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ML invariant of the symmetric algebra of a projective module over a one dimensional affine domain

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ABSTRACT

Let k be an algebraically closed field and let A be an affine domain of dimension one over k. Let P be a finitely generated projective A-module of rank d and let $R = \mathrm{Sym}_A(P)$ be the symmetric algebra of P over A. Assume that A is not a polynomial algebra over k. In this article we show that, under these assumptions, the Makar-Limanov invariant $\mathrm{ML}(R) = A$.

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1. Introduction

Let k be an algebraically closed field and let A be an affine domain of dimension one over k. Let B be a k-domain. Assume that A is not a polynomial algebra over k. In this set up, Crachiola and Makar-Limanov proved that the Makar-Limanov invariant $\mathrm{ML}(A \otimes_k B) = \mathrm{ML}(A) \otimes_k \mathrm{ML}(B)$ (see [3]). As a consequence, $\mathrm{ML}(A^{[d]}) = A$ where $A^{[d]}$ denotes a polynomial algebra in d variables over A. A word of caution: in their paper the authors have used notation AK in place of ML.

A crucial step in their proof is to show that A is a k-subalgebra of $\mathrm{ML}(A \otimes_k B)$ using the fact that $A \otimes_k B$ is faithfully flat over A. Now it can be easily shown that $\mathrm{ML}(B)$ is a subring of $\mathrm{ML}(A \otimes_k B)$ once we notice that, as k is algebraically closed, for every maximal ideal m of A, the k-algebra homomorphism $B \to A/m \otimes_k B$ is an isomorphism.

In this paper we first prove the following generalization (of their crucial step):

Proposition. *Let k be an algebraically closed field and let A be an affine domain of dimension one over k such that $A \neq k^{[1]}$. Let R be an overdomain of A which is flat over A. Then $A \subset \mathrm{ML}(R)$.*

We give an example to show that the above proposition is not always true if R is not flat over A.

We have the following result (Theorem 3.4) as a consequence of the above proposition.

Theorem. *Let k be an algebraically closed field and let A be an affine domain of dimension one over k such that $A \neq k^{[1]}$. Let P be a finitely generated projective A-module of rank d and let $R = \mathrm{Sym}_A(P)$ be the symmetric algebra of P over A. Then $\mathrm{ML}(R) = A$.*

Note that, for an affine domain A of dimension one over an algebraically closed field k, $\mathrm{ML}(A) = A$ if and only if $A \neq k^{[1]}$.

An example due to Dubouloz shows that the above theorem is not always valid in the case $\dim(A) = 2$ even if A is smooth over k and $\mathrm{ML}(A) = A$ (see [5]).

We conclude the introduction by stating the following corollary which is a nice offshoot of the above theorem (Corollary 3.5):

Corollary. *Let k be an algebraically closed field of characteristic 0 and let A be an affine normal domain of dimension one over k such that $A \neq k^{[1]}$. Let P be a finitely generated projective A-module of rank $d \leq 3$ and let $R = \mathrm{Sym}_A(P)$ be the symmetric algebra of P over A. Let D be a non-zero locally nilpotent derivation on R then $Ker(D)$ is affine over k.*

2. Preliminaries

Throughout the paper we will assume the rings to be commutative. A polynomial algebra in d variables over a ring S will be denoted by $S^{[d]}$.

In this section we collect some definitions and results for later use. We begin with definitions of an exponential map and some related concepts.

Definition 2.1. Let R be a domain. A locally finite iterative higher derivation on R is a sequence $d_i : R \to R$ $(0 \leq i < \infty)$ of maps satisfying following properties.

1. $d_0 = id$.
2. $d_n(a+b) = d_n(a) + d_n(b) \qquad a, b \in R$.
3. $d_n(a.b) = \sum_{i=0}^{n} d_i(a) d_{n-i}(b)$.
4. $d_i d_j = \binom{i+j}{i} d_{i+j}$.
5. For $b \in R \ \exists \ \ n_0 \in \mathbb{N}$ (depending on b) such that $d_n(b) = 0$ for $\forall\ n \geq n_0$.

Definition 2.2. Let R be a domain and $d_i : R \to R$ $(0 \leq i < \infty)$ be a locally finite iterative higher derivation. A ring morphism $\phi : R \to R[T]$ defined by

$$\phi(b) = \sum d_i(b) T^i$$

is called the *exponential map* (associated to a sequence d_i $(0 \leq i < \infty)$).

Definition 2.3. Let R be a domain and let $\phi : R \to R[T]$ be the exponential map associated to a locally finite iterative higher derivation d_i $(0 \leq i < \infty)$. Let $R^\phi = \{a \mid a \in R, \phi(a) = a\}$ (equivalently $R^\phi = \{a \mid \ d_i(a) = 0 \ \forall\ i \geq 1\}$). Then R^ϕ is called the ring of ϕ-invariants. We say ϕ is non-trivial if $R^\phi \neq R$.

Remark 2.4. Let R, ϕ, R^ϕ be as above. Since $R^\phi = \phi^{-1}(R)$ it is easy to see that R^ϕ is factorially and hence algebraically closed in R.

Definition 2.5. Let R be a domain and let $EXP(R)$ denote the set of all exponential maps on R. We define the Makar-Limanov invariant $\mathrm{ML}(R)$ as $\mathrm{ML}(R) = \cap_{\phi \in EXP(R)} R^\phi$ where R^ϕ is the ring of ϕ-invariants.

Remark 2.6. Note that $ML(R) = R$ if and only if R does not admit any non-trivial exponential map.

The following result is proved in [4].

Lemma 2.7. *Let ϕ be a non-trivial exponential map on a domain R and let $C = R^\phi$ be the ring of ϕ-invariants. Let L be the field of fractions of C. Then $R \otimes_C L = L^{[1]}$.*

The following result is known to experts (for example see [3]). For the sake of completeness we present a proof.

Lemma 2.8. *Let A be an affine domain of dimension 1 over an algebraically closed field k. Let L be a field extension of k such that A is a k-subalgebra of $L[T]$ ($L^{[1]}$). If $A \not\subset L$ then the normalization $\bar{A}$ of A is $k^{[1]}$.*

Proof. Since A is affine over k, it is easy to see that there exists an affine k-domain E such that $E \subset L$ and $A \subset E[T]$. Since $A \not\subset L$, $A \not\subset E$ and hence A contains an element say $g(T) \in E[T]$ such that $\deg_T(g(T)) = n > 0$. Let e be the leading coefficient of $g(T)$. As E is an affine domain over an algebraically closed field k, there exists a maximal ideal m of E such that $e \notin m$. Therefore the image of $g(T)$ in $E/m[T] = k[T]$ is a polynomial in T of degree $n > 0$. Hence, as A is an affine domain of dimension one over k, the induced map $A \to E/m[T] = k[T]$ is injective. Therefore, by [1], its normalization $\bar{A}$ is $k^{[1]}$. □

Corollary 2.9. *Let A be an affine domain of dimension one over an algebraically closed field k such that its normalization $\bar{A} \neq k^{[1]}$. Let R be an overdomain of A then $A \subseteq ML(R)$.*

Proof. If $\mathrm{ML}(R) = R$ then there is nothing to prove. So we assume that R admits non-trivial exponential maps.

Let ϕ be a non-trivial exponential map on R and let $C = R^{\phi}$ be the ring of ϕ-invariants. We want to show that $A \subset C$.

Let L be the field of fractions of C. Then, $A \subset R \subset R \otimes_C L = L^{[1]}$. Since $\bar{A} \neq k^{[1]}$, by Lemma 2.8, $A \subset L \cap R = C$. □

3. Main results

In this section we prove results mentioned in the introduction.

Let $R \hookrightarrow S$ be an extension of rings. Recall that $I = \mathrm{Ann}_R(S/R)$ is called the conductor ideal I of S over R i.e. I is the largest ideal of S such that $I \subseteq R$.

Let $A \hookrightarrow B$ be an extension of domains. Let K denote the field of fractions of A. If $A \subset B \subset K$ (i.e. K is also the field of fractions of B) we say that $A \hookrightarrow B$ is a birational extension. Moreover if B is a finite A-module then we say that the extension is finite.

Remark 3.1. Let $A \hookrightarrow B$ be an extension of domains and let I be the conductor ideal of B over A. Assume that A is Noetherian. Then $I \neq 0$ if and only if $A \hookrightarrow B$ is a finite birational extension.

Following lemma tells us about behavior of a conductor ideal under flat extensions.

Lemma 3.2. *Let A be a Noetherian domain and $A \hookrightarrow B$ be a proper finite birational extension (i.e. $A \neq B$). Let I be the conductor ideal of B over A. Let $A \hookrightarrow R$ be an extension of domains such that R is a flat A-module. Then $R \hookrightarrow R \otimes_A B$ is a finite birational extension and RI is the conductor ideal of $S = R \otimes_A B$ over R.*

Proof. Let K be the field of fractions of A. Then $R \hookrightarrow R \otimes_A K$ is a birational extension. Moreover, since R is flat over A, we have

$$R \hookrightarrow R \otimes_A B = S \hookrightarrow R \otimes_A K.$$

Therefore S is a finite birational extension of R as B is a finite extension of A.

Now consider the following exact sequence of A-modules

$$0 \to A \to B \to B/A \to 0.$$

Since R is a flat A-module, we get the following exact sequence of R-modules

$$0 \to R \to R \otimes_A B(= S) \to R \otimes_A B/A \to 0.$$

Let J denote the conductor ideal of S over R. Our aim is to prove that $RI = J$.

Note that $RI \subseteq J$. Indeed, if $a \in I$ then $Sa = (R \otimes_A B)a \subseteq R$ as $Ba \subseteq A$. Now it remains to prove that $J \subseteq RI$. If $RI = R$ then there is nothing to prove. So we assume that RI is a proper ideal of R.

Let $M = B/A$. As B is a finite A-module, M is a finitely generated module over A. Let $\{m_1, m_2, ..., m_n\}$ be generators of M.

Let N denote a direct sum of n copies of M. Let $x = (m_1, \cdots, m_n) \in N$. Then, since $I = \mathrm{Ann}_A(M)$ and $M = \sum_{1 \le i \le n} Am_i$, we get that $\mathrm{Ann}_A Ax = I$. Hence, as $Ax \subset N$, we get an injective map $A/I \hookrightarrow N$ of A-modules. Therefore, as R is flat over A, we get an injective map $R/RI = R \otimes_A A/I \hookrightarrow R \otimes_A N$ of R-modules. As a consequence, we get that $\mathrm{Ann}_R(R \otimes_A N) \subset RI$. Since N is a direct sum of n-copies of M, we have $\mathrm{Ann}_R(R \otimes_A N) = \mathrm{Ann}_R(R \otimes_A M) = J$. Thus $J \subseteq RI$. □

As a consequence we have the following result.

Proposition 3.3. *Let k be an algebraically closed field and let A be an affine domain of dimension one over k. Let R be an overdomain of A which is flat over A. Assume that $A \neq k^{[1]}$. Then $A \subset \mathrm{ML}(R)$.*

Proof. Let $\bar{A}$ denote the normalization of A. If $\bar{A} \neq k^{[1]}$ then the result follows in view of Corollary 2.9. So we assume that $\bar{A} = k^{[1]}$.

Note that by our assumption, $A \hookrightarrow \bar{A}$ is a proper finite birational extension. Hence the conductor ideal I of $\bar{A}$ over A is a proper non-zero ideal of $\bar{A}$. Since $\bar{A} = k^{[1]}$, $I = f\bar{A}$, $f \notin k$.

If $\mathrm{ML}(R) = R$, then there is nothing to prove. So assume that R has non-trivial exponential maps. Let ϕ be a non-trivial exponential map on R and let $C = R^\phi$ be the ring of ϕ-invariants. We want to show that $A \subset C$.

Let $S = R \otimes_A \bar{A}$. Then, as R is a flat A-module, by Lemma 3.2, S is a finite birational extension of R and RI is the conductor ideal of S over R.

Let L be the field of fractions of C. Then $L^{[1]} = R \otimes_C L$ is an integrally closed domain. Hence, as $S \otimes_C L$ is a finite birational extension of $R \otimes_C L$, we see that $R \otimes_C L = S \otimes_C L$. This equality implies that $RI \cap C$ is a non-zero ideal of C.

Note that $I = f\bar{A}$. Therefore

$$I^2 = f^2\bar{A} = f(f\bar{A}) = fI \subset fA.$$

Let $u \in RI \cap C$ be a non-zero element. Then, as $RI^2 \subset fR$, we see that $u^2 \in C$ is a multiple of f in R. Therefore, as C is factorially closed in R, we get that $f \in A \cap C$. Since $\dim(A) = 1$, $f \notin k$ and $A \cap C$ is algebraically closed in A, we get that $A \cap C = A$.

Thus $A \subset C$. □

The following generalization of a result of Crachiola and Makar-Limanov [3] is an easy consequence of the above proposition.

Theorem 3.4. *Let k be an algebraically closed field and let A be an affine domain of dimension one over k such that $A \neq k^{[1]}$. Let P be a finitely generated projective A-module of rank d and let $R = \mathrm{Sym}_A(P)$ be the symmetric algebra of P over A. Then $\mathrm{ML}(R) = A$.*

Proof. Let K denote the field of fractions of A. Then $R \otimes_A K = K^{[d]}$ and hence $\mathrm{ML}(R \otimes_A K) = K$. Since R is a finitely generated A-algebra, it is easy to see that every exponential map on $R \otimes_A K$ is an extension of an exponential map on R and hence $\mathrm{ML}(R) \subset \mathrm{ML}(R \otimes_A K) \cap R = K \cap R = A$. By the above proposition, $A \subset \mathrm{ML}(R)$. Thus $\mathrm{ML}(R) = A$. □

Corollary 3.5. *Let k be an algebraically closed field of characteristic 0 and let A be an affine normal domain of dimension one over k such that $A \neq k^{[1]}$. Let P be a finitely generated projective A-module of rank $d \leq 3$ and let $R = \mathrm{Sym}_A(P)$ be the symmetric algebra of P over A. Let D be a non-zero locally nilpotent derivation on R then $Ker(D)$ is affine over k.*

Proof. As $\mathrm{char.}(k) = 0$, giving a locally nilpotent derivation D on R is equivalent to defining an exponential map ϕ on R such that $R^\phi = \ker(D)$. Therefore, as $A \neq k^{[1]}$, by Theorem 3.4, a locally nilpotent derivation D on R is A-derivation. Hence if m is maximal ideal of A then D induces a locally nilpotent derivation D_m on $R \otimes_A A_m = {A_m}^{[d]}$ such that $\ker(D)_m = \ker(D_m)$. Since A is normal, A_m is a DVR. Therefore, as $\mathrm{rank}(P) = d \leq 3$, by [2, Theorem 4.15], $\ker(D_m)$ is a finitely generated A_m-algebra. Now we are through in view of [2, Lemma 2.2]. □

Remark 3.6. If A is not normal then there exists a locally nilpotent derivation D on $A^{[2]}$ such that $\ker(D)$ is not affine over k (see [2, Lemma 3.1]).

The following general form of the cancellation theorem of [1] is a simple consequence of Theorem 3.4.

Theorem 3.7. *Let A, B be an affine domain of dimension one over an algebraically closed field k. Let P, Q be projective modules of rank d over A, B respectively. If $Sym_A(P) \simeq Sym_B(Q)$ then $A \simeq B$.*

Now we conclude this paper by exhibiting an example which shows that the Proposition 3.3 is not always true if R is not flat over A. Note that, in view of Corollary 2.9, in this example the normalization $\bar{A}$ of A is necessarily $k^{[1]}$.

Example 3.8. Let A be an affine domain of dimension one over an algebraically closed field k of characteristic zero such that:

1. A is not normal.
2. Its normalization $\bar{A} = k[X]$.

Let $S = k[X, Y]$ $(= k^{[2]})$ and let $R = A + YS \subset S$.

Claim. $\mathrm{ML}(R) = k$.

Proof of the claim. Since k is a field of characteristic zero, giving an exponential map ϕ on R is equivalent to giving a locally nilpotent derivation D on R such that $R^{\phi} = \ker(D)$.

Let D_1 be a locally nilpotent derivation on S given by $D_1(X) = Y, D_1(Y) = 0$. Then $\ker(D_1) = k[Y] \subset R$. Since $D_1(S) \subset YS \subset R$ it is easy to see that $D_1(R) \subset R$ and hence D_1 induces a locally nilpotent derivation say $\bar{D}_1$ on R such that $k[Y] = \ker(\bar{D}_1)$.

Let I denote the conductor ideal of $\bar{A} = k[X]$ over A. Then $I = fk[X]$. Note that $fS \subset R = A + YS$.

Let D_2 be a locally nilpotent derivation on S given by $D_2(X) = 0, D_2(Y) = f$. Then $\ker(D_2) = k[X]$. Since $D_2(S) \subset fS \subset R$ we see that D_2 induces a locally nilpotent derivation say $\bar{D}_2$ on R such that $\ker(\bar{D}_2) = R \cap \ker(D_2) = A$.

Therefore

$$k \subset \mathrm{ML}(R) \subset \ker(\bar{D}_1) \cap \ker(\bar{D}_2) \subset k[Y] \cap k[X] = k.$$

Thus the claim is proved. □

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