Equivariant isomorphisms of Ext and Tor modules

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In this article we establish equivariant isomorphisms of Ext and Tor modules over different relative complete intersections. More precisely, for a commutative ring Q, this paper investigates how $\operatorname{Ext}^*_{Q/(\boldsymbol{f})}(M, N)$ and $\operatorname{Tor}_*^{Q/(\boldsymbol{f})}(M, N)$ change when one varies $\boldsymbol{f}$ among all Koszul-regular sequences of a fixed length such that $\boldsymbol{f}M = 0$ and $\boldsymbol{f}N = 0$. Of notable interest is how the theory of perturbations is used to establish isomorphisms of certain DG modules.

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1. Introduction

Fix a commutative ring Q and a pair of Q-modules M and N. In this paper, we study the following problem: *How do $\operatorname{Ext}^*_{Q/(\boldsymbol{f})}(M, N)$ and $\operatorname{Tor}_*^{Q/(\boldsymbol{f})}(M, N)$ change as we vary $\boldsymbol{f}$ among all Q-regular sequences of a fixed length such that $\boldsymbol{f}M = \boldsymbol{f}N = 0$?*

This has been studied when Q is local with residue field k, $\boldsymbol{f}$ is a single Q-regular element and $M = k$ (see [1], [4], [6], and [10]). The strongest result in this direction is the following theorem due to Avramov and Iyengar [6, 2.1(2)]: *Let $(Q, \mathfrak{n}, k)$ be a commutative noetherian local ring and I an ideal of Q. If f and g are Q-regular elements in I such that $f - g \in \mathfrak{n}I$, then there is an isomorphism of graded k-vector spaces*

$$\mathrm{Tor}_*^{Q/(f)}(k, N) \cong \mathrm{Tor}_*^{Q/(g)}(k, N)$$

for each complex of Q/I-modules N.

One of the main results of this article, which can be found in Theorem 5.7, is the following generalization of [6, 2.1(2)]:

Theorem. *Let Q be a commutative ring, $\boldsymbol{f} = f_1, \dots, f_n$ and $\boldsymbol{f}' = f_1' \dots, f_n'$ be Q-regular sequences in an ideal I of Q, and M a Q/I-module. If $f_i - f_i' \in \mathrm{ann}_Q(M)I$ for each i, then for each complex of Q/I-modules N we have isomorphisms of graded Q/I-modules:*

(1) $\mathrm{Ext}_{Q/(\boldsymbol{f})}^*(M, N) \cong \mathrm{Ext}_{Q/(\boldsymbol{f}')}^*(M, N)$
(2) $\mathrm{Tor}_*^{Q/(\boldsymbol{f})}(M, N) \cong \mathrm{Tor}_*^{Q/(\boldsymbol{f}')}(M, N)$

In fact, more is shown in Theorem 5.7. Loosely speaking, the isomorphisms in (1) and (2), above, respect the cohomology operators which were first introduced by Gulliksen in [9] and later studied by Avramov and Buchweitz [3], Eisenbud [7], and many others. As a consequence, even of the weaker result displayed above, if Q is local then the Bass series (or Poincaré series) of the pair (M, N) is the same over $Q/(\boldsymbol{f})$ as over $Q/(\boldsymbol{f}')$ (cf. 5.8 and Corollary 5.9 for precise statements). Hence, other homological invariants, like complexity (or Tor-complexity), are the same for the pair (M, N) when computed over the ring $Q/(\boldsymbol{f})$ and computed over $Q/(\boldsymbol{f}')$. Furthermore, Theorem 5.10 and Corollary 5.11 recover results of Shamash in [11] by specializing to the case where $\boldsymbol{f}$ consists of a single element and N is fixed as the residue field.

Secondly, it is worth remarking that the techniques developed, and used, in this document differ from those used in [6, 2.1(2)]; these techniques certainly add value to this paper by providing a framework to better understand perturbations of a complex. It is the author's hope that the techniques of Section 4 will be of particular interest to researchers in various fields, not just commutative algebraist.

In this paper, we also give a generalization of [6, 2.1(3)]. Again, as a consequence of the next theorem below, we obtain information regarding the numerical data of (co)homology modules associated to pairs of modules over a relative complete intersection (i.e., a commutative ring modulo a regular sequence).

Theorem. *Let Q be a commutative ring and let $\boldsymbol{f} = f_1, \dots, f_n$ be a Q-regular sequence in an ideal I of Q, and M a Q/I-module. If $(\boldsymbol{f}) \subseteq I \mathrm{ann}_Q(M)$, we have isomorphisms of graded Q/I-modules:*

(3) $\mathrm{Ext}^*_{Q/(\boldsymbol{f})}(M,N) \cong \mathcal{P} \otimes_Q \mathrm{Ext}^*_Q(M,N)$
(4) $\mathrm{Tor}^{Q/(\boldsymbol{f})}_*(M,N) \cong \mathcal{P}^* \otimes_Q \mathrm{Tor}^Q_*(M,N)$

where $\mathcal{P} = Q/I[\chi_1,\ldots,\chi_n]$, *each* χ_i *has cohomological degree two and* $\mathcal{P}^*$ *is the graded* Q*-linear dual of* $\mathcal{P}$.

In what follows, we give a brief outline of this paper. In Section 2, we review notation and tools from DG homological algebra which will be needed to discuss the content in the rest of the article. In Section 3, we discuss *universal resolutions.* This is mostly a summary of work in [3]. Universal resolutions allow one to equip projective resolutions of $Q/(\boldsymbol{f})$-modules with a structure of DG $\mathcal{S}$-module where $\mathcal{S}$ is a polynomial ring with n variables of cohomological degree two. We leverage this extra structure to obtain a finer result than the theorems mentioned above (see Section 5).

Section 4 contains most of the new ideas in this article. In [6, 2.1], the authors exploit that the minimal free resolution of the residue field over a local ring has a system of divided powers. Since the main result in this article is for arbitrary $Q/(\boldsymbol{f})$-modules, we do not have access to this tool in such a general situation. This is one of the major differences in the proof of [6, 2.1] and Theorems 5.7 and 5.10. Instead, Section 4 examines when two perturbations yield isomorphic DG modules. The main result in this section is Theorem 4.9. Finally, we apply the results in the previous sections in Section 5.

2. Homological preliminaries

Many results in this article depend on several homological constructions. In this section we set terminology and conventions, and list some basic properties regarding these constructions. This section can be skipped and referred to as needed. See [2], [5] or [8] as references.

Fix a commutative ring Q. Let $A = \{A_i\}_{i\in\mathbb{Z}}$ denote a DG Q-algebra. In this article, we will always assume that A is graded-commutative. By a DG A-module, we mean a *left* DG A-module.

2.1. Let M be a complex of Q-modules. The differential of M is denoted by ∂^M. The *boundaries* and *cycles of* M are denoted by $\mathrm{B}(M) := \{\mathrm{Im}\,\partial^M_{i+1}\}_{i\in\mathbb{Z}}$ and $\mathrm{Z}(M) := \{\mathrm{Ker}\,\partial^M_i\}_{i\in\mathbb{Z}}$, respectively. The *homology of* M is defined to be

$$\mathrm{H}(M) := \mathrm{Z}(M)/\mathrm{B}(M) = \{\mathrm{H}_i(M)\}_{i \in \mathbb{Z}}$$

which is a graded Q-module.

2.2. Let M be a DG A-module. We let $M^\natural$ denote the underlying graded Q-module. Note that $A^\natural$ is a graded Q-algebra and $M^\natural$ is a graded $A^\natural$-module. Also, $\mathrm{H}(M)$ is a graded $\mathrm{H}(A)$-module.

2.3. We say that $\alpha : M \to N$ is a degree d-map from M to N is a family of Q-linear maps $\alpha = \{\alpha_i : M_i \to N_{i+d}\}_{i \in \mathbb{Z}}$ such that

$$\alpha(am) = (-1)^{d|a|} a\alpha(m)$$

for all $a \in A$ and $m \in M$.

2.4. We define $\mathrm{Hom}_A(M, N)$ to be the DG A-module $\mathrm{Hom}_A(M, N) = \{\mathrm{Hom}_A(M, N)_i\}_{i \in \mathbb{Z}}$ determined by

$$\mathrm{Hom}_A(M, N)_d := \{\alpha : M \to N : \alpha \text{ is a degree } d \text{ map}\},$$
$$\partial^{\mathrm{Hom}_A(M,N)} := \mathrm{Hom}(M, \partial^N) - \mathrm{Hom}(\partial^M, N), \text{ and}$$
$$a \cdot \alpha := a\alpha(-) = (-1)^{d|a|}\alpha(a \cdot -).$$

We remark that $\mathrm{Hom}_A(M, N)$ is a subcomplex of $\mathrm{Hom}_Q(M, N)$.

2.5. Let $\alpha \in \mathrm{Hom}_A(M, N)_0$. We say that α is a morphism of DG A-modules if $\alpha \in \mathrm{Z}_0(\mathrm{Hom}_A(M, N))$. Equivalently,

$$\alpha(am) = a\alpha(m)$$

for all $a \in A$ and $m \in M$.

2.6. Let M and N be DG A-modules. We say that degree d maps α and β from M to N are homotopic, denoted $\alpha \sim \beta$, if $\alpha - \beta \in \mathrm{B}_d(\mathrm{Hom}_A(M, N))$. That is, there exists $\tau \in \mathrm{Hom}_A(M, N)_{d+1}$ such that

$$\partial^N \tau - (-1)^{d+1} \tau \partial^N = \alpha - \beta.$$

A morphism of DG A-modules $\alpha : M \to N$ is a homotopy equivalence if there exists a morphism of DG A-modules $\beta : N \to M$ such that

$$\beta\alpha \sim \mathrm{id}^M \text{ and } \alpha\beta \sim \mathrm{id}^N.$$

A morphism of DG A-modules $\alpha : M \to N$ is a quasi-isomorphism if $\mathrm{H}(\alpha) : \mathrm{H}(M) \to \mathrm{H}(N)$ is an isomorphism of graded $\mathrm{H}(A)$-modules.

2.7. A DG A-module P is *semiprojective* if for every morphism of DG A-modules $\alpha : P \to N$ and each surjective quasi-isomorphism of DG A-modules $\gamma : M \to N$ there exists a unique up to homotopy morphism of DG A-modules $\beta : P \to M$ such that $\alpha = \gamma\beta$. Equivalently, $P^{\natural}$ is a projective graded $A^{\natural}$-module and $\mathrm{Hom}_A(P, -)$ preserves quasi-isomorphisms.

2.8. A *semiprojective resolution* of a DG A-module M is a surjective quasi-isomorphism of DG A-modules $\epsilon : P \to M$ where P is a semiprojective DG A-module. Semiprojective resolutions exist and any two semiprojective resolutions of M are unique up to homotopy equivalence.

3. Universal resolutions

Fix a commutative ring Q. Let $\boldsymbol{f} = f_1, \ldots, f_n$ be a list of elements in Q. Let

$$E := Q\langle \xi_1, \ldots, \xi_n | \partial \xi_i = f_i \rangle$$

be the Koszul complex on $\boldsymbol{f}$ over Q. That is, E is the DG Q-algebra with $E^{\natural}$ the exterior algebra on a free Q-module with basis $\xi_1, \ldots, \xi_n$ of homological degree 1, and differential induced by $\partial \xi_i = f_i$. Finally, as indexing is important for several of the constructions it is worth mentioning that in this article $\mathbb{N}$ denotes $\{0, 1, 2, 3, \ldots\}$.

Set $R := Q/(\boldsymbol{f})$. Via the augmentation map $E \to R$, every complex of R-modules is a DG E-module. Finally, let $\mathcal{S} := R[\chi_1, \ldots, \chi_n]$ be a graded polynomial ring where each χ_i has homological degree -2.

We will also need to refer to the graded R-linear dual of $\mathcal{S}$ throughout Section 3. Let Γ denote the graded R-linear dual of $\mathcal{S}$ and let $\{y^{(H)}\}_{H \in \mathbb{N}^n}$ be the R-basis of Γ dual to $\{\chi^H := \chi_1^{h_1} \ldots \chi_n^{h_n}\}_{H \in \mathbb{N}^n}$ the standard R-basis of $\mathcal{S}$. Then Γ is a graded $\mathcal{S}$-module via the action

$$\chi_i \cdot y^{(H)} := \begin{cases} y^{(h_1, \ldots, h_{i-1}, h_i - 1, h_{i+1}, \ldots, h_n)} & h_i \geq 1 \\ 0 & h_i = 0 \end{cases}$$

3.1. Let M be a DG E-module. We let λ_i^M denote left multiplication by ξ_i on M, and when M is clear from context we simply write λ_i. As E is graded-commutative it follows that $\lambda_i \in \mathrm{Hom}_E(M, M)_1$. Moreover, λ_i is a null-homotopy for $f \, \mathrm{id}^M$.

3.2. Let M be a DG E-module. Define $U_E(M)$ to be the DG $\mathcal{S} \otimes_Q E$-module with

$$U_E(M)^{\natural} \cong (\Gamma \otimes_Q M)^{\natural}$$

and differential given by the formula

$$\partial = 1 \otimes \partial^M + \sum_{i=1}^{n} \chi_i \otimes \lambda_i.$$

It is a straightforward check that $U_E(M)$ is a DG $\mathcal{S} \otimes_Q E$-module.

3.3. We say that $\boldsymbol{f}$ is *Koszul-regular* if the augmentation map $E \to R$ is a quasi-isomorphism. Equivalently,

$$\mathrm{H}_i(E) = \begin{cases} R & i = 0 \\ 0 & i \neq 0 \end{cases}$$

If $\boldsymbol{f}$ is a Q-regular sequence, then it is Koszul-regular. When Q is local and $\boldsymbol{f}$ is contained in the maximal ideal of Q, the converse holds.

3.4. Assume that $\boldsymbol{f}$ is a Koszul-regular sequence and fix a complex of R-modules M. Let $\epsilon : F \xrightarrow{\simeq} M$ be a semiprojective resolution of M over E. By [3, 2.4], $U_E(F) \to M$ is a semiprojective resolution over R where the augmentation map is given by

$$y^{(H)} \otimes x \mapsto \begin{cases} \epsilon(x) & |H| = 0 \\ 0 & |H| > 1 \end{cases}$$

In particular, for any complex of R-modules N

$$\mathrm{Ext}_R^*(M, N) \cong \mathrm{H}(\mathrm{Hom}_R(U_E(F), N))$$
$$\mathrm{Tor}_*^R(M, N) \cong \mathrm{H}(U_E(F) \otimes_R N)$$

are graded $\mathcal{S}$-modules. When we refer to $\mathrm{Ext}_R^*(M, N)$ and $\mathrm{Tor}_*^R(M, N)$ as graded $\mathcal{S}$-modules, we are considering the $\mathcal{S}$-module structures determined by the isomorphisms above. Moreover, the $\mathcal{S}$-module structure on $\mathrm{Ext}_R^*(M, N)$ or $\mathrm{Tor}_*^R(M, N)$ is independent of choice F and is natural in both M and N (see [3, 3.1]).

4. Perturbations

Fix a DG algebra A over a commutative ring Q. Fix a DG A-module X. Recall that for $\alpha, \beta \in \mathrm{Hom}_A(X, X)$ the commutator of α and β is

$$[\alpha, \beta] := \alpha\beta - (-1)^{|\alpha||\beta|}\beta\alpha.$$

4.1. Let $\gamma \in \mathrm{Hom}_A(X, X)$. We say that γ is a *central map* if for each $\sigma \in \mathrm{Hom}_A(X, X)$,

$$\gamma\sigma = (-1)^{|\sigma||\gamma|}\sigma\gamma.$$

That is, $[\gamma, -] = 0$ on $\mathrm{Hom}_A(X, X)$.

4.2. For $\alpha \in \mathrm{Hom}_A(X, X)$,

$$\partial^{\mathrm{Hom}_A(X,X)}(\alpha) = [\partial^X, \alpha].$$

In particular, $[\partial^X, \alpha] = 0$ if and only if $\alpha \in \mathrm{Z}(\mathrm{Hom}_A(X, X))$.

4.3. Assume $\delta \in \mathrm{Z}_{-1}(\mathrm{Hom}_A(X, X))$ satisfies $\delta^2 = 0$. Define X^δ to be the perturbation of X by δ. That is, X^δ is the DG A-module where $(X^\delta)^\natural$ is $X^\natural$ as a graded A-module and differential $\partial^X + \delta$.

4.4. For $H = (h_1, \ldots, h_n) \in \mathbb{N}^n$, we let

$$|H| := h_1 + \ldots + h_n.$$

Also, define

$$H_i := \begin{cases} (h_1, \ldots, h_{i-1}, h_i - 1, h_{i+1}, \ldots, h_n) & h_i \geq 1 \\ 0 & h_i = 0 \end{cases}$$

and

$$H_{i,j} := (H_i)_j = (H_j)_i.$$

4.5. Fix $d \in \mathbb{Z}$. Let $\boldsymbol{\alpha} = \alpha_1, \ldots, \alpha_n$ and $\boldsymbol{\beta} = \beta_1, \ldots, \beta_n$ be sequences of elements in $\mathrm{Hom}_A(X, X)_d$. We say that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are *homotopic*, denoted $\boldsymbol{\alpha} \sim \boldsymbol{\beta}$, provided that

$$\alpha_i \sim \beta_i$$

for each $1 \leq i \leq n$.

We say that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are *strongly homotopic*, denoted $\boldsymbol{\alpha} \approx \boldsymbol{\beta}$, if there exists a family of maps $\boldsymbol{\tau} := \{\tau^{(H)}\}_{H \in \mathbb{N}^n}$ in $\mathrm{Hom}_A(X, X)$ satisfying the following:

(1) $|\tau^{(H)}| = |H|(d + 1)$,
(2) $\tau^{(\mathbf{0})} = \mathrm{id}^X$, and
(3) $[\partial^X, \tau^{(H)}] = \sum_{i=1}^n \tau^{(H_i)} \alpha_i - \beta_i \tau^{(H_i)}$ for each $|H| > 0$. In particular, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are homotopic, as $\tau^{(\boldsymbol{e}_i)}$ is a homotopy for α_i and β_i where $\boldsymbol{e}_i$ is the n-tuple with a 1 in the ith spot and 0's everywhere else.

In this case, we say that $\boldsymbol{\tau}$ is a *system of higher strong homotopies from $\boldsymbol{\alpha}$ to $\boldsymbol{\beta}$*.

Proposition 4.6. *Let X be a DG A-module and suppose $\boldsymbol{\alpha} = \alpha_1, \ldots, \alpha_n$ and $\boldsymbol{\beta} = \beta_1, \ldots, \beta_n$ are homotopic sequences of central maps on X of positive odd degree d. If $\mathrm{H}_i(\mathrm{Hom}_A(X, X)) = 0$ for all $i \geq d$, then $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are strongly homotopic.*

Proof. We set $\tau^{(0)} := \mathrm{id}^X$ and construct the rest by induction. Suppose $|H| > 0$ and assume we have constructed $\{\tau^{(H')} : |H'| < |H|\}$ satisfying (1)-(3). For each i, notice that

$$[\partial^X, \tau^{(H_i)}(\alpha_i - \beta_i)] = [\partial^X, \tau^{(H_i)}](\alpha_i - \beta_i) + \tau^{(H_i)}[\partial^X, \alpha_i - \beta_i].$$

As the homotopic maps α_i and β_i are central, it follows that $[\partial^X, \alpha_i - \beta_i] = 0$ and hence,

$$\tau^{(H_i)}[\partial^X, \alpha_i - \beta_i] = 0.$$

Moreover,

$$[\partial^X, \tau^{(H_i)}](\alpha_i - \beta_i) = \left(\sum_{j=1}^n \tau^{(H_{i,j})} \alpha_j - \beta_j \tau^{(H_{i,j})} \right) (\alpha_i - \beta_i)$$

$$= \sum_{j=1}^n \tau^{(H_{i,j})}(\alpha_j - \beta_j)(\alpha_i - \beta_i)$$

Now using that α_i and β_i are central maps of the same odd degree it follows that

$$\sum_{i=1}^n \sum_{j=1}^n \tau^{(H_{i,j})}(\alpha_j - \beta_j)(\alpha_i - \beta_i) = 0.$$

Thus,

$$\left[\partial^X, \sum_{i=1}^n \tau^{(H_i)}(\alpha_i - \beta_i) \right] = 0,$$

and hence,

$$\sum_{i=1}^n \tau^{(H_i)}(\alpha_i - \beta_i) \in \mathrm{Z}(\mathrm{Hom}_A(X, X)).$$

As $\mathrm{H}_i(\mathrm{Hom}_A(X, X)) = 0$ for all $i \geq d$ and

$$\left| \sum_{i=1}^n \tau^{(H_i)}(\alpha_i - \beta_i) \right| = |H_i|(d + 1) + d = (|H| - 1)(d + 1) + d \geq d,$$

it follows that there exists a map $\tau^{(H)}$ in $\mathrm{Hom}_A(X, X)$ of degree

$$|H_i|(d + 1) + d + 1 = (|H| - 1)(d + 1) + d + 1 = |H|(d + 1)$$

satisfying

$$[\partial^X, \tau^{(H)}] = \sum_{i=1}^{n} \tau^{(H_i)}(\alpha_i - \beta_i).$$

Finally, as β_i is central for each i, condition (3) in 4.5 is satisfied. Thus, by induction we are done. $\square$

Definition 4.7. Let $\boldsymbol{\alpha} = \alpha_1, \ldots, \alpha_n$ and $\boldsymbol{\beta} = \beta_1, \ldots, \beta_n$ be sequences of maps of positive odd degree d on X, $\boldsymbol{\gamma} = \gamma_1, \ldots, \gamma_n$ is a sequence of central maps on X of degree $|\gamma_i| = -d - 1$, and set

$$\delta := \sum_{i=1}^{n} \gamma_i \alpha_i \quad \text{and} \quad \epsilon := \sum_{i=1}^{n} \gamma_i \beta_i.$$

We say that $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\tau})$ is a *perturbing system on X* if the following hold

(1) $\boldsymbol{\tau} := \{\tau^{(H)}\}_{H \in \mathbb{N}^n}$ is a system of higher strong homotopies from $\boldsymbol{\alpha}$ to $\boldsymbol{\beta}$,
(2) $\delta, \epsilon \in \mathrm{Z}_{-1}(\mathrm{Hom}_A(X, X))$
(3) $\delta^2 = 0$ and $\epsilon^2 = 0$
(4) For each $x \in X$, $\gamma^H(x) = 0$ for all $|H| \gg 0$ where $\gamma^H = \gamma_1^{h_1} \ldots \gamma_n^{h_n}$

Construction 4.8. Let $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\tau})$ be a perturbing system on X. Using conditions (2) and (3) from Definition 4.7, X^δ and X^ϵ are well-defined DG A-modules (see 4.3). Define $\boldsymbol{\gamma}\boldsymbol{\tau} : X^\delta \to X^\epsilon$ by

$$x \mapsto \sum_{H \in \mathbb{N}^n} \gamma^H \tau^{(H)}(x)$$

where, by convention, $\gamma^{\mathbf{0}} = \mathrm{id}^X$. As each γ_i is a central map,

$$\gamma^H \tau^{(H)}(x) = \tau^{(H)} \gamma^H(x)$$

for each $H \in \mathbb{N}^n$. Hence, condition (4) implies that

$$\sum_{H \in \mathbb{N}^n} \gamma^H \tau^{(H)}(x)$$

is a finite sum. Moreover, $\boldsymbol{\gamma}\boldsymbol{\tau}$ is A-linear. Since $|\gamma_i| = -d - 1$ it follows that

$$|\gamma^H \tau^{(H)}| = |\gamma^H| + |\tau^{(H)}| = |H|(-d - 1) + H(d + 1) = 0$$

and so $\boldsymbol{\gamma}\boldsymbol{\tau}$ is a degree 0 map in $\mathrm{Hom}_A(X, X)$.

Finally, we claim that $\boldsymbol{\gamma}\boldsymbol{\tau}$ is a morphism of DG A-modules. It suffices to check that $\boldsymbol{\gamma}\boldsymbol{\tau}$ is a chain map. The proof is given with the following string of equalities. We note

that we still adopt the convention $\gamma^{\mathbf{0}} = \mathrm{id}^X$ and the fifth equality below follows from 4.5(3):

$$
\begin{aligned}
(\partial^X + \epsilon)\boldsymbol{\gamma}\boldsymbol{\tau} &= (\partial^X + \epsilon) \sum_{H \in \mathbb{N}^n} \gamma^H \tau^{(H)} \\
&= \sum_{H \in \mathbb{N}^n} \gamma^H \partial^X \tau^{(H)} + \sum_{i=1}^n \sum_{H \in \mathbb{N}^n} \gamma_i \gamma^H \beta_i \tau^{(H)} \\
&= \partial^X + \sum_{|H|>0} \gamma^H \partial^X \tau^{(H)} + \sum_{|H|>0} \sum_{i=1}^n \gamma^H \beta_i \tau^{(H_i)} \\
&= \partial^X + \sum_{|H|>0} \left(\gamma^H \partial^X \tau^{(H)} + \sum_{i=1}^n \gamma^H \beta_i \tau^{(H_i)} \right) \\
&= \partial^X + \sum_{|H|>0} \left(\gamma^H \tau^{(H)} \partial^X + \sum_{i=1}^n \gamma^H \tau^{(H_i)} \alpha_i \right) \\
&= \partial^X + \sum_{|H|>0} \gamma^H \tau^{(H)} \partial^X + \sum_{H \in \mathbb{N}^n} \sum_{i=1}^n \gamma_i \gamma^H \tau^{(H)} \alpha_i \\
&= \sum_{H \in \mathbb{N}^n} \gamma^H \tau^{(H)} \partial^X + \sum_{H \in \mathbb{N}^n} \gamma^H \tau^{(H)} \sum_{i=1}^n \gamma_i \alpha_i \\
&= \boldsymbol{\gamma}\boldsymbol{\tau}(\partial^X + \delta).
\end{aligned}
$$

Thus, $\boldsymbol{\gamma}\boldsymbol{\tau}$ is a morphism of DG A-modules.

Theorem 4.9. *Let X be a DG A-module and suppose $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\tau})$ is a perturbing system on X. Using the notation from Construction 4.8, the morphism of DG A-modules $\boldsymbol{\gamma}\boldsymbol{\tau} : X^\delta \to X^\epsilon$ is an isomorphism.*

Proof. For $n \in \mathbb{N}$, define

$$
X(n) := \{x \in X^\delta : \gamma^H(x) = 0 \text{ for all } |H| > n\}.
$$

Since each γ_i is central, it follows that $X(n)$ is a DG A-submodule of X^δ. Indeed, for $x \in X(n)$ and $|H| > n$ we have that

$$
\gamma^H(\partial^X + \delta)(x) = (\partial^X + \delta)\gamma^H(x) = 0.
$$

Similarly, $Y(n)$ is a DG A-submodule of X^ϵ where $Y(n)^\natural = X(n)^\natural$. Again using that γ_i is central for each i, for each $H \in \mathbb{N}^n$ it follows that

$$\gamma^H \circ \boldsymbol{\gamma\tau} = \boldsymbol{\gamma\tau} \circ \gamma^H.$$

Thus, $\boldsymbol{\gamma\tau}(X(n)) \subseteq Y(n)$. Set $\boldsymbol{\gamma\tau}(n) := \boldsymbol{\gamma\tau}|_{X(n)} : X(n) \to Y(n)$.

We have a chain of DG A-submodules

$$0 = X(0) \subseteq X(1) \subseteq X(2) \dots \quad \text{and} \quad 0 = Y(0) \subseteq Y(1) \subseteq Y(2) \dots$$

of X^δ and X^ϵ, respectively. Moreover, by assuming condition (4) it follows that

$$\varinjlim X(n) = X^\delta, \ \varinjlim Y(n) = X^\epsilon, \ \text{and} \ \varinjlim \boldsymbol{\gamma\tau}(n) = \boldsymbol{\gamma\tau}. \tag{4.1}$$

Finally, it is clear that $\boldsymbol{\gamma\tau}(0)$ is the identity map and that $\boldsymbol{\gamma\tau}(n)$ induces the identity map

$$X(n)/X(n-1) \to Y(n)/Y(n-1)$$

for each $n \in \mathbb{N}$. Hence, by induction $\boldsymbol{\gamma\tau}(n)$ is an isomorphism for each $n \in \mathbb{N}$. Thus, by (4.1) it follows that $\boldsymbol{\gamma\tau}$ is an isomorphism. $\quad\square$

5. Equivariant isomorphisms

5.1. Let $\varphi : A \to A'$ be a morphism of DG Q-algebras. Suppose M is a DG A-module and M' is a DG A'-module. A morphism of complexes $\psi : M \to M'$ is φ-*equivariant* if

$$\psi(am) = \varphi(a)\psi(m)$$

for all $a \in A$ and $m \in M$. Similarly, a morphism of complexes $\psi' : M' \to M$ is φ-*equivariant* if

$$\psi'(\varphi(a)m') = a\psi'(m')$$

for all $a \in A$ and $m' \in M$.

Let ψ be a φ-equivariant map. Then ψ is a morphism of DG A-modules when M' is regarded as an A-module via restriction of scalars along φ. If ψ is an isomorphism of complexes, we say that ψ is a φ-*equivariant isomorphism*.

First, we fix some notation which will be used throughout the section.

Notation 5.2. Let Q be a commutative ring and fix two lists of elements $\boldsymbol{f} = f_1, \dots, f_n$ and $\boldsymbol{f}' = f'_1 \dots, f'_n$ in Q. Set

$$E := Q\langle \xi_1, \dots, \xi_n | \partial \xi_i = f_i \rangle \qquad E' := Q\langle \xi'_1, \dots, \xi'_n | \partial \xi'_i = f'_i \rangle$$

$$\mathcal{S} := Q/(\boldsymbol{f})[\chi_1, \dots, \chi_n] \qquad \mathcal{S}' := Q/(\boldsymbol{f}')[\chi'_1, \dots, \chi'_n]$$

$$\Gamma := \mathrm{Hom}_{Q/(\boldsymbol{f})}(\mathcal{S}, Q/(\boldsymbol{f})) \qquad \Gamma' := \mathrm{Hom}_{Q/(\boldsymbol{f}')}(\mathcal{S}', Q/(\boldsymbol{f}'))$$

where each χ_i and χ_i' have homological degree -2. Also, let $\Lambda := E \otimes_Q E'$. For a DG Λ-module X we let λ_i and λ_i' denote left multiplication on X by ξ_i and ξ_i', respectively.

Suppose that I is an ideal of Q containing $(\boldsymbol{f}, \boldsymbol{f}')$, set $R = Q/I$ and

$$\mathcal{P} := R \otimes_Q \mathcal{S} = R[\chi_1, \ldots, \chi_n] \qquad \mathcal{P}' := R \otimes_Q \mathcal{S}' = R[\chi_1', \ldots, \chi_n']$$

$$\mathcal{D} := \mathrm{Hom}_R(\mathcal{P}, R) \qquad\qquad \mathcal{D}' := \mathrm{Hom}_R(\mathcal{P}', R)$$

It is worth noting that Γ, Γ', $\mathcal{D}$, and $\mathcal{D}'$ are defined above using the *graded* duals in each context.

Remark 5.3. Set $\overline{Q} := Q/(\boldsymbol{f})$ and $\overline{Q}' := Q/(\boldsymbol{f}')$ and assume that $\boldsymbol{f}$ and $\boldsymbol{f}'$ are Koszul-regular sequences. For complexes of R-modules M and N,

$$\mathrm{Ext}^*_{\overline{Q}}(M, N) = \mathrm{H}(\mathrm{Hom}_{\overline{Q}}(U_E(F), N))$$

is a graded module over $\mathcal{S}$ where F is a semiprojective resolution of M over E (see 3.4). As I annihilates $\mathrm{Ext}^*_{\overline{Q}}(M, N)$, $\mathrm{Ext}^*_{\overline{Q}}(M, N)$ is a graded $\mathcal{P}$-module. Moreover, since N is an R-module, using adjunction we have the following isomorphism of graded $\mathcal{P}$-modules

$$\mathrm{Ext}^*_{\overline{Q}}(M, N) \cong \mathrm{H}(\mathrm{Hom}_R(R \otimes_{\overline{Q}} U_E(F), N)). \tag{5.1}$$

Similarly, there is an isomorphism of graded $\mathcal{P}'$-modules

$$\mathrm{Ext}^*_{\overline{Q}'}(M, N) \cong \mathrm{H}(\mathrm{Hom}_R(R \otimes_{\overline{Q}'} U_{E'}(F'), N)) \tag{5.2}$$

where F' is a semiprojective resolution of M over E'. There are analogous statements for $\mathrm{Tor}^{\overline{Q}}_*(M, N)$ and $\mathrm{Tor}^{\overline{Q}'}(M, N)$ as graded $\mathcal{P}$- and $\mathcal{P}'$-modules, respectively.

In this section, first, we will be comparing $\mathrm{Ext}^*_{\overline{Q}}(M, N)$ with $\mathrm{Ext}^*_{\overline{Q}'}(M, N)$ and $\mathrm{Tor}^{\overline{Q}}(M, N)$ with $\mathrm{Tor}^{\overline{Q}'}(M, N)$ for R-modules M and N when $\boldsymbol{f}$ and $\boldsymbol{f}'$ are assumed to be Koszul-regular sequences. Roughly speaking, if each λ_i and λ_i' act similarly on a semiprojective resolution of M over Λ, then we have isomorphisms between the corresponding Ext and Tor modules. Later in the section, we will be interested in the case when $\boldsymbol{f}$ is Koszul-regular and $\boldsymbol{f}'$ is the zero sequence.

Lemma 5.4. *For any DG E-module Y and DG E'-module Y',*

$$U_E(Y) = (\Gamma \otimes_Q Y)^\delta \quad and \quad U_{E'}(Y') = (\Gamma' \otimes_Q Y')^{\delta'}$$

where $\delta := \sum_{i=1}^n \chi_i \otimes \lambda_i$ and $\delta' := \sum_{i=1}^n \chi_i' \otimes \lambda_i'$.

Proof. This follows immediately by examining the underlying graded modules and the differentials (see 3.2 and 4.3). $\quad\square$

5.5. Let M be an R-module. Suppose that $f_i - f'_i \in I \operatorname{ann}_Q(M)$ for each i, that is,

$$f_i - f'_i = \sum_j x_{i,j} g_{i,j}$$

where $x_{i,j} \in I$ and $g_{i,j} \in \operatorname{ann}_Q(M)$. We set $B := \Lambda\langle \zeta_{i,j} | \partial \zeta_{i,j} = g_{i,j} \rangle$ and as $f_i, f'_i, g_{i,j} \in \operatorname{ann}_Q(M)$ it follows that M is a DG B-module. A bounded below semiprojective resolution of M over B is called a *strong Λ-resolution of M*. Since B is semiprojective over Λ, a strong Λ-resolution of M is a semiprojective resolution of M over Λ.

By 2.7, strong Λ-resolutions of M exist. Moreover, any two strong Λ-resolutions of M are homotopy equivalent DG Λ-modules (cf. 2.8).

Lemma 5.6 was inspired by, and generalizes, Claim 2 in the proof of [6, 2.1]. The proof of Lemma 5.6 relies heavily on the ideas in Section 4.

Lemma 5.6. *Let M be an R-module. Suppose that $f_i - f'_i \in I \operatorname{ann}_Q(M)$ for each i. For any strong Λ-resolution $F \xrightarrow{\simeq} M$, there is an isomorphism of DG $\mathcal{P} \otimes_Q \Lambda$-modules*

$$(\mathcal{D} \otimes_Q F)^\delta \cong (\mathcal{D} \otimes_Q F)^\epsilon,$$

where $\delta = \sum_{i=1}^n \chi_i \otimes \lambda_i$ and $\epsilon = \sum_{i=1}^n \chi_i \otimes \lambda'_i$.

Proof. We let $\sigma_{i,j} \in \operatorname{Hom}_B(F, F)_1$ denote left multiplication by $\zeta_{i,j}$. By assumption,

$$[\partial^F, \lambda_i - \lambda'_i] = \left[\partial^F, \sum_j x_{i,j} \sigma_{i,j} \right]$$

and hence,

$$\lambda_i - \lambda'_i - \sum_j x_{i,j} \sigma_{i,j} \in \mathrm{Z}_1(\operatorname{Hom}_B(F, F)).$$

As $F \xrightarrow{\simeq} M$ and F is semiprojective over B, it follows that

$$\operatorname{Hom}_B(F, F) \xrightarrow{\simeq} \operatorname{Hom}_B(F, M).$$

Since F is nonnegatively graded and M is concentrated in degree 0,

$$\operatorname{Hom}_B(F, M)_1 = 0$$

and hence,

$$0 = \mathrm{H}_1(\operatorname{Hom}_B(F, M)) = \mathrm{H}_1(\operatorname{Hom}_B(F, F)).$$

Thus, $Z_1(\mathrm{Hom}_B(F,F)) = B_1(\mathrm{Hom}_B(F,F))$ and so there exists $\tau \in \mathrm{Hom}_B(F,F)_2$ such that

$$[\partial^F, \tau] = \lambda_i - \lambda_i' - \sum_j x_{i,j}\sigma_{i,j}.$$

Thus,

$$\lambda_i - \sum_j x_{i,j}\sigma_{i,j} \sim \lambda_i',$$

for all i. Since λ_i, λ_i' and σ_i' are central on F over B, we apply Proposition 4.6 to conclude that

$$\tilde{\boldsymbol{\alpha}} \approx \tilde{\boldsymbol{\beta}}$$

where

$$\tilde{\alpha}_i := \lambda_i - \sum_j x_{i,j}\sigma_{i,j} \text{ and } \tilde{\beta}_i := \lambda_i'$$

for each $1 \le i \le n$. Also, as maps on $\mathcal{D} \otimes_Q F$,

$$1 \otimes \lambda_i = 1 \otimes \lambda_i - 1 \otimes \sum_j x_{i,j}\sigma_{i,j}$$

since each $x_{i,j}$ is in I which annihilates $\mathcal{D}$. Therefore, the sequences $(1 \otimes \lambda_i)_{i=1}^n$ and $(1 \otimes \lambda_i')_{i=1}^n$ are strongly homotopic maps of degree 1 on $\mathcal{D} \otimes_Q F$.

In the notation of Section 4, set

$$A = \mathcal{P} \otimes_Q B$$
$$X = \mathcal{D} \otimes_Q F$$
$$\boldsymbol{\alpha} = 1 \otimes \lambda_1, \ldots, 1 \otimes \lambda_n$$
$$\boldsymbol{\beta} = 1 \otimes \lambda_1', \ldots, 1 \otimes \lambda_n'$$
$$\boldsymbol{\gamma} = \chi_1 \otimes 1, \ldots, \chi_n \otimes 1$$

and let $\boldsymbol{\tau}$ denote the strong homotopy from $\boldsymbol{\alpha}$ to $\boldsymbol{\beta}$ (the existence of $\boldsymbol{\tau}$ was justified above). It is easily checked that $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\tau})$ is a perturbing system on X (see Definition 4.7). Therefore, Theorem 4.9 yields an isomorphism of DG $\mathcal{P} \otimes_Q B$-modules

$$(\mathcal{D} \otimes_Q F)^\delta \cong (\mathcal{D} \otimes_Q F)^\epsilon,$$

where recall that $\delta = \sum_{i=1}^n \chi_i \otimes \lambda_i$ and $\epsilon = \sum_{i=1}^n \chi_i \otimes \lambda_i'$. $\quad\square$

We are now equipped to prove the main results of this paper. Theorem 5.7 generalizes [6, 2.1(2)], as discussed in the introduction.

Theorem 5.7. *We adopt the notation from Notation 5.2. Further assume that $\boldsymbol{f}$ and $\boldsymbol{f}'$ are Koszul-regular sequences. Let $\varphi : \mathcal{P} \to \mathcal{P}'$ be the morphism of DG R-algebras given by*

$$\chi_i \mapsto \chi_i'.$$

Assume that M is an R-module such that for each i

$$f_i - f_i' \in I \operatorname{ann}_Q(M).$$

For each complex of R-modules N, we have the following φ-equivariant isomorphisms:

(1) $\operatorname{Ext}_{Q/(\boldsymbol{f})}^*(M, N) \cong \operatorname{Ext}_{Q/(\boldsymbol{f}')}^*(M, N)$
(2) $\operatorname{Tor}_*^{Q/(\boldsymbol{f})}(M, N) \cong \operatorname{Tor}_*^{Q/(\boldsymbol{f}')}(M, N)$

Proof. The proof of (2) is similar to the proof of (1). So we only prove (1) and leave translating the proof of (1) to the context of (2) as an exercise for the reader.

By 5.5 and Lemma 5.6, there exists a semiprojective resolution $F \xrightarrow{\simeq} M$ over Λ and a canonical isomorphism of DG $\mathcal{P} \otimes_Q \Lambda$-modules

$$(\mathcal{D} \otimes_Q F)^\delta \cong (\mathcal{D} \otimes_Q F)^\epsilon, \tag{5.3}$$

where $\delta = \sum_{i=1}^n \chi_i \otimes \lambda_i$ and $\epsilon = \sum_{i=1}^n \chi_i \otimes \lambda_i'$. Moreover, φ induces the $\varphi \otimes \Lambda$-equivariant isomorphism

$$(\mathcal{D}' \otimes_Q F)^{\delta'} \xrightarrow{\operatorname{Hom}_R(\varphi, R) \otimes \Lambda} (\mathcal{D} \otimes_Q F)^\epsilon \tag{5.4}$$

where $\delta' = \sum_{i=1}^n \chi_i' \otimes \lambda_i'$. Composing the isomorphisms in (5.3) and (5.4), we obtain a $\varphi \otimes \Lambda$-equivariant isomorphism

$$(\mathcal{D} \otimes_Q F)^\delta \cong (\mathcal{D}' \otimes_Q F)^{\delta'}. \tag{5.5}$$

Using that $\boldsymbol{f}$ and $\boldsymbol{f}'$ are Koszul-regular and Lemma 5.4, we obtain the following isomorphism of DG $\mathcal{P} \otimes_Q \Lambda$-modules and isomorphism of DG $\mathcal{P}' \otimes_Q \Lambda$-modules

$$R \otimes_{Q/(\boldsymbol{f})} U_E(F) \cong (\mathcal{D} \otimes_Q F)^\delta \quad \text{and} \quad R \otimes_{Q'/(\boldsymbol{f}')} U_{E'}(F) \cong (\mathcal{D}' \otimes_Q F)^{\delta'},$$

respectively. Therefore, we have a $\varphi \otimes \Lambda$-equivariant isomorphism

$$R \otimes_{Q/(\boldsymbol{f})} U_E(F) \cong R \otimes_{Q/(\boldsymbol{f}')} U_{E'}(F)$$

using these isomorphisms and (5.5). Applying $\mathrm{Hom}_R(-, N)$ provides us with a $\varphi \otimes \Lambda$-equivariant isomorphism

$$\mathrm{Hom}_R(R \otimes_{Q/(\boldsymbol{f})} U_E(F), N) \cong \mathrm{Hom}_R(R \otimes_{Q/(\boldsymbol{f}')} U_{E'}(F), N).$$

Now using (5.1) and (5.2) from Remark 5.3, we obtain (1). $\quad\square$

5.8. Let Q be local and M a Q-module. We let $\nu(M)$ denote the minimal number of generators of M. For a complex of Q-modules N, the *Bass series of M and N* is

$$\mathsf{I}_Q^{M,N}(t) = \sum_{i=0}^{\infty} \nu(\mathrm{Ext}_Q^i(M, N)) t^i.$$

When $M = k$, this recovers the classical Bass series of N.

The *Poincaré series of M and N* is

$$\mathsf{P}_{M,N}^Q(t) = \sum_{i=0}^{\infty} \nu(\mathrm{Tor}_i^Q(M, N)) t^i.$$

When $M = k$, this recovers the classical Poincaré series of N.

From Theorem 5.7 we get the following immediate corollary.

Corollary 5.9. *Further assume that Q is local. For each complex of R-modules N, the following hold:*

(1) $\mathsf{I}_{Q/(\boldsymbol{f})}^{M,N}(t) = \mathsf{I}_{Q/(\boldsymbol{f}')}^{M,N}(t)$
(2) $\mathsf{P}_{M,N}^{Q/(\boldsymbol{f})}(t) = \mathsf{P}_{M,N}^{Q/(\boldsymbol{f}')}(t)$

Our last main theorem of this article is given below in Theorem 5.10. The techniques from Section 4 are again applied to establish this result. Corollary 5.11 is an immediate consequence of Theorem 5.10. Finally, as mentioned in the introduction, Theorem 5.10 and Corollary 5.11 greatly generalize work of Shamash in [11].

Theorem 5.10. *We adopt the notation from Notation 5.2. If $(\boldsymbol{f}) \subseteq I \,\mathrm{ann}_Q(M)$ is a Koszul-regular sequence, we have isomorphisms of graded $\mathcal{P}$-modules:*

(1) $\mathrm{Ext}_{Q/(\boldsymbol{f})}^*(M, N) \cong \mathcal{P} \otimes_R \mathrm{Ext}_Q^*(M, N)$
(2) $\mathrm{Tor}_*^{Q/(\boldsymbol{f})}(M, N) \cong \mathcal{P}^* \otimes_R \mathrm{Tor}_*^Q(M, N)$

Proof. The proof of (1) and (2) are similar, and hence, we only show (1).

Let $f_i' = 0$ for all $1 \leq i \leq n$ and fix a strong Λ-resolution $F \xrightarrow{\simeq} M$. By Lemma 5.6, we have an isomorphism of DG $\mathcal{P} \otimes_Q \Lambda$-modules

$$(\mathcal{D} \otimes_Q F)^\delta \cong \mathcal{D} \otimes_Q F.$$

Therefore, we have the following isomorphisms of graded $\mathcal{P}$-modules

$$\begin{aligned}
\mathrm{Ext}^*_{Q/(\boldsymbol{f})}(M, N) &= \mathrm{H}(\mathrm{Hom}_R((\mathcal{D} \otimes_Q F)^\delta, N)) \\
&\cong \mathrm{H}(\mathrm{Hom}_R(\mathcal{D} \otimes_Q F, N)) \\
&\cong \mathrm{H}(\mathrm{Hom}_R(\mathcal{D}, R) \otimes_Q \mathrm{Hom}_Q(F, N)) \\
&\cong \mathrm{H}(\mathcal{P} \otimes_R \mathrm{Hom}_Q(F, N)) \\
&\cong \mathcal{P} \otimes_R \mathrm{H}(\mathrm{Hom}_Q(F, N)).
\end{aligned}$$

As F is semiprojective over E, and E is semiprojective over Q, it follows that $F \xrightarrow{\simeq} M$ is a semiprojective resolution of M over Q. Thus, (3) holds by the isomorphisms exhibited. $\square$

Corollary 5.11. *Further assume that Q is local.*

(1) $\mathsf{I}^{M,N}_{Q/(\boldsymbol{f})}(t) = \dfrac{\mathsf{I}^{M,N}_Q(t)}{(1 - t^2)^n}.$

(2) $\mathsf{P}^{Q/(\boldsymbol{f})}_{M,N}(t) = \dfrac{\mathsf{P}^Q_{M,N}(t)}{(1 - t^2)^n}.$

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