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SKOLEM-NOETHER ALGEBRAS

MATEJ BREŠAR¹, CHRISTOPH HANSELKA², IGOR KLEP³, AND JURIJ VOLČIČ⁴

ABSTRACT. An algebra S is called a Skolem-Noether algebra (SN algebra for short) if for every central simple algebra R , every homomorphism $R \rightarrow R \otimes S$ extends to an inner automorphism of $R \otimes S$. One of the important properties of such an algebra is that each automorphism of a matrix algebra over S is the composition of an inner automorphism with an automorphism of S . The bulk of the paper is devoted to finding properties and examples of SN algebras. The classical Skolem-Noether theorem implies that every central simple algebra is SN. In this article it is shown that actually so is every semilocal, and hence every finite-dimensional algebra. Not every domain is SN, but, for instance, unique factorization domains, polynomial algebras and free algebras are. Further, an algebra S is SN if and only if the power series algebra $S[[\xi]]$ is SN.

1. INTRODUCTION

Our main motivation for this work is the celebrated Skolem-Noether theorem. We will state its version as given, for example, in [Her68]. But first, a word on conventions. All our algebras are assumed to be unital algebras over a fixed field F , subalgebras are assumed to contain the same unity, and all homomorphisms send 1 to 1.

Theorem 1.1. (Skolem-Noether) *Let A be simple Artinian algebra with center F . If R is a finite-dimensional simple F -subalgebra of A and φ is an F -algebra homomorphism from R into A , then there exists an invertible element $c \in A$ such that $\varphi(x) = cxc^{-1}$ for all $x \in R$. (In other words, φ can be extended to an inner automorphism of A .)*

Recall that an algebra is said to be **central** if its center consists of scalar multiples of unity. As usual, we will use the term **central simple algebra** for an algebra that is central, simple, and also finite-dimensional.

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Definition 1.2. An algebra S is a **Skolem-Noether algebra** (SN algebra for short) if for every central simple algebra R and every homomorphism $\varphi : R \rightarrow R \otimes S$ there exists an invertible element $c \in R \otimes S$ such that $\varphi(x) = cxc^{-1}$ for every $x \in R$. (Here, R is identified with $R \otimes 1$).

The Skolem-Noether theorem, together with the well-known fact that the class of central simple algebras is closed under tensor products, implies that every central simple algebra S is an SN algebra. A partial converse is also true: the assertion that central simple algebras are SN algebras implies an important special case of the Skolem-Noether theorem where A is a central simple algebra and R is its central simple subalgebra. This is because, under these assumptions, A is isomorphic to $R \otimes S$ where S is also a central simple subalgebra of A [Bre14, Corollary 4.49].

SN algebras naturally arise from the problem of understanding automorphism groups of tensor products of algebras. Unlike the case of derivations on tensor products [Bre17], the general solution to this problem seems far out of reach. For instance, while automorphisms of univariate and bivariate polynomial algebras are well understood [Jun42], already the trivariate case is wild [SU03]. In another direction, functional analysts consider the question when the flip automorphism $A \otimes A \rightarrow A \otimes A$ is (approximately) inner for operator algebras A , see [Sak75, ER78, Izu17]. In this paper we settle the following special case of the above problem. If S is an SN algebra and R is a central simple algebra, then automorphisms of $R \otimes S$ are just compositions of inner automorphisms and automorphisms of S ; see Proposition 3.3. While the class of SN algebras looks restrictive, our main results show that various classical and important families of algebras satisfy the SN property, for example semilocal (in particular Artinian and finite-dimensional) algebras, unique factorization domains, free algebras, etc.

Some of the readers might be interested only in the case where $R = M_n(F)$, the algebra of $n \times n$ matrices with entries in F . Let us therefore mention that since $M_n(F) \otimes S$ can be identified with $M_n(S)$, the condition that S is an SN algebra implies that every homomorphism from $M_n(F)$ into $M_n(S)$ can be extended to an inner automorphism of $M_n(S)$. Moreover, we show in Proposition 2.1 that the latter condition implies the SN property. However, this does not lead to any simplifications of our proofs, so we persist with central simple algebras as in Definition 1.2.

Main results and guide to the paper. The short Section 2 on preliminaries includes Proposition 2.1: S is an SN algebra if and only if all homomorphisms $M_n(F) \rightarrow M_n(S)$ extend to inner automorphisms. Section 3 positions SN algebras into a wider context of automorphisms of tensor products. For instance, Proposition 3.3 proves that given an SN algebra S and a central simple algebra R , every automorphism of $R \otimes S$ is the composition of an inner automorphism and an automorphism of S . In particular, this applies to matrix algebras over SN algebras.

We then identify classes of algebras which satisfy the SN property. In Section 4 we derive Lemma 4.1, which is the main technical tool for proving subsequent results. Section 5 culminates in Theorem 5.5 showing that semilocal algebras are SN. Hence

all Artinian algebras and thus all finite-dimensional algebras are SN. Section 6 refines the latter result. Namely, every homomorphism from a central simple subalgebra R of a finite-dimensional algebra A into A extends to an inner automorphism of A (see Theorem 6.1). In Section 7 we give examples of domains which are SN algebras, such as unique factorization domains (UFDs) and free algebras, see Corollary 7.2 and Corollary 7.4. Section 8 uses the Quillen-Suslin theorem to prove that matrix algebras over polynomial algebras are SN. The paper concludes with Section 9, where we show that an algebra S is SN if and only if the formal power series algebra $S[[\xi]]$ is SN.

2. PRELIMINARIES

The purpose of this section is to introduce the notation and terminology, and prove a proposition that yields a characterization of SN algebras.

Let R be a central simple algebra. Given $w, z \in R$, we define the left and right multiplication operators $L_w, R_z : R \rightarrow R$ by

$$L_w(x) = wx \quad \text{and} \quad R_z(x) = xz.$$

As is well-known, every linear map from R into R can be written as a sum of maps of the form $L_w R_z$, $w, z \in R$ [Bre14, Lemma 1.25]. Accordingly, given a basis $\{r_1, \dots, r_d\}$ of R , there exists $w_j, z_j \in R$ such that $h = \sum_j L_{w_j} R_{z_j}$ satisfies $h(r_1) = 1$ and $h(r_k) = 0$, $k \neq 1$. That is,

$$(2.1) \quad \sum_j w_j r_1 z_j = 1 \quad \text{and} \quad \sum_j w_j r_k z_j = 0 \quad \text{if } k > 1.$$

We will be mostly concerned with tensor product algebras $R \otimes S$. Here R, S are algebras over a field F and the tensor product is taken over F . As usual, we identify R with $R \otimes 1$, and, accordingly, often write $r \otimes 1 \in R \otimes 1$ simply as r . Let us point out an elementary fact that will be used freely without further reference. If the r_i -s are linearly independent elements in R , then for all $p_j \in R$ and $s_j, t_i \in S$,

$$(2.2) \quad \sum_i r_i \otimes t_i = \sum_j p_j \otimes s_j$$

implies that each t_i lies in the linear span of the s_j -s [Bre14, Lemma 4.9]. Similarly, assuming that the t_i -s are linearly independent, it follows from (2.2) that each r_i lies in the linear span of the p_j -s.

By $\text{rad}(S)$ we denote the **Jacobson radical** of the algebra S . Recall that S is called a **semilocal algebra** if $S/\text{rad}(S)$ is a semisimple algebra, i.e., isomorphic to a finite direct product of simple Artinian algebras. In the special case where $S/\text{rad}(S)$ is a division algebra, S is called a **local algebra**. Finally, we say that S is a **stably finite algebra** if for all $n \geq 1$ and all $x, y \in M_n(S)$, $xy = 1$ implies $yx = 1$.

To conclude the section we give an alternative characterization of the SN property. In order to show that S is an SN algebra it suffices to verify the condition of Definition 1.2 for $R = M_n(F)$, i.e., all F -algebra homomorphisms $M_n(F) \rightarrow M_n(S)$ are given by conjugation.

Proposition 2.1. *An algebra S is SN if and only if for every $n \in \mathbb{N}$ and a homomorphism $\varphi : M_n(F) \rightarrow M_n(S)$ there exists $c \in M_n(S)$ such that $\varphi(x) = cxc^{-1}$ for every $x \in R$.*

Proof. The implication (\Rightarrow) holds by definition, so consider the direction (\Leftarrow). Let R be a central simple algebra and $\varphi : R \rightarrow R \otimes S$ a homomorphism. Let R^{op} denote the opposite algebra of R and consider

$$\tilde{\varphi} = \text{id} \otimes \varphi : R^{\text{op}} \otimes R \rightarrow R^{\text{op}} \otimes R \otimes S.$$

Since

$$R^{\text{op}} \otimes R \cong M_{(\dim R)}(F),$$

by the assumption there exists $c \in R^{\text{op}} \otimes R \otimes S$ such that $\tilde{\varphi}(x) = cxc^{-1}$ for every $x \in R^{\text{op}} \otimes R$. We can write c as

$$c = \sum_{i,j} a_{ij} \otimes r_i \otimes s_j$$

for some $a_{ij} \in R^{\text{op}}$ and linearly independent sets $\{r_i\}_i \subset R$ and $\{s_j\}_j \subset S$. If $x = x_1 \otimes 1$ for $x_1 \in R^{\text{op}}$, then $\tilde{\varphi}(x)c - cx = 0$ becomes

$$\sum_{i,j} (x_1 a_{ij} - a_{ij} x_1) \otimes r_i \otimes s_j = 0.$$

Since the elements $r_i \otimes s_j$ form a linearly independent set in $R \otimes S$, we conclude that $x_1 a_{ij} = a_{ij} x_1$ for all $a_{ij} \in R^{\text{op}}$ and $x_1 \in R^{\text{op}}$. As R^{op} is central we have $a_{ij} \in F$ and therefore $c \in 1 \otimes R \otimes S \cong R \otimes S$. Consequently $\varphi(x) = cxc^{-1}$ for every $x \in R$. Hence S is an SN algebra. \square

While Proposition 2.1 seems to facilitate the process of showing that a certain algebra S is an SN algebra, it does not simplify our proofs in the sequel.

3. SN ALGEBRAS AND AUTOMORPHISMS

In this section we give a few motivating results and prove that every automorphism of a matrix algebra over an SN algebra S is an inner automorphism composed with an automorphism of S , see Corollary 3.4.

We begin with a proposition which explains the requirement in Definition 1.2 that the algebra R is central simple.

Proposition 3.1. *Let R be a subalgebra of an algebra S . If the homomorphism $x \mapsto 1 \otimes x$ from $R = R \otimes 1$ into $R \otimes S$ can be extended to an inner automorphism of $R \otimes S$, then R is a central simple algebra.*

Proof. By the assumption, there exists an invertible element $a \in R \otimes S$ such that

$$1 \otimes x = a(x \otimes 1)a^{-1}$$

for all $x \in R$. Let us write $a = \sum_{i=1}^m u_i \otimes v_i$ and $a^{-1} = \sum_{j=1}^n w_j \otimes z_j$. Accordingly,

$$(3.1) \quad \begin{aligned} 1 \otimes x &= \left(\sum_{i=1}^m u_i \otimes v_i \right) (x \otimes 1) \left(\sum_{j=1}^n w_j \otimes z_j \right) \\ &= \sum_{i=1}^m \sum_{j=1}^n u_i x w_j \otimes v_i z_j. \end{aligned}$$

This implies that every $x \in R$ lies in the linear span of all $v_i z_j$, $i = 1, \dots, m$, $j = 1, \dots, n$. Thus, R is finite-dimensional. On the other hand, (3.1) implies that for every nonzero $x \in R$, 1 lies in RxR . This means that R is simple. Finally, if z lies in the center of R , then $1 \otimes z = a(z \otimes 1)a^{-1} = z \otimes 1$, which readily implies that z is a scalar multiple of 1 , as desired. \square

The question of when the automorphism $x \otimes y \mapsto y \otimes x$ of $R \otimes R$ is inner was raised by Sakai [Sak75] in the C*-algebra context, and investigated further by Bunce [Bun74]. The following corollary is an extension of [Bun74, Theorem 2].

Corollary 3.2. *Let R be an arbitrary algebra. The homomorphism $x \otimes 1 \mapsto 1 \otimes x$ from $R = R \otimes 1$ into $R \otimes R$ can be extended to an inner automorphism of $R \otimes R$ if and only if R is a central simple algebra.*

Proof. If R is a central simple algebra, then so is $R \otimes R$ [Bre14, Corollary 4.44], and so every homomorphism from R into $R \otimes R$ can be extended to an inner automorphism by the Skolem-Noether theorem. The converse follows from Proposition 3.1. \square

The next proposition yields another motivation for considering SN algebras.

Proposition 3.3. *Let R be a central simple algebra and let S be an SN algebra. Then every automorphism φ of $R \otimes S$ is the composition of an inner automorphism and an automorphism of the form $\text{id}_R \otimes \sigma$ where σ is an automorphism of S .*

Proof. By the assumption, the restriction of φ to R can be extended to an inner automorphism $x \mapsto cxc^{-1}$ of $R \otimes S$. Considering the automorphism $x \mapsto c^{-1}\varphi(x)c$ we thus see that there is no loss of generality in assuming that φ acts as the identity on R . The proposition will be proved by showing that φ maps $1 \otimes S$ into itself. Pick $s \in S$. We can write $\varphi(1 \otimes s)$ as $\sum_j p_j \otimes s_j$ where the s_j -s are linearly independent. Since $1 \otimes s$ commutes with every $x \in R$ it follows that so does $\varphi(1 \otimes s)$. This implies that

$$\sum_j (p_j x - x p_j) \otimes s_j = 0.$$

As the s_j -s are linearly independent it follows that $p_j x - x p_j = 0$ for each j and each $x \in R$. Hence, since R is central, each p_j is a scalar multiple of 1 . Consequently, $\varphi(1 \otimes s) \in 1 \otimes S$. \square

If $R = M_n(F)$, then $R \otimes S$ can be identified with $M_n(S)$, and the proposition gets the following form.

Corollary 3.4. *If S is an SN algebra, then every automorphism φ of $M_n(S)$ is of the form*

$$\varphi((s_{ij})) = c(\sigma(s_{ij}))c^{-1}$$

where c is an invertible element in $M_n(S)$ and σ is an automorphism of S .

This result is known in the case where S is either an Artinian algebra [BO81, Theorem 3.13], a UFD [Isa80, Corollary 15], or a commutative local algebra (see, e.g., [Kov73, p. 163]). As we will see, all these algebras are SN algebras. On the other hand, [Isa80] shows that the commutative domain $\mathbb{Z}[\sqrt{-5}]$ does not satisfy the conclusion of Corollary 3.4. We give an algebra with the same property in Example 7.6.

While we identify several classes of SN algebras, the following question is still open: is a matrix algebra over an SN algebra again an SN algebra? Or, more generally, is a tensor product of an SN algebra with a central simple algebra again an SN algebra?

4. BASIC LEMMA

All our main results will be derived from the following technical lemma. Its proof will use some ideas from the proof of the (special case of) Skolem-Noether theorem given in [Bre14, pp. 13–14].

Lemma 4.1. *Let R be a central simple algebra with basis $\{r_1, \dots, r_d\}$ and let S be an arbitrary algebra. Then $\varphi : R \rightarrow R \otimes S$ is a homomorphism if and only if there exist $c_1, \dots, c_d \in R \otimes S$ such that*

- (a) $\varphi(x) = \sum_{k=1}^d c_k x r_k$ for all $x \in R$,
- (b) $\varphi(x)c_k = c_k x$ for all $x \in R$ and all $1 \leq k \leq d$, and
- (c) $\sum_{k=1}^d c_k r_k = 1$.

Moreover, writing $c_k = \sum_{\ell=1}^d r_\ell \otimes s_{k\ell}$, we have that for each k and ℓ there exists $b_{k\ell} \in R \otimes S$ such that

$$b_{k\ell} c_k = 1 \otimes s_{k\ell}.$$

Consequently, if S is stably finite and there exist k and ℓ such that $s_{k\ell}$ is invertible in S , then $c = c_k$ is invertible in $R \otimes S$ and

$$\varphi(x) = cxc^{-1}$$

for all $x \in R$.

Proof. Since R is finite-dimensional, there exist finitely many $s_i \in S$ and linear maps $f_i : R \rightarrow R$ such that

$$\varphi(x) = \sum_i f_i(x) \otimes s_i$$

for all $x \in S$. By [Bre14, Lemma 1.25] there exist $w_{ij}, z_{ij} \in R$ such that

$$f_i = \sum_j L_{w_{ij}} R_{z_{ij}}.$$

Consequently, for every $x \in R$ we have

$$\begin{aligned}\varphi(x) &= \sum_i \left(\sum_j w_{ij} x z_{ij} \right) \otimes s_i \\ &= \sum_i \sum_j (w_{ij} \otimes s_i) x z_{ij}.\end{aligned}$$

Writing each z_{ij} as a linear combination of r_1, \dots, r_d we see that φ is of the form described in (a).

We now use the multiplicativity of φ , i.e., $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in R$. In view of (a) we can rewrite this as

$$(4.1) \quad \sum_{k=1}^d c_k x y r_k = \sum_{k=1}^d \varphi(x) c_k y r_k.$$

Pick $w_j, z_j \in R$ such that (2.1) holds. Setting $y = w_j$ in (4.1), multiplying the equality, so obtained, from the right by z_j , and then summing up over all j we get

$$\sum_j \sum_{k=1}^d c_k x w_j r_k z_j = \sum_j \sum_{k=1}^d \varphi(x) c_k w_j r_k z_j,$$

that is,

$$\sum_{k=1}^d c_k x \left(\sum_j w_j r_k z_j \right) = \sum_{k=1}^d \varphi(x) c_k \left(\sum_j w_j r_k z_j \right).$$

By (2.1) this reduces to $c_1 x = \varphi(x) c_1$. Of course, the same proof applies to every c_k , so (b) holds. Finally, (c) follows from $\varphi(1) = 1$.

A direct verification shows that (a), (b), and (c) imply that φ is a homomorphism.

Let us write $c_1 = \sum_\ell r_\ell \otimes s_{1\ell}$, and let w_j, z_j be as above. Using (b) we obtain

$$\begin{aligned}\sum_j w_j \varphi(z_j) c_1 &= \sum_j w_j c_1 z_j \\ &= \sum_j \sum_\ell w_j (r_\ell \otimes s_{1\ell}) z_j \\ &= \sum_\ell \left(\sum_j w_j r_\ell z_j \right) \otimes s_{1\ell} \\ &= 1 \otimes s_{11}.\end{aligned}$$

Thus, $b_{11} = \sum_j w_j \varphi(z_j)$ satisfies $b_{11} c_1 = 1 \otimes s_{11}$. Similarly we find other $b_{k\ell}$ -s.

Finally, assume that $s_{k\ell}$ is invertible in S for some k and ℓ . Then $(1 \otimes s_{k\ell}^{-1}) b_{k\ell}$ is a left inverse of c_k . If S is stably finite, then the result by Montgomery [Mon83, Theorem 1] implies that this element is also a right inverse. Therefore, (b) shows that $c = c_k$ satisfies $\varphi(x) = c x c^{-1}$ for all $x \in R$. \square

We continue with a simple application of Lemma 4.1, showing that local algebras are SN. This result will be generalized to semilocal algebras (with a considerably more involved proof) in the next section, see Theorem 5.5.

Corollary 4.2. *Every local algebra is an SN algebra.*

Proof. Let $r_k, s_{k\ell}$ be elements from Lemma 4.1. From (c) it follows that

$$\sum_{k,\ell} r_\ell r_k \otimes s_{k\ell} = 1 \otimes 1.$$

This implies that 1 lies in the linear span of $s_{k\ell}$. Consequently, at least one $s_{k\ell}$ does not lie in $\text{rad}(S)$. Since S is local it follows that $s_{k\ell}$ is invertible in S . As S is stably finite [Lam01, Theorem 20.13], the last assertion of Lemma 4.1 shows that there exists $c \in R \otimes S$ such that $\varphi(x) = cxc^{-1}$ for all $x \in R$. \square

5. SEMILOCAL ALGEBRAS

The main result of this section is Theorem 5.5 showing that semilocal algebras are SN. We begin with a simple lemma.

Lemma 5.1. *If S_1 and S_2 are SN algebras, then so is their direct product $S_1 \times S_2$.*

Proof. Recall that $R \otimes (S_1 \times S_2)$ can be identified with $(R \otimes S_1) \times (R \otimes S_2)$. Take a homomorphism

$$\varphi : R \rightarrow (R \otimes S_1) \times (R \otimes S_2).$$

Writing

$$\varphi(x) = (\varphi_1(x), \varphi_2(x))$$

it is immediate that φ_i is a homomorphism from R into $R \otimes S_i$, $i = 1, 2$. By the assumption, there exist $c_i \in R \otimes S_i$ such that $\varphi_i(x) = c_i x c_i^{-1}$ for all $x \in R$, $i = 1, 2$. Hence,

$$c = (c_1, c_2) \in (R \otimes S_1) \times (R \otimes S_2)$$

satisfies $\varphi(x) = cxc^{-1}$ for all $x \in R$. \square

As mentioned in the introduction, the Skolem-Noether theorem implies that every central simple algebra is an SN algebra. Consequently every finite-dimensional semisimple algebra is an SN algebra by Lemma 5.1. With a little extra effort we can extend this to (possibly infinite-dimensional) semisimple algebras.

Lemma 5.2. *Every semisimple algebra is an SN algebra.*

Proof. In view of Lemma 5.1 it suffices to consider the case where S is simple Artinian. Let R be a central simple algebra. The algebra $R \otimes S$ is then simple [Bre14, Theorem 4.42]. We claim that it is also Artinian. Indeed, considering $R \otimes S$ as a left S -module in the natural way we see that it is isomorphic to the left S -module S^d where d is the dimension of R , and that a descending chain of left ideals of $R \otimes S$ is also a descending chain of left S -submodules. The desired conclusion thus follows from the fact that S^d is Artinian.

Let K be the center of S . The center of $R \otimes S$ is equal to $1 \otimes K$ [Bre14, Corollary 4.32], which we identify with K . Consider $R \otimes K$ as an algebra over K in the usual way. Clearly, it is finite-dimensional and, again by [Bre14, Theorem 4.42], simple.

Now, given a homomorphism $\varphi : R \rightarrow R \otimes S$, we define

$$\Phi : R \otimes K \rightarrow R \otimes S$$

by

$$\Phi(x \otimes k) = \varphi(x)(1 \otimes k).$$

Note that Φ is a K -algebra homomorphism. The Skolem-Noether theorem thus tells us that there exists $c \in R \otimes S$ such that $\Phi(x \otimes k) = c(x \otimes k)c^{-1}$ for all $x \in R$, $k \in K$. Setting $k = 1$ we get the desired conclusion. \square

Our goal is to show that semilocal algebras are SN algebras by reducing the general case to the semisimple case. We will actually prove a general reduction theorem whose possible applications are not limited to semilocal algebras. To this end, we need the following lemma. From its nature one would expect that it is known, but we were unable to find a good reference. We include a short proof for the sake of completeness.

Lemma 5.3. *If R is a central simple algebra, then $\text{rad}(R \otimes S) = R \otimes \text{rad}(S)$ for every algebra S .*

Proof. As an ideal of $R \otimes S$, $\text{rad}(R \otimes S)$ is necessarily of the form $R \otimes I$ for some ideal I of S [Bre14, Theorem 4.42]. We will show that $I \subseteq \text{rad}(S)$, by making use of the following characterization of $\text{rad}(A)$: $v \in \text{rad}(A)$ if and only if $1 - vx$ is invertible for every $x \in A$. Take $u \in I$. Since $1 \otimes u \in \text{rad}(R \otimes S)$ it follows that

$$1 \otimes (1 - ux) = 1 \otimes 1 - 1 \otimes ux = 1 \otimes 1 - (1 \otimes u)(1 \otimes x)$$

is invertible in $R \otimes S$ for every $x \in S$. However, this is possible only if $1 - ux$ is invertible, implying that $u \in \text{rad}(S)$. Thus, $I \subseteq \text{rad}(S)$, and so $\text{rad}(R \otimes S) \subseteq R \otimes \text{rad}(S)$.

As the lemma is well-known if $R = M_n(F)$ (see, e.g., [Lam01, pp. 57-58]), we will establish the converse inclusion by reducing the general case to this one. Take a splitting field K for R which is a finite separable extension of F (see, e.g., [GS06, Proposition 4.5.4]). Then $K \otimes R$ may be identified with $M_n(K)$ for some $n \geq 1$, and, therefore, $K \otimes R \otimes S$ may be identified with $M_n(K \otimes S)$. Thus, by what we pointed out at the beginning of the paragraph, we have

$$\text{rad}(K \otimes R \otimes S) = M_n(\text{rad}(K \otimes S)).$$

By [Lam01, Theorem 5.17], $\text{rad}(K \otimes S) = K \otimes \text{rad}(S)$, and so

$$(5.1) \quad \text{rad}(K \otimes R \otimes S) = M_n(K) \otimes \text{rad}(S).$$

According to [Lam01, Theorem 5.14],

$$\text{rad}(R \otimes S) = (R \otimes S) \cap \text{rad}(K \otimes R \otimes S),$$

and hence, by (5.1),

$$\text{rad}(R \otimes S) = (R \otimes S) \cap (M_n(K) \otimes \text{rad}(S)).$$

Since both $R \otimes S$ and $M_n(K) \otimes \text{rad}(S)$ readily contain $R \otimes \text{rad}(S)$, it follows that $R \otimes \text{rad}(S) \subseteq \text{rad}(R \otimes S)$. \square

We can now prove the announced reduction theorem.

Theorem 5.4. *If an algebra S is stably finite and $S/\text{rad}(S)$ is an SN algebra, then S is an SN algebra.*

Proof. Let R be central simple and write

$$J = R \otimes \text{rad}(S).$$

Take a homomorphism $\varphi : R \rightarrow R \otimes S$. We define

$$\Phi : R \rightarrow (R \otimes S)/J$$

by

$$\Phi(x) = \varphi(x) + J.$$

Since $(R \otimes S)/J$ is canonically isomorphic to $R \otimes (S/\text{rad}(S))$, and $S/\text{rad}(S)$ is assumed to be an SN algebra, it follows that there exists an invertible element $a \in (R \otimes S)/J$ such that

$$\Phi(x) = a(x + J)a^{-1} \text{ for all } x \in R.$$

As J is, by Lemma 5.3, the Jacobson radical of $R \otimes S$, it follows that there exists an invertible element $b \in R \otimes S$ such that $a = b + J$. Obviously, we have

$$\varphi(x) - bxb^{-1} \in J \text{ for all } x \in R,$$

that is,

$$b^{-1}\varphi(x)b - x \in J \text{ for all } x \in R.$$

Replacing the role of φ by the homomorphism $x \mapsto b^{-1}\varphi(x)b$ we see that without loss of generality we may assume that $b = 1$. Thus,

$$(5.2) \quad \varphi(x) - x \in J \text{ for all } x \in R.$$

Now apply Lemma 4.1. Picking a basis $\{r_1, \dots, r_d\}$ of R , we can thus find $s_{k\ell} \in S$, $k, \ell = 1, \dots, p$, such that

$$(5.3) \quad \varphi(x) = \sum_{k=1}^d \sum_{\ell=1}^d r_\ell x r_k \otimes s_{k\ell} \text{ for all } x \in R,$$

and our goal is to show that at least one $s_{k\ell}$ is invertible in S .

Let $\lambda_k \in F$ be such that $1 = \sum_{k=1}^d \lambda_k r_k$. Then

$$x = 1 \cdot x \cdot 1 = \sum_{k=1}^d \sum_{\ell=1}^d (\lambda_k \lambda_\ell) r_\ell x r_k \text{ for all } x \in R.$$

Using (5.2) and (5.3) we thus obtain

$$(5.4) \quad \sum_{k=1}^d \sum_{\ell=1}^d r_{\ell} x r_k \otimes (s_{k\ell} - \lambda_k \lambda_{\ell} \cdot 1) \in J \text{ for all } x \in R.$$

We may assume that $\lambda_1 \neq 0$. Choose $w_j, z_j \in R$ that satisfy (2.1). Denoting the expression in (5.4) by $\rho(x)$, we have

$$\begin{aligned} \sum_j w_j \rho(z_j) &= \sum_j \sum_{k=1}^d \sum_{\ell=1}^d w_j r_{\ell} z_j r_k \otimes (s_{k\ell} - \lambda_k \lambda_{\ell} \cdot 1) \\ &= \sum_{k=1}^d \sum_{\ell=1}^d \left(\sum_j w_j r_{\ell} z_j \right) r_k \otimes (s_{k\ell} - \lambda_k \lambda_{\ell} \cdot 1) \\ &= \sum_{k=1}^d r_k \otimes (s_{k1} - \lambda_k \lambda_1 \cdot 1). \end{aligned}$$

As ρ maps into J it follows that

$$\sum_{k=1}^d r_k \otimes (s_{k1} - \lambda_k \lambda_1 \cdot 1) \in J = R \otimes \text{rad}(S).$$

Since the r_k -s are linearly independent, we must have $s_{k1} - \lambda_k \lambda_1 \cdot 1 \in \text{rad}(S)$ for each k . In particular, $s_{11} = \lambda_1^2 \cdot 1 + u$ for some $u \in \text{rad}(S)$. Since $\lambda_1 \neq 0$ it follows that s_{11} is invertible, as desired. \square

We are now in a position to give our main result.

Theorem 5.5. *Every semilocal algebra S is an SN algebra.*

Proof. Since S is stably finite [Lam01, Theorem 2.13] and the algebra $S/\text{rad}(S)$ is semisimple, the theorem follows from Lemma 5.2 and Theorem 5.4. \square

Corollary 5.6. *Every Artinian algebra is an SN algebra.*

6. FINITE-DIMENSIONAL ALGEBRAS

Corollary 5.6 shows that every finite-dimensional algebra is an SN algebra. The next result strengthens this statement.

Theorem 6.1. *Let A be a finite-dimensional algebra and let R be a central simple subalgebra of A . Then every homomorphism from R into A can be extended to an inner automorphism of A .*

Proof. Assume first that $R = M_n(F)$. Then A contains a set of $n \times n$ matrix units and is therefore isomorphic to $M_n(S) \cong R \otimes S$ for some subalgebra S of A [Bre14, Lemma 2.52]. Since S is also finite-dimensional, the desired conclusion follows from Corollary 5.6.

Now let R be an arbitrary central simple algebra. We may assume that the field F is infinite, for otherwise $R \cong M_n(F)$ by Wedderburn's theorem on finite division rings. Let φ be a homomorphism from R into A . Take a splitting field K for R . Identifying $K \otimes R$ with $M_n(K)$, $n \geq 1$, it follows from the preceding paragraph that there exists $b \in K \otimes A$ such that

$$(\text{id}_K \otimes \varphi)(y) = byb^{-1}$$

for all $y \in K \otimes R$. In particular,

$$(1 \otimes \varphi(x))b = b(1 \otimes x)$$

for all $x \in R$. Writing $b = \sum_{i=1}^m k_i \otimes a_i$ with the k_i -s linearly independent, it follows that

$$\sum_{i=1}^m k_i \otimes (\varphi(x)a_i - a_i x) = 0,$$

and so $\varphi(x)a_i = a_i x$ for all $x \in R$ and every i . Hence it suffices to show that $\text{span}_F\{a_1, \dots, a_m\}$ contains an element which is invertible in A .

As a finite-dimensional algebra, A can be considered a subalgebra of $M_N(F)$ for some $N \geq 1$. Take the polynomial

$$f(\xi_1, \dots, \xi_m) = \det \left(\sum_{i=1}^m \xi_i a_i \right) \in F[\xi_1, \dots, \xi_m].$$

Note that $K \otimes A$ can be viewed as a subalgebra of $M_N(K)$. Since b is invertible in $K \otimes A$, we know that $\text{span}_K\{a_1, \dots, a_m\}$ contains an invertible element in $K \otimes A$. This clearly implies that f is a nonzero polynomial. As F is infinite, there exist $\lambda_i \in F$ such that $f(\lambda_1, \dots, \lambda_m) \neq 0$. That is, $\text{span}_F\{a_1, \dots, a_m\}$ contains an element c which is invertible in $M_N(F)$. However, since we are in finite dimensions, c^{-1} is a polynomial in c . Thus, c is invertible in A . \square

Using the standard homomorphism construction we will now see that Theorem 6.1 can be used for showing that all derivations from R into any R -bimodule M are inner (in accordance with the conventions mentioned at the very beginning of the paper, we assume that our bimodules are unital). This is, of course, a well-known result. Another way of stating it is that central simple algebras are separable.

Corollary 6.2. *Every derivation from a central simple algebra R into an arbitrary R -bimodule M is inner.*

Proof. Let $d : R \rightarrow M$ be a derivation. As a finite-dimensional subspace of M , $d(R)$ generates a finite-dimensional subbimodule of M . Therefore, there is no loss of generality in assuming that M itself is finite-dimensional.

Let \tilde{A} be the set of all matrices of the form $\begin{bmatrix} x & u \\ 0 & x \end{bmatrix}$, where $x \in R$ and $u \in M$. Note that \tilde{A} is a (finite-dimensional!) algebra under the standard matrix operations. Let \tilde{R} be its subalgebra consisting of all matrices of the form $\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}$, $x \in R$. Of course, $\tilde{R} \cong R$.

Define $\varphi : \tilde{R} \rightarrow \tilde{A}$ by

$$\varphi \left(\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} \right) = \begin{bmatrix} x & d(x) \\ 0 & x \end{bmatrix}.$$

One immediately checks that φ is a homomorphism. By Theorem 6.1 there exists an invertible element $c = \begin{bmatrix} t & v \\ 0 & t \end{bmatrix} \in \tilde{A}$ such that $\varphi(\tilde{x}) = c\tilde{x}c^{-1}$ for all $\tilde{x} \in \tilde{R}$. Consequently, $\varphi(\tilde{x})c = c\tilde{x}$, that is,

$$\begin{bmatrix} x & d(x) \\ 0 & x \end{bmatrix} \cdot \begin{bmatrix} t & v \\ 0 & t \end{bmatrix} = \begin{bmatrix} t & v \\ 0 & t \end{bmatrix} \cdot \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}$$

for all $x \in R$. This yields

$$xt = tx \quad \text{and} \quad xv + d(x)t = vx$$

for all $x \in R$. Since R is central, the first identity shows that $t \in F$. Moreover, $t \neq 0$ for c is invertible. Hence $w = t^{-1}v$ satisfies $d(x) = wx - xw$ by the second identity. \square

7. DOMAINS

In this section we give classes of domains which are SN algebras. For instance, UFDs and free algebras are SN algebras (Corollaries 7.2 and 7.4).

As the coordinate ring of an elliptic curve demonstrates (see Example 7.6), not every commutative domain is an SN algebra. However, the following proposition shows that every domain S embedded into a division algebra satisfies a certain weaker condition.

Proposition 7.1. *Let R be a central simple algebra, and let an algebra S be a domain which can be embedded into a division algebra D . If φ is a homomorphism from R into $R \otimes S$, then there exists $c \in R \otimes S$ which is invertible in $R \otimes D$, in fact*

$$c^{-1} = (1 \otimes s^{-1})b \in R \otimes D$$

for some nonzero $s \in S$ and $b \in R \otimes S$, such that

$$\varphi(x) = cxc^{-1}$$

for all $x \in R$. Moreover, if $\{r_1, \dots, r_d\}$ is a basis of R , $b = \sum_{k=1}^d r_k \otimes s_k$ for some $s_k \in S$, and $c = \sum_{\ell=1}^d r_\ell \otimes t_\ell$ for some $t_\ell \in S$, then

$$t_\ell s^{-1} s_k \in S$$

for all k and ℓ .

Proof. Not every c_k from Lemma 4.1 can be 0 (in view of (c)), and so $s_{k\ell} \neq 0$ for some k and ℓ . Set $c = c_k$ and $s = s_{k\ell}$. By the lemma we have $\varphi(x)c = cx$ for all $x \in R$ and $bc = 1 \otimes s$ for some $b \in R \otimes S$. Of course, s is invertible in D . Therefore $(1 \otimes s^{-1})b$ is a left inverse of c in $R \otimes D$. By [Mon83, Theorem 1], a left inverse in $R \otimes D$ is also a right inverse, so $c^{-1} = (1 \otimes s^{-1})b$.

Now take a basis $\{r_1, \dots, r_d\}$ of R , and let us write $b = \sum_{k=1}^d r_k \otimes s_k$ and $c = \sum_{\ell=1}^d r_\ell \otimes t_\ell$. Then

$$\varphi(x) = cxc^{-1} = \sum_{k=1}^d \sum_{\ell=1}^d r_\ell x r_k \otimes t_\ell s^{-1} s_k.$$

for all $x \in R$. Pick $w_j, z_j \in R$ satisfying (2.1). We have

$$\begin{aligned} \sum_j w_j \varphi(z_j) &= \sum_j \sum_{k=1}^d \sum_{\ell=1}^d w_j r_\ell z_j r_k \otimes t_\ell s^{-1} s_k \\ &= \sum_{k=1}^d \sum_{\ell=1}^d \left(\sum_j w_j r_\ell z_j \right) r_k \otimes t_\ell s^{-1} s_k \\ &= \sum_{k=1}^d r_k \otimes t_1 s^{-1} s_k. \end{aligned}$$

Since the left hand side, i.e. $\sum_j w_j \varphi(z_j)$, lies in $R \otimes S$, so does the right hand side. This readily yields that $t_1 s^{-1} s_k \in S$. \square

Corollary 7.2. *Every UFD is an SN algebra.*

Proof. Let S be a UFD and let $R, \varphi, b, c, s, s_k, t_\ell$ be as in Proposition 7.1. Since S is a UFD, t_1, \dots, t_d have a greatest common divisor e and c can be replaced with $(1 \otimes e^{-1})c$, we can without loss of generality assume that t_1, \dots, t_d are coprime. Then it suffices to prove that $s^{-1} s_k \in S$ for every k . Since $t_\ell s^{-1} s_k \in S$ for every k, ℓ , we see that s divides $t_\ell s_k$ for every ℓ, k . Let p be a prime such that p^n divides s . Suppose that p^n does not divide s_{k_0} for some k_0 . Since p^n divides $t_\ell s_{k_0}$ for every ℓ , we conclude that p divides t_ℓ for every ℓ , which contradicts the assumption about t_ℓ being coprime. Hence s divides s_k for every k . \square

We now move to the noncommutative setting. Since embeddings of noncommutative domains into division rings can be ill-behaved or nonexistent, one needs stronger assumptions than in Corollary 7.2. Let S be an arbitrary ring. The **inner rank** of $A \in S^{m \times n}$ is the least r such that $A = BC$ for some $B \in S^{m \times r}$ and $C \in S^{r \times n}$. We write $\rho A = r$. For example, if S is a division ring, then ρA is just the rank of A . We say that S is a **Sylvester domain** [Coh06, Section 5.5] if for any $P \in S^{\ell \times m}$ and $Q \in S^{m \times n}$ such that $PQ = 0$, it follows that $\rho P + \rho Q \leq m$.

We say that an element $s \in S$ **right divides** $a \in S$ if $a = a's$ for some $a' \in S$. If S is a domain and $a, b \in S \setminus \{0\}$, then s is a **highest common right factor** (HCRF) of a and b if s right divides a, b and every $s' \in S$ that right divides a, b also right divides s . We say that S is an **HCRF domain** if every pair of nonzero elements in S admits a HCRF. Special examples of HCRF domains are filtered rings satisfying the 2-term weak algorithm [Coh06, Section 2.8] or more generally, 2-firs with right ACC₁ (ascending chain condition on principal right ideals) [Coh06, Exercise 3.2.2].

Theorem 7.3. *If S is an HCRF domain and a Sylvester domain, then S is an SN algebra.*

Proof. Since S is a Sylvester domain, it admits a universal skew field of fractions D and this embedding preserves the inner rank by [Coh06, Theorem 7.5.13]. Let R be a central simple algebra and $\varphi : R \rightarrow R \otimes S$ a homomorphism. By Proposition 7.1 there exists $c \in R \otimes S$ invertible in $R \otimes D$ such that $\varphi(x) = cxc^{-1}$ for all $x \in R$. Furthermore, if $\{r_1, \dots, r_d\}$ is a basis of R and

$$c = \sum_{\ell} r_{\ell} \otimes t_{\ell}, \quad c^{-1} = \sum_k r_k \otimes u_k$$

for $t_{\ell} \in S$ and $u_k \in D$, then $t_{\ell}u_k = s_{\ell k} \in S$ for all $1 \leq \ell, k \leq d$ (here $u_k = s^{-1}s_k$ from Proposition 7.1). Since S is an HCRF domain, we can assume that t_1, \dots, t_d have no non-trivial common right factors (otherwise they have a nontrivial HCRF e and we can replace c with $c(1 \otimes e^{-1})$). Fix k such that $u_k \neq 0$. Then $(u_k, -1)^t \in D^2$ belongs to the right kernel of the matrix

$$\begin{pmatrix} t_1 & s_{1k} \\ \vdots & \vdots \\ t_d & s_{dk} \end{pmatrix} \in S^{d \times 2}$$

which is therefore of (inner) rank 1 over D . Since the embedding $S \subseteq D$ is inner rank preserving, this matrix is also of inner rank 1 over S , so

$$\begin{pmatrix} t_1 & s_{1k} \\ \vdots & \vdots \\ t_d & s_{dk} \end{pmatrix} = \begin{pmatrix} v_1 \\ \vdots \\ v_d \end{pmatrix} (w_1 \ w_2)$$

for some $v_i, w_j \in S$. Since w_1 right divides t_{ℓ} for every ℓ and t_1, \dots, t_d are right coprime by the assumption, we conclude that w_1 is invertible in S . By taking some $t_{\ell} \neq 0$ we get

$$u_k = t_{\ell}^{-1}s_{\ell k} = w_1^{-1}v_{\ell}^{-1}v_{\ell}w_2 = w_1^{-1}w_2 \in S.$$

Consequently $c^{-1} \in R \otimes S$. □

Corollary 7.4. *Every free algebra $F\langle X \rangle$ is an SN algebra.*

Proof. A free algebra is a filtered ring with a weak algorithm [Coh06, Theorem 2.5.3], so it is a HCRF domain and a fir (free ideal ring) by [Coh06, Theorem 2.4.6] and hence a Sylvester domain by [Coh06, Proposition 5.5.1]. □

Theorem 7.3 has the following form for commutative rings.

Corollary 7.5. *Every Bézout domain is an SN algebra.*

Proof. Every Bézout domain is a GCD domain, which is just a commutative HCRF domain. Moreover, by [Coh06, Proposition 2.3.17] it is also a semifir and hence a Sylvester domain by [Coh06, Proposition 5.5.1]. Therefore Theorem 7.3 applies. □

In the next example we present a domain that is not an SN algebra; cf. [RZ61, Theorem 15].

Example 7.6. Let $S = F[x, y]/(y^2 - x^3 - x)$. Then S is a domain,

$$a = \begin{pmatrix} y & x \\ x^2 & y \end{pmatrix} \in M_2(S)$$

is invertible as a matrix over the field of fractions of S and

$$a^{-1} = \begin{pmatrix} \frac{y}{x} & -1 \\ -x & \frac{y}{x} \end{pmatrix}.$$

Since every product of an entry in a and an entry in a^{-1} lies in S , it follows that

$$\varphi: M_2(F) \rightarrow M_2(F) \otimes S, \quad u \mapsto au a^{-1}$$

is a well-defined homomorphism. Suppose that S is an SN algebra. Then there exists an invertible $c \in M_2(S)$ such that $\varphi(u) = cuc^{-1}$ for all $u \in M_2(F)$. Then $\gamma = \det(c)$ is invertible in S and it is easy to see that this implies $\gamma \in F \setminus \{0\}$. Since $c^{-1}a$ commutes with every $u \in M_2(F)$ by the definition of φ , we have $c^{-1}a = fI_2$ for some $f \in S$. But then

$$\gamma f^2 = \gamma \det(c^{-1}a) = \det(a) = x$$

contradicts the irreducibility of x in S .

8. POLYNOMIAL MATRIX ALGEBRAS

In this section we prove that matrix algebras over polynomial algebras are SN.

Theorem 8.1. $M_n(F[\xi_1, \dots, \xi_s])$ is an SN algebra.

Besides the Quillen-Suslin theorem, saying that over $F[\xi_1, \dots, \xi_s]$ every finitely generated projective module is free, see [Qui76, Su76], the proof of this theorem is mostly based on the following simple factorization lemma.

Lemma 8.2. *Let A be a commutative algebra, which is a domain with field of fractions K , R a central simple algebra and $a \in R \otimes M_n(A)$. Suppose that a is invertible in $R \otimes M_n(K)$ and that $axa^{-1} \in R \otimes M_n(A)$ for all $x \in R$. Then for $c \in M_n(A)$ the following are equivalent:*

- (i) *There exists a factorization $a = u(1 \otimes c)$ for some invertible $u \in R \otimes M_n(A)$;*
- (ii) *The left ideal*

$$I(a) := \{ m \in M_n(A) \mid (1 \otimes m)a^{-1} \in R \otimes M_n(A) \} \subseteq M_n(A)$$

is generated by c ;

- (iii) *The rows of c form a basis of the A -module*

$$M(a) := \{ r \in A^{1 \times n} \mid (1 \otimes r)a^{-1} \in R \otimes A^{1 \times n} \}.$$

Proof. (i) \Rightarrow (ii): Let $u \in R \otimes M_n(A)$ be invertible such that $a = u(1 \otimes c)$. Then for any $m \in M_n(A)$ we have that $(1 \otimes m)a^{-1} = (1 \otimes mc^{-1})u^{-1}$ lies in $R \otimes M_n(A)$ if and only if $mc^{-1} \in M_n(A)$, i.e., $m \in M_n(A)c$.

(ii) \Rightarrow (i): Let $\{r_1, \dots, r_d\}$ be a basis of R and for fixed i take $w_k, z_k \in R$ such that $\sum_k w_k r_j z_k = \delta_{ij}$, see (2.1) at the beginning of Section 2. If $a = \sum_j r_j \otimes a_j$, then

$$(1 \otimes a_i)a^{-1} = \sum_k (w_k \otimes 1)a(z_k \otimes 1)a^{-1} \in R \otimes M_n(A).$$

Since i was arbitrary, this shows that all coefficients a_i of a lie in $I(a)$. Assuming that $I(a) = M_n(A)c$, we can thus factor $a = u(1 \otimes c)$ for some $u \in R \otimes M_n(A)$. But then $u^{-1} = (1 \otimes c)a^{-1} \in M_n(A)$, i.e., u is invertible.

(ii) \Rightarrow (iii): Suppose $I(a) = M_n(A)c$. Since a is invertible over K , there exists a nonzero $e \in A$ such that $e \cdot 1 \in I(a)$. Hence the rows of c are clearly linearly independent. Given any $r \in M(a)$, we can extend r by zero to form a matrix $m \in I(a)$ which has r as one of its rows. By the assumption $m \in M_n(A)c$. In particular, r is a linear combination of the rows of c , which shows that they form a basis of $M(a)$.

(iii) \Rightarrow (ii): Conversely, suppose that the rows of c form a basis of $M(a)$. In particular, $c \in I(a)$. Moreover, for any $m \in I(a)$ the rows of m lie in $M(a)$ and are, therefore, linear combinations of the rows of c , which implies that $m \in M_n(A)c$. \square

Proof of Theorem 8.1. Let $A := F[\xi_1, \dots, \xi_s]$, K its field of fractions, R a central simple algebra and $\varphi: R \rightarrow R \otimes M_n(A)$ a homomorphism. Since $M_n(K)$ is SN, there exists (after clearing denominators) $a \in R \otimes M_n(A)$, invertible in $R \otimes M_n(K)$, such that $\varphi(x) = axa^{-1}$ for all $x \in R$.

Fix any prime ideal P of A . Then $M_n(A_P)/\text{rad}(M_n(A_P))$ is canonically isomorphic to the simple algebra $M_n(F) \otimes A_P/PA_P$, see Lemma 5.3. Therefore, $M_n(A_P)$ is semilocal and by Theorem 5.5 it is also an SN algebra.

It follows that there exists an invertible $u_P \in R \otimes M_n(A_P)$ such that $\varphi(x) = u_P x u_P^{-1}$. Then $u_P^{-1}a$ commutes with all elements of R and thus lies in $1 \otimes M_n(A)$. This means, a can be factored as $a = u_P(1 \otimes c_P)$ for some $c_P \in M_n(A)$. By Lemma 8.2, this implies that the A_P -module $A_P M(a)$ is free of rank n . Since the prime ideal P was arbitrary, this shows that $M(a)$ is locally free of rank n . As being projective is a local property, this implies $M(a)$ is projective. By the Quillen-Suslin theorem $M(a)$ is free of rank n . We choose $c \in M_n(A)$ such that its rows form a basis of $M(a)$. Then again from Lemma 8.2, we get a factorization $a = u(1 \otimes c)$ where $u \in R \otimes M_n(A)$ is invertible. Now $\varphi(x) = u x u^{-1}$ for all $x \in R$. \square

9. FORMAL POWER SERIES

The aim of this section is to show that having (or not having) the SN property is invariant under extension to formal power series.

Theorem 9.1. *S is an SN algebra if and only if $S[[\xi]]$ is an SN algebra.*

Proof. (\Rightarrow) Let R be a central simple algebra and let $\varphi : R \rightarrow R \otimes S[[\xi]]$ be a homomorphism. Since R is finite-dimensional, we can identify $R \otimes S[[\xi]]$ with $(R \otimes S)[[\xi]]$ and write

$$\varphi(x) = \varphi_0(x) + \varphi_1(x)\xi + \varphi_2(x)\xi^2 + \dots$$

where $\varphi_i : R \rightarrow R \otimes S$. Note that φ_0 is an algebra homomorphism. By the assumption, there exists an invertible element $a \in R \otimes S$ such that $\varphi_0(x) = axa^{-1}$ for all $x \in R$. Considering the map $x \mapsto a^{-1}\varphi(x)a$ we see that without loss of generality we may assume that $\varphi_0(x) = x$ for all $x \in R$, so that

$$(9.1) \quad \varphi(x) = x + \varphi_1(x)\xi + \varphi_2(x)\xi^2 + \dots$$

Now apply Lemma 4.1. Thus, let $\{r_1, \dots, r_d\}$ be a basis of R and let $c_1, \dots, c_d \in R \otimes S[[\xi]]$ be such that

$$(9.2) \quad \sum_{k=1}^d c_k r_k = 1 \quad \text{and} \quad \varphi(x)c_k = c_k x$$

for all $x \in R$ and all k . Writing

$$c_k = \sum_{j=0}^{\infty} c_{kj} \xi^j,$$

where $c_{kj} \in R \otimes S$, it follows from (9.1) and (9.2) that

$$(9.3) \quad x c_{k0} = c_{k0} x$$

for all $x \in R$. Let us write $c_{k0} = \sum_j p_{kj} \otimes s_{kj}$ with the s_{kj} -s linearly independent. From (9.3) we infer that

$$\sum_j (x p_{kj} - p_{kj} x) \otimes s_{kj} = 0$$

for all $x \in R$, yielding $x p_{kj} - p_{kj} x = 0$. Since R is central this means that each p_{kj} is a scalar multiple of 1. Accordingly, each c_{k0} is of the form $1 \otimes t_k$ for some $t_k \in S$. From the first identity in (9.2) one easily deduces that $\sum_{k=1}^d r_k \otimes t_k = 1 \otimes 1$. Writing $1 = \sum_{k=1}^d \lambda_k r_k$, where $\lambda_k \in F$, it follows that $t_k = \lambda_k 1$. We may assume that $\lambda_1 \neq 0$. Accordingly, c_{10} is a nonzero scalar multiple of unity of $R \otimes S$, implying that c_1 is invertible in $(R \otimes S)[[\xi]]$. Applying (9.2) we arrive at $\varphi(x) = c_1 x c_1^{-1}$ for all $x \in R$.

(\Leftarrow) Straightforward; more generally, the SN property is clearly preserved by retractions. Here an algebra S' is a retract of S if $S' \subset S$ and there exists a homomorphism $\pi : S \rightarrow S'$ that restricts to the identity map on S' . \square

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CONTENTS

1. Introduction	1
Main results and guide to the paper	2
2. Preliminaries	3
3. SN algebras and automorphisms	4
4. Basic lemma	6
5. Semilocal algebras	8
6. Finite-dimensional algebras	11
7. Domains	13
8. Polynomial matrix algebras	16
9. Formal power series	17
References	18