *be defined by*

$$\overline{\beta}_r - n_{r-1}\overline{\beta}_{r-1} = \beta_r - \beta_{r-1}, \tag{9}$$

hence we have

$$\begin{array}{l} \overline{\beta}_r > n_{r-1}\ \overline{\beta}_{r-1} \quad \text{for } 1 \leq r \leq g, \quad \text{ and } \quad \overline{\beta}_{g+1} \geq n_g\ \overline{\beta}_g; \\ n_r\overline{\beta}_r \ \text{ belongs to the semigroup generated by } \overline{\beta}_0, \ldots, \overline{\beta}_{r-1},\ 1 \leq r \leq g+1. \end{array} \tag{10}$$

Then, there exist an open subset U *of* Y *containing* ξ_E *and a sequence of integers* $\{i_s\}_{s=1}^N$ *such that*

(i) $i_1 < i_2 < \ldots < i_N = \overline{\beta}_{g+1}$ *and* $\{i_s\}_{s=1}^N \subset \{\overline{\beta}_0\} \cup \cup_{r=1}^{g+1}(n_{r-1}\overline{\beta}_{r-1},\ \overline{\beta}_r]$,

(ii) $\{\overline{\beta}_r\}_{r=1}^{g+1}$ *is contained in* $\{i_1, \ldots, i_N\}$, *that is, there exist* $s_1 < s_2 < \ldots < s_{g+1} := N$ *such that* $i_{s_r} = \overline{\beta}_r$ *for* $1 \leq r \leq g+1$,

(iii) *for each closed point* y_0' *in* $U \cap E$ *there exist a regular system of parameters* $\{u_1, v_2', \ldots, v_{j-1}', u_j', \ldots, u_d'\}$ *of* $\mathcal{O}_{Y,y_0'}$, *where* $v_i' = v_i + c_i$, $u_i' = u_i + c_i$, $(c_i)_i \in k^{d-1}$, *and there exist* $\{h_1 = y, h_2, \ldots, h_N\}$ *satisfying: given* s, *let* r, $1 \leq r \leq g+1$, *be such that* $n_{r-1}\overline{\beta}_{r-1} < i_s \leq \overline{\beta}_r$ *(or* $r = 1$ *if* $s = 1$ *and* $i_1 = \overline{\beta}_0$*), then*

(a) $h_s \in T_{r-1}^{-1} \ldots T_0^{-1} S_{j-1}^{-1} A[x, y]$, *where* $T_{r'}$ *is the multiplicative part generated by* $q_{r'} := h_{s_{r'}}$ *(resp.* $q_0 := x_1$*) for* $1 \leq r' \leq r-1$ *(resp.* $r' = 0$*),*

(b) *the image of* h_s *in* $K(\mathcal{O}_{Y,y_0'})$ *belongs to* $\mathcal{O}_{Y,y_0'}$, *and if we identify* h_s *with its image in* $\mathcal{O}_{Y,y_0'}$ *then*

$$\begin{array}{l} h_s = \sum_{i_s \leq i \leq m^{(r)}} \lambda_{s,i}(v_2', \ldots, v_{j-1}')\ u_1^i \\ \qquad + \gamma_{s,m^{(r)}}(v_2', \ldots, v_{j-1}')\ u_1^{m^{(r)}}\ \varrho \quad \mod (u_1)^{m^{(r)}+1} \end{array} \tag{11}$$

where $\lambda_{s,i}, \gamma_{s,m^{(r)}} \in k < v_2', \ldots v_{j-1}' >$, $\lambda_{s,i_s} \neq 0$, $\gamma_{s,m^{(r)}}$ *is a unit and* $m^{(r)} := m + (n_1 - 1)\overline{\beta}_1 + \ldots + (n_{r-1} - 1)\overline{\beta}_{r-1}$. *Moreover, for* $r \leq r' \leq g$, *let* $\beta_{r'}^{(r)} := \beta_{r'} + (n_1 - 1)\overline{\beta}_1 + \ldots + (n_{r-1} - 1)\overline{\beta}_{r-1}$ *then we have*

$$min\left\{ i \;/\; \lambda_{s,i} \neq 0 \text{ and } g.c.d.\{e_{r-1}, \beta_r^{(r)}, \ldots, \beta_{r'-1}^{(r)}, i\} < e_{r'-1}\ \right\} = \beta_{r'}^{(r)} \tag{12}$$

and $\lambda_{s,\beta_{r'}^{(r)}}$ *is a unit.*

(c) *For* $s \geq 2$, *if* $s = s_{r-1} + 1$ *(resp.* $s_{r-1} + 1 < s$*), then*

$$h_s := q_0^{b_0^s} \cdots q_\rho^{b_\rho^s}\ P_s\left(\frac{\overline{\mu}_s h}{q_0^{b_0^s} \cdots q_\rho^{b_\rho^s}},\ l_2, \ldots, l_{j-1} \right)$$

where $h = (q_{r-1})^{n_{r-1}}$ (resp. $h = h_{s-1}$), $\rho = r-2$ (resp. $\rho = r-1$), the integers $\{b^s_{r'}\}_{r'=0}^{\rho}$ are the unique nonnegative integers satisfying $b^s_{r'} < n_{r'}$, $1 \leq r' \leq \rho$, and $n_{r-1}\overline{\beta}_{r-1} = \sum_{0 \leq r' \leq r-2} b^s_{r'}\overline{\beta}_{r'}$ (resp. $i_{j,s-1} = \sum_{0 \leq r' \leq r-1} b^s_{j,r'}\overline{\beta}_{j,r'}$), $\overline{\mu}_s = (\lambda_{s_1,\overline{\beta}_1})^{b^s_1} \cdots (\lambda_{s_\rho,\overline{\beta}_\rho})^{b^s_\rho}$ is a unit, and $P_s \in k[z, v'_2, \ldots, v'_{j-1}]$ is such that

$$P_s(\lambda, v'_2, \ldots, v'_{j-1}) = 0, \ \frac{\partial P_s}{\partial z}(\lambda, v'_2, \ldots, v'_{j-1}) \text{ is a unit in } k < v'_2, \ldots, v'_{j-1} >, \tag{13}$$

where $\lambda = (\lambda_{s-1,i_{s-1}})^{n_{r-1}}$ (resp. $\lambda = \lambda_{j,s-1,i_{j,s-1}}$).

Proof. First note that (10) follows from (8) and (9) (see [28] 2.2.1 in the Appendix). Note also that there exists an open neighborhood of y_0 in E such that if y'_0 is a closed point on it and $\{u_1, v'_2, \ldots, v'_{j-1}, u'_j, \ldots, u'_d\}$ is a regular system of parameters of $\mathcal{O}_{Y,y'_0}$, where $v'_i = v + c_i$, $u'_i = u_i + c_i$, $(c_i)_i \in k^{d-1}$, then the integers defined by (8) and (9) for the expression of the image of y in terms of $\{u_1, v'_2, \ldots, v'_{j-1}, u'_j, \ldots, u'_d\}$ are the same as the ones defined for the expression in (7). Thus, to prove the lemma, it suffices to show that, after a possible replacement of y_0 in an open subset $U \cap E$ of E, there exist $\{i_s\}_{s=1}^N$ and $\{h_s\}_{s=1}^N$ satisfying (i), (ii) and (a), (b) for the image of h_s in $K(\mathcal{O}_{Y,y_0})$ (hence $v'_2 = v_2, \ldots, v'_{j-1} = v_{j-1}$ in (11)) and (c).

We will define $\{i_s\}_{s=1}^N$ and $\{h_s\}_{s=1}^N$ by induction on s. First, after a possible replacement of y_0 in an open subset of E, we may suppose that, for every i such that $\lambda_i \neq 0$ in (7), λ_i is a unit in the ring

$$R_{j-1} := k < v_2, \ldots, v_{j-1} > .$$

Then, for $s = 1$, let $i_1 := \min\{i \ / \ \lambda_i(v_2, \ldots, v_{j-1}) \neq 0\}$ and $h_1 := y$. It is clear that $\overline{\beta}_0 \leq i_1 \leq \overline{\beta}_1$ and that (a) and (b) hold for $s = 1$. Now, let $s \geq 2$ and suppose that $i_1 < \ldots < i_{s-1}$ and $h_1, \ldots, h_{s-1}$ are defined and satisfy the required conditions. If $i_{s-1} = \overline{\beta}_{g+1}$ then set $N := s-1$. If not, then $i_{s-1} < \overline{\beta}_{g+1}$. Thus, there exist r, $1 \leq r \leq g+1$ such that $i_{s-1} \in \{\overline{\beta}_{r-1}\} \cup (n_{r-1}\overline{\beta}_{r-1}, \overline{\beta}_r)$. Let $s_1 < s_2 < \ldots < s_{r-1} \leq s-1$ be such that $i_{s_{r'}} = \overline{\beta}_{r'}$ for $1 \leq r' \leq r-1$ and let $q_0 := x$, $q_{r'} := h_{s_{r'}}$ for $1 \leq r' \leq r-1$.

If $i_{s-1} = \overline{\beta}_{r-1}$, recall that $\lambda_{s-1,\overline{\beta}_{r-1}}(v_2, \ldots, v_{j-1}) \in R_{j-1} \setminus \{0\}$, thus $(\lambda_{s-1,\overline{\beta}_{r-1}})^{n_r}$ belongs to $R_{j-1} \setminus \{0\}$ and hence there exists an irreducible monic polynomial $P_s \in k[z, v_2, \ldots, v_{j-1}]$ such that

$$P_s((\lambda_{s-1,\overline{\beta}_{r-1}})^{n_r}, v_2, \ldots, v_{j-1}) = 0 \ \text{ and } \ \frac{\partial P_s}{\partial z}((\lambda_{s-1,\overline{\beta}_{r-1}})^{n_r}, v_2, \ldots, v_{j-1}) \neq 0$$

Moreover, after a possible replacement of y_0 in an open subset of E, we may suppose that

$$\begin{aligned} &P_s((\lambda_{s-1,\overline{\beta}_{r-1}})^{n_r}, v_2, \ldots, v_{j-1}) = 0 \quad \text{and} \\ &\frac{\partial P_s}{\partial z}((\lambda_{s-1,\overline{\beta}_{r-1}})^{n_r}, \ldots, v_{j-1}) \text{ is a unit in } R_{j-1}. \end{aligned} \tag{14}$$

Analogously, if $i_{s-1} \in (n_{r-1}\overline{\beta}_{r-1}, \overline{\beta}_r)$, then after a possible replacement of y_0 in an open subset of E, we may suppose that there exists an irreducible monic polynomial $P_s \in k[z, v_2, \ldots, v_{j-1}]$ such that

$$P_s(\lambda_{s-1,i_{s-1}}, v_2, \ldots, v_{j-1}) = 0, \quad \frac{\partial P_s}{\partial z}(\lambda_{s-1,i_{s-1}}, \ldots, v_{j-1}) \text{ is a unit in } R_{j-1}. \tag{15}$$

If $i_{s-1} = \overline{\beta}_{r-1}$, let $b^s_{r'} = b_{r-1,r'}$, $0 \le r' \le r-2$, be the unique nonnegative integers satisfying $b_{r-1,r'} < n_{r'}$ for $1 \le r' \le r-2$, and $n_{r-1}\overline{\beta}_{r-1} = \sum_{0 \le r' \le r-2} b_{r-1,r'}\overline{\beta}_{r'}$, and let $\overline{\mu}_s := (\lambda_{s_1,\overline{\beta}_1})^{b^s_1} \cdots (\lambda_{s_{r-2},\overline{\beta}_{r-2}})^{b^s_{r-2}}$, which is a unit in R_{j-1}, such that the image of $q_0^{b^s_0} \cdots q_{r-2}^{b^s_{r-2}}$ by $\theta^\sharp$ is equal to $\overline{\mu}_s u_1^{n_{r-1}\overline{\beta}_{r-1}} \bmod (u_1)^{n_{r-1}\overline{\beta}_{r-1}+1}$. Set

$$h_s := q_0^{b_{r-1,0}} \cdots q_{r-2}^{b_{r-1,r-2}}\ P_s\left(\frac{\overline{\mu}_s\ (q_{r-1})^{n_{r-1}}}{q_0^{b_{r-1,0}} \cdots q_{r-2}^{b_{r-1,r-2}}},\ \ l_2, \ldots, l_{j-1} \right) \tag{16}$$

and $i_s := (n_{r-1}-1)\overline{\beta}_{r-1} \ + \ min\left\{ i \ / \ i > \overline{\beta}_{r-1},\ \lambda_{s-1,i} \neq 0 \right\}$, unless we have $\lambda_{s-1,i} = 0$ for all $i > \overline{\beta}_{r-1}$, which implies $r-1 = g$, then set $i_s := \overline{\beta}_{g+1}$. From (14), (16) and Taylor's development for P_s it follows that, if $s < N$ (resp. $s = N$) then the ν_E-value of the image $\theta^\sharp(h_s)$ of h_s in $\mathcal{O}_{Y,y_0}$ is $i_s > n_{r-1}\overline{\beta}_{r-1}$ (resp. $i_s \ge i_N = \overline{\beta}_{g+1}$), and the exponents of u_1 in $\theta^\sharp(h_s)$ with nonzero coefficient (see the left hand side of (11)) are determined by the ones in $\theta^\sharp(h_{s-1})$ by adding $(n_{r-1}-1)\overline{\beta}_{r-1}$, therefore $n_{r-1}\overline{\beta}_{r-1} < i_s \le \overline{\beta}_r$ and (11) and (12) hold for s. Moreover, for $r \le r' \le g$, the coefficient $\lambda_{s,\beta^{(r)}_{r'}}$ in $u_1^{\beta^{(r)}_{r'}}$ of $\theta^\sharp(h_s)$ is equal, modulo product by a unit, to $(\lambda_{s-1,\overline{\beta}_{r-1}})^{n_{r-1}-1}\ \lambda_{s-1,\beta^{(r-1)}_{r'}}$, therefore it is a unit, and (b) is satisfied. Besides, $h_s \in T^{-1}_{r-2} \ldots T^{-1}_0 S^{-1}_{j-1} A[x,y]$, hence (a) also holds.

If $n_{r-1}\overline{\beta}_{r-1} < i_{s-1} < \overline{\beta}_r$ then e_{r-1} divides i_{s-1} (by (b) applied to $s-1$) and there exist unique nonnegative integers $\{b^s_{r'}\}_{r'=0}^{r-1}$ satisfying $b^s_{r'} < n_{r'}$ for $1 \le r' \le r-1$ and $i_{s-1} = \sum_{0 \le r' \le r-1} b^s_{r'}\overline{\beta}_{r'}$ (because $n_{r-1}\overline{\beta}_{r-1} \le i_{s-1}$). Then, let $\overline{\mu}_s := (\lambda_{s_1,\overline{\beta}_1})^{b^s_1} \cdots (\lambda_{s_{r-1},\overline{\beta}_{r-1}})^{b^s_{r-1}}$, which is a unit in R_{j-1}, such that the image of $q_0^{b^s_0} \cdots q_{r-1}^{b^s_{r-1}}$ by $\theta^\sharp$ is equal to $\overline{\mu}_s u_1^{i_{s-1}} \bmod (u_1)^{i_{s-1}+1}$, and set

$$h_s := q_0^{b^s_0} \cdots q_{r-1}^{b^s_{r-1}}\ P_s\left(\frac{\overline{\mu}_s\ h_{s-1}}{q_0^{b^s_0} \cdots q_{r-1}^{b^s_{r-1}}},\ l_2, \ldots, l_{j-1} \right) \tag{17}$$

and $i_s := \ min\{ i \ / \ i > i_{s-1},\ \lambda_{s,i} \neq 0\}$, unless we have $\lambda_{s-1,i} = 0$ for all $i > \overline{\beta}_{r-1}$, which implies $r-1 = g$ and then we set $i_s := \overline{\beta}_{g+1}$. It is clear that (a) holds and, from (15) and (17), it follows that, if $s < N$ (resp. $s = N$), then the ν_E-value of the image $\theta^\sharp(h_s)$ of h_s in $\mathcal{O}_{Y,y_0}$ is $i_s > i_{s-1} > n_{r-1}\overline{\beta}_{r-1}$ (resp. $\ge i_N = \overline{\beta}_{g+1} > n_g\overline{\beta}_g$), and the exponents of u_1 in $\theta^\sharp(h_s)$ with nonzero coefficient are the same as the ones for by $\theta^\sharp(h_{s-1})$, hence $n_{r-1}\overline{\beta}_r < i_s \leq \overline{\beta}_r$ and (11) and (12) hold for s. Moreover, for $r \leq r' \leq g$, the coefficient $\lambda_{s,\beta^{(r)}_{r'}}$ in $u_1^{\beta^{(r)}_{r'}}$ of $\theta^\sharp(h_s)$ is the same, modulo product by a unit, as the coefficient $\lambda_{s-1,\beta^{(r)}_{r'}}$ of $\theta^\sharp(h_{s-1})$, therefore it is a unit, and (b) is satisfied. Besides note that $\beta^{(r)}_r = \overline{\beta}_r$ for $1 \leq r \leq g+1$, hence from the previous construction it follows that $\{\overline{\beta}_r\}_{r=1}^{g+1} \subset \{i_s\}_{s=1}^N$, hence the result is proved.

Corollary 3.2. *Let j, $2 \leq j \leq \delta$. Set $A := k[x_2, \ldots, x_{j-1}]$, $x = x_1$, $y = x_j$, and let $\theta : Y \to$ Spec $A[x_1, x_j]$ be the composition of $\eta : Y \to \mathbb{A}^d$ with the projection $\mathbb{A}^d \to$ Spec $A[x_1, x_j]$. Suppose that the hypothesis in Lemma 3.1 holds and let the image by $\eta^\sharp$ of x_j be given by*

$$x_j = \sum_{m_1 \leq i \leq m_j} \lambda'_{j,i}(v_2, \ldots, v_{j-1})\ u_1^i + u_1^{m_j}\ u_j \quad mod\ (u_1)^{m_j+1}, \tag{18}$$

where $\lambda'_{j,i}(v_2, \ldots, v_{j-1}) \in R_{j-1} = k < v_2, \ldots, v_{j-1} >$. Let $\{\beta_{j,r}\}_{r=0}^{g_j+1}$, $\{e_{j,r}\}_{r=0}^{g_j}$, $\{n_{j,r}\}_{r=0}^{g_j}$ and $\{\overline{\beta}_{j,r}\}_{r=0}^{g_j+1}$ be the integers defined by (8) and (9). Then there exist an open subset U of Y and, for each point y'_0 in $U \cap E$, a regular system of parameters $\{u_1, v'_2, \ldots, v'_{j-1}, u'_j, \ldots, u'_d\}$ of $\mathcal{O}_{Y,y'_0}$, where $v'_i = v + c_i$, $u'_i = u_i + c_i$, $(c_i)_i \in k^{d-1}$, and there exist elements $\{q_{j,0} = x_1, q_{j,1}, \ldots, q_{j,g_j+1}\}$ where

$$q_{j,r} \in T_{r-1}^{-1} \cdots T_0^{-1} S_{j-1}^{-1}[x_1, x_2, \ldots, x_{j-1}, x_j]$$

being $T_{r'}$ the multiplicative part generated by $q_{j,r'}$, such that the images of $\{q_{j,r}\}_{r=0}^{g_j+1}$ in $\mathcal{O}_{Y,y'_0}$ are given by

$$\begin{aligned} q_{j,r} &= \mu_{j,r}(v'_2, \ldots, v'_{j-1})\ u_1^{\overline{\beta}_{j,r}} && mod\ (u_1)^{\overline{\beta}_{j,r}+1} \quad for\ 0 \leq r \leq g_j \\ q_{j,g_j+1} &= \mu_{j,g_j+1}(v'_2, \ldots, v'_{j-1})\ u_1^{\overline{\beta}_{j,g_j+1}}\ u_j && mod\ (u_1)^{\overline{\beta}_{j,g_1+1}+1} \end{aligned} \tag{19}$$

where $\mu_{j,r}(v'_2, \ldots, v'_{j-1})$ is a unit in $k < v'_2, \ldots, v'_{j-1} >$ for $0 \leq r \leq g_j + 1$.

Proof. This is consequence of Lemma 3.1. In fact, after a possible replacement of y_0 in an open subset of E, we may suppose that there exist $\{i_s\}_{s=1}^N$ and $\{h_s\}_{s=1}^N$ satisfying (i), (ii) and (a), (b) in Lemma 3.1. Let $q_{j,0} := x_1$, $q_{j,1} := h_{s_1}, \ldots, q_{j,g_j} := h_{s_{g_j}}$. If $\lambda_{s_{g_j+1},\overline{\beta}_{g+1}} = 0$ in the expression (11) for $\eta^\sharp(h_{s_{g_j+1}})$ then let $q_{j,g_j+1} := h_{s_{g_j+1}}$. Otherwise, after a possible replacement of y_0 in an open subset of E, we may suppose that there exists an irreducible monic polynomial $P \in k[z, v_2, \ldots, v_{j-1}]$ such that $P(\lambda_{s_{g_j+1},\overline{\beta}_{j,g_j+1}}, v_2, \ldots, v_{j-1}) = 0$ and $\frac{\partial P}{\partial z}(\lambda_{s_{g_j+1},\overline{\beta}_{g_j+1}})$ is a unit in R_{j-1}. Then we proceed as in (17), that is we set

$$q_{j,g_j+1} := q_{j,0}^{b_{j,0}} \cdots q_{j,g_j}^{b_{j,g_j}}\ P\left(\frac{\overline{\mu}\ h_{s_{g_j+1}}}{q_{j,0}^{b_{j,0}} \cdots q_{j,g_j}^{b_{j,g_j}}},\ l_2, \ldots, l_{j-1}\right)$$

where $b_{j,0}, \ldots, b_{j,g_j}$ are nonnegative integers satisfying $b_{j,r} < n_{j,r}$, $1 \leq r \leq g_j$, and $\overline{\beta}_{j,g_j+1} = \sum_{0 \leq r \leq g_j} b_{j,r}\overline{\beta}_{j,r}$, and $\overline{\mu} = (\lambda_{s_1,\overline{\beta}_{j,1}})^{b_{j,1}} \cdots (\lambda_{s_{g_j},\overline{\beta}_{j,1}})^{b_{j,g_j}}$. Then $\{q_{j,r}\}_{r=0}^{g_j+1}$ satisfy the required condition.

Proposition 3.3. *There exist a point $y_0 \in E$, a regular system of parameters $\{u, v_2, \ldots, v_d\}$ of $\mathcal{O}_{Y,y_0}$ and a regular system of parameters $\{x_1, \ldots, x_d\}$ of $\mathcal{O}_{\mathbb{A}^d,\eta(y_0)}$ such that the following holds:*

(i) If we identify $x_1, \ldots, x_d$ with their images in $\mathcal{O}_{Y,y_0}$ then

$$\begin{aligned} &x_1 = u^{m_1} \\ &x_j = \textstyle\sum_{m_1 \leq i \leq m_j} \lambda_{j,i}(v_2, \ldots, v_{j-1})\ u^i + u^{m_j}\ v_j \quad mod\ (u)^{m_j+1},\ for\ 2 \leq j \leq \delta \\ &x_r = v_r \quad for\ \delta + 1 \leq r \leq d, \end{aligned}$$

where $0 < m_1 \leq m_2 \leq \ldots \leq m_d$ and, for $2 \leq j \leq \delta$, if we set $R_{j-1} := k < v_2, \ldots, v_{j-1} >$, then $\lambda_{j,i}(v_2, \ldots, v_{j-1}) \in R_{j-1}$, $\lambda_{j,i} \neq 0$ implies that it is a unit in R_{j-1}, $\lambda_{j,m_j}(v_2, \ldots, v_{j-1})$ is a unit in R_{j-1} and

$$if \quad i < m_{j'}, j' < j,\ then\ \lambda_{j,i} \in R_{j'-1}. \tag{20}$$

(ii) For $2 \leq j \leq \delta$, let $B_j := R_{j-1}[x_1, x_j]_{(x_1,x_j)}$, let ν_j be the restriction of ν_E to B_j, let $\overline{\beta}_{j,0} = m_1, \overline{\beta}_{j,1}, \ldots, \overline{\beta}_{j,g_j}$ be a minimal system of generators of the semigroup $\nu_j(B_j \setminus \{0\})$ and $\overline{\beta}_{j,g_j+1} = \nu_j(I_j)$, where I_j is the complete ideal defined by the restriction of ν_j to a general fibre of Spec $B_j \to$ Spec R_{j-1}. Set

$$\mathcal{J}^* := \{(1,0)\} \cup \{(j,r)\ /\ 2 \leq j \leq \delta,\ 1 \leq r \leq g_j\}, \quad \mathcal{J} := \mathcal{J}^* \cup \{(j, g_j+1)\ /\ 2 \leq j \leq \delta\}$$

let us consider the lexicographical order in $\mathcal{J}$ and, for $(j,r) \in \mathcal{J}$, let

$$\mathcal{J}^*_{j,r} := \{(j',r') \in \mathcal{J}^*\ /\ (j',r') < (j,r)\}, \quad \mathcal{J}_{j,r} := \{(j',r') \in \mathcal{J}\ /\ (j',r') < (j,r)\}\ .$$

Then, there exist elements $\{q_{j,r}\}_{(j,r) \in \mathcal{J}}$ in $k(x_1, \ldots, x_j)$, more precisely,

$$q_{j,r} \in \prod_{(j',r') \in \mathcal{J}^*_{j,r}} T^{-1}_{j',r'}\ k[x_1, \ldots, x_j] \tag{21}$$

where, for $(j'r') \in \mathcal{J}^*_{j,r}$, $T_{j',r'}$ is the multiplicative system generated by $q_{j',r'}$, such that:

(a.2) $q_{1,0} := x_1$ and, for $2 \leq j \leq \delta$, $0 \leq r \leq g_j + 1$, the image of $q_{j,r}$ in the fraction field $K(\mathcal{O}_{Y,y_0})$ of $\mathcal{O}_{Y,y_0}$ belongs to $\mathcal{O}_{Y,y_0}$ and, if we identify $q_{j,r}$ with its image, then

$$\begin{aligned} q_{j,r} &= \mu_{j,r}(v_2,\ldots,v_{j-1})\ u^{\overline{\beta}_{j,r}} \qquad \mathit{mod}\ (u)^{\overline{\beta}_{j,r}+1} \text{ for } 1\le r\le g_j \\ q_{j,g_j+1} &= \mu_{j,g_j+1}(v_2,\ldots,v_{j-1})\ u^{\overline{\beta}_{j,g_j+1}}\ v_j \qquad \mathit{mod}\ (u)^{\overline{\beta}_{j,g_j+1}+1} \end{aligned} \tag{22}$$

where $\mu_{j,r}(v_2,\ldots,v_{j-1})$ *is a unit in* R_{j-1} *for* $1\le r\le g_j+1$.

(b.2) *For* $2\le j\le\delta$, *set* $q_{j,0}:=q_{1,0}=x_1$, $e_{j,r}:=g.c.d.\{\beta_{j,0},\ldots,\beta_{j,r}\}$, $n_{j,r}:=\frac{e_{j,r-1}}{e_{j,r}}$ *for* $1\le r\le g_j$, *and let* $b_{j,0},\ldots,b_{j,g_j}$ *be the unique nonnegative integers satisfying*

$$b_{j,r}<n_{j,r} \text{ for } 1\le r\le g_j \quad \text{and} \quad \overline{\beta}_{j,g_j+1}=\sum_{0\le i\le g_j} b_{j,r}\overline{\beta}_{j,r}, \tag{23}$$

then, identifying $q_{j,r}$ *with its image in* $\mathcal{O}_{Y,y_0}$, *we have*

$$\frac{q_{j,g_j+1}}{q_{j,0}^{b_{j,0}}\cdots q_{j,g_j}^{b_{j,g_j}}}\ =\ v_j\ \in\ \mathcal{O}_{Y,y_0}.$$

(iii) *Even more, for* $2\le j\le\delta$, *there exist nonnegative integers* N_j *and* $s_{j,1}<s_{j,2}<\ldots<s_{j,g_j+1}=N_j$, *and elements* $\{h_{j,s}\}_{s=1}^{N_j}$, *such that* $q_{j,r}=h_{j,s_{j,r}}$ *for* $1\le r\le g_j+1$, *and besides the following holds: given* s, *let* r, $1\le r\le g_j+1$, *be such that* $s_{j,r-1}<s\le s_{j,r}$ *(resp.* $r=1$ *if* $s\le s_{j,1}$*), then we have:*

(a.3) $h_{j,s}\ \in\ \prod_{(j',r')\in\mathcal{J}^*_{j,r}} T^{-1}_{j',r'}\ k[x_1,\ldots,x_j]$

(b.3) *the image of* $h_{j,s}$ *in* $K(\mathcal{O}_{Y,y_0})$ *belongs to* $\mathcal{O}_{Y,y_0}$ *and, if we identify* $h_{j,s}$ *with its image in* $\mathcal{O}_{Y,y_0}$ *then*

$$\begin{aligned} h_{j,s} = &\sum_{i_{j,s}\le i\le m_j^{(r)}} \lambda_{j,s,i}(v_2,\ldots,v_{j-1})\ u^i \\ &+\gamma_{j,s,m_j^{(r)}}(v_2,\ldots,v_{j-1})u^{m_j^{(r)}}\ u_j \quad \mathit{mod}\ (u)^{m_j^{(r)}+1} \end{aligned}$$

where $n_{j,r-1}\overline{\beta}_{j,r-1}<i_{j,s}\le\overline{\beta}_{j,r}$, $i_{j,s-1}<i_{j,s}$, $i_{j,s}=\overline{\beta}_{j,r}$ *iff* $s=s_r$, $\lambda_{j,s,i},\gamma_{j,s,m_j^{(r)}}\in R_{j-1}$, $\lambda_{j,s,i_{j,s}},\gamma_{j,s,m_j^{(r)}}$ *is a unit, and* $m_j^{(r)}:=m_j+(n_{j,1}-1)\overline{\beta}_{j,1}+\ldots+(n_{j,r-1}-1)\overline{\beta}_{j,r-1}$.

(c.3) *If* $s=s_{j,r-1}+1$ *(resp.* $s_{r-1}+1<s$*), then* $h_{j,s}$ *is equal to*

$$q_{j,0}^{b^s_{j,0}}\cdots q_{j,\rho}^{b^s_{j,\rho}}\ P_{j,s}\left(\frac{\overline{\mu}_{j,s}h}{q_{j,0}^{b^s_{j,0}}\cdots q_{j,\rho}^{b^s_{j,\rho}}},\ \frac{q_{2,g_2+1}}{q_{2,0}^{b_{2,0}}\cdots q_{2,g_2}^{b_{2,g_2}}},\ldots,\frac{q_{j-1,g_{j-1}+1}}{q_{j-1,0}^{b_{j-1,0}}\cdots q_{j-1,g_{j-1}}^{b_{j-1,g_{j-1}}}}\right)$$

where $h=q_{j,r-1}^{n_{j,r-1}}$ *(resp.* $h=h_{j,s-1}$*),* $\rho=r-2$ *(resp.* $\rho=r-1$*), the integers* $\{b^s_{j,r'}\}_{r'=0}^{\rho}$ *satisfy* $b^s_{j,r'}<n_{j,r'}$, $1\le r'\le\rho$, *and* $n_{j,r-1}\overline{\beta}_{j,r-1}=\sum_{0\le r'\le r-2} b^s_{j,r'}\overline{\beta}_{j,r'}$ *(resp.* $i_{j,s-1}=\sum_{r'\le r-1} b^s_{j,r'}\overline{\beta}_{j,r'}$*),* $\overline{\mu}_{j,s}=\mu_{j,0}^{b^s_{j,0}}\cdots\mu_{j,\rho}^{b^s_{j,\rho}}$ *is a unit, and* $P_{j,s}\in k[z,v_2,\ldots,v_{j-1}]$ *is irreducible and satisfies*

$$P_{j,s}(\lambda, v_2, \ldots, v_{j-1}) = 0, \quad \frac{\partial P_{j,s}}{\partial z}(\lambda, v_2, \ldots, v_{j-1}) \textit{ is a unit in } R_{j-1}; \tag{24}$$

where $\lambda = (\lambda_{j,s-1,i_{j,s-1}})^{n_{j,r-1}}$ *(resp.* $\lambda = \lambda_{j,s-1,i_{j,s-1}}$*).*

Proof. The result is a consequence of Lemma 3.1 and its Corollary 3.2. First note that, given j, $2 \leq j \leq \delta$, if there exist $\{q_{j,r}\}_{r=1}^{g_j+1}$ in $k(x_1, \ldots, x_j)$ satisfying (22) and we define

$$l_j := \frac{q_{j,g_j+1}}{q_{j,0}^{b_{j,0}} \cdots q_{j,g_j}^{b_{j,g_j}}} \;\in\; k(x_1, \ldots, x_j) \tag{25}$$

where $q_{j,0} = x_1$ and $\{b_{j,r}\}_{r=0}^{g_j}$ satisfy (23), and v_j to be the image of l_j, then v_j belongs to $\mathcal{O}_{Y,y_0}$ and besides

$$v_j = \gamma_j \; u_j \mod (u) \quad \text{where } \gamma_j \text{ is a unit in } R_{j-1}. \tag{26}$$

In fact, with the notation in (22) we may take $\gamma_j = \frac{\mu_{j,g_1+1}}{\mu_{j,1}^{b_{j,1}} \cdots \mu_{j,g_j}^{b_{j,g_j}}}$.

Note also that, fixed j, $2 \leq j \leq \delta$, if (26) holds for every $j' \leq j-1$, then the image of x_j in $\mathcal{O}_{Y,y_0}$ is given by

$$x_j = \sum_{m_1 \leq i \leq m_j} \lambda_{j,i}(v_2, \ldots, v_{j-1}) \; u^i + u^{m_j} \; u_j \mod (u)^{m_j+1}$$

where $\lambda_{j,i} \in R_{j-1}$, $u := u_1$, m_j is the integer in (4), $\lambda_{j,i} \neq 0$ implies that it is a unit in R_{j-1}, λ_{j,m_j} is a unit in R_{j-1} and (20) holds (recall the conditions in (5)). Moreover, the integers $\{\overline{\beta}_{j,r}\}_{r=0}^{g_j}$ (resp. $\overline{\beta}_{j,g_j+1}$) defined in (8) and (9) for the image of x_j are a minimal system of generators of the semigroup $\nu_j(B_j \setminus 0)$ (resp. equal to $\nu_j(I_j)$). From this, and defining v_r to be the image of x_r for $\delta+1 \leq r \leq d$ (i.e. $v_r = u_r$, $\delta+1 \leq r \leq d$, with the notation in (4)), (i) would follow.

Hence, in order to prove (i) and (ii), it suffices to show that, for $2 \leq j \leq \delta$, there exist $\{q_{j,r}\}_{r=1}^{g_j+1}$ satisfying (21) and (22), where R_{j-1} is defined taking $v_{j'}$ to be the image of $l_{j'}$ for $2 \leq j' \leq j-1$ (see (25)). We argue by induction on j. For $j=2$ the hypothesis in Corollary 3.2 is clearly satisfied (we may take $S_1 = \{1\}$). Thus, by Corollary 3.2, there exist $\{q_{2,r}\}_{r=1}^{g_2+1}$ satisfying (21) and (22). Now, let j, $2 \leq j \leq \delta$ and suppose that, for $2 \leq j' \leq j-1$, there exist $\{q_{j',r}\}_{r=0}^{g_{j'}+1}$ satisfying (21) and (22). Since $v_{j'}$ is defined to be the image of $l_{j'}$, $2 \leq j' \leq j-1$, the hypothesis of Corollary 3.2 is satisfied. In fact, there exists a multiplicative part S_{j-1} of $k[x_1, \ldots, x_{j-1}]$ such that $\prod_{(j',r') \in \mathcal{J}_{j,1}^*} T_{j',r'}^{-1} \; k[x_1, \ldots, x_{j-1}] \cong S_{j-1}^{-1} k[x_1, \ldots, x_{j-1}]$, hence $l_{j'} \in S_{j-1}^{-1} k[x_1, \ldots, x_{j-1}]$ for $2 \leq j' \leq j-1$. Thus, Corollary 3.2 assures the existence of $\{q_{j,r}\}_{r=1}^{g_j+1}$ satisfying (21) and (22). From this, we conclude (i) and (ii). Besides, from the proof of Corollary 3.2 (see the proof of Lemma 3.1), (iii) follows.

Definition 3.4. The local expression in Proposition 3.3 (i) (or in (4) at the beginning of this section) will be called *a general transverse expression of* $\eta : Y \to \mathbb{A}^d_k$ *with respect to* E. The elements $\{q_{j,r}\}_{(j,r)\in\mathcal{J}}$ obtained in Proposition 3.3 (ii) will be called a *system of transverse generators for* $\eta : Y \to \mathbb{A}^d_k$ *with respect to* E.

Remark 3.5. For $j = 2$, $B_2 = k[x_1, x_2]_{(x_1,x_2)}$ is a two-dimensional regular local ring. Then $q_{2,0}, q_{2,1}, \ldots, q_{2,g_2}, q_{2,g_2+1} \in B_2$ is a minimal generating sequence for ν_2 ([26], theorem 8.6). In fact, since $R_1 = k$, if we apply Lemma 3.1 to $y = x_2$ then all the $\lambda_{s,i}$'s in (11) belong to k, hence we can take $P_s(z) = z - (\lambda_{s-1,i_{s-1}})^{n_r}$ (resp. $P_s(z) = z - \lambda_{s-1,i_{s-1}}$) in (13). Hence $q_{2,r} \in k[x_1, x_2]$ for $0 \le r \le g_2 + 1$, moreover we have $q_{2,0} = x_1$, $q_{2,1} = x_2 - \sum_{i<\overline{\beta}_{2,1}} \lambda_{2,i}\, q_{2,0}^{\frac{i}{\overline{\beta}_{2,0}}}$ and, for $1 \le r \le g_2$,

$$q_{2,r+1} = q_{2,r}^{n_{2,r}} - c_{2,r}\, q_{2,0}^{b_{2,r,0}} \cdots q_{2,r-1}^{b_{2,r,r-1}} - \sum_{\gamma=(\gamma_0,\ldots,\gamma_r)} c_{2,\gamma}\, q_{2,0}^{\gamma_0} \cdots q_{2,r}^{\gamma_r}$$

where the $b_{2,r,i}$'s are the unique nonnegative integers satisfying $b_{2,r,i} < n_{2,i}$ for $1 \le i \le r-1$, and $n_{2,r}\overline{\beta}_{2,r} = \sum_{0\le i<r} b_{2,r,i}\overline{\beta}_{2,i}$, the γ's are nonnegative integers satisfying $\gamma_i < n_{2,i}$ for $1 \le i \le r$ and $n_{2,r}\overline{\beta}_{2,r} < \sum_i \gamma_i\overline{\beta}_{2,i}$, and $c_{2,r}, c_{2,\gamma} \in k$, $c_{2,r} \neq 0$ and $c_{2,\gamma} \neq 0$ only for a finite number of γ's.

Remark 3.6. Let j, $2 \le j \le \delta$. Set $A := k[v_2, \ldots, v_{j-1}]$, $x = x_1$, $y = x_j$ and let $\theta : Y \to \operatorname{Spec} A[x_1, x_j]$ be defined by the morphism of k-algebras given by $v_{j'} \mapsto v_{j'}, 2 \le j' \le j-1$, $x_i \mapsto \eta^\sharp(x_i)$, $i = 1, j$ (see (18)). Setting $l_{j'} = v_{j'}$, $2 \le j' \le j-1$, and $S_{j-1} = \{1\}$, the hypothesis in Lemma 3.1 is satisfied. Let us apply Lemma 3.1, then the integers defined in (8) and (9) are $\{\beta_{j,r}\}_{r=0}^{g_j+1}$, $\{e_{j,r}\}_{r=0}^{g_j}$, $\{n_{j,r}\}_{r=0}^{g_j}$ and $\{\overline{\beta}_{j,r}\}_{r=0}^{g_j+1}$ (see Proposition 3.3 or Corollary 3.2). We denote by $\{q'_{j,r}\}_{r=0}^{g_j+1}$ the elements $\{q_r = h_{s_r}\}_{r=0}^{g_j+1}$ in 3.1 (iii).(a), hence satisfying

$$q'_{j,r} \in {T'}_{j,r-1}^{-1} \cdots {T'}_{j,0}^{-1} k[v_2, \ldots, v_{j-1}, x_1, x_j]$$

being $T'_{j,r'}$ the multiplicative part generated by $q'_{j,r'}$, and such that the images by $\theta^\sharp$ of $\{q'_{j,r}\}_{r=0}^{g_j+1}$ are $\{\eta^\sharp(q_{j,r})\}_{r=0}^{g_j+1}$, thus given in (19). In fact, note that $q_{j,r}$ is obtained from ${q'}_{j,r}$ by replacing v_i by $\frac{q_{i,g_i+1}}{q_{i,0}^{b_{i,0}} \cdots q_{i,g_i}^{b_{i,g_i}}}$, for $1 \le i \le j-1$.

On the other hand, for $2 \le j \le \delta$, there exists a domain B_{j-1} which is an étale extension of $k[v_2, \ldots, v_{j-1}]$ and contains $\lambda_{j,i}(v_2, \ldots, v_{j-1})$, $m_1 \le i \le m_j$ (see (i) in Proposition 3.3). Let $\widetilde{\nu}_j$ be the valuation on $B_{j-1}[x_1, x_j]$ extending ν_j and such that $\widetilde{\nu}_j(\ell) = 0$ for all $\ell \in B_{j-1}$ (see (ii) in Proposition 3.3). Let $\widetilde{q}_{j,1}, \ldots, \widetilde{q}_{j,g_j+1} \in B_{j-1}[x_1, x_j]$ be a minimal generating sequence for $\widetilde{\nu}_j$ defined as in Remark 3.5, i.e. $\widetilde{q}_{j,0} = x_1$, $\widetilde{q}_{j,1} = x_j - \sum_{i<\overline{\beta}_{j,1}} \lambda'_{j,i}\, (\widetilde{q}_{j,0})^{\frac{i}{\overline{\beta}_{j,0}}}$ and, for $1 \le r \le g_j$,

$$\widetilde{q}_{j,r+1} = \widetilde{q}_{j,r}^{n_{j,r}} - \widetilde{c}_{j,r}\, \widetilde{q}_{j,0}^{b_{j,r,0}} \cdots \widetilde{q}_{j,r-1}^{b_{j,r,r-1}} - \sum_{\gamma=(\gamma_0,\ldots,\gamma_r)} \widetilde{c}_{j,\gamma}\, \widetilde{q}_{j,0}^{\gamma_0} \cdots \widetilde{q}_{j,r}^{\gamma_r}, \quad 1 \le r \le g_j \tag{27}$$

where $b_{j,r,i} = b_{j,i}^{s_{j,r}+1}$, $1 \le i \le r-1$, i.e. $b_{j,r,i} < n_{j,i}$ and $n_{j,r}\overline{\beta}_{j,r} = \sum_{0\le i<r} b_{j,r,i}\overline{\beta}_{j,i}$, we have $\widetilde{\nu}_{j-1}(\widetilde{q}_{j,0}^{\gamma_0}\cdots\widetilde{q}_{j,r}^{\gamma_r}) > n_{j,r}\overline{\beta}_{j,r}$ for each sequence γ of nonnegative integers in the right hand side, and $\widetilde{c}_{j,r}, \widetilde{c}_{j,\gamma} \in B_{j-1}$, $\widetilde{c}_{j,r} \neq 0$ and $\widetilde{c}_{j,\gamma} \neq 0$ only for a finite number of γ's.

Note that, for $1 \le r \le g_j + 1$, in the ring $\prod_{r'=0}^{r-1} T'^{-1}_{j,r'} B_{j-1}[x_1, x_j]$ we have

$$q'_{j,r} \;=\; \widetilde{q}_{j,r}\cdot\widetilde{\ell} + \widetilde{h} \tag{28}$$

where $\widetilde{\ell}, \widetilde{h} \in \prod_{s=0}^{r-1} T'^{-1}_{j,s} B_{j-1}[x_1, x_j]$, $\widetilde{\ell}$ is a unit and $\widetilde{\nu}(\widetilde{h}) > \overline{\beta}_{j,r}$.

3.7. Now, let X be a smooth k-scheme and let ν be a divisorial valuation on an irreducible component X_0 of X. Let P_0 be the center of ν on X and let $R := \mathcal{O}_{X,P_0}$. We consider the graded algebra associated with ν, that is, $gr_\nu R := \otimes_{n\in\Phi^+} \wp_n/\wp_n^+$ where $\Phi^+ := \nu(R \setminus \{0\})$ is the semigroup of the valuation and, for $n \in \Phi^+$,

$$\wp_n = \{h \in R \;/\; \nu(h) \ge n\}, \qquad \wp_n^+ = \{h \in R \;/\; \nu(h) > n\}.$$

Let $\pi : Y \to X_0$ be a proper and birational morphism such that the center of ν on Y is a divisor E, and let $\eta : Y \to \mathbb{A}^d_k$ be the composition of π with an étale morphism $X_0 \to \mathbb{A}^d_k$, where $d = \dim X_0$. Let us consider the notation introduced in this section for the morphism η, in particular, let $\{q_{j,r}\}_{(j,r)\in\mathcal{J}}$ be a system of transverse generators for $\eta : Y \to \mathbb{A}^d_k$ with respect to E, (Proposition 3.3 (ii)). Recall that the center of ν on $\mathbb{A}^d_k$ is $(x_1, \ldots, x_\delta)$ and let $S := k[x_1, \ldots, x_d]_{(x_1,\ldots,x_\delta)}$.

There exists a proper and birational morphism $Z \to \mathbb{A}^d_k$ with Z smooth such that the center of ν on Z is a divisor F. Since $\mathcal{O}_{Z,F}$ is the valuation ring of the restriction of ν to $K(S)$, we have that $\mathcal{O}_{Z,F} \prec \mathcal{O}_{Y,E}$, i.e. $\mathcal{O}_{Y,E}$ dominates $\mathcal{O}_{Z,F}$, hence, after restricting to some open subset of Y, we may suppose that Y dominates Z, let $\sigma : Y \to Z$ denote the corresponding morphism. Note that we have

$$\frac{q_{j,g_j+1}}{q_{j,0}^{b_{j,0}}\cdots q_{j,g_j}^{b_{j,g_j}}} \;\in\; \mathcal{O}_{Z,F} \qquad \text{for } 2 \le j \le \delta,$$

because these elements belong to $K(S)$ and have ν-value equal to 0; we also denote by v_j the element $\frac{q_{j,g_j+1}}{q_{j,0}^{b_{j,0}}\cdots q_{j,g_j}^{b_{j,g_j}}}$ of $\mathcal{O}_{Z,F}$ (see Proposition 3.3 (ii)). Besides, the ramification index $\mathfrak{e}$ of $\mathcal{O}_{Y,E}$ over $\mathcal{O}_{Z,F}$ is equal to $g.c.d.(\{\overline{\beta}_{j',r'}\}_{(j',r')\in\mathcal{J}^*})$. Thus there exist $\{\mathfrak{a}_{j,r}\}_{(j,r)\in\mathcal{J}^*}$, $\mathfrak{a}_{j,r} \in \mathbb{Z}$, such that

$$z := \prod_{(j',r')\in\mathcal{J}^*} q_{j,r}^{\mathfrak{a}_{j,r}} \;\in\; \mathcal{O}_{Z,F} \qquad \text{and} \qquad \nu(z) = \sum_{(j',r')\in\mathcal{J}^*} \mathfrak{a}_{j,r}\overline{\beta}_{j,r} \;=\; \mathfrak{e}. \tag{29}$$

Then,

$$\nu(\sigma^*(dz \wedge dv_2 \wedge \ldots \wedge dv_d)) = \mathfrak{e} - 1$$

and hence, if $k_F(\mathbb{A}^d)$ denotes the discrepancy of $\mathbb{A}^d$ with respect ro F, we have

$$a_E = \mathfrak{e} k_F(\mathbb{A}^d) + \mathfrak{e} - 1 \tag{30}$$

Since $S \prec R$, the initial forms of the elements of $k[x_1, \ldots, x_d]$ are well defined elements in $gr_\nu R$, and since $q_{1,0} = x_1$, applying (21) in Proposition 3.3, by recurrence on (j, r) we can define the initial form $\mathbf{q_{j,r}}$ of $q_{j,r}$ for every $(j, r) \in \mathcal{J}$. We have

$$\mathbf{q}_{j,r} \in \prod_{(j',r') \in \mathcal{J}^*_{j,r}} \mathbf{T}^{-1}_{j',r'} \, (gr_\nu R)$$

where, for $(j', r') \in \mathcal{J}^*_{j,r}$, $\mathbf{T}_{j',r'}$ is the multiplicative system generated by $\mathbf{q_{j,r}}$. Let $k[\{\mathbf{q}_{j,r}\}_{(j,r)\in\mathcal{J}}]$ be the k-subalgebra of the fraction field $K(gr_\nu R)$ of $gr_\nu R$ generated by the $\mathbf{q}_{j,r}$'s and, for $\delta + 1 \leq j \leq d$, let $\mathbf{x}_j$ be the initial form of x_j. With this notation, the following holds:

Theorem 3.8. *The initial forms $\{\mathbf{q}_{j,r}\}_{(j,r)\in\mathcal{J}}$ of the system of transverse generators satisfy the following properties:*

(i) We have an isomorphism of graded rings

$$G := \prod_{(j,r)\in\mathcal{J}^*} \mathbf{T}^{-1}_{j,r} \; k\left[\{\mathbf{q}_{j,r}\}_{(j,r)\in\mathcal{J}}, \; \mathbf{x}_{\delta+1}, \ldots, \mathbf{x}_d\right] \overset{\Phi}{\cong} \; A[\mathbf{u}^{\mathfrak{e}}, \mathbf{u}^{-\mathfrak{e}}]$$

where $deg(\mathbf{u}) = 1$, and A is a k-algebra which is étale over the polynomial ring in $d-1$ variables $k[\mathbf{v}_2, \ldots, \mathbf{v}_d]$, with $deg(\mathbf{v}_j) = 0$, $2 \leq j \leq d$.

(ii) We have an isomorphism

$$\prod_{(j,r)\in\mathcal{J}^*} \mathbf{T}^{-1}_{j,r} \; gr_\nu R \;\cong\; B[\mathbf{u}^{\mathfrak{e}}, \mathbf{u}^{-\mathfrak{e}}]$$

whose restriction to G is Φ, where $A \otimes_k \kappa(P_0) \subseteq B$ and the extension is étale. Besides, the fraction field $K(B)$ of B is $\kappa(E)$.

(iii) For $2 \leq j \leq \delta$, the isomorphism Φ in (i) restricts to

$$G_j := \prod_{(j,r)\in\mathcal{J}^*_{j,g_j+1}} \mathbf{T}^{-1}_{j,r} \; k\left[\{\mathbf{q}_{j',r'}\}_{(j',r')\in\mathcal{J}_{j,g_j+1}\cup\{(j,g_j+1)\}}\right] \overset{\Phi}{\cong} A_{j-1}[\mathbf{v}_j][\mathbf{u}^{\mathfrak{e}_j}, \mathbf{u}^{-\mathfrak{e}_j}]$$

where $\mathfrak{e}_j := g.c.d.\{\overline{\beta}_{j',r'} \; / \; (j', r') \in \mathcal{J}^*_{j,g_j+1}\}$, $A_1 = k$ and A_{j-1} is étale over $k[\mathbf{v}_2, \ldots, \mathbf{v}_{j-1}]$ for $2 < j \leq \delta$.

(iv) For $2 \leq j \leq \delta$, there exists a domain B_{j-1} étale over A_{j-1} such that

$$B_{j-1}\left[\{\mathbf{q}_{1,0}\} \cup \{\mathbf{q}_{j,r}\}_{r=1}^{g_j+1}\right] \cong \; B_{j-1}\left[y_{1,0}, y_{j,2} \ldots, y_{j,g_j+1}\right] \; / \; J_j$$

where the $y_{j,r}$'s are indeterminacies and J_j is a prime ideal which is generated by $\{y_{j,r}^{n_{j,r}} - \widetilde{c}_{j,r}\ y_{1,0}^{b_{j,r,0}} \cdot y_{j,1}^{b_{j,r,1}} \dots y_{j,r-1}^{b_{j,r,r-1}}\}_{r=1}^{g_j}$, $\widetilde{c}_{j,r} \in B_{j-1}$. In particular, the previous ring is a domain which is a complete intersection over B_{j-1}.
Moreover, for any domain C, any ideal in $C[y_{1,0}, y_{j,2}, \dots, y_{j,g_j+1}]$ generated by $\{y_{j,r}^{n_{j,r}} - c_{j,r}\ y_{1,0}^{b_{j,r,0}} \cdot y_{j,1}^{b_{j,r,1}} \dots y_{j,r-1}^{b_{j,rr-1}}\}_{r=1}^{g_j}$, $c_{j,r} \in C$, is a prime ideal.

Proof. First, we have that $R = \mathcal{O}_{X,P_0} \supseteq k[x_1, \dots, x_d]_{(x_1,\dots,x_\delta)} =: S$ is étale, hence $\widehat{R} \cong \widehat{S} \otimes_k \kappa(P_0)$ where we denote by $\widehat{R}$ (resp. $\widehat{S}$) the completion with respect to the maximal ideal. Since the valuation ν on R (resp. on S) can be extended to a valuation $\widehat{\nu}$ on $\widehat{R}$ (resp. on $\widehat{S}$) and we have $gr_\nu R = gr_{\widehat{\nu}}\widehat{R}$ (resp. $gr_\nu S = gr_{\widehat{\nu}}\widehat{S}$) we conclude that $gr_\nu R \cong gr_\nu S \otimes_k \kappa(P_0)$. Therefore, in (ii) we may suppose that $X = \mathbb{A}_k^d$, i.e. $R = S$.

Keep the notation in Proposition 3.3. The morphism $S \hookrightarrow \mathcal{O}_{Z,F}$ induces an inclusion

$$\Phi :\ gr_\nu S\ \hookrightarrow\ gr_\nu \mathcal{O}_{Z,F}\ \cong\ \kappa(F)\ [\mathbf{u}^{\mathfrak{e}}]$$

where $\kappa(F)$ is the residue field of F on Y, which contains $k(\mathbf{v}_2, \dots, \mathbf{v}_d)$, and $\mathbf{u}, \mathbf{v}_2, \dots, \mathbf{v}_d$ are indeterminacies, $\mathbf{v}_j$, $2 \le j \le d$ (resp. $\mathbf{u}$) denotes the initial form of v_j (resp. u), hence $\deg(\mathbf{v}_j) = 0$, $\deg(\mathbf{u}) = 1$. We have

$$\prod_{r'=0}^{g_2} \mathbf{T}_{2,r'}^{-1}\ k\left[\{\mathbf{q}_{1,0}\} \cup \{\mathbf{q}_{2,r'}\}_{r'=1}^{g_2}\right] \overset{\Phi}{\cong}\ k[\mathbf{u}^{e_{2,g_2}}, \mathbf{u}^{-e_{2,g_2}}]$$

and hence $G_2 \overset{\Phi}{\cong}\ k[\mathbf{v}_2][\mathbf{u}^{e_{2,g_2}}, \mathbf{u}^{-e_{2,g_2}}]$. More precisely, the image of the ring in the left hand side in the fraction field $K(gr_\nu \mathcal{O}_{Z,F})$ of $gr_\nu \mathcal{O}_{Z,F}$ is in fact in $gr_\nu \mathcal{O}_{Z,F}$ and is equal to the ring in the hand side. Arguing by recurrence on j, $2 \le j \le \delta$, it follows that

$$\prod_{(j',r') \in \mathcal{J}^*_{j,g_j+1}} \mathbf{T}_{j',r'}^{-1}\ k\left[\{\mathbf{q}_{j',r'}\}_{(j',r') \in \mathcal{J}_{j,g_j+1}}\right] \overset{\Phi}{\cong}\ A_{j-1}[\mathbf{u}^{\mathfrak{e}_j}, \mathbf{u}^{-\mathfrak{e}_j}]$$

where $\mathfrak{e}_j := g.c.d.\{e_{2,g_2}, \dots, e_{j,g_j}\} = g.c.d.\{\overline{\beta}_{j',r'}\ /\ (j',r') \in \mathcal{J}^*_{j,g_j+1}\}$ and A_{j-1} is the minimal subring of $\kappa(F)$ containing $k[\mathbf{v}_2, \dots, \mathbf{v}_{j-1}]$ and $\mu_{j',r'}(\mathbf{v}_2, \dots, \mathbf{v}_{j'-1})$, $\mu_{j',r'}(\mathbf{v}_2, \dots, \mathbf{v}_{j'-1})^{-1}$ for $(j',r') \in \mathcal{J}^*_{j,g_j+1}$, hence A_{j-1} is étale over $k[\mathbf{v}_2, \dots, \mathbf{v}_{j-1}]$. Therefore

$$G_j \overset{\Phi}{\cong}\ A_{j-1}[\mathbf{v}_j][\mathbf{u}^{\mathfrak{e}_j}, \mathbf{u}^{-\mathfrak{e}_j}] \quad \text{and} \quad G = G_\delta \otimes_k k[x_{\delta+1}, \dots, x_d] \overset{\Phi}{\cong} A[\mathbf{u}^{\mathfrak{e}_\delta}, \mathbf{u}^{-\mathfrak{e}_\delta}]$$

where $A = A_{\delta-1}[\mathbf{v}_\delta, \dots, \mathbf{v}_d]$, hence (i) and (iii) hold.

In order to prove (ii), let B be the minimal subring of $\kappa(F)$ containing $k[\mathbf{v}_2, \dots \mathbf{v}_d]$ and $\{\lambda_{j,i}(\mathbf{v}_2, \dots, \mathbf{v}_{j-1})\}_{2 \le j \le d, m_1 \le i \le m_j}$. From the construction of the $h_{j,s}$'s in Proposition 3.3 (iii) (see the proof of (iii) in Lemma 3.1) it follows that, for every (j,i), $2 \le j \le d, m_1 \le i \le m_j$, there exists $h \in \prod_{(j,r) \in \mathcal{J}^*} T_{j,r}^{-1} S$ such that the initial form of

h is $\lambda_{j,i}(\mathbf{v}_2,\ldots,\mathbf{v}_d)\mathbf{u}^{\mathfrak{e}}$. Now, let $h \in S = k[x_1,\ldots,x_\delta]_{(x_1,\ldots,x_\delta)}$ and let $a := \nu(h)$. Then $\mathfrak{e}_\delta$ divides a and the image of h in $\mathcal{O}_{Y,y_0}$ is equal to $\lambda(v_2,\ldots,v_\delta)u^a$ modulo u^{a+1}, where $\lambda(\mathbf{v}_2,\ldots,\mathbf{v}_\delta) \in B$. Hence the initial form of h belongs to $B[\mathbf{u}^{\mathfrak{e}_\delta}]$. Besides, it follows that the set of elements of $K(S)$ of degree 0 is precisely $K(B)$, that is, $\kappa(F) = K(B)$. From this (ii) follows.

For (iv), recall that, given $n \in \mathbb{N}$, a field F containing a primitive n-th root of unity ξ and an element $b \in F^* = F \setminus \{0\}$, if the class of b in F^*/F^{*n} has order m, then there exists $d \in F$ such that $X^m - d$ is an irreducible polynomial in $F[X]$ and moreover $X^n - b = \prod_{i=0}^{n/m}(X^m - \xi^i d)$ is the decomposition in $F[x]$ of $X^n - b$ in irreducible factors (see for instance prop. 9.6 in [18]). In particular, if A is a domain containing a primitive n-th root of unity and $b \in A$ is such that

$$b^{\frac{1}{n'}} \notin A \text{ for every } n' > 1,\ n'|n, \quad \text{then } X^n - b \text{ is irreducible in } A[x]. \tag{31}$$

For $j = 2$, with the notation in Remark 3.5, let J_2 is the ideal of $k\left[y_{1,0}, y_{2,1}, \ldots y_{2,g_2}\right]$ generated by $\{y_{2,r}^{n_{2,r}} - c_{2,r}\ y_{1,0}^{b_{2,r,0}}\ y_{2,1}^{b_{2,r,1}} \ldots y_{2,r-1}^{b_{2,r,r-1}}\}_{r=1}^{g_2}$, where the $y_{2,r}$'s are indeterminacies. Let $B_1 = A_1 = k$ and let us consider the morphism of k-algebras

$$k[y_{1,0}, y_{2,1}, \ldots, y_{2,g_2+1}]\ /\ J_2 \to k[\{\mathbf{q}_{1,0}\} \cup \{\mathbf{q}_{2,r}\}_{r=1}^{g_2+1}]$$

sending $y_{2,r}$, $1 \leq r \leq g_2+1$ (resp. $y_{1,0}$) to $\mathbf{q}_{2,r}$ (resp. $\mathbf{q}_{1,0}$). Since $k[\{\mathbf{q}_{1,0}\} \cup \{\mathbf{q}_{2,r}\}_{r=1}^{g_2+1}]$ is a 2-dimensional domain, to prove the isomorphism it suffices to show that for $1 \leq r \leq g_2$ the element $y_{2,r}^{n_{2,r}} - c_{2,r}\ y_{1,0}^{b_{2,r,0}}\ y_{2,1}^{b_{2,r,1}} \ldots y_{2,r-1}^{b_{2,r,r-1}}$ is irreducible in

$$\left(k\left[y_{1,0}, \ldots, y_{2,r-1}\right] / \left(\{y_{2,r'}^{n_{2,r'}} - c_{2,r'}\ y_{1,0}^{b_{2,r',0}} \ldots y_{2,r'-1}^{b_{2,r',r'-1}}\}_{r'=1}^{r-1}\right)\right)[y_{2,r}]$$

i.e. $y_{1,0}^{b_{2,r,0}} \ldots y_{2,r-1}^{b_{2,rr-1}}$ does not have a n'-root for any $n' > 1$ dividing $n_{2,r}$. In fact, suppose that

$$\begin{gathered} y_{1,0}^{b_{2,r,0}} \ldots y_{2,r-1}^{b_{2,r,r-1}} = \left(\sum_{\underline{a} \in \mathbb{Z}_{\geq 0}^r} \lambda_{\underline{a}}\ y_{1,0}^{a_0} \ldots y_{2,r-1}^{a_{r-1}}\right)^{n'} \\ \mod \left(\{y_{2,r'}^{n_{2,r'}} - c_{2,r'}\ y_{1,0}^{b_{2,r',0}} \ldots y_{2,r'-1}^{b_{2,r',r'-1}}\}_{r'=1}^{r-1}\right) \end{gathered} \tag{32}$$

where $n'|n_{2,r}$, $\lambda_{\underline{a}} \in k$, the sum in the right hand side term is finite, then we may suppose that (32) is homogeneous with respect to the degree, that is, for each $\underline{a}$ in (32), we have $n'\ \left(\sum_{i=0}^{r-1} a_i\ \overline{\beta}_{2,i}\right) = n_{2,r}\ \overline{\beta}_{2,r}$. Since there exists at least one $\underline{a}$ in (32) and we have $n_{2,r} = \frac{e_{2,r-1}}{e_{2,r}}$ where $e_{2,l} = g.c.d.(\overline{\beta}_{1,0},\ \ldots\ ,\overline{\beta}_{2,l})$, $l = r-1, r$, and $n'|n_{2,r}$, we conclude that $n'e_{2,r}$ divides $\overline{\beta}_{2,r}$ and also $e_{2,r-1}$, hence $n'e_{2,r}$ divides $e_{2,r}$, that is $n' = 1$.

Now, let j, $2 < j \leq \delta$. Let us consider the notation in Remark 3.6. We have $B_{j-1}\left[\{\mathbf{q}_{1,0}\} \cup \{\mathbf{q}_{j,r}\}_{r=1}^{g_j+1}\right] \cong B_{j-1}\left[\{\mathbf{q}'_{1,0}\} \cup \{\mathbf{q}'_{j,r}\}_{r=1}^{g_j+1}\right]$. Besides, from (28) it follows that, for $1 \leq r \leq g_j + 1$, the initial form $\mathbf{q}'_{j,r}$ of $q'_{j,r}$ belongs to $gr_{\widetilde{\nu}_j}(B_{j-1}[x_1, x_j])$, although $q'_{j,r} \in \prod_{r'=0}^{r-1} T'^{-1}_{j,r'} B_{j-1}[x_1, x_j]$. It also follows that

$$B_{j-1}\left[\{\mathbf{q}'_{1,0}\} \cup \{\mathbf{q}'_{j,r}\}_{r=1}^{g_j+1}\right] \cong \; B_{j-1}\left[y_{1,0}, y_{j,1}, \ldots, y_{j,g_j+1}\right] \; / \; J_j$$

where J_j is the ideal generated by $\{y_{j,r}^{n_{j,r}} - \widetilde{c}_{j,r}\; y_{1,0}^{b_{j,r,0}} \ldots y_{j,r-1}^{b_{j,r,r-1}}\}_{r=1}^{g_j}$. In fact, from the same argument as in before it follows that, for $1 \leq r \leq g_j$ and for any n' dividing $n_{j,r}$, $\widetilde{c}_{j,r} y_{1,0}^{b_{j,r,0}} \cdots y_{j,r-2}^{b_{j,r,r-2}}$ does not have a n'-root in the ring

$$B_{j-1}\left[y_{1,0}, \ldots, y_{j,r-2}\right] / \left(\{y_{j,r'}^{n_{j,r'}} - \widetilde{c}_{j,r'}\; y_{1,0}^{b_{j,r',0}} \ldots y_{j,r'-1}^{b_{j,r',r'-1}}\}_{r'=1}^{r-1}\right)$$

More precisely, $(b_{j,r,0}, \ldots, b_{j,r,r-2}) \neq (0, \ldots, 0)$ and $y_{1,0}^{b_{j,r,0}} \cdots y_{j,r-2}^{b_{j,r,r-2}}$ does not have a n'-root in any ring of the form

$$C\left[y_{1,0}, \ldots, y_{j,r-2}\right] / \left(\{y_{j,r'}^{n_{j,r'}} - c_{j,r'}\; y_{1,0}^{b_{j,r',0}} \ldots y_{j,r'-1}^{b_{j,r',r'-1}}\}_{r'=1}^{r-1}\right)$$

where C is a domain and the $c_{j,r'}$'s are in C. Hence J_j is a prime ideal and (iv) holds. This concludes the proof.

Remark 3.9. Similar ideas to the ones in (ii) in theorem 3.8 appear in [21], proof of th. 1.3.8.

Restricting to dimension 3, but considering any valuation ν of rational rank 1 and dimension 3, i.e. ν centered in a regular 3-dimensional ring R, in [15] an (infinite) generating sequence $\{q_n\}_{n\in\mathbb{N}}$ of ν in R is constructed. Our construction in Proposition 3.3 is different to the one in [15] and we do not reach a generating sequence. Generating sequences in higher dimensional complete local rings are considered in [17].

4. Defining coordinates at stable points of the space of arcs

Let $\eta : Y \to \mathbb{A}_k^d$ be a k-morphism dominant and generically finite, where Y is a nonsingular k-scheme, let E be a divisor on Y and $e \geq 1$, and keep the notation in section 3.

Let P_{eE}^{Y} be the generic point of Y_∞^{eE} (see 2.7), and let $P_{eE}^{\mathbb{A}^d}$ be the image by η_∞ of P_{eE}^{Y}, which is a stable point of $(\mathbb{A}^d)_\infty$ ([25] prop. 4.5). We will first prove (Proposition 4.5) that a system of transverse generators for η with respect to E induces a regular system of parameters of $\mathcal{O}_{(\mathbb{A}^d)_\infty, P_{eE}^{\mathbb{A}^d}}$. Then we will conclude Theorem 4.8 and Corollary 4.10.

Given a finitely generated k-algebra A, let us denote by A_∞ the ring of $(\mathrm{Spec}\ A)_\infty$. Given $l \in A$, we denote by $\sum_{n=0}^{\infty} L_n\; t^n$ the image of l by the morphism of k-algebras $A \to A_\infty[[t]]$.

Lemma 4.1. *([25] proof of prop. 4.1 (iii)) Let $A \subseteq B$ be finitely generated k-algebras and let θ : Spec $B \to$ Spec A be the induced dominant morphism. Let P' be a stable point of*

Spec B_∞ and let P be its image by θ_∞ in Spec A_∞. Let $h \in B$ belonging to the fraction field $K(A)$ of A, $h = l/q$ where $l, q \in A$. Then, there exist $\{\overline{H}_n\}_{n \geq 0}$ in $(A_\infty)_P$ such that

$$H_n \equiv \overline{H}_n \quad \mod P' \tag{33}$$

(recall that $H_n \in B_\infty$ for $n \geq 0$). Even more, there exists $c \in \mathbb{N}$ such that $Q_0, \ldots, Q_{c-1} \in P$, $Q_c \notin P$ and there exist polynomials S_n on $2(n+1)$ indeterminacies with coefficients in k, for $n \geq 0$, such that,

$$\overline{H}_n := \frac{S_n(L_c, \ldots, L_{n+c}, Q_c, \ldots, Q_{n+c})}{(Q_c)^{n+1}} \in (A_\infty)_P$$

satisfies (33).

Proof. First note that P is a stable point of Spec A_∞ ([25] prop. 4.5), hence the existence of c such that $Q_0, \ldots, Q_{c-1} \in P$, $Q_c \notin P$ ([25], prop. 3.7 (i)). Then, the result follows from the following observation: given $h = l/q$, $l, q \in A$, if $Q_0, \ldots, Q_{c-1} \in P$, $Q_c \notin P$, then we have

$$Q_c H_n + \ldots + Q_{n+c} H_0 \equiv L_{n+c} \mod P' \quad \text{for } n \geq 0.$$

([25] proof of prop. 4.1).

Lemma 4.2. *Suppose that the assumptions in Lemma 3.1 hold and suppose besides that $\theta : Y \to$ Spec $A[x, y]$ is dominant. Let $P = P_{eE}^{A[x,y]}$ be the image of P_{eE}^{Y} by θ_∞, which is a stable point of Spec $A[x, y]_\infty$. Let y_0, the regular system of parameters $\{u, v_2, \ldots, v_d\}$ of $\mathcal{O}_{Y,y_0}$ and $\{h_1 = y, h_2, \ldots, h_N\}$ satisfy (a) to (c) in 3.1. For $2 \leq j' \leq j-1$, let $\{\overline{L}_{j';n}\}_{n \geq 0}$ in $(A[x, y]_\infty)_P$ be such that $L_{j';n} \equiv \overline{L}_{j';n}$ mod P_{eE}^{Y} (see Lemma 4.1). Then, there exists a multiplicative system $\overline{S}_{j-1}$ of $A[x]_\infty$ such that $\overline{L}_{j';n} \in \overline{S}_{j-1}^{-1} A[x]_\infty$ for $2 \leq j' \leq j-1, n \geq 0$ and there exist elements $\{\overline{H}_{s;n}\}_{1 \leq s \leq N, n \geq 0}$ in $(A[x, y]_\infty)_P$, $n \geq 0$, satisfying:*

(i) *$H_{s;n} \equiv \overline{H}_{s;n}$ mod P_{eE}^{Y}, therefore*

$$\overline{H}_{s;n} \in P(A[x, y]_\infty)_P \quad \text{for } 0 \leq n \leq ei_s - 1 \quad \text{and} \quad \overline{H}_{s;ei_s} \notin P(A[x, y]_\infty)_P.$$

(ii) *Let r, $1 \leq r \leq g+1$ be such that $n_{r-1}\overline{\beta}_{r-1} < i_s \leq \overline{\beta}_r$ (resp. $r = 1$ if $s = 1$ and $i_1 = \overline{\beta}_0$). Set $\overline{Q}_{0;n} := X_n$ for $n \geq 0$, $\overline{Q}_{r';n} := \overline{H}_{s_{r'};n}$, for $1 \leq r' < r, n \geq 0$ and let $\overline{T}_{r'}$ is the multiplicative part generated by $\overline{Q}_{r';e\overline{\beta}_{r'}}$, $0 \leq r' < r$. Then, for $n \geq e(\overline{\beta}_r - \beta_r)$, we have:*

$$\overline{H}_{s;n} \in \overline{T}_{r-1}^{-1} \ldots \overline{T}_0^{-1} \overline{S}_{j-1}^{-1} A_\infty[X_0, X_1, \ldots, X_n, \ldots, Y_0, Y_1, \ldots, Y_{n-e(\overline{\beta}_r - \beta_r)}].$$

(iii) *If $s=1$ then $\overline{H}_{1;n}=Y_n$ for $n\geq 0$. If $s>1$ then*

$$\overline{H}_{s;n} \in (\{\overline{Q}_{r';n}\}_{\substack{r'\leq r-1\\ n<e\overline{\beta}_{r'}}} \cup \{\overline{H}_{s-1;n}\}_{n<ei_{s-1}}) \quad \textit{for } 0\leq n< \max\{en_{r-1}\overline{\beta}_{r-1}, ei_{s-1}\}$$

$$\overline{H}_{s;n} = u_{s,n}\, Y_{n-(\overline{\beta}_r\,-\,\beta_r)} + \rho_{s,n} \qquad \textit{for } n>\max\{en_{r-1}\overline{\beta}_{r-1}, ei_{s-1}\}$$

where $u_{s,n},\ \rho_{s,n} \in \overline{T}_{r-1}^{-1}\cdots\overline{T}_0^{-1}\overline{S}_{j-1}^{-1}A_\infty[X_0,\ldots,X_n,\ldots,Y_0,\ldots,Y_{n-e(\overline{\beta}_r-\beta_r)-1}]$ *and $u_{s,n}$ is a unit.*

(iv) *Suppose that $s>1$. If $i_{s-1}=\overline{\beta}_{r-1}$ (resp. $i_{s-1}\in(n_{r-1}\overline{\beta}_{r-1},\overline{\beta}_r)$) then $\overline{H}_{s;en_{r-1}\overline{\beta}_{r-1}}$ (resp. $\overline{H}_{s;ei_{s-1}}$) is equal to*

$$\overline{Q}_{0;e\overline{\beta}_0}^{b_0^s}\cdots\overline{Q}_{\rho;e\overline{\beta}_\rho}^{b_\rho^s}\cdot P_s\left(\frac{c_s\,\overline{H}}{\overline{Q}_{0;e\overline{\beta}_0}^{b_0^s}\cdots\overline{Q}_{\rho;e\overline{\beta}_\rho}^{b_\rho^s}},\ \overline{L}_{2;0},\ \ldots\ ,\overline{L}_{j-1;0}\right)$$

where $\overline{H}=(\overline{Q}_{r-1;e\overline{\beta}_{r-1}})^{n_{r-1}}$ (resp. $\overline{H}=\overline{H}_{s-1;ei_{s-1}}$), $c_s\in k\setminus\{0\}$ and ρ, $\{b_{r'}^s\}_{r'=0}^{\rho}$ and P_s are as in (c) in 3.1.

(v) *Fixed r, $1\leq r\leq g+1$, the following ideals in $\overline{T}_{r-1}^{-1}\cdots\overline{T}_0^{-1}\overline{S}_{j-1}^{-1}A[x,y]_\infty$ are equal:*

$$\left(\{\overline{Q}_{r';n}\}_{\substack{0\leq r'\leq r\\ 0\leq n\leq e\overline{\beta}_{r'}-1}}\right) = \left(\{\overline{Q}_{r';n}\}_{\substack{0\leq r'\leq 1\\ 0\leq n\leq e\overline{\beta}_{r'}-1}} \cup \{\overline{Q}_{r';n}\}_{\substack{2\leq r'\leq r\\ en_{r'-1}\overline{\beta}_{r'-1}\leq n\leq e\overline{\beta}_{r'}-1}}\right)$$

and also the ideal generated by

$$\{\overline{Q}_{0;n}\}_{n=0}^{em_1-1}\cup\{\overline{H}_{1;n}\}_{n=0}^{ei_1-1}\ \cup\ \left(\cup_{s=2}^{s_1}\{\overline{H}_{s;n}\}_{e\ i_{s-1}}^{ei_s-1}\right)\cup$$
$$\cup_{r'=2}^{r}\left(\{\overline{H}_{s_{r'-1}+1;n}\}_{n=en_{r'-1}\overline{\beta}_{r'-1}}^{ei_{s_{r'-1}+1}-1}\cup\left(\cup_{s=s_{r'-1}+2}^{s'_r}\{\overline{H}_{s;n}\}_{n=ei_{s-1}}^{ei_s-1}\right)\right).$$

Proof. The existence of $\overline{S}_{j-1}$ follows from Lemma 4.1; in fact, it suffices to ask $\overline{S}_{j-1}$ to contain the elements Q_c where $q\in S_{j-1}$ and c is such that $Q_0,\ldots,Q_{c-1}\in P$ and $Q_c\notin P$. Now, let us define the elements $\{\overline{H}_{s;n}\}_{n\geq 0}$, $1\leq s\leq N$, by induction on s. For $s=1$, $h_1=y\in A[x,y]$, so $H_{1;n}\in A[x,y]_\infty$ for $n\geq 0$. We set $\overline{H}_{1;n}:=H_{1;n}=Y_n\in A[x,y]_\infty$ for $n\geq 0$. It is clear that (i) to (iii) are satisfied. Now, let s, $2\leq s\leq N$, and suppose that $\overline{H}_{s';n}\in(A[x,y]_\infty)_P$ are defined, for $1\leq s'<s$, $n\geq 0$, and satisfy the conditions. Let r, $1\leq r\leq g_j+1$ be such that $i_{s-1}\in\{\overline{\beta}_{r-1}\}\cup(n_{r-1}\overline{\beta}_{r-1},\overline{\beta}_r)$. Therefore $\{\overline{Q}_{r';n}\}_{0\leq r'<r,n\geq 0}$ in $(A[x,y]_\infty)_P$ are defined, and satisfy:

$$\overline{Q}_{r';n}\in P(A[x,y]_\infty)_P \quad \text{for } 0\leq n\leq e\overline{\beta}_{r'}-1 \quad \text{and} \quad \overline{Q}_{r';e\overline{\beta}_{r'}}\notin P(A[x,y]_\infty)_P.$$

Hence, for every l in the k-algebra $k[\{q_{r'}\}_{0\leq r'<r}\cup\{h_{s-1}\}]$ generated by $q_{r'}$, $0\leq r'<r$, and h_{s-1}, and for every $n\geq 0$, there exists a polynomial function $\overline{L}_n$ on $\{\overline{Q}_{r';n}\}_{r'<r,n\geq e\overline{\beta}_{r'}}\cup\{\overline{H}_{s-1;n}\}_{n\geq ei_{s-1}}$ such that $L_n\equiv\overline{L}_n \bmod P_{eE}^{Y}$. Moreover, given

$$h = \frac{l}{q} \in \mathcal{O}_{Y,y_0} \quad \text{where } l \in k[\{q_{r'}\}_{0\le r'<r} \cup \{h_{s-1}\}], \ \ q = \prod_{0\le r'<r} q_{r'}^{a_{r'}} \tag{34}$$

$a_{r'} \in \mathbb{N}\cup\{0\}$, let $c = \sum_{0\le r'<r} a_{r'} e\overline{\beta}_{r'}$, so that $\overline{Q}_0, \dots, \overline{Q}_{c-1} \in P$, $\overline{Q}_c \notin P$ and set

$$\overline{H}_n := \frac{S_n(\overline{L}_c, \overline{L}_{c+1}, \dots, \overline{L}_{n+c}, \overline{Q}_c, \overline{Q}_{c+1}, \dots, \overline{Q}_{n+c})}{(\overline{Q}_c)^{n+1}} \ \ \in (A[x,y]_\infty)_P$$

where S_n is the polynomial in Lemma 4.1; then $H_n \equiv \overline{H}_n \mod P^Y_{eE}$. From this and (c) in Lemma 3.1, which expresses h_s as a polynomial in elements of the form (34), the definition of $\{\overline{H}_{s;n}\}_{n\ge 0} \subset (A[x,y]_\infty)_P$ follows. They satisfy (i) and, from the expression in 3.1 (c) and the induction hypothesis, it follows that (ii) holds and that the first statement in (iii) and also (iv) are satisfied. In (iv), c_s is the class of $\overline{\mu}_s \in R_{j-1}$, hence $c_s \neq 0$. The second statement in (iii) is obtained from the expression in 3.1 (c) and the induction hypothesis, applying also (13) in Lemma 3.1. Finally, (v) can also be proved by induction, applying the same argument as before.

Let $A^{1,0}_\infty := k$ and, for $2 \le j \le \delta$, let

$$A^{j,1}_\infty := k[\underline{X}^{j-1}_0, \dots \underline{X}^{j-1}_n, \dots], \quad A^{j,r}_\infty := A^{j,1}_\infty[X_{j;0}, \dots, X_{j,e\beta_{j,r-1}}] \ , \ 2 \le r \le g_j + 1,$$

where $\underline{X}^{j-1}_n := (X_{1;n}, \dots, X_{j-1;n})$. Let $\{q_{j,r}\}_{(j,r)\in\mathcal{J}}$ be a system of transverse generators for $\eta : Y \to \mathbb{A}^d_k$ with respect to E, as in 3.3 (ii). Even more, for $2 \le j \le \delta$, let us consider the elements $\{h_{j,s}\}_{s=1}^{N_j}$ in 3.3 (iii) and set $h_{1,0} := q_{1,0} = x_1 \in A$. Let

$$\mathcal{I} := \{(1,0)\} \cup \{(j,s) \ / \ 2 \le j \le \delta, 1 \le s \le N_j\}.$$

Then we have:

Lemma 4.3. *There exist elements $\{\overline{H}_{j,s;n}\}_{(j,s)\in\mathcal{I}, n\ge 0}$ in $\mathcal{O}_{(\mathbb{A}^d)_\infty, P^{\mathbb{A}^d}_{eE}}$, $n \ge 0$, satisfying:*

(i) $H_{j,s;n} \equiv \overline{H}_{j,s;n} \mod P^Y_{eE}$, therefore $\overline{H}_{j,s;n} \in P^{\mathbb{A}^d}_{eE}$ for $0 \le n \le ei_{j,s} - 1$ and $\overline{H}_{j,s;ei_{j,s}} \notin P^{\mathbb{A}^d}_{eE}$.

(ii) We have $\overline{H}_{1,0;n} = X_{1;n}$ for $n \ge 0$. For $2 \le j \le \delta$, let r, $1 \le r \le g_j + 1$ be such that $n_{j,r-1}\overline{\beta}_{j,r-1} < i_{j,s} \le \overline{\beta}_{j,r}$ (resp. $r = 1$ if $s = 1$ and $i_{j,1} = \overline{\beta}_{j,0}$). For $(j',r') \in \mathcal{J}_{j,r}$, set $\overline{Q}_{j',r';n} := \overline{H}_{j',s_{j',r'};n}$, $n \ge 0$ and, for $(j',r') \in \mathcal{J}^*_{j,r}$, let $\overline{T}_{j',r'}$ be the multiplicative system generated by $\overline{Q}_{j',r';e\overline{\beta}_{j',r'}}$. Then, for $n \ge e(\overline{\beta}_{j,r} - \beta_{j,r})$ we have:

$$\overline{H}_{j,s;n} \in \prod_{(j',r')\in\mathcal{J}^*_{j,r}} \overline{T}^{-1}_{j',r'} \ A^{j,r}_\infty[X_{j;e\beta_{j,r-1}+1}, \dots, X_{j;n-e(\overline{\beta}_{j,r}-\beta_{j,r})}]$$

(if $r = 1$, replace $X_{j;e\beta_{j,r-1}+1}$ by $X_{j;0}$ in the previous equality).

(iii) *For* $2 \leq j \leq \delta$, *if* $s = 1$ *then* $\overline{H}_{j,s;n} = X_{j;n}$ *for* $n \geq 0$. *If* $s > 1$ *then:*

$$\overline{H}_{j,s;n} \in \left(\{\overline{Q}_{j,r';n}\}_{\substack{r' \leq r-1 \\ n < e\overline{\beta}_{j,r'}}} \cup \{\overline{H}_{j,s-1;n}\}_{n<ei_{s-1}} \right)$$

for $0 \leq n <$ *max* $\{en_{j,r-1}\overline{\beta}_{j,r-1}, ei_{j,s-1}\}$ *and*

$$\overline{H}_{j,s;n} = u_{j,s,n}\ X_{j;n-e(\overline{\beta}_{j,r}\ -\ \beta_{j,r})} + \rho_{j,s,n} \qquad \textit{for } n > \textit{max}\ \{en_{j,r-1}\overline{\beta}_{j,r-1}, ei_{j,s-1}\}$$

where $u_{j,s,n},\ \rho_{j,s,n} \in \prod_{(j',r') \in \mathcal{J}^*_{j,r}} \overline{T}^{-1}_{j',r'}\ A^{j,r}_{\infty}[X_{j;e\beta_{j,r-1}+1}, \ldots X_{j;n-e(\overline{\beta}_{j,r}-\beta_{j,r})-1}]$ *and* $u_{j,s,n}$ *is a unit.*

(iv) *Suppose that* $j, s \geq 2$. *If* $i_{j,s-1} = \overline{\beta}_{j,r-1}$ *(resp.* $i_{j,s-1} \in (n_{j,r-1}\overline{\beta}_{j,r-1}, \overline{\beta}_{j,r})$*) then* $\overline{H}_{j,s;en_{j,r-1}\overline{\beta}_{j,r-1}}$ *(resp.* $\overline{H}_{j,s;ei_{j,s-1}}$*) is equal to*

$$\overline{Q}^{b^s_{j,0}}_{1,0;e\overline{\beta}_{j,0}} \cdot \overline{Q}^{b^s_{j,1}}_{j,1;e\overline{\beta}_{j,1}} \cdots \overline{Q}^{b^s_{j,\rho}}_{j,\rho;e\overline{\beta}_{j,\rho}} \cdot$$
$$\cdot P_{j,s}\left(\frac{c_{j,s}\overline{H}}{\overline{Q}^{b^s_{j,0}}_{1,0;e\overline{\beta}_{j,0}} \cdots \overline{Q}^{b^s_{j,\rho}}_{j,\rho;e\overline{\beta}_{j,\rho}}}, \ \ldots\ , \frac{\overline{Q}_{j-1,g_{j-1}+1;e\overline{\beta}_{j-1,g_{j-1}+1}}}{\overline{Q}^{b_{j-1,0}}_{1,0;e\overline{\beta}_{j-1,0}} \cdots \overline{Q}^{b_{j-1,g_{j-1}}}_{j-1,g_{j-1};e\overline{\beta}_{j-1,g_{j-1}}}} \right)$$

where $\overline{H} = (\overline{Q}_{j,r-1;e\overline{\beta}_{j,r-1}})^{n_{j,r-1}}$ *(resp.* $\overline{H} = \overline{H}_{j,s-1;ei_{j,s-1}}$*),* $c_{j,s} \in k \setminus \{0\}$ *and* ρ, $\{b^s_{j,r'}\}^{\rho}_{r'=0}$ *and* P_s *are as in (c.3) in 3.3.*

(v) *Set* $\mathcal{G}_{1,0} := \{\overline{H}_{1,0;n}\ /\ 0 \leq n \leq e\ m_1 - 1\}$ *and, for* $2 \leq j \leq \delta$,

$$\begin{aligned} \mathcal{G}_{j,1} &:= \{\overline{H}_{j,1;n}\ /\ 0 \leq n \leq e\ i_{j,1} - 1\}\ \cup\ \cup^{s_1}_{s=2}\{\overline{H}_{j,s;n}\ /\ e\ i_{j,s-1} \leq n \leq e\ i_{j,s} - 1\} \\ \mathcal{G}_{j,r} &:= \{\overline{H}_{j,s_{r-1}+1;n}\ /\ e\ n_{j,r-1}\overline{\beta}_{j,r-1} \leq n \leq e\ i_{j,s_{r-1}+1} - 1\} \cup \\ &\quad \cup^{s_r}_{s=s_{r-1}+2}\{\overline{H}_{j,s;n}\ /\ e\ i_{j,s-1} \leq n \leq e\ i_{j,s} - 1\} \qquad \textit{for } 2 \leq r \leq g_j + 1. \end{aligned}$$

Then, for $2 \leq j \leq \delta$ *and* $1 \leq r \leq g_1 + 1$, *we have*

$$\left(\{\overline{Q}_{j,r';n}\}_{\substack{0 \leq r' \leq 1 \\ 0 \leq n \leq e\overline{\beta}_{j,r'}-1}} \cup \{\overline{Q}_{j,r';n}\}_{\substack{2 \leq r' \leq r \\ en_{j,r'-1}\overline{\beta}_{j,r'-1} \leq n \leq e\overline{\beta}_{j,r'}-1}} \right) \prod_{(j',r') \in \mathcal{J}^*_{j,r}} \overline{T}^{-1}_{j',r'}\ A^{j+1,1}_{\infty} =$$
$$= \ (\mathcal{G}_{1,0} \cup \mathcal{G}_{j,1} \cup \ldots \cup \mathcal{G}_{j,r}) \prod_{(j',r') \in \mathcal{J}^*_{j,r}} \overline{T}^{-1}_{j',r'}\ A^{j+1,1}_{\infty}.$$

Proof. Let us prove, by induction on j, $1 \leq j \leq \delta$, the existence of $\{\overline{H}_{j',s';n}\}_{\substack{(j',s') \in \mathcal{I} \\ j' \leq j, n \geq 0}}$ satisfying the required conditions. For $j = 1$, $(j, s) = (1, 0)$, $h_{1,0} := q_{1,0} = x_1 \in \mathcal{O}_{\mathbb{A}^d,\eta(y_0)}$, so, if we set $\overline{H}_{1,0;n} := H_{1,0;n} = X_{1;n} \in \mathcal{O}_{(\mathbb{A}^d)_\infty, P^{\mathbb{A}^d}_{eE}}$ for $n \geq 0$ then it is clear that (i) to (iii) are satisfied. Now, let j, $2 \leq j \leq \delta$, and suppose that $\overline{H}_{j',s';n} \in \mathcal{O}_{(\mathbb{A}^d)_\infty, P^{\mathbb{A}^d}_{eE}}$ are defined, for $j' < j$, $(j', s') \in \mathcal{I}$, $n \geq 0$, and satisfy the conditions. Then the result follows applying Lemma 4.2 to $Y \to \operatorname{Spec} A[x_1, x_j]$, where $A = k[x_2, \ldots, x_{j-1}]$, and the following remark: since

$$l_{j'} = \frac{q_{j',g_{j'}+1}}{q_{1,0}^{b_{j',0}} q_{j',1}^{b_{j',1}} \cdots q_{j',g_{j'}}^{b_{j',g_{j'}}}} \quad \text{for } 2 \le j' \le j-1$$

we may take $\overline{S}_{j-1} = \{\overline{Q}_{j',r';e\overline{\beta}_{j',r'}}\}_{(j',r') \in \mathcal{J}^*_{j-1,g_{j-1}+1}}$ and

$$\overline{L}_{j';0} = \frac{\overline{Q}_{j',g_{j'}+1;e\overline{\beta}_{j',g_{j'}+1}}}{\overline{Q}^{b_{j',0}}_{1,0;e\overline{\beta}_{j',0}} \cdot \overline{Q}^{b_{j',1}}_{j',1;e\overline{\beta}_{j',1}} \cdots \overline{Q}^{b_{j',g_{j'}}}_{j',g_{j'};e\overline{\beta}_{j',g_{j'}}}}.$$

From this, (i) to (iv) follow for j. This concludes the proof.

Remark 4.4. Let j, $2 \le j \le \delta$. Let $\{\widetilde{q}_{j,r}\}_{r=0}^{g_j+1}$ in $B_{j-1}[x_1, x_j]$ be as in Remark 3.6, and $\widetilde{Q}_{j,r;n} \in B_{j-1}[x_1, x_j]_\infty$, $n \ge 0$, as in the beginning of this section. Arguing by recurrence and applying (27), we obtain that, for $1 \le r \le g_j + 1$,

$$\widetilde{Q}_{j,r;n} \in \left(\{\widetilde{Q}_{j',r';n}\}_{\substack{0 \le r' \le r-1 \\ 0 \le n \le e\overline{\beta}_{j',r'}-1}} \right)^2 B_{j-1}[x_1, x_j]_\infty$$

for $0 \le n < \epsilon(\widetilde{q}_{j,r}) := e\ \left((n_{r-1}-1)\overline{\beta}_{r-1} + \ldots + (n_1 - 1)\overline{\beta}_1\right) = e(\overline{\beta}_{j,r} - \beta_{j,r})$ and

$$\widetilde{Q}_{j,r;n} \in \left(\{\widetilde{Q}_{j',r';n}\}_{\substack{0 \le r' \le r-1 \\ 0 \le n \le e\overline{\beta}_{j',r'}-1}} \right) B_{j-1}[x_1, x_j]_\infty$$

for $\epsilon(\widetilde{q}_{j,r}) \le n < \epsilon(\widetilde{q}_{j,r}) + e\ \beta_{j,r-1} = e\ n_{j,r-1}\overline{\beta}_{j,r-1}$.

Analogously, for $\{q'_{j,r}\}_{r=0}^{g_j+1}$, $q'_{j,r} \in T'^{-1}_{j,r-1} \cdots T'^{-1}_{j,0} k[v_2, \ldots, v_{j-1}, x_1, x_j]$ (see Remark 3.6), let $\{\overline{Q'}_{j,r;n}\}_{n \ge 0}$ in $\prod_{0 \le s \le r-1} \overline{T'}^{-1}_{j,s} k[v_2, \ldots, v_{j-1}, x_1, x_j]_\infty$ be obtained applying Lemma 4.2. Given r, $0 \le r \le g_j + 1$, let $\{a_s\}_{0 \le s \le r-1}$ be nonnegative integers such that

$$z'_{j,r} := q'_{j,r} \cdot \prod_{0 \le s \le r-1} {q'_{j,s}}^{a_s} \in k[v_2, \ldots, v_{j-1}, x_1, x_j],$$

and let $Z'_{j,r;n} \in k[v_2, \ldots, v_{j-1}, x_1, x_j]_\infty[[t]]$, $n \ge 0$, as before. Arguing by recurrence, from (c) in Lemma 3.1 it follows that, for $0 \le n < \epsilon(z'_{j,r}) := e\left(\nu(z'_{j,r}) - \beta_{j,r}\right)$,

$$Z'_{j,r;n} \in \left(\{\overline{Q'}_{j,s;n}\}_{\substack{0 \le s \le r-1 \\ 0 \le n \le e\overline{\beta}_{j,s}-1}} \right)^2 \prod_{0 \le s \le r-1} \overline{T'}^{-1}_{j,s}\ k[v_2, \ldots, v_{j-1}, x_1, x_j]_\infty \qquad (35)$$

and, for $\epsilon(z'_{j,r}) \leq n < \epsilon(z'_{j,r}) + e\ \beta_{j,r-1} = e\ (\nu(z'_{j,r}) - \beta_{j,r} + \beta_{j,r-1})$, we have

$$Z'_{j,r;n} \in \left(\{\overline{Q'}_{j,s;n}\}_{\substack{0\leq s\leq r-1 \\ 0\leq n\leq e\overline{\beta}_{j,s}-1}} \right) \prod_{0\leq s\leq r-1} \overline{T'}^{-1}_{j,s}\ k[v_2,\ldots,v_{j-1},x_1,x_j]_\infty$$

Now, with the assumptions and notation in Lemma 4.3, given $(j,r) \in \mathcal{J}$, let $\{a_{j',r'}(j,r)\}_{(j',r')\in\mathcal{J}^*_{j,r}}$ be any sequence of nonnegative integers such that

$$z_{j,r} := q_{j,r} \cdot \prod_{(j',r')\in\mathcal{J}^*_{j,r}} q_{j',r'}^{a_{j',r'}(j,r)} \in k[x_1,\ldots,x_j]$$

let $\overline{\alpha}_{j,r} := \nu(z_{j,r})$ and let $Z_{j,r;n} \in k[x_1,\ldots,x_j]_\infty$, $n \geq 0$, as before. Then we have

$$\left(\{Z_{j',r';n}\}_{\substack{(j',r')\in\mathcal{J}_{j,r} \\ 0\leq n\leq e\overline{\alpha}_{j',r'}-1}} \right) \prod_{(j',r')\in\mathcal{J}^*_{j,r}} S^{-1}_{j',r'}\ k[x_1,\ldots,x_j]_\infty =$$

$$= \left(\{\overline{Q}_{j',r';n}\}_{\substack{(j',r')\in\mathcal{J}_{j,r} \\ 0\leq n\leq e\overline{\beta}_{j',r'}-1}} \right) \prod_{(j',r')\in\mathcal{J}^*_{j,r}} T^{-1}_{j',r'}\ k[x_1,\ldots,x_j]_\infty$$

where $S_{j',r'}$ is the multiplicative part generated by $Z_{j',r';e\overline{\alpha}_{j',r'}}$. Moreover, arguing by recurrence and applying (c.2) in Proposition 3.3 and the condition (20), it follows that

$$Z_{j,r;n} \in \left(\{Z_{j',r';n}\}_{\substack{(j',r')\in\mathcal{J}_{j,r} \\ 0\leq n\leq e\overline{\alpha}_{j',r'}-1}} \right)^2 \prod_{(j',r')\in\mathcal{J}^*_{j,r}} S^{-1}_{j',r'}\ k[x_1,\ldots,x_j]_\infty \tag{36}$$

for $0 \leq n < \epsilon(z_{j,r}) := e\ \ (\nu(z_{j,r}) - \beta_{j,r})$. In fact, the proof is based on the one for (35), taking into account condition (20). We also obtain that

$$Z_{j,r;n} \in \left(\{Z_{j',r';n}\}_{\substack{(j',r')\in\mathcal{J}_{j,r} \\ 0\leq n\leq e\overline{\alpha}_{j',r'}-1}} \right) \prod_{(j',r')\in\mathcal{J}^*_{j,r}} S^{-1}_{j',r'}\ k[x_1,\ldots,x_j]_\infty \tag{37}$$

for $\epsilon(z_{j,r}) \leq n < e\ \ (\nu(z_{j,r}) - \beta_{j,r} + \beta_{j,r-1})$.

Let $\mathcal{G} := \cup_{(j,r)\in\mathcal{J}}\mathcal{G}_{j,r}$ where the $\mathcal{G}_{j,r}$'s are defined in Lemma 4.3 (v). Note that the cardinal of $\mathcal{G}_{1,1}$ is em_1 and, for $2 \leq j \leq \delta$,

$$\begin{aligned} \sharp\left(\bigcup_{r=1}^{g_j+1}\mathcal{G}_{j,r}\right) &= e\ \overline{\beta}_{j,1} + (e\ \overline{\beta}_{j,2} - e\ n_{j,1}\overline{\beta}_{j,1}) + \ldots + (e\ \overline{\beta}_{j,g_j+1} - e\ n_{j,g_j}\overline{\beta}_{j,g_j}) \\ &= e\left(\beta_{j,1} + (\ \beta_{j,2} - \beta_{j,1}) + \ldots + (\beta_{j,g_j+1} - \beta_{j,g_j})\right) = e\ \beta_{j,g_j+1} = e\ m_j. \end{aligned}$$

Hence, applying (6) and (30) we obtain

$$\sharp\mathcal{G} = e\ (a_E + 1) = e\ \mathfrak{e}\ (k_F(\mathbb{A}^d) + 1).$$

Proposition 4.5. *We have*

$$P_{eE}^{\mathbb{A}^d}\ \mathcal{O}_{(\mathbb{A}^d)_\infty, P_{eE}^{\mathbb{A}^d}} = (\mathcal{G})\ \mathcal{O}_{(\mathbb{A}^d)_\infty, P_{eE}^{\mathbb{A}^d}}$$

moreover, there exists $L \in \mathcal{O}_{(\mathbb{A}^d)_\infty} \setminus P_{eE}^{\mathbb{A}^d}$ *such that* $P_{eE}^{\mathbb{A}^d}\ (\mathcal{O}_{(\mathbb{A}^d)_\infty})_L = (\mathcal{G})\ (\mathcal{O}_{(\mathbb{A}^d)_\infty})_L$. *Besides, the images of the elements of* $\mathcal{G}$ *in* $P_{eE}^{\mathbb{A}^d}/(P_{eE}^{\mathbb{A}^d})^2\ \mathcal{O}_{(\mathbb{A}^d)_\infty, P_{eE}^{\mathbb{A}^d}}$ *are independent, hence define a basis as* $\kappa(P_{eE}^{\mathbb{A}^d})$*-vector space. In particular, we obtain* $\dim \mathcal{O}_{(\mathbb{A}^d)_\infty, P_{eE}^{\mathbb{A}^d}} = \sharp\mathcal{G} = e\ (a_E + 1)$.

Proof. First note that, by (i) in Lemma 4.3, we have $\mathcal{G} \subset P_{eE}^{\mathbb{A}^d}$. Let us prove that $(\mathcal{G})\ \mathcal{O}_{(\mathbb{A}^d)_\infty, P_{eE}^{\mathbb{A}^d}}$ is a prime ideal. By (ii) in Lemma 4.3, for $(j,r) \in \mathcal{J}$, we have

$$\mathcal{G}_{j,r} \ \subset \prod_{(j',r') \in \mathcal{J}^*_{j,r}} \overline{T}_{j',r'}^{-1}\ A_\infty^{j,r}[X_{j;e\beta_{j,r-1}+1}, \ldots, X_{j;e\beta_{j,r}-1}]$$

(if $r = 0$ or 1, replace $X_{j;e\beta_{j,r-1}+1}$ by $X_{j;0}$ and set $\beta_{1,0} := m_1$). Then, for each j, $2 \le j \le \delta$, there exists $M_j \in \mathbb{N}$ such that

$$\overline{Q}_{j',r'} \in \prod_{(j'',r'') \in \mathcal{J}^*_{j',r'}} \overline{T}_{j'',r''}^{-1}\ k[\underline{X}_0^j, \ldots, \underline{X}_{M_j}^j] \quad \text{for every } (j',r') \in \mathcal{J}^*_{j,g_j+1}$$

and, if we set

$$B_\infty^j := \prod_{(j',r') \in \mathcal{J}^*_{j,g_j+1}} \overline{T}_{j',r'}^{-1}\ k[\underline{X}_0^j, \ldots, \underline{X}_{M_j}^j]$$

then

$$\mathcal{G}_j := \bigcup_{(j',r') \in \mathcal{J}_{j,g_j+1} \cup \{(j,g_j+1)\}} \mathcal{G}_{j',r'} \quad \subset \quad B_\infty^j$$

(in fact, M_j can be taken to be equal to em_j). Let P_j be the contraction of $P_{eE}^{\mathbb{A}^d}$ to B_∞^j. We will prove, by induction on j, $2 \le j \le \delta$, that there exists $L_j \in B_\infty^j \setminus P_j$ such that the ring $(B_\infty^j)_{L_j}/(\mathcal{G}_j)$ is a domain. For $j = 2$, we have $h_{1,0} = x_1$, thus $\mathcal{G}_{1,0} = \{X_{1;0}, \ldots, X_{1;em_1-1}\}$ and, applying Remark 3.5 and (iii) in Lemma 4.3 to $\overline{Q}_{2,r;n}$, $en_{j,r-1}\overline{\beta}_{2,r-1} < n < e\overline{\beta}_{2,r}$ and (iv) in Lemma 4.3 to $\overline{Q}_{2,r;e\overline{\beta}_{2,r}}$, we obtain that $B_\infty^2\ /\ (\mathcal{G}_2)$ is isomorphic to

$$\left(S_2^{-1}k\,[y_{2,0}, y_{2,2} \ldots, y_{2,g_2+1}]\ /\ J_2\right)\ [\{X_{1;n}\}_{em_1 < n \le M_2} \cup \{X_{2;n}\}_{e\beta_{2,g_2+1} < n \le M_2}]$$

where the image of $y_{2,r}$, $1 \le r \le g_2 + 1$ (resp. $y_{2,0}$) is $Q_{2,r;e\overline{\beta}_{2,r}}$ (resp. $X_{1;em_1}$), J_2 is the ideal in Theorem 3.8 (iv) and S_2 is the multiplicative part generated by $\{y_{2,r}\}_{r=0}^{g_2}$, therefore $B_\infty^2/(\mathcal{G}_2)$ is a domain by Theorem 3.8.

Let j, $3 \leq j \leq \delta$, and suppose that the result holds for $j-1$. Applying (iii) in Lemma 4.3 to $\overline{H}_{j,s;n}$, for $ei_{j,s-1} < n \leq ei_{j,s} - 1$ (resp. $en_{j,r-1}\overline{\beta}_{j,r-1} < n \leq ei_{j,s_{r-1}+1} - 1$) if $s_{r-1} + 2 \leq s \leq s_r$ (resp. $s = s_{r-1} + 1$) and applying (iv) in 4.3 to $\overline{H}_{j,s;ei_{j,s-1}}$ (resp. $\overline{H}_{j,s;en_{j,r-1}\overline{\beta}_{j,r-1}}$), we obtain that there exists an étale extension $\widetilde{B}^j_\infty$ of B^j_∞ containing the image of P^Y_{eE}, i.e. the contraction of P^Y_{eE} to $\widetilde{B}^j_\infty$ is a prime ideal $\widetilde{P}_j \neq \widetilde{B}^j_\infty$, and such that $\widetilde{B}^j_\infty/(\mathcal{G}_j)\widetilde{B}^j_\infty$ is isomorphic to a localization of

$$\left(S_j^{-1}\widetilde{D}_{j-1}[y_{j,1},\ldots,y_{j,g_j+1}] \ / \ J_j\right)\left[\{X_{j;n}\}_{e\beta_{j,g_j+1}<n\leq M_j}\right]$$

where $\widetilde{D}_{j-1}$ is a domain which is an étale extension of $B^{j-1}_\infty/(\mathcal{G}_{j-1})$, S_j is the multiplicative part generated by $\{y_{j,r}\}_{r=1}^{g_j}$ and J_j is an ideal generated by $\{y_{j,r}^{n_{j,r}} - \widetilde{c}_{j,r}\ y_{1,0}^{b_{j,r,0}} \cdot y_{j,1}^{b_{j,r,1}} \ldots y_{j,r-1}^{b_{j,r,r-1}}\}_{r=1}^{g_j}$, $\widetilde{c}_{j,r} \in \widetilde{D}_{j-1}$ and $y_{1,0} = X_{1;em_1} \in \widetilde{D}_{j-1}$. Here $y_{j,r}$ is identified with $\overline{Q}_{j,r;e\overline{\beta}_{j,r}}$. Applying Theorem 3.8 (iv) we conclude that $\widetilde{B}^j_\infty/(\mathcal{G}_j)$ is a domain. Since the morphism $(B^j_\infty)_{P_j}/(\mathcal{G}_j) \to (\widetilde{B}^j_\infty)_{\widetilde{P}_j}/(\mathcal{G}_j)\widetilde{B}^j_\infty$ is local étale, hence an inclusion of local rings, we conclude that $(B^j_\infty)_{P_j}/(\mathcal{G}_j)$ is a domain. Therefore, there exists $L_j \in B^j_\infty \setminus P_j$ such that $(B^j_\infty)_{L_j}/(\mathcal{G}_j)$ is a domain (recall that B^j_∞ is the localization of a finitely generated k-algebra).

In particular, it follows that there exists $L_\delta \in B^\delta_\infty \setminus P_\delta \ \subset \ \prod_{(j,r)\in\mathcal{J}^*} \overline{T}_{j,r}^{-1}\mathcal{O}_{(\mathbb{A}^d)_\infty} \setminus P^{\mathbb{A}^d}_{eE}$ such that the ideal generated by $\mathcal{G}$ in $(\prod_{(j,r)\in\mathcal{J}^*} \overline{T}_{j,r}^{-1}\mathcal{O}_{(\mathbb{A}^d)_\infty})_{L_\delta}$ is a prime ideal. From this it follows that there exists $L \in \mathcal{O}_{(\mathbb{A}^d)_\infty} \setminus P^{\mathbb{A}^d}_{eE}$ such that $(\mathcal{G})\ (\mathcal{O}_{(\mathbb{A}^d)_\infty})_L$ is a prime ideal, in fact, we may take $L = L_\delta \cdot \prod_{(j,r)\in\mathcal{J}^*} \overline{Q}^{a_{j,r}}_{j,r;e\overline{\beta}_{j,r}}$ for some positive integers $\{a_{j,r}\}_{(j,r)\in\mathcal{J}^*}$. Hence $(\mathcal{G})\ \mathcal{O}_{(\mathbb{A}^d)_\infty, P^{\mathbb{A}^d}_{eE}}$ is a prime ideal.

Let us denote by P' the prime ideal of $\mathcal{O}_{\mathbb{A}^d_\infty}$ such that $(\mathcal{G})\ \mathcal{O}_{(\mathbb{A}^d)_\infty, P^{\mathbb{A}^d}_{eE}} = P'\mathcal{O}_{(\mathbb{A}^d)_\infty, P^{\mathbb{A}^d}_{eE}}$. We will next prove that $P' = P^{\mathbb{A}^d}_{eE}$. In fact, with the notation in 2.7 and 3.7, let $P^Z_{e\mathfrak{e}F}$ be the generic point of $Z^{e\mathfrak{e}F}_\infty$ and let $P^{\mathbb{A}^d}_{e\mathfrak{e}F}$ be the image of $P^Z_{e\mathfrak{e}F}$ by the morphism $Z_\infty \to (\mathbb{A}^d)_\infty$. Since $\mathfrak{e}$ is the ramification index of $\mathcal{O}_{Y,E}$ over $\mathcal{O}_{Z,F}$, $P^Z_{e\mathfrak{e}F}$ is the image of P^Y_{eE} by $\sigma_\infty : Y_\infty \to Z_\infty$ and hence $P^{\mathbb{A}^d}_{e\mathfrak{e}F} = P^{\mathbb{A}^d}_{eE}$. Now, by the definition of $\mathcal{G}$, and since $P' \subseteq P^{\mathbb{A}^d}_{eE}$, we have

$$e\overline{\beta}_{j,r} \leq \nu_{P'}(q_{j,r}) \leq e\ \nu(q_{j,r}) = e\overline{\beta}_{j,r} \quad \text{ for } (j,r) \in \mathcal{J}.$$

Therefore $\nu_{P'}(q_{j,r}) = e\overline{\beta}_{j,r}$ for every $(j,r) \in \mathcal{J}$ and hence

$$\nu_{P'}\left(\frac{q_{j,g_j+1}}{q_{j,0}^{b_{j,0}} \ldots q_{j,g_j}^{b_{j,g_j}}}\right) = 0 \quad \text{for } 2 \leq j \leq \delta \quad \text{and} \quad \nu_{P'}(z) \ = \sum_{(j',r')\in\mathcal{J}^*} \mathfrak{a}_{j,r} e\ \overline{\beta}_{j,r} = e\ \mathfrak{e}$$

(recall (29) in 3.7). From this it follows that the morphism of k-algebras $h^\sharp_{P'} : \mathcal{O}_{X,P_0} \to \kappa(P')[[t]]$ induced by the arc $h_{P'}$ extends to $\mathcal{O}_{Z,F}$. That is, $h_{P'} : \operatorname{Spec} \kappa(P')[[t]] \to X$ lifts to (Z, F), more precisely, since $\nu_{P'}(z) = e\mathfrak{e}$, this lifting defines a point in $Z^{e\mathfrak{e}F}_\infty$. Therefore $P' \in \overline{\{P^{\mathbb{A}^d}_{e\mathfrak{e}F}\}}$, hence we conclude that $P' = P^{\mathbb{A}^d}_{e\mathfrak{e}F} = P^{\mathbb{A}^d}_{eE}$.

Finally, since $\sharp\mathcal{G} = e\ \mathfrak{e}\ (k_F(\mathbb{A}^d)+1)$, the end of the proof follows from Proposition 2.6 and equality (3), which is in fact lemma 3.4 in [4].

Remark 4.6. Alternatively, in the proof of Proposition 4.5 it can be proved by induction on j, $2 \leq j \leq \delta$, applying (iii) and (iv) in Lemma 4.3, not only that $(\mathcal{G}_j)$ is a prime ideal of $(B^j_\infty)_{L_j}$, but also that the elements in $\mathcal{G}_j$ are independent in $(\mathcal{G}_j)\ /\ (\mathcal{G}_j)^2$. Then, lemma 3.4 in [4] can be recovered (at least for X smooth) from Propositions 4.5 and 2.6. Therefore, Proposition 4.5 can be seen as a new version of lemma 3.4 in [4], which is in fact the change of variables theorem in the motivic integration.

Definition 4.7. Let $\eta : Y \to \mathbb{A}^d_k$ be a k-morphism dominant and generically finite, where Y is a nonsingular k-scheme, let E be a divisor on Y and let $e \geq 1$. Let $\{q_{j,r}\}_{(j,r)\in\mathcal{J}}$ be a system of transverse generators for η with respect to E (Definition 3.4), and let $\{\overline{Q}_{j,r;n}\}_{(j,r)\in\mathcal{J},n\geq 0}$ defined as in Lemma 4.3. We call

$$\mathcal{Q} := \{\overline{Q}_{j,r;n}\}_{(j,r)\in\mathcal{J},\ en_{j,r-1}\overline{\beta}_{j,r-1}\leq n\leq e\overline{\beta}_{j,r}-1}$$

a regular system of parameters of $\mathcal{O}_{(\mathbb{A}^d)_\infty,P^{\mathbb{A}^d}_{eE}}$ *associated to* $\{q_{j,r}\}_{(j,r)\in\mathcal{J}}$.

In fact, note that by Proposition 4.5 (see also Lemma 4.3 (v)), $\mathcal{O}_{(\mathbb{A}^d)_\infty,P^{\mathbb{A}^d}_{eE}}$ is a regular local ring of dimension the cardinal of $\mathcal{Q}$ whose maximal ideal $P^{\mathbb{A}^d}_{eE}\mathcal{O}_{(\mathbb{A}^d)_\infty,P^{\mathbb{A}^d}_{eE}}$ is generated by $\mathcal{Q}$.

Theorem 4.8. *Assume that char $k = 0$. Let X be a nonsingular k-scheme, let ν be a divisorial valuation on an irreducible component X_0 of X, and let $e \in \mathbb{N}$. Let $\pi : Y \to X_0$ be a proper and birational morphism such that the center of ν on Y is a divisor E, and let $\eta : Y \to \mathbb{A}^d_k$ be the composition of π with an étale morphism $X_0 \to \mathbb{A}^d_k$, where $d = \dim X_0$. Let $\mathcal{Q} = \{\overline{Q}_{j,r;n}\}_{(j,r)\in\mathcal{J},\ 0\leq n\leq e\overline{\beta}_{j,r}-1}$ be a regular system of parameters of $\mathcal{O}_{(\mathbb{A}^d)_\infty,P^{\mathbb{A}^d}_{eE}}$ associated to a system of transverse generators for η with respect to E. Then $\mathcal{Q}$ is also a regular system of parameters of $\mathcal{O}_{X_\infty,P^X_{eE}}$, that is*

$$P^X_{eE}\ \mathcal{O}_{X_\infty,P^X_{eE}} = \left(\{\overline{Q}_{j,r;n}\}_{(j,r)\in\mathcal{J},\ en_{j,r-1}\overline{\beta}_{j,r-1}\leq n\leq e\overline{\beta}_{j,r}-1}\right)\mathcal{O}_{X_\infty,P^X_{eE}},$$

and $\mathcal{O}_{X_\infty,P^X_{eE}}$ is a regular local ring of dimension

$$\dim\mathcal{O}_{X_\infty,P^X_{eE}} = \sharp\mathcal{Q} = e\ (k_E+1),$$

where k_E is the discrepancy of X with respect to E.

Moreover, there exist elements $z_{j,r} \in \mathcal{O}_{X,P_0}$, $(j,r) \in \mathcal{J}$, and $L \in \mathcal{O}_{X_\infty} \setminus P^X_{eE}$ such that

$$P^X_{eE}(\mathcal{O}_{X_\infty})_L \;=\; \left(\{Z_{j,r;n}\}_{(j,r)\in\mathcal{J},0\leq n<e\overline{\alpha}_{j,r}}\right)(\mathcal{O}_{X_\infty})_L \tag{38}$$

where $\overline{\alpha}_{j,r} = \nu(z_{j,r})$ for $(j,r) \in \mathcal{J}$.

Proof. Recall that P^Y_{eE} is the generic point of Y^{eE}_∞ (see 2.7) and that P^X_{eE} (resp. $P^{\mathbb{A}^d}_{eE}$) is the image of P^Y_{eE} by π_∞ (resp. η_∞). By Proposition 2.5 (see also Corollary. 2.9) it suffices to prove the result for the point $P^{\mathbb{A}^d}_{eE}$ in $(\mathbb{A}^d)_\infty$. Then it follows from Proposition 4.5. In fact, for the first assertion note that in this case $k_E(\mathbb{A}^d_k)$ is equal to the discrepancy k_E of X with respect to E. For the second assertion, let $\{q_{j,r}\}_{(j,r)\in\mathcal{J}}$ be a system of transverse generators for η with respect to E. For each $(j,r)\in\mathcal{J}$ there exists a sequence of nonnegative integers $\{a_{j',r'}(j,r)\}_{(j',r')\in\mathcal{J}^*_{j,r}}$, such that

$$z_{j,r} := q_{j,r}\cdot \prod_{(j',r')\in\mathcal{J}^*_{j,r}} q_{j',r'}^{a_{j',r'}(j,r)} \quad \in \mathcal{O}_{\mathbb{A}^d,P_0}.$$

(see Proposition 3.3). Then, from Proposition 4.5, (38) follows. This concludes the proof.

Remark 4.9. Let P be any stable point of X_∞ and suppose that X is nonsingular at the center P_0 of P and that P_0 is not the generic point of X. There exists a birational and proper morphism $\pi : Y \to X$ such that the center of ν_P on Y is a divisor E, and $e\in\mathbb{N}$ such that $\nu_P = e\nu_E$ ([25], (vii) in prop. 3.7). Let $P^Y\in Y_\infty$ whose image by π_∞ is P, then we have $\dim\mathcal{O}_{X_\infty,P} = ek_E + \dim\mathcal{O}_{Y_\infty,P^Y}$ (corol. 2.9). Since $P^Y \supseteq P^Y_{eE}$ and $P\supseteq P^X_{eE}$, with the notation in Theorem 4.8 and Proposition 3.3, $\{U_0,\dots,U_{e-1}\}$ is part of a regular system of parameters of $\mathcal{O}_{Y_\infty,P^Y}$ and $\mathcal{Q} = \{\overline{Q}_{j,r;n}\}_{(j,r)\in\mathcal{J},\ en_{j,r-1}\overline{\beta}_{j,r-1}\le n\le e\overline{\beta}_{j,r}-1}$ is part of a regular system of parameters of $\mathcal{O}_{X_\infty,P}$. Moreover, suppose that $\{U_0,\dots,U_{e-1},G_1,\dots,G_s\}$ is a regular system of parameters of $\mathcal{O}_{Y_\infty,P^Y}$. To describe a regular system of parameters of $\mathcal{O}_{X_\infty,P}$ we add to $\mathcal{Q}$ the following elements: By Lemma 4.1 and since π is birational, for each $y\in\mathcal{O}_Y$ and for each n, there exists $\overline{Y}_n\in\mathcal{O}_{X_\infty,P}$ such that

$$Y_n \equiv \overline{Y}_n \mod P.$$

Then, let $\overline{G}_i\in\mathcal{O}_{X_\infty,P}$, $1\le i\le s$ be obtained from G_i by replacing U_n and $V_{j;n}$ by $\overline{U}_n$ and $\overline{V}_{j;n}$, for $n\ge 0$, $2\le j\le d$. We have

$$G_i \equiv \overline{G}_i \mod P.$$

and $\mathcal{Q}\cup\{\overline{G}_1,\dots,\overline{G}_s\}$ is a regular system of parameters of $\mathcal{O}_{X_\infty,P}$.

Now let us consider a reduced separated k-scheme of finite type X and a divisorial valuation ν on X centered on Sing X. There exists a resolution of singularities $\pi : Y \to X$ (i.e. π is a proper, birational k-morphism, with Y smooth, such that the induced morphism $Y\setminus\pi^{-1}(Sing\ X)\to X\setminus Sing\ X$ is an isomorphism) such that the center of ν on Y is a divisor E.

Corollary 4.10. *Assume that char $k=0$. Let X be a reduced separated k-scheme of finite type, let ν be a divisorial valuation on an irreducible component X_0 of X centered on*

Sing X and let $e \in \mathbb{N}$. Let $\pi : Y \to X$ be a resolution of singularities such that the center of ν on Y is a divisor E, and let $\eta : Y \to \mathbb{A}^d_k$ be the composition of π with a general projection $\mu : X_0 \to \mathbb{A}^d$, where $d = \dim X_0$. Let $\mathcal{Q} = \{\overline{Q}_{j,r;n}\}_{(j,r)\in\mathcal{J},\ en_{j,r-1}\overline{\beta}_{j,r-1}\le n\le e\overline{\beta}_{j,r}-1}$ be a regular system of parameters of $\mathcal{O}_{(\mathbb{A}^d)_\infty, P^{\mathbb{A}^d}_{eE}}$ associated to a system of transverse generators for η with respect to E. Then $\mathcal{Q}$ is a system of coordinates of $((X_\infty)_{red}, P^X_{eE})$, that is,

$$P^X_{eE}\mathcal{O}_{(X_\infty)_{red},P^X_{eE}} = \left(\{\overline{Q}_{j,r;n}\}_{(j,r)\in\mathcal{J},\ en_{j,r-1}\overline{\beta}_{j,r-1}\le n\le e\overline{\beta}_{j,r}-1}\right)\mathcal{O}_{(X_\infty)_{red},P^X_{eE}}.$$

Therefore

$$\text{embdim } \widehat{\mathcal{O}_{X_\infty,P_{eE}}} \;=\; \text{embdim } \mathcal{O}_{(X_\infty)_{\rm red},P_{eE}} \;\le\; \sharp\mathcal{Q} = e\ (\widehat{k}_E + 1),$$

where $\widehat{k}_E$ is the Mather discrepancy of X with respect to E.

Moreover, there exist elements $\{z_{j,r}\}_{(j,r)\in\mathcal{J}}$ in $\mathcal{O}_{X,P_0}$ and $L \in \mathcal{O}_{X_\infty}\setminus P^X_{eE}$ such that

$$\begin{aligned} P^X_{eE}(\mathcal{O}_{(X_\infty)_{red}})_L &= \left(\{Z_{j,r;n}\}_{(j,r)\in\mathcal{J},n\le e\overline{\alpha}_{j,r}-1}\right)(\mathcal{O}_{(X_\infty)_{red}})_L \\ &= \left(\{Z_{j,r;n}\}_{(j,r)\in\mathcal{J},e\overline{\alpha}_{j,r}-e(\beta_{j,r}-\beta_{j,r-1})\le n\le e\overline{\alpha}_{j,r}-1}\right)(\mathcal{O}_{(X_\infty)_{red}})_L \end{aligned}$$

where $\overline{\alpha}_{j,r} = \nu(z_{j,r})$ for $(j,r)\in\mathcal{J}$.

Proof. We may suppose that $\pi : Y \to X$ dominates the Nash blowing up of X. We may suppose that X is affine, let $X \subseteq \mathbb{A}^N_k = \operatorname{Spec} k[y_1,\ldots,y_N]$. Then, a general projection $\rho : X \subseteq \mathbb{A}^N_k \to \mathbb{A}^d_k$, $\underline{y} \to (x_1,\ldots,x_d)$ satisfies

$$\operatorname{ord}_E\ \pi^*(dx_1\wedge\ldots\wedge dx_d) = \widehat{k}_E. \tag{39}$$

Let $P^{\mathbb{A}^d}_{eE}$ be the image of P^Y_{eE} by η_∞. Then the result follows from Proposition 4.5 applied to $P^{\mathbb{A}^d}_{eE}$ (see also (37) in Remark 4.4), Proposition 4.5 (iii) in [25] applied to $\rho : X \to \mathbb{A}^d_k$ and the finiteness property of the stable points in [24] th. 4.1 (see 2.4).

From Corollary 4.10 it follows that $\widehat{\mathcal{O}_{X_\infty,P_{eE}}}$ is a quotient of the ring $\kappa(P_{eE})[[\{X_{j,r;n}\}_{(j,r)\in\mathcal{J},\ e\overline{\alpha}_{j,r}-e(\beta_{j,r}-\beta_{j,r-1})\le n\le e\overline{\alpha}_{j,r}-1}]]$ where $X_{j,r;n}$ is sent to $Z_{j,r;n}$. Moreover, we may suppose that X is affine, let $X \subseteq \mathbb{A}^N_k$, and let $X \to \mathbb{A}^d_k$, $(x_1,\ldots,x_N) \mapsto (x_1,\ldots x_d)$, be a general projection. Then, there exist series $\widetilde{X}_{l;n} \in \kappa(P_{eE})\left[\left[\{X_{j,r;n}\}_{(j,r;n)}\right]\right]$ whose image in $\widehat{\mathcal{O}_{X_\infty,P_{eE}}}$ is $X_{l;n}$, $d+1 \le l \le N$, $n \ge 0$. For $d+1 \le l \le N$, let $f_l \in k[x_1,\ldots,x_d,x_l]$ be such that $f_l(x_1,\ldots,x_d,\overline{x}_l) = 0$ where $\overline{x}_l$ denotes the class of x_l in $\mathcal{O}_X$, hence X is contained in the complete intersection X' of dimension $d = \dim X$ defined by $\{f_l\}_{d+1\le l\le N}$. We have that $\widehat{\mathcal{O}_{X_\infty,P_{eE}}} \cong \widehat{\mathcal{O}_{X'_\infty,P^{X'}_{eE}}}$ ([25], Proposition 3.7 (ii) and Theorem 3.13). Therefore we conclude:

Corollary 4.11. *With the notation as before,*

$$\widehat{\mathcal{O}_{X_\infty,P_{eE}}} \;\cong\; \kappa(P_{eE})\left[\left[\{X_{j,r;n}\}_{(j,r;n)\in\mathcal{C}}\right]\right] \Big/ \;\widetilde{I}$$

where $\mathcal{C} = \{(j,r;n)\ /\ (j,r)\in\mathcal{J},\ e\overline{\alpha}_{j,r} - e(\beta_{j,r}-\beta_{j,r-1}) \leq n < e\overline{\alpha}_{j,r}\}$ *has* $e(\widehat{k}_E+1)$ *elements and*

$$\widetilde{I} \;=\; \left(\{\widetilde{F}_{l;n}\}_{d+1\leq l\leq N, n\geq 0}\right)$$

where $\widetilde{F}_{l,n}$ *is obtained from* $F_{l;n}$ *by substituting* $X_{l;n'}$ *by* $\widetilde{X}_{l;n'}$*, for* $0\leq n'\leq n$.

Remark 4.12. Note that $\widetilde{I}$ is a finitely generated ideal. Moreover, since $X\to\mathbb{A}^d$ is a general projection (39) holds, and hence $\epsilon_l := \nu_E(\mathrm{Jac}(f_l)) = \nu_E(\frac{\partial f_l}{\partial x_l})$. Then, from [23] proof of lemma 3.2 (see also [24], proof of lemma 4.2) it follows that $\widetilde{F}_{l;\epsilon_l+n} = 0$ for $n > \epsilon_l$. In fact, this is the effective way of constructing the series $\widetilde{X}_{l;n}$, $n\geq \epsilon_l+1$ (see [25], corol. 5.6). Analogously, imposing $\widetilde{F}_{l;\epsilon_l+n} = 0$ for $0\leq n\leq\epsilon_l$ is the way to construct $\widetilde{X}_{l;n}$, $0\leq n\leq \epsilon_l$, and hence to describe the ring $\widehat{\mathcal{O}_{X_\infty,P_{eE}}}$ (see example below).

Example 4.13. The following examples were given in [14]: For $m\geq 3$ let us consider the singular threefold X_m given by $z^2 = xy + w^m$ in $\mathbb{A}^4_k$ where k is a field of characteristic 0. It has an isolated singularity at the origin O. Let $X_{m,1}\to X_m$ be the blowing up of X_m at O. Then $X_{m,1}$ has an isolated singularity which is locally isomorphic to X_{m-2}. If we continue in this way $r := \lfloor\frac{m}{2}\rfloor$ steps we obtain a resolution of singularities of X_m. Let

$$Y = X_{m,r}\to X_{m,r-1}\to\cdots\to X_{m,1}\to X_m,$$

be the chain of point blowing ups and let E_i, $1\leq i\leq r$, be the strict transform in Y of the exceptional locus of the blowing up $X_{m,i}\to X_{m,i-1}$, which is irreducible. Let ν_{E_i} be the divisorial valuation defined by E_i, hence $\nu_{E_i}(x) = \nu_{E_i}(y) = \nu_{E_i}(z) = i$ and $\nu_{E_i}(w) = 1$. Then $k_{E_i}(X_m) = i$ for $1\leq i\leq r$ ([14] lemma 12) and we can check that $\widehat{k}_{E_i}(X_m) = 2i$, $1\leq i\leq r$.

Set $X := X_m$ and $f := z^2 - xy - w^m$. For $1\leq i\leq r$, let P_{E_i} be the stable point of X_∞ defined by ν_{E_i}. The projection $X\to\mathbb{A}^3_k$, $(x,y,z,w)\mapsto(x,y,w)$ is general, it satisfies $\mathrm{ord}_{E_i}\pi^*(dx\wedge dy\wedge dw) = \widehat{k}_{E_i}(X)$. Applying Corollary 4.10, in this case we obtain $P_{E_i}\left(\mathcal{O}_{(X_\infty)_{\mathrm{red}}}\right)_{P_{E_i}} = (X_0,\ldots,X_{i-1},Y_0,\ldots,Y_{i-1},W_0)\ \left(\mathcal{O}_{(X_\infty)_{\mathrm{red}}}\right)_{P_{E_i}}$. Moreover

$$P_{E_i}\left(\mathcal{O}_{(X_\infty)_{\mathrm{red}}}\right)_{W_1X_r} = (X_0,\ldots,X_{i-1},Y_0,\ldots,Y_{i-1},Z_0,\ldots,Z_{i-1},W_0)\left(\mathcal{O}_{(X_\infty)_{\mathrm{red}}}\right)_{W_1X_r}. \tag{40}$$

Since the open subset $W_1X_r\neq 0$ of X_∞ has nonempty intersection with $\overline{\{P_{E_i}\}}$ for all i, $1\leq i\leq r$, it follows that $P_{E_1}\subset P_{E_2}\subset\ldots\subset P_{E_r}$ and, in particular, $X_\infty^{Sing} = \overline{\{P_{E_1}\}}$.

This gives another proof of theorem 1 (1) in [14]. Besides, from (37) it follows that $\widehat{\mathcal{O}_{X_\infty,P_{E_i}}}$ is a quotient of $\kappa(P_{E_i})[[X_0,\ldots,X_{i-1},Y_0,\ldots,Y_{i-1},W_0]]$. For $i=1$, applying Corollary 4.11 we obtain

$$\widehat{\mathcal{O}_{X_\infty,P_{E_1}}} \;\cong\; \kappa(P_{E_1})[[X_0,Y_0,W_0]] \;/\; ((\widetilde{Z}_0)^2 - X_0Y_0 - W_0^m) \tag{41}$$

where

$$\kappa(P_{E_1}) \cong k(X_1,X_2,\ldots,Y_1,Y_2,\ldots,W_1,W_2,\ldots)[Z_1]/(Z_1^2 - X_1Y_1) \tag{42}$$

and $\widetilde{Z}_0 \in \kappa(P_{E_1})[[X_0,Y_0,W_0]]$ is defined from the F_n's, $n \geq 1 = \nu_{E_1}(\text{Jac } f)$. Precisely, the isomorphism (42) defines, for each $n \geq 0$, $Z_n^{(0)} \in \kappa(P_{E_1})$ such that $Z_n - Z_n^{(0)} \in (X_0,Y_0,W_0)$. Arguing recursively on $n' \geq 1$ and $n \geq 0$, with the lexicographical order on (n',n), that is, reasoning as in corol. 5.6 in [25], it follows that, for $n',n \geq 0$, there exists $Z_n^{(n')} \in \kappa(P_{E_1})[X_0,Y_0,W_0]$ such that,

$$\begin{aligned} F_{1+n} &\equiv 2Z_1\;(Z_n - Z_n^{(n')}) \quad \mod\;(X_0,Y_0,W_0)^{n'+1} \quad \text{for } n \geq 0,\; n \neq 1 \\ F_2 &\equiv (Z_1)^2 - (Z_1^{(n')})^2 \quad \mod\;(X_0,Y_0,W_0)^{n'+1} \end{aligned}$$

hence $Z_n^{(n'+1)} \equiv Z_n^{(n')} \mod (X_0,Y_0,W_0)^{n'+1}$ and this defines series $\widetilde{Z}_n$ in $\kappa(P_{E_1})[[X_0,Y_0,W_0]]$ such that $\widetilde{F}_{1+n} = 0$ for $n \geq 0$ (see Remark 4.12). In particular, it defines $\widetilde{Z}_0$ in (41). From these computations it follows that the ring $\widehat{\mathcal{O}_{X_\infty,P_{E_1}}}$ has dimension 2. Even more, for m odd (resp. m even) $\widehat{\mathcal{O}_{X_\infty,P_{E_1}}}$ has a 2-dimensional singularity whose normalization has a D_4- singularity (resp. A_1-singularity). This in particular implies that $\widehat{\mathcal{O}_{X_\infty,P_{E_1}}}$ is reduced and not regular.

From analogous computations applying Corollary 4.11 to describe the ring $\widehat{\mathcal{O}_{X_\infty,P_{E_i}}}$, $2 \leq i \leq r$, we obtain that $\widehat{\mathcal{O}_{X_\infty,P_{E_i}}}$ is a complete intersection local ring of dimension

$$\dim \widehat{\mathcal{O}_{X_\infty,P_{E_i}}} \;=\; i+1 = k_{E_i}(X)+1 \qquad \text{for } 1 \leq i \leq r.$$

In [14] prop. 9 it is proved that if $m \geq 5$ is odd (resp. m is even or $m=3$) then ν_{E_1} and ν_{E_2} (resp. ν_{E_1}) are the only essential valuations. Since $(X_m)_\infty^{Sing}$ is irreducible for every m, in case that $m \geq 5$ is odd, E_2 defines an essential valuation whose family of arcs $\overline{\{P_{E_2}\}}$ is not an irreducible component of $(X_m)_\infty^{Sing}$ ([14] example 1). Let us consider

$$\mathcal{X} := \text{Spec } \widehat{\mathcal{O}_{X_\infty,P_{E_1}}}$$

so that the set $\mathcal{X}_\infty^O$ of arcs $\varphi : \text{Spec } K[[\xi]] \to \mathcal{X}$ centered at the closed point O of $\mathcal{X}$ is precisely the set of wedges, or k-morphisms $\phi : \text{Spec } K[[\xi,t]] \to X$, whose special arc is P_{E_1}. By special arc (resp. generic arc) of ϕ we mean the image by φ of the closed (resp. generic) point of $K[[\xi]]$. If m is even then $\mathcal{X}$ is a surface singularity whose normalization has a A_1- singularity, hence $\mathcal{X}_\infty^O$ is irreducible. If m is odd then $\mathcal{X}$ is a surface singularity whose normalization has a D_4-singularity, hence $\mathcal{X}_\infty^O$ has 4 irreducible components. These families of arcs of $\mathcal{X}_\infty^O$ correspond to wedges on X whose special arc is P_{E_1} and whose generic arc is not centered at Sing X, i.e. it does not belong to X_∞^{Sing}. On the other hand, if $m \geq 5$ then

$$\widehat{\mathcal{O}_{(X_{m,1})_\infty, P_{E_2}}} \cong \widehat{\mathcal{O}_{X_\infty, P_{E_1}}},$$

therefore $\mathcal{X} = \mathrm{Spec}\ \widehat{\mathcal{O}_{(X_{m,1})_\infty, P_{E_2}}}$. In this ring P_{E_1} is defined by (W_0), which contains the closed point of $\mathcal{X}$ and is contained in Sing $\mathcal{X}$. From this one can compute explicitly a wedge on $X_{m,1}$ whose projection in X is a wedge whose special arc is P_{E_2} and whose generic arc is P_{E_1} (see [14], proof of th. 1).

Acknowledgment

I thank O. Piltant for his suggestions and comments.

References

[1] S. Abhyankar, On the valuations centered in a local domain, Amer. J. Math. 78 (1956) 321–348.
[2] N. Bourbaki, Commutative Algebra, Chapter III, Springer, 1989.
[3] C. Bouvier, G. Gonzalez-Sprinberg, Système générateur minimal, diviseurs essentiels et G-désingularisations de variétés toriques, Tohoku Math. J. (2) 47 (1) (1995) 125–149.
[4] J. Denef, F. Loeser, Germs of arcs on singular algebraic varieties and motivic integration, Invent. Math. 135 (1999) 201–232.
[5] L. Ein, S. Ishii, Singularities with respect to Mather–Jacobian discrepancies, in: Commutative Algebra and Noncommutative Algebraic Geometry, vol. II, in: Math. Sci. Res. Inst. Publ., vol. 68, Cambridge Univ. Press, New York, 2015, pp. 125–168.
[6] L. Ein, M. Mustata, Jet schemes and singularities, Proc. Sympos. Pure Math. 80 (2) (2009) 505–546.
[7] T. de Fernex, Three dimensional counter-examples to the Nash problem, Compos. Math. 149 (9) (2013) 1519–1534.
[8] T. de Fernex, R. Docampo, Terminal valuations and the Nash problem, Invent. Math. 203 (1) (2016) 303–331.
[9] T. de Fernex, L. Ein, S. Ishii, Divisorial valuations via arcs, Publ. Res. Inst. Math. Sci. 44 (2) (2008) 425–448.
[10] R. Goldin, B. Teissier, Resolving singularities of plane analytic branches with one toric morphism, in: Resolution of Singularities, Obergurgl, 1997, in: Progr. Math., vol. 181, Birkhauser, Basel, 2000, pp. 315–340.
[11] R. Hartshorne, Algebraic Geometry, Springer-Verlag, 1977.
[12] S. Ishii, Mather discrepancy and the arc spaces, Ann. Inst. Fourier 63 (1) (2013) 89–111.
[13] S. Ishii, J. Kollár, The Nash problem on arc families of singularities, Duke Math. J. 120 (3) (2003) 601–620.
[14] J. Johnson, J. Kollár, Arc spaces of cA-type singularities, J. Singul. 7 (2013) 238–252.
[15] O. Kashcheyeva, A construction of generating sequences for valuations centered in dimension 3 regular local rings, preprint.
[16] M. Lejeune-Jalabert, A. Reguera, Exceptional divisors which are nor uniruled belong to the image of the Nash map, J. Inst. Math. Jussieu 11 (2) (2012) 273–287.
[17] F. Lucas, J. Madden, D. Schaub, M. Spivakovsky, Approximate roots of a valuation and the Pierce-Birkhoff conjecture, Ann. Fac. Sci. Toulouse Math. (6) 21 (2) (2012) 259–342.
[18] P. Morandi, Field and Galois Theory, Grad. Texts in Math., vol. 167, Springer-Verlag, 1996.
[19] H. Mourtada, A. Reguera, Mather discrepancy as an embedded dimension in the space of arcs, Publ. Res. Inst. Math. Sci. (2017), in press.

[20] J. Nash, Arc structure of singularities, Duke Math. J. 81 (1995) 207–212.
[21] O. Piltant, Gradués associés aux valuations Frobenius pour les sommes de Selberg, Thesis Ecole Polytechnique, 1994.
[22] M. Raynaud, Anneaux Locaux Henséliens, Lecture Notes in Math., vol. 169, Springer-Verlag, 1956.
[23] A.J. Reguera, Image of the Nash map in terms of wedges, C. R. Math. Acad. Sci. Paris 338 (5) (2004) 385–390.
[24] A.J. Reguera, A curve selection lemma in spaces of arcs and the image of the Nash map, Compos. Math. 142 (2006) 119–130.
[25] A.J. Reguera, Towards the singular locus of the space of arcs, Amer. J. Math. 131 (2) (2009) 313–350.
[26] M. Spivakovsky, Valuations in function fields of surfaces, Amer. J. Math. 112 (1990) 107–156.
[27] B. Teissier, Overweight deformations of affine toric varieties and local uniformization, in: A. Campillo, F-V. Kuhlmann, B. Teissier (Eds.), Valuation Theory in Interaction, Proceedings of the second International Conference on Valuation Theory, Segovia–El Escorial, 2011, in: European Math. Soc. Publishing House, Congress Reports Series, Sept. 2014, pp. 474–565.
[28] O. Zariski, The moduli problem for plane branches, Amer. Math. Soc. 39 (2006), with an appendix by B. Teissier.