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Depth of ideals of $C(X)$

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ABSTRACT

We show that every regular sequence in $C(X)$ has length ≤ 1. This shows that depth$(C(X)) \leq 1$. We also show that the depth of each maximal ideal of $C(X)$ is either zero or one. In fact we observe that X is an almost P-space if and only if the depth of each maximal ideal of $C(X)$ is zero and X contains at least one non-almost P-point if and only if depth$(M) = 1$ for each maximal ideal M of $C(X)$. Using this it turns out that for a given topological space X, there are no maximal ideals in $C(X)$ with different depths. Regular sequences are also investigated in the factor rings of $C(X)$ and we observe that such sequences in the factor rings of $C(X)$ modulo principal z-ideals have also length ≤ 1. Finally, we obtain some topological conditions for which the depth of factor rings of $C(X)$ modulo some closed ideals are zero.

1. Introduction

In this article, all rings are assumed to be commutative rings with identity, ideals are proper and all topological spaces are considered to be completely regular Hausdorff

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(Tychonoff) spaces. We denote by $C(X)$ the ring of all real-valued continuous functions on a space X and $C^*(X)$ is the subring of $C(X)$ consisting of bounded functions. For each $f \in C(X)$ the *zero-set* $Z(f)$ is the set of zeros of f and its complement $\operatorname{coz} f$, is called the *cozero-set* of f. It is well known that a Hausdorff space X is completely regular if and only if the set of all zero-sets is a base for closed subsets of X, or equivalently the set of all cozero-sets is a base for open subsets of X; see Theorem 3.2 in [6].

We recall that a commutative ring is a reduced ring if it does not contain any nonzero nilpotents. It is well known that a prime ideal P in a reduced ring is minimal if and only if for each $a \in P$ there exists $b \notin P$ such that $ab = 0$. The set of all minimal prime (resp., maximal) ideals of a ring R is denoted by $Min(R)$ (resp., $Max(R)$). An ideal I in a commutative ring is called a *z-ideal* if $M_a \subseteq I$ for each $a \in I$, where M_a is the intersection of all maximal ideals of the ring containing a. It is easy to see that an ideal I in $C(X)$ is a z-ideal if and only if whenever $f \in I$, $g \in C(X)$ and $Z(f) \subseteq Z(g)$, then $g \in I$; see Problem 4A in [6]. For each ideal I of $C(X)$, the ideal $I_z = \{g \in C(X) : Z(f) = Z(g) \text{ for some } f \in I\}$ is a z-ideal and it is in fact the smallest z-ideal containing I; see [6] and [9]. In particular, for each $f \in C(X)$, $M_f = \{g \in C(X) : Z(f) \subseteq Z(g)\}$ is the smallest z-ideal containing f. Similarly, an ideal I in a commutative ring is called a *z°-ideal* if $P_a \subseteq I$ for each $a \in I$, where P_a is the intersection of all minimal prime ideals of the ring containing a. Whenever the ring is reduced, the ideal P_a is clearly the smallest z°-ideal containing a and $P_a = \{b \in R : \operatorname{Ann}(a) \subseteq \operatorname{Ann}(b)\}$; see Proposition 1.5 in [2]. Since for each $f, g \in C(X)$, $\operatorname{Ann}(f) \subseteq \operatorname{Ann}(g)$ is equivalent to $\operatorname{int}_X Z(f) \subseteq \operatorname{int}_X Z(g)$, we have $P_f = \{g \in C(X) : \operatorname{int}_X Z(f) \subseteq \operatorname{int}_X Z(g)\}$. Using this, an ideal I in $C(X)$ is a z°-ideal if and only if whenever $f \in I$, $g \in C(X)$ and $\operatorname{int}_X Z(f) \subseteq \operatorname{int}_X Z(g)$, then $g \in I$; see [2] for more details of z°-ideals.

An ideal I in $C(X)$ is called *fixed* if $\bigcap Z[I] := \bigcap_{f\in I} Z(f) \neq \emptyset$, otherwise it is called *free*. A nonzero ideal in a ring is said to be *essential* if it intersects every nonzero ideal non-trivially. The following topological characterization of essential ideals in $C(X)$ is given in [1].

Lemma 1.1. *An ideal E in $C(X)$ is essential if and only if* $\operatorname{int}_X \bigcap Z[E] = \emptyset$.

The spaces βX and υX are the *Stone-Čech compactification* and the *Hewitt realcompactification* of X, respectively. For any $p \in \beta X$, M^p (resp., O^p) is the set of all $f \in C(X)$ for which $p \in \operatorname{cl}_{\beta X} Z(f)$ (resp., $p \in \operatorname{int}_{\beta X} \operatorname{cl}_{\beta X} Z(f)$). For each $p \in \beta X$, M^p is a maximal ideal of $C(X)$ and also, every maximal ideal of $C(X)$ is precisely of the form M^p, for some $p \in \beta X$. The maximal ideal M^p (resp., O^p) is free or fixed according as $p \in \beta X \setminus X$ or $p \in X$; and in the latter case, M^p (resp., O^p) is denoted by M_p (resp., O_p); see § 7 in [6]. Clearly, $O^p \subseteq M^p$, for each $p \in \beta X$. Moreover, every prime ideal of $C(X)$ is contained in a unique maximal ideal M^p for some $p \in \beta X$ and contains the unique O^p; see Theorems 2.11 and 7.13 in [6]. Using this, it is evident that for each $p \in \beta X$, we have $O^p = \bigcap_{P\in Min(O^p)} P$, where $Min(O^p)$ is the set of all prime ideals in $C(X)$ minimal over O^p. Note that $Min(O^p)$ is exactly the set of all minimal prime ideals of $C(X)$ contained in M^p. For every $f \in C^*(X)$ the unique extension of f to a function in $C(\beta X)$ is denoted by f^β and for undefined terms and notations we refer the reader to [5] and [6].

We denote by $U(R)$ (resp., $U(X)$) the set of all unit (invertible) elements of a ring R (resp., $C(X)$). The set of all non-zerodivisor (regular) elements of a ring R (resp., $C(X)$) is denoted by $r(R)$ (resp., $r(X)$). The following lemma gives a well known topological characterization of units and also non-zerodivisors of $C(X)$.

Lemma 1.2. *The following statements hold.*

(1) *An element $f \in C(X)$ is a unit if and only if $Z(f) = \emptyset$.*
(2) *An element $f \in C(X)$ is a regular (non-zerodivisor) if and only if $\mathrm{int}_X Z(f) = \emptyset$, or equivalently $\mathrm{coz} f$ is dense in X.*

Recall that a space X is a *P-space* if every G_δ-set or equivalently every zero-set in X is open, and it is an *almost P-space* if every nonempty G_δ-set or equivalently every nonempty zero-set in X has a nonempty interior. Hence, every P-space is an almost P-space but the converse is not true, for instance, the one-point compactification of an uncountable discrete space is an almost P-space which is not a P-space; see Example 2 in [8] and Problem 4K(1) in [6]. By the above lemma, it is easy to see that X is an almost P-space if and only if $r(X) = U(X)$. Using this, X is an almost P-space if and only if $f \in C(X)$ and $\mathrm{cl}_X \mathrm{coz}\, f = X$ imply that $\mathrm{coz}\, f = X$, or equivalently whenever $f \in C(X)$ is nonzero at every point of some dense subset of X, then f is nonzero at every point of X. A point $x \in X$ is called an *almost P-point* if for every $f \in C(X)$, $x \in Z(f)$ implies that $\mathrm{int}_X Z(f) \neq \emptyset$. Hence, whenever $x \in X$ is not an almost P-point, then there exists a regular element $r \in C(X)$ such that $x \in Z(r)$. Clearly, a space X is an almost P-space if and only if every point of X is an almost P-point. We need the following lemma in the sequel which gives another characterization of an almost P-point of X.

Lemma 1.3. *The following statements are equivalent.*

(1) *A point $x \in X$ is an almost P-point.*
(2) *Every element of the maximal ideal M_x is a zerodivisor.*
(3) *The maximal ideal M_x is a z°-ideal.*
(4) *If $f \in C(X)$ and $x \in Z(f)$, then $x \in \mathrm{cl}_X \mathrm{int}_X Z(f)$.*

Proof. Clearly, (1) and (2) are equivalent by the above lemma. The equivalence of (2) and (3) follows from Theorem 1.21 in [2]. By Corollary 3.3 in [3], $x \in \mathrm{cl}_X \mathrm{int}_X Z(f)$ if and only if f belongs to a maximal z°-ideal of $C(X)$ contained in M_x. Then (3) is equivalent to (4) and we are done. □

If Y is a subspace of X, then almost P-points of Y are not necessarily the almost P-points of X and every point of Y which is an almost P-point of X need not be an almost P-point of Y. For instance every point of a discrete subspace D of $\mathbb{R}$ is an isolated point of D and so it is an almost P-point of D. But, clearly no point of D is an almost P-point of $\mathbb{R}$. Also if $X = Y \cup \{\sigma\}$ is the one-point compactification of an uncountable discrete space Y, and if we take a countable subspace Y' of X containing σ, then σ is not an almost P-point of Y', whereas it is an almost P-point of X. But whenever Y is an open subspace of X, then a point of Y is an almost P-point of Y if and only if it is an almost P-point of X. This is evident because every G_δ-set in Y is also a G_δ-set in X and the intersection of Y with a G_δ-set in X is also a G_δ-set in Y. We cite this fact as a lemma which is needed for later use.

Lemma 1.4. *Let Y be an open subset of X and let $x \in Y$. Then x is an almost P-point of Y if and only if it is an almost P-point of X.*

We also need the following lemma which topologically characterizes the unit elements of the factor rings of $C(X)$. First, for an ideal I in $C(X)$, we recall that $\theta(I) = \{p \in \beta X : I \subseteq M^p\}$ and from 7O in [6], $\theta(I) = \bigcap_{f\in I} \mathrm{cl}_{\beta X} Z(f)$. We have also $\theta(I) = \bigcap_{f\in I^*} Z(f^\beta)$, where $I^* = I \cap C^*(X)$. For the sake of completeness, we give a short proof for the latter equality. Clearly, $\theta(I) \subseteq \bigcap_{f\in I^*} Z(f^\beta)$, since $f \in I$ if and only if $(-1 \vee f) \wedge 1 \in I^*$, and also $Z(f) = Z((-1 \vee f) \wedge 1)$; see 1E in [6]. To prove the reverse inclusion, let $p \notin \theta(I)$. Then there exists $g \in I$ such that $p \notin \mathrm{cl}_{\beta X} Z(g)$. Thus, there is $h \in C^*(X)$ such that $h^\beta(p) = 2$ and $h^\beta(\mathrm{cl}_{\beta X} Z(g)) = 0$. Now, if we take the function $k \in C^*(X)$ such that $k^\beta = (h^\beta - 1) \vee 0$, then

$$\begin{gathered}\mathrm{cl}_{\beta X} Z(g) \subseteq \{x \in \beta X : h^\beta(x) < 1\} \subseteq \{x \in \beta X : h^\beta(x) \leq 1\} = \\ \{x \in \beta X : ((h^\beta - 1) \vee 0)(x) = 0\} = Z(k^\beta).\end{gathered}$$

Therefore, $Z(g) \subseteq \mathrm{int}_X Z(k)$ and hence $k \in I^*$, by 1D in [6]. But $k^\beta(p) = 1$ implies that $p \notin \bigcap_{f\in I^*} Z(f^\beta)$.

Lemma 1.5. *Let I be an ideal of $C(X)$ and $f \in C(X)$. Then the following statements are equivalent.*

(1) *$f + I$ is a unit in $C(X)/I$.*
(2) *$Z(g) \cap Z(f) = \emptyset$, for some $g \in I$, i.e., $(f, I) = C(X)$.*
(3) *$\theta(I) \cap \mathrm{cl}_{\beta X} Z(f) = \emptyset$.*

In particular, if $f, g \in C(X)$, then $f + (g)$ (resp., $f + M_g$) is a unit in $C(X)/(g)$ (resp., $C(X)/M_g$) if and only if $Z(f) \cap Z(g) = \emptyset$.

Proof. (1)$\Leftrightarrow$ (2). If $f + I$ is a unit in $C(X)/I$, then there exists $h \in C(X)$ such that $(h + I)(f + I) = 1 + I$, i.e., $fh - 1 \in I$ or equivalently, $fh + g = 1$ for some $g \in I$.

This implies that $Z(f) \cap Z(g) = \emptyset$. Conversely, let $Z(f) \cap Z(g) = \emptyset$ for some $g \in I$. Then $Z(f^2 + g^2) = \emptyset$ implies that $f^2 + g^2 = u$ is a unit in $C(X)$ by Lemma 1.2. Now, $\frac{1}{u}f^2 + \frac{1}{u}g^2 = 1$ implies that $(\frac{1}{u}f + I)(f + I) = 1 + I$, i.e., $f + I$ is a unit.

(2) $\Leftrightarrow$ (3). If $\theta(I) \cap \mathrm{cl}_{\beta X} Z(f) \neq \emptyset$, then $\mathrm{cl}_{\beta X} Z(f)$ intersects $\mathrm{cl}_{\beta X} Z(g)$, for every $g \in I$ and thus $Z(g) \cap Z(f) \neq \emptyset$, for every $g \in I$. To prove the converse, suppose $Z(g) \cap Z(f) \neq \emptyset$, for every $g \in I$. The set $\{\mathrm{cl}_{\beta X} Z(g) \cap \mathrm{cl}_{\beta X} Z(f) : g \in I\}$ has the finite intersection property and thus using the compactness of βX, we conclude that $\theta(I) \cap \mathrm{cl}_{\beta X} Z(f) = (\cap_{g \in I} \mathrm{cl}_{\beta X} Z(g)) \cap \mathrm{cl}_{\beta X} Z(f) \neq \emptyset$. $\square$

2. Regular sequences in $C(X)$

Let R be a ring and M be an R-module. An element $a \in R$ is called *M-regular* if a is not a zerodivisor on M. In other words, a is M-regular if $am \neq 0$ for all $0 \neq m \in M$. A sequence $a_1, \cdots, a_n$ of elements of R is said to be an M-regular sequence (or briefly, M-sequence) of length n, if the following statements hold.

(1) a_1 is M-regular, a_2 is M/a_1M-regular, a_3 is $M/(a_1M + a_2M)$-regular, etc.
(2) $M \neq \sum_{i=1}^{n} a_iM$.

The maximum length of all M-regular sequences, if exists, is called the *depth* of M and it is denoted by $\mathrm{depth}(M)$. The depth of an ideal of a ring R (or R itself) is defined similarly when we consider it as an R-module.

Our main result in this section is based on a question which was raised by Roger Wiegand at a seminar in Iran (Ardabil) that "does $C((0,1))$ (or $C([0,1])$) contain a regular sequence of length 2?" In this section we show that $C(X)$, for every completely regular Hausdorff space X, has no sequence with length 2 and hence we conclude that $\mathrm{depth}(C(X)) \leq 1$. Nevertheless, we may give a negative answer to the aforementioned question by our main result of this section, but before to present the main result, we prefer to settle this case directly, because we think that the simple proof of this special case will be useful.

Proposition 2.1. *Let $r \in C((0,1))$ be a regular element. Then every nonzero element of $C((0,1))/(r)$ is either a unit or a zerodivisor.*

Proof. First, for every $g \in C((0,1))$ and each $0 < \alpha < 1$ with $g(\alpha) = 0$, we define a function g_α with $g_\alpha(x) = g(x)$ for $x \geq \alpha$ and $g_\alpha(x) = -g(x)$ for $x < \alpha$. Clearly, $g_\alpha \in C((0,1))$. Now, suppose that $f + (r) \neq 0$ is not a unit in $C((0,1))/(r)$. By Lemma 1.5, $Z(f) \cap Z(r) \neq \emptyset$. Take $\alpha \in Z(f) \cap Z(r)$, then $r_\alpha \notin (r)$. In fact, if $r_\alpha = rg$ for some $g \in C((0,1))$, then $\lim_{x \to \alpha} g(x)$ does not exist which contradicts the continuity of g (note, using Lemma 1.2, $\mathrm{int}_X Z(r) = \emptyset$, so there exists a sequence $x_n \notin Z(r)$ such that $x_{2n} \geq \alpha$ and $x_{2n-1} < \alpha$, for each $n \in \mathbb{N}$ and $x_n \to \alpha$. Now, we have $g(x_{2n}) \to 1$ and $g(x_{2n-1}) \to -1$). Therefore, $r_\alpha + (r) \neq 0$. But $fr_\alpha = f_\alpha r \in (r)$ because for $x \geq \alpha$,

$f(x)r_\alpha(x) = f(x)r(x) = f_\alpha(x)r(x)$ and for $x < \alpha$, $f(x)r_\alpha(x) = -f(x)r(x) = f_\alpha(x)r(x)$. Hence, $(f + (r))(r_\alpha + (r)) = fr_\alpha + (r) = (r)$, i.e., $f + (r)$ is a zerodivisor. □

Now, to prove our main result of this section, we need the following lemmas and corollary.

Lemma 2.2. *Let I be an ideal of $C(X)$. Let $\theta(I) \cap \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} Z(g) \neq \emptyset$ and $g \notin I$. Then $g + I$ is a zerodivisor in $C(X)/I$. In particular, if $(\bigcap Z[I]) \cap \mathrm{int}_X Z(g) \neq \emptyset$, then $g + I$ is a zerodivisor in $C(X)/I$.*

Proof. Without loss of generality, we assume that $g \in C^*(X)$; see 2A in [6]. Define $h \in C^*(X)$ with $h^\beta(\beta X \setminus \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} Z(g)) = 0$ and $h^\beta(x_0) = 1$ for some $x_0 \in \theta(I) \cap \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} Z(g)$. Hence, $g^\beta h^\beta = 0$, so $gh = 0$ and thus $(g + I)(h + I) = I$. On the other hand, $h \notin I$, for $x_0 \in \theta(I)$ and $h^\beta(x_0) = 1$; see the argument preceding Lemma 1.5. This implies that $h + I \neq I$. Hence, $g + I \neq I$ is a zerodivisor. □

The following corollary is an immediate consequence of Lemma 2.2.

Corollary 2.3. *If $Z(f) \cap \mathrm{int}_X Z(g) \neq \emptyset$, then $g + (f)$ is a zerodivisor in $C(X)/(f)$.*

The converse of the above result is not necessarily true. To see this, take $r \in r(X) \setminus U(X)$, then $0 \neq r^{\frac{1}{3}} + (r) \in C(X)/(r)$ is a zerodivisor since $(r^{\frac{1}{3}} + (r))(r^{\frac{2}{3}} + (r)) = (r)$. But $Z(r^{\frac{1}{3}}) \cap \mathrm{int}_X Z(r) = \emptyset$.

Lemma 2.4. *Let $f \in C(X)$. Then $Z(f)$ is open if and only if $C(Z(f)) \cong C(X)/(f)$.*

Proof. Suppose $Z(f)$ is open. We define $\varphi : C(X) \to C(Z(f))$ with $\varphi(g) = g|_{Z(f)}$, for all $g \in C(X)$. Clearly, φ is a homomorphism with $Ker\varphi = (f)$. In fact, $g \in Ker\varphi$ if and only if $g|_{Z(f)} = 0$, and this is equivalent to saying that $Z(f) \subseteq Z(g)$. But $Z(f)$ is open, hence $g \in Ker\varphi$ if and only if $Z(f) \subseteq \mathrm{int}Z(g)$, which implies that $g \in (f)$ by 1D in [6]. Therefore, $C(X)/(f) \cong C(Z(f))$. Conversely, if $C(Z(f)) \cong C(X)/(f)$, then $f + (f) = (f)$ implies that $f^{\frac{1}{3}} + (f) = (f)$ since $C(Z(f))$ is a reduced ring. Hence, $f^{\frac{1}{3}} \in (f)$ and this implies that $Z(f)$ is open. □

Now, we are ready to prove our main result of this section which states that whenever every non-almost P-point in a zero-set $Z(f)$ is contained in $\partial Z(f)$ (in particular, if $\mathrm{int}_X Z(f) = \emptyset$), then every element of $C(X)/(f)$ is either a unit or a zerodivisor. Note that, if A is a subset of a space X, then ∂A means the boundary of A.

Theorem 2.5. *Let f be a non-unit element of $C(X)$. Then the following statements hold.*

(1) *If every element of $\mathrm{int}_X Z(f)$ is an almost P-point (or equivalently $\mathrm{int}_X Z(f)$ is an almost P-space), then every non-unit element of $C(X)/(f)$ is a zerodivisor.*

(2) *If every non-unit element of* $C(X)/(f)$ *is a zerodivisor, then either every point of* $\text{int}_X Z(f)$ *is an almost* P*-point of* X *or* $\partial Z(f) \neq \emptyset$.

Proof. (1). Suppose that $g + (f) \neq 0$ is a non-unit element of $C(X)/(f)$. We show that $g+(f)$ is a zerodivisor. By Lemma 1.5, $Z(f)\cap Z(g) \neq \emptyset$. Whenever $\text{int}_X Z(f)\cap \text{int}_X Z(g) \neq \emptyset$, then by Corollary 2.3, $g+(f)$ is a zerodivisor. Now, suppose that $\text{int}_X Z(f)\cap \text{int}_X Z(g) = \emptyset$. If we take $y \in Z(f) \cap Z(g)$ and $r = f^2 + g^2$, then $y \in Z(r)$ and $r \in r(X)$. This means that y is a non-almost P-point of X, and hence $y \in Z(f) \setminus \text{int}_X Z(f) = \partial Z(f)$, by our hypothesis. Now, we define

$$h(x) = \begin{cases} \frac{g(x)}{f^{\frac{2}{3}}(x)+g^{\frac{2}{3}}(x)} & x \notin Z(g) \cap Z(f) \\ 0 & x \in Z(g) \cap Z(f), \end{cases}$$

$$k(x) = \begin{cases} \frac{f(x)}{f^{\frac{2}{3}}(x)+g^{\frac{2}{3}}(x)} & x \notin Z(g) \cap Z(f) \\ 0 & x \in Z(g) \cap Z(f). \end{cases}$$

Clearly, $h, k \in C(X)$ (note, $|h| \leq |g^{\frac{1}{3}}|$ on $X\setminus(Z(g)\cap Z(f)))$ and for each $x \notin Z(g)\cap Z(f)$, we have $f(x)h(x) = \frac{f(x)g(x)}{f^{\frac{2}{3}}(x)+g^{\frac{2}{3}}(x)} = g(x)k(x)$. This implies that fh and gk coincide on X, so $(g + (f))(k + (f)) = (f)$. Now, it is enough to show that $k \notin (f)$. Whenever $k \in (f)$, then $k = ft$ for some $t \in C(X)$. Hence, for each $x \notin Z(f)$, we have $t(x) = \frac{1}{f^{\frac{2}{3}}(x)+g^{\frac{2}{3}}(x)}$. Using this, we observe that t is not continuous at y. In fact, since $y \in \partial Z(f)$, we may take a net (y_λ) in $X \setminus Z(f)$ such that $y_\lambda \to y$ and hence we get $t(y_\lambda) \to \infty$. Therefore, $h \notin (f)$.

(2). Let every element of $C(X)/(f)$ be a unit or a zerodivisor. Suppose on the contrary, that $\partial Z(f) = \emptyset$ and there exists $x \in \text{int}_X Z(f)$ which is not an almost P-point of X. Hence, $Z(f) = \text{int}_X Z(f)$ and using Lemma 1.4, x is not an almost P-point of $Z(f)$ as well. Therefore, $C(Z(f))$ contains a non-unit regular element. But $C(Z(f)) \cong C(X)/(f)$ by Lemma 2.4, so $C(X)/(f)$ also contains a non-unit regular element, a contradiction. □

Now, using Theoerm 2.5(1), the following corollaries are evident.

Corollary 2.6. *Let* $r \in r(X)$ *be a non-unit. Then every element of* $C(X)/(r)$ *is either a unit or a zerodivisor.*

Corollary 2.7. $C(X)$ *does not contain a regular sequence of length* ≥ 2.

By definition, a space X is an almost P-space if and only if $C(X)$ has no regular non-units, and this is equivalent to saying that $C(X)$ has no regular sequence. This means that $\text{depth}(C(X)) = 0$ if and only if X is an almost P-space. Now, using Corollary 2.6, $C(X)$ has a sequence of length 1 (i.e., $\text{depth}(C(X)) = 1$) if and only if the space X contains at least one non-almost P-point.

Remark 2.8. In contrast to the ring $C(X)/(r)$, every non-unit element of a factor ring of $C(X)$ modulo any arbitrary ideal I of $C(X)$ is not necessarily a zerodivisor. That is, Corollary 2.6 is not true in general. For instance, $C(\mathbb{R})/O_0$ contains both a zerodivisor and a non-unit regular element. Whenever we take the identity function $i \in C(\mathbb{R})$ and if $g \in C(\mathbb{R})$ such that $ig \in O_0$, then $g \in O_0$ which means that $i + O_0$ is not a zerodivisor. On the other hand if $ig - 1 \in O_0$, then $ig = 1$ on a neighborhood of 0 which is impossible, for $i(0) = 0$, so $i + O_0$ is not a unit. This implies that $i + O_0$ is a non-unit regular in $C(\mathbb{R})/O_0$.

For an example of a zerodivisor in $C(\mathbb{R})/O_0$, we may take $f \in C(\mathbb{R})$ with $Z(f) = [0, 1]$. We also take $g \in C(\mathbb{R})$ with $Z(g) = [-1, 0]$. Clearly, $f, g \notin O_0$, i.e., $f + O_0$ and $g + O_0$ are nonzero and since $fg \in O_0$, we have $(f + O_0)(g + O_0) = O_0$, i.e., $f + O_0$ is a zerodivisor in $C(\mathbb{R})/O_0$. Moreover, using Lemma 1.5, the set of all $h + O_0$, $h \in C(\mathbb{R})$ such that $0 \notin Z(h)$ is exactly the set of all units of $C(\mathbb{R})/O_0$. More generally, whenever $x \in X$ is not an almost P-point and O_x is not prime, then $C(X)/O_x$ contains both a zerodivisor and a non-unit regular element. To this end, let $r \in r(X)$ and $x \in Z(r)$. Then clearly $r + O_x$ is a regular element for if $g \in C(X)$, then $rg \in O_x$ implies $x \in \mathrm{int}_X(Z(r) \cup Z(g)) = \mathrm{int}_X Z(g)$, i.e., $g \in O_x$. On the other hand, since O_x is not prime, $C(X)/O_x$ is not an integral domain and thus it contains a nonzero zerodivisor element.

Corollary 2.6 is not even true for a factor ring of $C(X)$ modulo the smallest z-ideal containing a principal ideal generated by a regular element $r \in C(X)$. For example, if we take $r, s \in C(\mathbb{R})$ with $Z(r) = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ and $Z(s) = \{0, -1, -\frac{1}{2}, -\frac{1}{3}, \ldots\}$, then both r and s are regular elements and the smallest z-ideal containing the principal ideal (r) is M_r. Now, $s + M_r$ is neither a unit nor a zerodivisor in $C(\mathbb{R})/M_r$. In fact, since $Z(s) \cap Z(r) \neq \emptyset$, $s + M_r$ is not a unit and for $g \in C(\mathbb{R})$, $sg \in M_r$ implies that $g \in M_r$, i.e., $s + M_r$ is not a zerodivisor.

In Remark 2.8 we observed that for $r \in r(\mathbb{R})$, the factor rings $C(\mathbb{R})/O_0$ and $C(\mathbb{R})/M_r$ contain a non-unit regular element. But it is not known for us that what is the maximum length of regular sequences in such factor rings? In the last section we investigate the depth and regular sequences of some special factor rings of $C(X)$. Also in Theorem 2.5, we have shown that whenever every point of $\mathrm{int}_X Z(f)$ is an almost P-point of X, then the factor ring $C(X)/(f)$ has no regular sequence of length 1, i.e., $\mathrm{depth}(C(X)/(f)) = 0$. Now, it is natural to ask what is the maximum length of the regular sequences in a factor ring of $C(X)$ modulo a principal ideal? In the following proposition, we prove that it is at most 1 if (f) is a z-ideal.

Proposition 2.9. *Let (f) be a principal z-ideal in $C(X)$. Then every regular sequence in $C(X)/(f)$ is of length 0 or 1.*

Proof. Since (f) is a z-ideal, $Z(f)$ is open. Therefore, using Corollary 2.7, $C(Z(f))$ has a regular sequence of length at most 1. Now, using Lemma 2.4, every regular sequence of $C(X)/(f)$ has length at most 1 as well. $\square$

3. Depth of maximal ideals of $C(X)$

Whenever an ideal I in $C(X)$ is considered as a $C(X)$-module, then by our definition in Section 2, $f \in C(X)$ is I-regular if it is not a zerodivisor on I, i.e., $if \neq 0$ for all $0 \neq i \in I$ or equivalently $I \cap \mathrm{Ann}(f) = 0$. I-regular sequences are also defined similar to M-regular sequences, where the module M is considered to be the $C(X)$-module I. We denote the set of all I-regular elements of $C(X)$ by $r_I(X)$. In this section we obtain the depth of maximal ideals of $C(X)$ and characterize almost P-spaces and P-spaces via the depth of some ideals of $C(X)$. First, we need the following lemmas and corollary.

Lemma 3.1. *Let R be a reduced ring. Then the following statements hold.*

(1) *If $a, b \in R$, then a is M_b-regular if and only if $b \in P_a$.*
(2) *If I is a z-ideal of R, then $a \in R$ is I-regular if and only if $I \subseteq P_a$.*

Proof. (1). By definition, a is M_b-regular if and only if $M_b \cap \mathrm{Ann}(a) = (0)$. But $M_b \cap \mathrm{Ann}(a) = (0)$ is equivalent to saying that $\mathrm{Ann}(a) \subseteq \mathrm{Ann}(b)$. In fact, if $c \in \mathrm{Ann}(a) \setminus \mathrm{Ann}(b)$ for some $c \in R$, then $0 \neq cb \in M_b \cap \mathrm{Ann}(a)$, a contradiction. Conversely, let $\mathrm{Ann}(a) \subseteq \mathrm{Ann}(b)$ and $c \in M_b \cap \mathrm{Ann}(a)$, $c \in R$. Since every minimal prime ideal in R is a z-ideal by Theorem 1.1 in [9], $c \in M_b \subseteq P_b$, which means $\mathrm{Ann}(b) \subseteq \mathrm{Ann}(c)$. Now, $c \in \mathrm{Ann}(c)$ implies that $c^2 = 0$, so $c = 0$ as well because R is a reduced ring. Thus, $M_b \cap \mathrm{Ann}(a) = (0)$.

(2). Clearly, a is I-regular if and only if a is M_b-regular for each $b \in I$, because $M_b \subseteq I$ as I is a z-ideal. Now, using (1), this is equivalent to saying that $b \in P_a$ for each $b \in I$, i.e., $I \subseteq P_a$. □

Corollary 3.2. *Let I be an ideal of $C(X)$ and $f \in C(X)$. Then the following statements hold.*

(1) *If $g \in C(X)$, then f is M_g-regular if and only if $\mathrm{int}_X Z(f) \subseteq Z(g)$, if and only if $g \in P_f$.*
(2) *The element f is I-regular if and only if it is I_z-regular.*
(3) *The element f is I-regular if and only if $\mathrm{int}_X Z(f) \subseteq \bigcap Z[I]$ if and only if $I \subseteq P_f$.*

Proof. (1). Since $g \in P_f$ if and only if $\mathrm{int}_X Z(f) \subseteq Z(g)$, the proof is evident by part (1) of the above lemma.

(2). Let f be I-regular and $0 \neq g \in I_z$. Then there is $h \in I$ such that $Z(g) = Z(h)$. Since $h \neq 0$, we have $fh \neq 0$. Hence, $fg \neq 0$, for $Z(fg) = Z(fh)$. Conversely, let f be I_z-regular. Then clearly f is I-regular because $I \subseteq I_z$.

(3). By part (2), we may consider I to be a z-ideal. Now, using part (2) of the above lemma, f is I-regular if and only if $I \subseteq P_f$. On the other hand, $I \subseteq P_f$ if and only if $\mathrm{int}_X Z(f) \subseteq Z(g)$ for each $g \in I$ which is equivalent to $\mathrm{int}_X Z(f) \subseteq \bigcap_{g \in I} Z(g) = \bigcap Z[I]$. □

In view of Corollary 3.2, $f \in C(X)$ is always P_f-regular and among the ideals I in $C(X)$ for which f is I-regular, the ideal P_f is the largest one. Moreover, for every $f \in C(X)$, we have f is (f)-regular and it is M_f-regular as well. More generally, for every $f \in C(X)$, the set of (f)-regular elements, the set of M_f-regular elements and the set of P_f-regular elements in $C(X)$ coincide.

Lemma 3.3. *The following statements hold.*

(1) *An ideal I in $C(X)$ is essential if and only if $r_I(X) = r(X)$.*
(2) *The depth of each ideal of $C(X)$ is zero if and only if X is a P-space.*
(3) *The depth of each essential ideal of $C(X)$ is zero if and only if X is an almost P-space.*

Proof. (1). Let $f \in C(X)$. If $f \in r_I(X) \setminus r(X)$, then $I \bigcap \mathrm{Ann}(f) = (0)$ but $\mathrm{Ann}(f) \neq (0)$ which means that I is not essential. Conversely, suppose that $r_I(X) = r(X)$, but I is not essential. Hence, $\mathrm{int}_X \bigcap Z[I] \neq \emptyset$ by Lemma 1.1. Take $x_0 \in \mathrm{int}_X \bigcap Z[I]$ and define $g \in C(X)$ such that $x_0 \in \mathrm{int}_X Z(g)$ and $g(X \setminus \mathrm{int}_X \bigcap Z[I]) = 1$. By Corollary 3.2, g is I-regular, i.e., $g \in r_I(X)$. But $g \notin r(X)$, for $\mathrm{int}_X Z(g) \neq \emptyset$, a contradiction.

(2). Let X be a P-space and I be an ideal in $C(X)$. Let a non-unit $f \in C(X)$ be I-regular. Then $Z(f)$ is open and for each $i \in I$, we may define $h_i \in C(X)$ as follows:

$$h_i(x) = \begin{cases} \frac{i(x)}{f(x)} & x \notin Z(f) \\ 0 & x \in Z(f). \end{cases}$$

Inasmuch as $Z(i) \subseteq Z(h_i)$ and I is a z-ideal, we infer that $h_i \in I$. We have also $i = fh_i$, which means $fI = I$, so $\mathrm{depth}(I) = 0$. Conversely, let the depth of every ideal of $C(X)$ is zero and let $f \in C(X)$ be a non-unit. By Corollary 3.2, f is M_f-regular and since $\mathrm{depth}(M_f) = 0$ by our hypothesis, we must have $fM_f = M_f$. Hence, there exists $g \in M_f$ (whence $Z(f) \subseteq Z(g)$) such that $f = fg$ or $f(1-g) = 0$. Therefore, $Z(f) \cup Z(1-g) = X$ and $Z(f) \cap Z(1-g) = \emptyset$ imply that $Z(f)$ is open and hence X is a P-space by 4J in [6].

(3). Let X be an almost P-space and I be an essential ideal in $C(X)$. Then $r(X) = U(X)$ and by part (1), we have $r_I(X) = U(X)$. This means that there is no non-unit I-regular and hence $\mathrm{depth}(I) = 0$. Conversely, let the depth of every essential ideal of $C(X)$ be zero. Suppose, on the contrary, that X is not an almost P-space. Thus, there exists a non-unit regular element $r \in r(X)$. If we consider the principal ideal $I = (r)$, then I is essential, since $\mathrm{Ann}(I) = 0$. Now, by part (1), $r_I(X) = r(X)$ and hence r is I-regular. But $\mathrm{depth}(I) = 0$ by our hypothesis, so we must have $(r) = I = rI = r(r)$. This means that $r = r^2k$ or $r(1-rk) = 0$ for some $k \in C(X)$. But r is a non-zerodivisor, then $rk = 1$ which is impossible, because r is a non-unit. □

Part (3) of the above lemma is also true in any reduced ring, in the sense that the depth of each essential ideal of a reduced ring R is zero if and only if R is a classical ring.

The proof of this algebraic case will be done by the arguments similar to those above word-for-word if we replace $C(X)$ with R, $U(X)$ with $U(R)$ and $r(X)$, $r_I(X)$ with $r(R)$, $r_I(R)$ respectively.

To prove our main result of this section, we need the following lemmas and corollary.

Lemma 3.4. *Let I be an ideal in $C(X)$ and $r, s \in C(X)$ such that $rI + sI \neq I$. Then $Z(r) \cap Z(s) \neq \emptyset$.*

Proof. Let $Z(r) \cap Z(s) = \emptyset$ and $f \in I$. Then $f = \frac{r^2}{r^2+s^2} f + \frac{s^2}{r^2+s^2} f \in rI + sI$, i.e., $rI + sI = I$, a contradiction. $\square$

Lemma 3.5. *Let I be an ideal of $C(X)$ and suppose $r, s \in C(X)$. Let there exists $f \in I$ such that $\partial Z(r) \cap Z(s) \setminus Z(f) \neq \emptyset$. Then s is not $\frac{I}{rI}$-regular.*

Proof. Suppose $f \in I$ and $\partial Z(r) \cap Z(s) \setminus Z(f) \neq \emptyset$. We define

$$\hat{r}(x) = \begin{cases} \frac{r(x)}{r^{\frac{2}{3}}(x)+s^{\frac{2}{3}}(x)} & x \notin Z(r) \cap Z(s) \\ 0 & x \in Z(r) \cap Z(s), \end{cases}$$

$$\hat{s}(x) = \begin{cases} \frac{s(x)}{r^{\frac{2}{3}}(x)+s^{\frac{2}{3}}(x)} & x \notin Z(r) \cap Z(s) \\ 0 & x \in Z(r) \cap Z(s). \end{cases}$$

Clearly, $\hat{r}, \hat{s} \in C(X)$ (note that, $|\hat{r}| \leq |r^{1/3}|$ and $|\hat{s}| \leq |s^{1/3}|$ on X) and $sf\hat{r} = rf\hat{s} \in rI$. To prove that s is not $\frac{I}{rI}$-regular, we show that $f\hat{r} \notin rI$. In fact, $f\hat{r} \in rI$ implies that $f\hat{r} = rt$ for some $t \in I$ and hence $t = \frac{f}{r^{\frac{2}{3}}+s^{\frac{2}{3}}}$ on $X \setminus Z(r)$. Now, if we take $y \in \partial Z(r) \cap Z(s) \setminus Z(f)$, then there exists a net (y_λ) in $(X \setminus Z(f)) \cap (X \setminus Z(r))$ such that $y_\lambda \to y$ (note that $(X \setminus Z(f)) \cap (X \setminus Z(r)) \neq \emptyset$, for $X \setminus Z(f)$ is a neighborhood of $y \in \partial Z(r)$). Clearly, $t(y_\lambda) \to \infty$ which implies that t is not continuous at y, a contradiction. $\square$

Corollary 3.6. *Let I be a free ideal of $C(X)$. Then* $\operatorname{depth}(I)$ *is either* 0 *or* 1.

Proof. If X is an almost P-space, then $\operatorname{depth}(I) = 0$ by Lemma 3.3(3), as a free ideal of $C(X)$ is essential. Whenever X has at least one non-almost P-point, then a non-unit $r \in r(X)$ exists. Clearly, r is I-regular and $rI \neq I$. In fact, if $x \in Z(r)$, there exists $f \in I$ such that $f(x) \neq 0$ since I is free. Thus, for each $i \in I$, $ri \neq f$ which means $f \in I \setminus rI$. Therefore, $\operatorname{depth}(I) \geq 1$. Now, we show that there is no any I-regular sequence in $C(X)$ of length more than 1. Suppose on the contrary, that r, s is an I-regular sequence in $C(X)$. Hence, r is I-regular, s is $\frac{I}{rI}$-regular and $rI + sI \neq I$. By Lemma 3.4, $Z(r) \cap Z(s) \neq \emptyset$. On the other hand, since r is I-regular and I is free, using Corollary 3.2, $\operatorname{int}_X Z(r) = \emptyset$ and so $\partial Z(r) = Z(r)$. Now, we have two cases and for each case we are to get a contradiction. Either for each $f \in I$ we have $Z(r) \cap Z(s) \subseteq Z(f)$, or there exists $f \in I$ such that $\partial Z(r) \cap Z(s) \setminus Z(f) = Z(r) \cap Z(s) \setminus Z(f) \neq \emptyset$. The first case implies

$Z(r) \cap Z(s) \subseteq \bigcap Z[I] = \emptyset$, a contradiction. The second case implies that s is not an $\frac{I}{rI}$-regular by Lemma 3.5 which is again a contradiction. $\square$

Now, we are ready to prove our main result of this section.

Theorem 3.7. *Let M be a maximal ideal of $C(X)$. Then* $\operatorname{depth}(M)$ *is either 0 or 1.*

Proof. If M is a free maximal ideal of $C(X)$, then by Corollary 3.6, $\operatorname{depth}(M)$ is either 0 or 1 and so we are done. Thus, we suppose M is fixed and show that $\operatorname{depth}(M) \leq 1$. Let $M = M_a$ for some $a \in X$. Suppose, on the contrary, that r, s is an M_a-regular sequence in $C(X)$. Hence, r is M_a-regular, s is $\frac{M_a}{rM_a}$-regular and $rM_a + sM_a \neq M_a$. Since r is M_a-regular, using Corollary 3.2, we have two cases: $\operatorname{int}_X Z(r) = \emptyset$ or $\operatorname{int}_X Z(r) = \{a\}$. We are to get a contradiction for each case.

First, let $\operatorname{int}_X Z(r) = \emptyset$. In this case if there exists $f \in M_a$ such that $Z(r) \cap Z(s) \setminus Z(f) \neq \emptyset$, then using Lemma 3.5, s is not $\frac{M_a}{rM_a}$-regular, a contradiction. Otherwise, if for every $f \in M_a$, $Z(r) \cap Z(s) \subseteq Z(f)$, then $Z(r) \cap Z(s) \subseteq \bigcap Z[M] = \{a\}$. By Lemma 3.4, $Z(r) \cap Z(s) \neq \emptyset$, and thus $Z(r) \cap Z(s) = \{a\}$. Therefore, $r, s \in M_a$ and so $sr \in rM_a$, which implies $r \in rM_a$ as s is $\frac{M_a}{rM_a}$-regular. Hence, there exists $m \in M_a$ such that $r = rm$. Now, since r is regular by Lemma 1.2, we have $m = 1$, a contradiction.

Next, let $\operatorname{int}_X Z(r) = \{a\}$. Then a is an isolated point. We claim that there exists $a \neq y \in Z(r) \cap Z(s)$. Suppose on the contrary, that $Z(r) \cap Z(s) = \{a\}$. We take $t \in M_a$ and define

$$g(x) = \begin{cases} \frac{t(x)}{r^2(x)+s^2(x)} & x \neq a \\ 0 & x = a. \end{cases}$$

Then $g \in C(X)$ and $t = r(rg) + s(sg) \in rM_a + sM_a$. Therefore, $M_a = rM_a + sM_a$ which is impossible. Therefore, there exists $a \neq y \in Z(r) \cap Z(s)$. Now, take an $f \in C(X)$ such that $Z(f) = \{a\}$. Thus, $f \in M_a$ and $y \in \partial Z(r) \cap Z(s) \setminus Z(f)$. Now, using Lemma 3.5, we conclude that s is not an $\frac{M_a}{rM_a}$-regular, a contradiction. $\square$

The following proposition characterizes almost P-spaces via the depth of maximal ideals of $C(X)$. Using Theorem 3.7, this proposition also states that for a given space X, there are no two maximal ideals in $C(X)$ with different depths.

Proposition 3.8. *The following statements are equivalent.*

(1) *X is an almost P-space.*
(2) *The depth of each maximal ideal of $C(X)$ is zero.*
(3) *There exists a maximal ideal M in $C(X)$ such that* $\operatorname{depth}(M) = 0$.

Proof. (1) $\Rightarrow$ (2). Suppose X is an almost P-space and M is a maximal ideal of $C(X)$. If M is free or M is a fixed maximal ideal M_a, where $a \in X$ is not an isolated point, then it is essential and hence using Lemma 3.3, we have $\operatorname{depth}(M) = 0$. Now, suppose that $M = M_a$, where a is an isolated point. Let $r \in C(X)$ be M_a-regular. Then $\operatorname{int}_X Z(r) \subseteq \{a\}$ by Corollary 3.2. Since X is an almost P-space, $\operatorname{int}_X Z(r) \neq \emptyset$, so $\operatorname{int}_X Z(r) = \{a\}$. If $t \in M_a$, we define $g \in C(X)$ such that $g(a) = 0$ and $g(x) = \frac{t(x)}{r(x)}$, for $x \neq a$. Hence, $t = rg$ implies that $M_a = rM_a$ and this means that $\operatorname{depth}(M_a) = 0$.

(2) ⇒ (3). It is evident.

(3) ⇒ (1). On the contrary, suppose X is not an almost P-space and let M be an arbitrary maximal ideal of $C(X)$ whose depth is zero. Since X is not an almost P-space, there is a non-unit $r \in r(X)$. Clearly, r is M-regular and we claim that $rM \neq M$. First, we let $rM = M$ and $r \in M$. Then $r = mr$ for some $m \in M$, so $r(1 - m) = 0$. But r is not a zerodivisor, hence $1 = m \in M$ which is impossible. Next, suppose $rM = M$ and $r \notin M$. Then there is $f \in M$ such that $Z(f) \cap Z(r) = \emptyset$ by Theorem 2.6 in [6]. Now, we must have $f = rm$ for some $m \in M$ which is again impossible because $Z(f) \cap Z(r) = \emptyset$. Therefore, $\operatorname{depth}(M) \geq 1$, a contradiction. □

The following result is an immediate corollary of Theorem 3.7 and Proposition 3.8.

Corollary 3.9. *The following statements are equivalent.*

(1) *X contains at least one non-almost P-point.*
(2) *The depth of each maximal ideal of $C(X)$ is 1.*
(3) *There exists a maximal ideal M in $C(X)$ such that* $\operatorname{depth}(M) = 1$.

The following proposition shows that the depth of each principal z-ideal of $C(X)$ is at most 1. Notice that for each $f \in C(X)$, the principal ideal (f) is a z-ideal if and only if $Z(f)$ is an open subset of X.

Proposition 3.10. *Let $f \in C(X)$ and $Z(f) \subseteq X$ be open. Then* $\operatorname{depth}((f)) \leq 1$.

Proof. Suppose, on the contrary, that r, s is (f)-sequence. Thus, r is (f)-regular, s is $\frac{(f)}{r(f)}$-regular and $r(f) + s(f) \neq (f)$. Since r is (f)-regular, $\operatorname{int}_X Z(r) \subseteq Z(f)$ by Corollary 3.2 and since s is $\frac{(f)}{r(f)}$-regular, $Z(r) \cap Z(s) \setminus Z(f) = \partial Z(r) \cap Z(s) \setminus Z(f) = \emptyset$ by Lemma 3.5. Therefore, $Z(r) \cap Z(s) \subseteq Z(f)$. Now, we take $t \in (f)$ and define

$$h(x) = \begin{cases} \frac{t(x)}{r^2(x)+s^2(x)} & x \notin Z(f) \\ 0 & x \in Z(f). \end{cases}$$

Since $Z(f)$ is open, $h \in C(X)$. Moreover, $h \in (f)$, for $Z(f) \subseteq Z(h)$ and (f) is a z-ideal. Clearly, $t = r(rh) + s(sh) \in r(f) + s(f)$ and this implies that $r(f) + s(f) = (f)$, a contradiction. □

Corollary 3.11. *Let $f \in C(X)$ and $Z(f)$ be open. Then* $\operatorname{depth}((f)) = 1$ *(resp.,* $\operatorname{depth}((f)) = 0$*) if and only if* $\operatorname{coz} f$ *contains a non-almost P-point of X (resp., every point of* $\operatorname{coz} f$ *is an almost P-point of X).*

Proof. Suppose that $\operatorname{coz} f$ contains a non-almost P-point of X which it will be also a non-almost P-point of $\operatorname{coz} f$ by Lemma 1.4. Hence, there exists a non-unit regular element $s \in C(\operatorname{coz} f)$. Take $r \in C(X)$ such that $\operatorname{int}_X Z(r) \subseteq Z(f)$ and define a function g with $g = r$ on $Z(f)$ and $g = s$ on $\operatorname{coz} f$. Clearly, $g \in C(X)$ and $\operatorname{int}_X Z(g) = \operatorname{int}_X Z(r) \cup \operatorname{int}_X Z(s) = \operatorname{int}_X Z(r) \subseteq Z(f)$ for $\operatorname{int}_X Z(s) = \operatorname{int}_{\operatorname{coz} f} Z(s) = \emptyset$, hence g is (f)-regular by Corollary 3.2. On the other hand, $g(f) \neq (f)$. In fact, if the equality $g(f) = (f)$ holds, then $f = gft$ for some $t \in C(X)$ and this implies that $f(1 - gt) = 0$. So $gt = 1$ on $\operatorname{coz} f$ which means that $g|_{\operatorname{coz} f} t|_{\operatorname{coz} f} = st|_{\operatorname{coz} f} = 1$ and this contradicts the fact that $s \in C(\operatorname{coz} f)$ is a non-unit. Now, using Proposition 3.10, $\operatorname{depth}((f))$=1.

Conversely, let $\operatorname{depth}((f)) = 1$. Suppose on the contrary, that every point of $\operatorname{coz} f$ is an almost P-point of X and $r \in C(X)$ is (f)-regular. We claim that $r(f) = (f)$ which implies $\operatorname{depth}((f)) = 0$ and we get a contradiction. By Corollary 3.2, we have $\operatorname{int}_X Z(r) \subseteq Z(f)$ and this implies that $Z(r) \subseteq Z(f)$. In fact, if $Z(r) \nsubseteq Z(f)$, then $Z(r) \cap \operatorname{coz} f$ is a nonempty G_δ-set whose points are almost P-points. Hence, its interior must be nonempty, but $\operatorname{int}_X (Z(r) \cap \operatorname{coz} f) = \operatorname{int}_X Z(r) \cap \operatorname{coz} f \subseteq Z(f) \cap \operatorname{coz} f = \emptyset$, a contradiction. Therefore, $Z(r^2) = Z(r) \subseteq Z(f)$ and since $Z(f)$ is open, $f = r^2 t$ for some $t \in C(X)$ by 1D in [6]. Since $Z(rt) = Z(f)$, we have $rt \in (f)$ again by 1D in [6]. Therefore, $rt = sf$ for some $s \in C(X)$, so $rsf = r^2 t = f$, i.e., $r(f) = (f)$. □

Proposition 3.12. *For every $x \in \beta X$,* $\operatorname{depth}(O^x) \leq 1$.

Proof. If $x \in \beta X \setminus X$, then O^x is free and hence $\operatorname{depth}(O^x) \leq 1$ by Corollary 3.6. Whenever $x \in X$ is an isolated point, then $O_x = M_x$, so $\operatorname{depth}(O_x) \leq 1$ by Theorem 3.7. Now, suppose that $x \in X$ is not an isolated point. We claim that $\operatorname{depth}(O_x) \leq 1$. To see this, let r, s be an O_x-sequence, i.e., r is O_x-regular, s is O_x/rO_x-regular and $rO_x + sO_x \neq O_x$. Using Corollary 3.2, $\operatorname{int}_X Z(r) \subseteq \{x\}$ which implies that $\operatorname{int}_X Z(r) = \emptyset$ and so $\partial Z(r) = Z(r)$. On the other hand, $Z(r) \cap Z(s) = \partial Z(r) \cap Z(s) \subseteq \bigcap Z[O_x] = \{x\}$ by Lemma 3.5. But $Z(r) \cap Z(s) \neq \emptyset$ by Lemma 3.4, so $Z(r) \cap Z(s) = \{x\}$. Take $f \in O_x$ and define

$$h(y) = \begin{cases} \frac{f(y)}{r^2(y)+s^2(y)} & y \neq x \\ 0 & y = x. \end{cases}$$

Clearly, $h \in C(X)$. To see this, it is enough to take a net (x_λ) in $\operatorname{int}_X Z(f)$ such that $x_\lambda \to x$. Evidently, we have $h(x_\lambda) \to 0$. Moreover, $h \in O_x$, for $Z(f) \subseteq Z(h)$ and O_x is a z-ideal. Now, $f = r^2 h + s^2 h$ implies that $O_x = rO_x + sO_x$, a contradiction. Therefore, $\operatorname{depth}(O_x) \leq 1$. □

As we have shown in this section, many of the ideals of $C(X)$ such as maximal ideals, free ideals and principal z-ideals have depth 0 or 1. These facts lead us to have a guess that the depth of every ideal of $C(X)$ is either 0 or 1. We could not settle our guess and so we cite it as a conjecture.

Conjecture. $\mathrm{depth}(I) \leq 1$ *for each ideal* I *of* $C(X)$.

4. When is the depth of a factor ring of $C(X)$ zero?

We recall from [7, page 320] that a ring R is a classical ring if its every non-unit element is a zerodivisor. It is clear that a ring R is a classical ring if and only if $\mathrm{depth}(R) = 0$. We observed in Theorem 2.5 that the factor ring of $C(X)$ modulo a principal ideal (f) may be a classical ring. In particular, whenever $r \in r(X)$, then $C(X)/(r)$ is a classical ring; see Corollary 2.6. As we also observed in Remark 2.8, the factor ring of $C(\mathbb{R})$ modulo O_0 is not a classical ring. It is also well known that $C(X)$ is a classical ring if and only if X is an almost P-space. In this section, motivated by these observations, we are going to obtain conditions on a space X or on a given ideal I for which $\mathrm{depth}(C(X)/I) = 0$ or equivalently $C(X)/I$ is a classical ring.

We remind that in a commutative reduced ring R, every element is either a unit or a zerodivisor if and only if $\bigcup_{M\in Max(R)} M = \bigcup_{P\in Min(R)} P$. In fact, $\bigcup_{P\in Min(R)} P$ is the set of all zerodivisors of R and $\bigcup_{M\in Max(R)} M$ is the set of all non-unit elements of R and the coincidence of these two sets is equivalent to saying that R is classical. In particular, $C(X)$ is a classical ring if and only if for every $x \in X$, we have $M_x \subseteq \bigcup_{P\in Min(X)} P$, and this is equivalent to saying that each point of X is an almost P-point; see Lemma 1.3. Thus, whenever I is an ideal of R, then $\mathrm{depth}(R/I) = 0$ is equivalent to the equality $\bigcup_{I\subseteq M\in Max(R)} M = \bigcup_{P\in Min(I)} P$, where $Min(I)$ is the set of all prime ideals minimal over I. We cite this fact for the ring $C(X)$ as a proposition for later use. Note that every maximal ideal of $C(X)$ containing an ideal I of $C(X)$ is precisely of the form M^p, where $p \in \theta(I)$.

Proposition 4.1. *Let I be an ideal of $C(X)$. Then $C(X)/I$ is a classical ring if and only if* $\bigcup_{p\in\theta(I)} M^p = \bigcup_{P\in Min(I)} P$.

If we consider $I = O^p$, for some $p \in \beta X$ in Proposition 4.1, then $C(X)/O^p$ is a classical ring if and only if M^p coincides with the union of all minimal prime ideals contained in M^p. Any point $p \in \beta X$ with this property is called a UMP-point by the authors in [4]. They also called a space Y a UMP-space if every maximal ideal M of $C(Y)$ is the union of minimal prime ideals contained in M. Clearly, every P-space is a UMP-space and every UMP-space is an almost P-space but neither of the converses is true; see [4] for more details.

In the next proposition we observe that X is a UMP-space if and only if $\mathrm{depth}(C(X)/O^p) = 0$ for every $p \in \beta X$. This proposition also gives a topological characterization of UMP-points. First, we need the following lemma

Lemma 4.2. *Let $x \in \beta X$ and $f \in C(X)$. Then $f \in \bigcup_{P \in Min(O^x)} P$ if and only if there exists $g \in C(X)$ such that $x \in \mathrm{cl}_{\beta X} \mathrm{coz}\, g \subseteq \mathrm{cl}_{\beta X} Z(f)$.*

Proof. Let $f \in \bigcup_{P \in Min(O^x)} P$. Then $f \in P$ for some $P \in Min(O^x)$ and hence $fg = 0$ for some $g \notin P$. Thus, $\mathrm{coz}\, g \subseteq Z(f)$ and $g \notin O^x$ as $O^x \subseteq P$. Now, using Lemma 2.3 in [4], we have $x \in \mathrm{cl}_{\beta X} \mathrm{coz}\, g \subseteq \mathrm{cl}_{\beta X} Z(f)$. Conversely, let there exists $g \in C(X)$ such that $x \in \mathrm{cl}_{\beta X} \mathrm{coz}\, g \subseteq \mathrm{cl}_{\beta X} Z(f)$. Then $\mathrm{coz}\, g \subseteq Z(f)$ implies that $fg = 0$ and $x \in \mathrm{cl}_{\beta X} \mathrm{coz} g$ implies that $g \notin O^x$, by the same lemma. Therefore, there exists $P \in Min(O^x)$ such that $g \notin P$. Now, $fg = 0 \in P$ implies $f \in \bigcup_{P \in Min(O^x)} P$. □

Using Lemma 4.2 and the argument preceding this lemma, the following proposition is now evident.

Proposition 4.3. *The following statements are equivalent for every $p \in \beta X$.*

(1) *The point p is a UMP-point.*
(2) *If $p \in \mathrm{cl}_{\beta X} Z(f)$ for some $f \in C(X)$, then there exists $g \in C(X)$ such that $p \in \mathrm{cl}_{\beta X} \mathrm{coz}\, g \subseteq \mathrm{cl}_{\beta X} Z(f)$.*
(3) *The factor ring $C(X)/O^p$ is classical, i.e.,* $\mathrm{depth}(C(X)/O^p) = 0$.

Corollary 4.4. *Let $x \in X$. Then $C(X)/O_x$ is a classical ring if and only if whenever $x \in Z(f)$ for some $f \in C(X)$, then there exists $g \in C(X)$ such that $x \in \mathrm{cl}_X \mathrm{coz}\, g \subseteq Z(f)$.*

In the rest of the paper, we are to obtain equivalent topological conditions for which some factor rings of $C(X)$ modulo some familiar fixed closed ideals of $C(X)$ such as annihilator of a principal ideal and the smallest z-ideal (or z°-ideal) containing a principal ideal are classical or equivalently have depth zero. Note that the aforementioned ideals are closed ideal, i.e., each of them is of the form of an intersection of maximal ideals. In fact, if $f \in C(X)$, then $\mathrm{Ann}(f) = M_{\mathrm{cl}_X \mathrm{coz} f}$, $M_f = M_{Z(f)}$ and $P_f = M_{\mathrm{cl}_X \mathrm{int}_X Z(f)}$. First, we focus our attention to the annihilator ideals. We need the following lemma and corollary before giving respective result.

Lemma 4.5. *Let $f, g \in C(X)$. Then $g + \mathrm{Ann}(f)$ is a zerodivisor in $C(X)/\mathrm{Ann}(f)$ if and only if $\mathrm{int}_X Z(g) \nsubseteq Z(f)$.*

Proof. Let $\mathrm{int}_X Z(g) \nsubseteq Z(f)$ and take $x \in \mathrm{int}_X Z(g) \setminus Z(f)$. Define $h \in C(X)$ such that $h(X \setminus \mathrm{int}_X Z(g)) = 0$ and $h(x) = 1$. Hence, $hf \neq 0$ and $hgf = 0$, i.e., $g + \mathrm{Ann}(f)$ is a zerodivisor in $C(X)/\mathrm{Ann}(f)$. Conversely, suppose that $g + \mathrm{Ann}(f)$ is a zerodivisor and $\mathrm{int}_X Z(g) \subseteq Z(f)$. Therefore, there exists $h \in C(X)$ such that $hf \neq 0$ and $hgf = 0$.

So $Z(f) \cup Z(g) \cup Z(h) = X$ and hence $Z(f) \cup Z(h) = Z(f) \cup \mathrm{int}_X Z(g) \cup Z(h) = X$ (note that $Z(t) \cup Z(s) = X$ implies $X \setminus Z(t) \subseteq Z(s)$, whence $X \setminus Z(t) \subseteq \mathrm{int}_X Z(s)$, so $Z(t) \cup \mathrm{int}_X Z(s) = X$) which implies that $fh = 0$, a contradiction. □

Corollary 4.6. *Let $f \in C(X)$. Then*

$$r(C(X)/\mathrm{Ann}(f)) = \{g + \mathrm{Ann}(f) : \mathrm{int}_X Z(g) \subseteq Z(f)\}.$$

By the following proposition, every cozero-set whose closure is an almost P-space must be closed.

Proposition 4.7. *Let $f \in C(X)$. Then the following statements are equivalent.*

(1) *The depth of $C(X)/\mathrm{Ann}(f)$ is zero.*
(2) *The zero-set $Z(f)$ is open and $\mathrm{coz} f$ is an almost P-space.*
(3) *The subspace $\mathrm{cl}_X \mathrm{coz} f$ is an almost P-space.*

Proof. (1) ⇒ (2). Since $C(X)/\mathrm{Ann}(f)$ is a classical ring by our hypothesis, we have

$$r(C(X)/\mathrm{Ann}(f)) = U(C(X)/\mathrm{Ann}(f)).$$

Since $f + \mathrm{Ann}(f) \in r(C(X)/\mathrm{Ann}(f))$ by Corollary 4.6, it must be a unit. Now, using Lemma 1.5, there exists $h \in \mathrm{Ann}(f)$ such that $Z(h) \cap Z(f) = \emptyset$. But $hf = 0$ implies that $Z(f) \cup Z(h) = X$. Hence, $Z(f)$ and $Z(h)$ are both open sets.

To prove that $\mathrm{coz} f$ is an almost P-space, suppose on the contrary, that there exists $x \in \mathrm{coz} f$ which is not an almost P-point of $\mathrm{coz} f$. Thus, using Lemma 1.4, x is not an almost P-point of X as well. Then there exists $r \in r(X)$ such that $x \in Z(r)$. Now, $r + \mathrm{Ann}(f)$ is not a unit, in fact, if there exists $g \in C(X)$ such that $rg - 1 \in \mathrm{Ann}(f)$, then $x \in Z(r) \subseteq X \setminus Z(rg - 1) \subseteq Z(f)$, i.e., $x \in \mathrm{coz} f \cap Z(f)$ which is impossible. Therefore, $r + \mathrm{Ann}(f)$ should be a zerodivisor by our hypothesis. Hence, using Lemma 4.5, $\emptyset = \mathrm{int}_X Z(r) \not\subseteq Z(f)$, a contradiction. So, $\mathrm{coz} f$ is an almost P-space.

(2) ⇒ (1). Since $Z(f)$ is open, $\mathrm{Ann}(f) = (g)$, for some $g \in C(X)$ with $Z(g) = \mathrm{coz} f$. But $Z(g)$ is an almost P-space, hence every point of $Z(g)$ is an almost P-point of X by Lemma 1.4. Now, using Theorem 2.5(1), $\mathrm{depth}(C(X)/\mathrm{Ann}(f))) = \mathrm{depth}(C(X)/(g)) = 0$

(2) ⇒ (3). It is evident.

(3) ⇒ (2). It is enough to show that $\mathrm{cl}_X \mathrm{coz}\, f = \mathrm{coz}\, f$. Let $\hat{f} := f|_{\mathrm{cl}_X \mathrm{coz}\, f} \in C(\mathrm{cl}_X \mathrm{coz}\, f)$. Since $\hat{f}$ is nonzero at every point of $\mathrm{coz}\, f$, then $\hat{f}$ is nonzero at every point of $\mathrm{cl}_X \mathrm{coz}\, f$ as $\mathrm{coz}\, f$ is a dense subset of the almost P-space $\mathrm{cl}_X \mathrm{coz}\, f$, by the argument preceding Lemma 1.3. This means that $\mathrm{cl}_X \mathrm{coz}\, f \subseteq \mathrm{coz}\, \hat{f} = \mathrm{coz}\, f$, and we are done. □

Now, we are to investigate functions $f \in C(X)$ and topological conditions on X for which $\mathrm{depth}(C(X)/M_f) = 0$. First, we need the following lemma which characterizes the set of all regular elements of the factor ring $C(X)/M_A$, where A is an arbitrary subset of X. Note that if Y is a subspace of a topological space X and $x \in A \subseteq Y \subseteq X$, then $x \in \mathrm{int}_Y A$ if and only if there exists an open set G in X containing x such that $G \cap (Y \setminus A) = \emptyset$. Thus, A has empty interior in Y if and only if for each element $x \in A$, every neighborhood of x intersects $Y \setminus A$.

Lemma 4.8. *Let A be a subset of X. Then*

$$r(C(X)/M_A) = \{g + M_A : \mathrm{int}_A(Z(g) \cap A) = \emptyset\}$$

Proof. Let $g \in C(X)$ and $\mathrm{int}_A(Z(g) \cap A) = \emptyset$. Suppose $h \in C(X)$ and $gh \in M_A$. Thus, $A \subseteq Z(g) \cup Z(h)$ and so $A \setminus Z(g) \subseteq Z(h)$. Since $A \setminus Z(g) = A \setminus (Z(g) \cap A)$ and $Z(g) \cap A$ has empty interior with respect to A, we conclude that $A \setminus Z(g)$ is a dense subset of A. Therefore,

$$A = \mathrm{cl}_A(A \setminus Z(g)) \subseteq \mathrm{cl}_X(A \setminus Z(g)) \subseteq Z(h).$$

Hence $h \in M_A$, which implies that $g + M_A$ is a regular element of $C(X)/M_A$.

Conversely, let $g \in C(X)$ and $\mathrm{int}_A(Z(g) \cap A) \neq \emptyset$. Take $a \in \mathrm{int}_A(Z(g) \cap A)$. There exists an open set V in X containing a such that $V \cap (A \setminus Z(g)) = \emptyset$, by the argument preceding the lemma. On the other hand, there is a function $h \in C(X)$ such that $h(a) = 1$ and $h(X \setminus V) = 0$. Thus, $A \subseteq Z(g) \cup Z(h)$, which means $gh \in M_A$. Now, as $h \notin M_A$ we conclude that $g + M_A$ is a zerodivisor. □

Proposition 4.9. *Let f be a non-unit element in $C(X)$. Then the following statements are equivalent.*

(1) *The depth of $C(X)/M_f$ is zero.*
(2) *Every nonempty G_δ-set contained in $Z(f)$ has nonempty interior in $Z(f)$, i.e., whenever $G \subseteq Z(f)$ is a G_δ-set in X, then $\mathrm{int}_{Z(f)}G \neq \emptyset$.*
(3) *If g is a non-unit element in $C(X)$ and $M_f \subseteq M_g$, then $g|_{Z(f)}$ is a zerodivisor in $C(Z(f))$.*
(4) *The zero-set $Z(f)$ is an almost P-space.*

Proof. Clearly, parts (2) and (3) are equivalent. Also, (2) is equivalent to (4) since every G_δ-subset of $Z(f)$ is a G_δ-subset of X contained in $Z(f)$. So, it is enough to show that (1) implies (2), and (4) implies (1).

(1) ⇒ (2). Let $\mathrm{depth}(C(X)/M_f) = 0$. Suppose, on the contrary, that there exists a nonempty G_δ-set G contained in $Z(f)$ with $\mathrm{int}_{Z(f)}G = \emptyset$. Clearly there is $g \in C(X)$ such that $\emptyset \neq Z(g) \subseteq G \subseteq Z(f)$. Thus, $\mathrm{int}_{Z(f)}Z(g) = \emptyset$ and hence $g + M_f \in r(C(X)/M_f)$ by Lemma 4.8. Since $Z(g) \cap Z(f) \neq \emptyset$, $g + M_f$ is a non-unit in $C(X)/M_f$ by Lemma 1.5. Therefore, $\mathrm{depth}(C(X)/M_f) \geq 1$, a contradiction.

(4) ⇒ (1). Let $g + M_f$ be a non-unit element of $C(X)/M_f$. Then by Lemma 1.5 we have $Z(g) \cap Z(f) \neq \emptyset$. Since $Z(g) \cap Z(f)$ is a nonempty zero-set in $Z(f)$ and $Z(f)$ is an

almost P-space, we have $\mathrm{int}_{Z(f)}(Z(g) \cap Z(f)) \neq \emptyset$. Therefore, $g + M_f$ is a zerodivisor, by Lemma 4.8, i.e., $\mathrm{depth}(C(X)/M_f) = 0$. $\square$

If every point of a zero-set $Z(f)$ is an almost P-point of X, then for every nonempty G_δ-set G of X contained in $Z(f)$, we have $\mathrm{int}_{Z(f)} G \neq \emptyset$. Thus, by Proposition 4.9, $\mathrm{depth}(C(X)/M_f) = 0$, or equivalently $Z(f)$ is an almost P-space. But the converse is not necessarily true. For instance, if we take the identity function $i \in C(\mathbb{R})$ and $f = i(i-1)$, then clearly each point of $Z(f) = \{0, 1\}$ is a non-almost P-point of $\mathbb{R}$. But $\mathrm{depth}(C(X)/M_f) = 0$ by the above proposition. More generally, if $Z(f)$ is a discrete subset of a topological space X, then $\mathrm{depth}(C(X)/M_f) = 0$.

Recall that a closed subset A of a topological space X is regular closed whenever $\mathrm{cl}_X \mathrm{int}_X A = A$. Notice that for every $f \in C(X)$, we have $M_f = M_{Z(f)}$ and $P_f = M_{\mathrm{cl}_X \mathrm{int}_X Z(f)}$. Thus, $Z(f)$ is regular closed if and only if $M_f = P_f$. Using Lemma 1.3, whenever every point of a zero-set Z is an almost P-point of X, then it is regular closed, but the converse is not true. For example, consider the zero-set $[0, 1] \subseteq \mathbb{R}$, which is regular closed, but non of its point is an almost P-point of $\mathbb{R}$. By the following corollary whenever $f \in C(X)$ and $Z(f)$ is regular closed containing at least one non-almost P-point, then we have $\mathrm{depth}(C(X)/M_f) \geq 1$

Corollary 4.10. *Let $f \in C(X)$. Then $Z(f)$ is regular closed and* $\mathrm{depth}(C(X)/M_f) = 0$ *if and only if every point of $Z(f)$ is an almost P-point of X.*

Proof. Let every point of $Z(f)$ be an almost P-point of X. Then by the argument preceding the corollary, we conclude that $Z(f)$ is regular closed and also $\mathrm{depth}(C(X)/M_f) = 0$.

Conversely, suppose on the contrary that there is $x \in Z(f)$ which is not an almost P-point of X. Then there exists $r \in r(X)$ such that $x \in Z(r)$. Hence, $\emptyset \neq Z(r) \cap Z(f) \subseteq Z(r) \cap Z(g)$, for each $g \in M_f$. Therefore, the element $r + M_f \in C(X)/M_f$ is not a unit by Lemma 1.5. Also $r + M_f$ is not a zerodivisor. For otherwise if $rg \in M_f = P_f$, for some $g \in C(X)$, then $\mathrm{int}_X Z(f) \subseteq \mathrm{int}_X (Z(r) \cup Z(g)) = \mathrm{int}_X Z(g)$, i.e., $g \in P_f = M_f$ and this implies that $\mathrm{depth}(C(X)/M_f) \geq 1$, a contradiction. $\square$

The following proposition shows that the implication (1) $\Rightarrow$ (4) in Proposition 4.9 also holds for each subset A of X, i.e., whenever the depth of $C(X)/M_A$ is zero, then A is an almost P-space. The converse is true if A is C-embedded, but it is not true if A is even C^*-embedded as we see in Example 4.12 below.

Proposition 4.11. *Let A be a subset of X. If* $\mathrm{depth}(C(X)/M_A) = 0$, *then A is an almost P-space. The converse is also true whenever the subset A is C-embedded.*

Proof. Let $\mathrm{depth}(C(X)/M_A) = 0$. Suppose, on the contrary, that there exists a non-almost P-point $a \in A$. Then there is a non-unit regular element r in $C(A)$ (i.e., $\mathrm{int}_A Z(r) = \emptyset$) such that $a \in Z(r)$. Since $Z(r)$ is a G_δ-subset of A, we have $Z(r) =$

$\bigcap_{n\in\mathbb{N}}(G_n \cap A)$, where G_n is an open subset of X, for every $n \in \mathbb{N}$. Now, there exists $g \in C(X)$ such that $a \in Z(g) \subseteq \bigcap_{n\in\mathbb{N}} G_n$. Therefore,

$$\emptyset \neq Z(g) \cap A \subseteq \left(\bigcap_{n\in\mathbb{N}} G_n\right) \cap A = \bigcap_{n\in\mathbb{N}} (G_n \cap A) = Z(r).$$

By Lemma 1.5, $g + M_A$ is a non-unit element of $C(X)/M_A$. On the other hand, since $\mathrm{int}_A(Z(g) \cap A) \subseteq \mathrm{int}_A Z(r) = \emptyset$, using Lemma 4.8 we conclude that $g + M_A$ is a regular element of $C(X)/M_A$, a contradiction.

To prove the second part of the proposition, let A be a C-embedded subset of X. Clearly the homomorphism $\eta : C(X) \to C(A)$ defined by $\eta(f) = f|_A$ is onto. Since $Ker\,\eta = \{g \in C(X) : A \subseteq Z(g)\} = M_A$, we have $C(X)/M_A \cong C(A)$. Now, $\mathrm{depth}(C(X)/M_A) = 0$ if and only if $\mathrm{depth}(C(A)) = 0$ which is equivalent to saying that A is an almost P-space. $\square$

The following example shows that the condition "A is C-embedded" is essential in Proposition 4.11.

Example 4.12. Consider the non-normal space $\Lambda = \beta\mathbb{R} \setminus (\beta\mathbb{N} \setminus \mathbb{N})$ presented in 6P in [6]. The subset $\mathbb{N}$ is C^*-embedded in Λ, but it is not C-embedded. Using Theorem 1.18 in [6], there exists a zero-set $Z(g)$ in Λ disjoint from $\mathbb{N}$ which is not completely separated from $\mathbb{N}$. Thus $Z(g) \cap Z(f) \neq \emptyset$, for every $f \in M_{\mathbb{N}}$ and so $g + M_{\mathbb{N}}$ is a non-unit element of $C(\Lambda)/M_{\mathbb{N}}$, by Lemma 1.5. On the other hand, $Z(g) \cap \mathbb{N} = \emptyset$ implies that $\mathrm{int}_{\mathbb{N}}(Z(g) \cap \mathbb{N}) = \emptyset$. Therefore, $g + M_{\mathbb{N}}$ is regular, by Lemma 4.8. Thus, $\mathrm{depth}(C(\Lambda)/M_{\mathbb{N}}) \geq 1$.

We conclude this section by investigating conditions for which $\mathrm{depth}(C(X)/P_f) = 0$, where $f \in C(X)$. If X is a compact (or a normal) space, we have $\mathrm{depth}(C(X)/P_f) = 0$ if and only if $\mathrm{cl}_X \mathrm{int}_X Z(f)$ is an almost P-space by Proposition 4.11 as $P_f = M_{\mathrm{cl}_X \mathrm{int}_X Z(f)}$. Also, whenever $Z(f)$ is regular closed, we observe by Corollary 4.10 that $\mathrm{depth}(C(X)/P_f) = 0$ if and only if every point of $Z(f)$ is an almost P-point of X, as $P_f = M_f$ in this case. For the general case, we need the following lemma which characterizes the zerodivisors of $C(X)/P_f$. First, let $\mathcal{P}$ be a family of minimal prime ideals of $C(X)$ and consider $I = \bigcap \mathcal{P}$. Then $\bigcup_{P\in Min(I)} P = \bigcup \mathcal{P}$. In fact, $\bigcup \mathcal{P} \subseteq \bigcup_{P\in Min(I)} P$ as $\mathcal{P} \subseteq Min(I)$, and to prove the reverse inclusion, let $f \notin \bigcup \mathcal{P}$ and suppose $fg \in I$ for some $g \in C(X)$. Thus, $g \in I$ which means $f + I$ is a non-zerodivisor in $C(X)/I$, so $f \notin \bigcup_{P\in Min(I)} P$.

Lemma 4.13. *Let $f, g \in C(X)$. Then $g + P_f$ is a zerodivisor element of $C(X)/P_f$ if and only if* $\mathrm{int}_X Z(f) \cap \mathrm{int}_X Z(g) \neq \emptyset$.

Proof. Let $g + P_f$ be a zerodivisor in $C(X)/P_f$. Then

$$g \in \bigcup_{Q \in Min(P_f)} Q = \bigcup_{f \in Q \in Min(C(X))} Q,$$

by the argument preceding the lemma, as $P_f = \bigcap_{f \in Q \in Min(C(X))} Q$. Thus, there exists a minimal prime ideal Q_0 of $C(X)$ containing g such that $P_f \subseteq Q_0$. Also, there is $h \notin Q_0$ such that $gh = 0$, as $C(X)$ is a reduced ring. Therefore, $h \notin P_f$ implies that $\mathrm{int}_X Z(f) \nsubseteq Z(h)$, so $\mathrm{int}_X Z(f) \cap \mathrm{coz}\, h \neq \emptyset$. Now, since $\mathrm{coz}\, h \subseteq Z(g)$ (whence $\mathrm{coz}\, h \subseteq \mathrm{int}_X Z(g)$), we conclude that $\mathrm{int}_X Z(f) \cap \mathrm{int}_X Z(g) \neq \emptyset$. Conversely, let $y \in \mathrm{int}_X Z(f) \cap \mathrm{int}_X Z(g)$. Define $h \in C(X)$ such that $h(y) = 1$ and $h(X \setminus (\mathrm{int}_X Z(f) \cap \mathrm{int}_X Z(g))) = 0$. Thus, $hg = 0 \in P_f$ but $h \notin P_f$, as $y \in \mathrm{int}_X Z(f) \setminus Z(h)$ which means $g + P_f$ is a zerodivisor in $C(X)/P_f$. □

Corollary 4.14. *Let $f \in C(X)$. Then*

$$r(C(X)/P_f) = \{g + P_f : \mathrm{int}_X Z(g) \cap \mathrm{int}_X Z(f) = \emptyset\}$$

Before presenting the next proposition, we should emphasize here that a maximal z°-ideal in $C(X)$ is an ideal which is maximal among the collection of all z°-ideals of $C(X)$ and it is not necessarily a maximal ideal. But a maximal ideal which is a z°-ideal is also a maximal z°-ideal; see [3] for more details of maximal z°-ideals of $C(X)$.

Proposition 4.15. *Let $f \in C(X)$. Then the following statements are equivalent.*

(1) *The depth of $C(X)/P_f$ is zero.*
(2) *Every maximal ideal of $C(X)$ containing P_f is a z°-ideal.*
(3) *If $x \in \theta(P_f)$, then $g \in C(X)$ and $x \in \mathrm{cl}_{\beta X} Z(g)$ imply that $x \in \mathrm{cl}_{\beta X} \mathrm{int}_X Z(g)$.*

Proof. (1) ⇒ (2). Let M be a maximal ideal of $C(X)$ containing P_f and $g \in M$. Then $g + P_f$ is a non-unit element of $C(X)/P_f$ and thus it is a zerodivisor by our hypothesis. Hence, using Lemma 4.13, we have $\emptyset \neq \mathrm{int}_X Z(f) \cap \mathrm{int}_X Z(g) \subseteq \mathrm{int}_X Z(g)$ which means g is a zerodivisor of $C(X)$, by Lemma 1.2. Therefore, every element of M is a zerodivisor and since M is maximal, it is a z°-ideal by Theorem 1.21 in [2].

(2) ⇒ (1). Let $r + P_f$ be a non-unit element of $C(X)/P_f$. Using Lemma 1.5, $Z(r) \cap Z(g) \neq \emptyset$ for every $g \in P_f$, and thus there is a maximal ideal M containing both r and P_f. Since $r^2 + f^2 \in M$ and M is a z°-ideal by (2), $r^2 + f^2$ is a zerodivisor and so by Lemma 1.2, $\mathrm{int}_X Z(f) \cap \mathrm{int}_X Z(r) = \mathrm{int}_X Z(f^2 + r^2) \neq \emptyset$. Now, using Lemma 4.13, $r + P_f$ is a zerodivisor of $C(X)/P_f$, which means $\mathrm{depth}(C(X)/P_f) = 0$.

(2) ⇔ (3). If (2) holds, then $x \in \theta(P_f) \cap \mathrm{cl}_{\beta X} Z(g)$ implies that $g \in M^x$ which contains P_f, so M^x is a z°-ideal. Therefore, M^x is a maximal z°-ideal and using Proposition 3.2 in [3], $x \in \mathrm{cl}_{\beta X} \mathrm{int}_X Z(g)$. Conversely, suppose (3) holds and M^x, $x \in \beta X$ is a maximal ideal containing P_f. Let $g \in M^x$. Then by the hypothesis we have $x \in \mathrm{cl}_{\beta X} \mathrm{int}_X Z(g)$

which implies that $\mathrm{int}_X Z(g) \neq \emptyset$, i.e., g is a zerodivisor, so every element of M^x is a zerodivisor and using Theorem 1.21 in [2], M^x is a $z°$-ideal. □

Corollary 4.16. *Let X be a compact space. Then the following statements hold for each $f \in C(X)$.*

(1) *The depth of $C(X)/P_f$ is zero.*
(2) *The subspace $\mathrm{cl}_X \mathrm{int}_X Z(f)$ is an almost P-space.*
(3) *Every element of $\mathrm{cl}_X \mathrm{int}_X Z(f)$ is an almost P-point of X.*

Proof. Using the argument preceding Lemma 4.13, the equivalence of (1) and (2) is clear. Since $X \cap \theta(P_f) = \mathrm{cl}_X \mathrm{int}_X Z(f)$, (1) implies (3) by Lemma 1.3 and part (3) of the above proposition. To complete the proof we show that (3) implies (1). Let every element of $\mathrm{cl}_X \mathrm{int}_X Z(f)$ be an almost P-point of X and $r + P_f$ be a non-unit in $C(X)/P_f$. Since X is compact, $Z(r) \cap \mathrm{cl}_X \mathrm{int}_X Z(f) \neq \emptyset$, by Lemma 1.5. If $x \in Z(r) \cap \mathrm{cl}_X \mathrm{int}_X Z(f)$, then $x \in Z(r) \cap Z(f)$ and since x is an almost P-point of X by our hypothesis, $\mathrm{int}_X Z(r) \cap \mathrm{int}_X Z(f) \neq \emptyset$. Now, using Lemma 4.13, we conclude that $r + P_f$ is a zerodivisor, so $\mathrm{depth}(C(X)/P_f) = 0$. □

Remark 4.17. Using the argument preceding Proposition 4.1, a reduced ring is classical if and only if every maximal ideal of the ring consists entirely of zerodivisors. Moreover, if R is a reduced ring with "property A" (i.e., every finitely generated ideal of R consisting of zerodivisors has a nonzero annihilator), then R is a classical ring if and only if every maximal ideal of R is a $z°$-ideal or equivalently every z-ideal of R is a $z°$-ideal. In fact, in such rings a maximal ideal M consists entirely of zerodivisors if and only if M is a $z°$-ideal; see Theorem 1.21 in [2]. Thus, whenever I is a semiprime ideal of $C(X)$, then $C(X)/I$ is a classical ring if and only if every maximal ideal M/I in $C(X)/I$ is a $z°$-ideal. In contrast to part (2) of the above proposition, if M is a maximal ideal containing I and if M/I is a $z°$-ideal of $C(X)/I$, then M is not necessarily a $z°$-ideal of $C(X)$. For example, whenever $r \in r(X)$, then every maximal ideal $M/(r)$ in $C(X)/(r)$ is a $z°$-ideal, by Corollary 2.6. But M is not a $z°$-ideal in $C(X)$. Moreover, a maximal ideal M containing an ideal $I \subseteq C(X)$, may be a $z°$-ideal, but M/I may not be a $z°$-ideal in $C(X)/I$. For instance, consider a $z°$-ideal M_x in $C(X)$, where x is an almost P-point of X which is not a P-point (resp., which is not a UMP-point). Let Q be an arbitrary prime ideal contained in M_x. Then M_x/Q (resp., M_x/O_x) is not a $z°$-ideal in $C(X)/Q$ (resp., $C(X)/O_x$).

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References

[1] F. Azarpanah, Intersection of essential ideals in $C(X)$, Proc. Amer. Math. Soc. 125 (7) (1997) 2149–2154.
[2] F. Azarpanah, O.A.S. Karamzadeh, A. Rezai Aliabad, On ideals consisting entirely of zero divisors, Comm. Algebra 28 (2) (2000) 1061–1073.
[3] F. Azarpanah, A.R. Salehi, Ideal structure of the classical ring of quotients of $C(X)$, Topology Appl. 209 (2016) 170–180.
[4] B. Banerjee, S.K. Ghosh, M. Henriksen, Union of minimal prime ideals in rings of continuous functions on compact spaces, Algebra Universalis 62 (2009) 239–246.
[5] R. Engelking, General Topology, Sigma Ser. Pure Math., vol. 6, Heldermann Verlag, Berlin, 1989.
[6] L. Gillman, M. Jerison, Rings of Continuous Functions, Springer-Verlag, 1976.
[7] T.Y. Lam, Lectures on Modules and Rings, Graduate Texts in Mathematics, vol. 189, Springer-Verlag, New York, 1999.
[8] R. Levy, Almost P-spaces, Canad. J. Math. XXIX (2) (1977) 284–288.
[9] G. Mason, z-ideals and prime ideals, J. Algebra 26 (1973) 280–297.