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Blocks with abelian defect groups of rank 2 and one simple module



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ABSTRACT

In this paper, we investigate the block that has an abelian defect group of rank 2 and its Brauer correspondent has only one simple module. We will get an isotypy between the block and its Brauer correspondent. It will generalize a result of Kessar and Linckelmann.

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1. Introduction

Let p be a prime and \mathcal{O} a complete discrete valuation ring having an algebraically closed residue field k of characteristic p and a quotient field \mathcal{K} of characteristic 0. We will always assume that \mathcal{K} is big enough for the finite groups below.

Let G be a finite group and b a block of $\mathcal{O}G$ with a defect group P . Denote by $\text{Irr}_{\mathcal{K}}(G, b)$ and $\text{IBr}(G, b)$ the set of irreducible ordinary characters in b and the set of irreducible Brauer characters in b respectively. Set $l_G(b) = |\text{IBr}(G, b)|$. Let c be the Brauer correspondent of b in $N_G(P)$. In [6], Kessar and Linckelmann investigated the block b under the assumptions that $l_{N_G(P)}(c) = 1$ and P is elementary abelian of rank 2.

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They showed that the inertial quotient of b is abelian and there is an isotypy between b and c all of whose signs are positive.

In this note, we will generalize these results to the blocks with defect groups of rank 2.

Theorem 1.1. *Keep the notation as above. Assume that P is abelian of rank 2 and $l_{N_G(P)}(c) = 1$. Then the inertial quotient of b is abelian and there is an isotypy between b and c .*

When the inertial quotient of b is trivial, it is clear that there is an isotypy between b and c since the block b is nilpotent (see [3, Theorem 1.2]). For the case $p = 2$, by [13, Theorem 8.1], the condition that $l_{N_G(P)}(c) = 1$ will force the block b is nilpotent. Therefore, we may assume that p is odd and the inertial quotient of b is non-trivial throughout this paper.

2. The structure of the block c

Keep the notation as above. In this section, we will investigate the structure of the inertial quotient of b and irreducible ordinary characters of the block c .

Given a positive integer a , denote by C_a the cyclic group of order a . We will use $[-, -]$ to represent the commutator. Assume that $P = C_{p^n} \times C_{p^m}$ for some positive integers n, m . We will fix a maximal b -Brauer pair (P, b_P) . For any $Q \leq P$, denote by (Q, b_Q) the unique b -Brauer pair contained in (P, b_P) . Let E be the inertial quotient of b associated with (P, b_P) , namely, $E = N_G(P, b_P)/C_G(P)$.

Lemma 2.1. *The inertial quotient E is abelian if $l_{N_G(P)}(c) = 1$.*

Proof. Let $\Phi(P)$ be the Frattini subgroup of P . So $P/\Phi(P)$ is $C_p \times C_p$. Set H to be $N_G(P, b_P)$. Then $\Phi(P) \leq H$ and denote $H/\Phi(P)$ by \bar{H} . For any subset X of $\mathcal{O}H$, \bar{X} denotes the image of X under the canonical map $\mathcal{O}H \rightarrow \mathcal{O}\bar{H}$.

Since $l_H(b_P) = 1$, $l_{\bar{H}}(\bar{b}_P) = 1$ and \bar{b}_P is a block of \bar{H} with defect group $\bar{P} = C_p \times C_p$ (see [9, Theorem 5.8.10]). Let \hat{C} be the subgroup of H containing $\Phi(P)$ such that $\hat{C}/\Phi(P) = C_{\bar{H}}(\bar{P})$. Hence, $\hat{C} = \{x \in H \mid [P, x] \subseteq \Phi(P)\}$. It is clear that $P = [P, \hat{C}] \times C_P(\hat{C})$. So $P = C_P(\hat{C})$ since $[P, \hat{C}] \leq \Phi(P)$. This means $C_{\bar{H}}(\bar{P}) = \bar{C}_G(P)$. Hence, (\bar{P}, \bar{b}_P) is a maximal \bar{b}_P -Brauer pair of $\mathcal{O}\bar{H}\bar{b}_P$. By [6, Proposition 5.2], $N_{\bar{H}}(\bar{P}, \bar{b}_P)/C_{\bar{H}}(\bar{P})$ is abelian. It is evident that E is isomorphic to $N_{\bar{H}}(\bar{P}, \bar{b}_P)/C_{\bar{H}}(\bar{P})$. We are done. \square

Now we will always assume that $l_{N_G(P)}(c) = 1$. Let's recall the structure theory of blocks with normal defect groups. By [14, Theorem (45.12)] and [10, Lemma 5.5, Proposition 5.15 (i)], there exists a central extension

$$1 \rightarrow Z \rightarrow \tilde{E} \rightarrow E \rightarrow 1$$

with Z cyclic p' -group such that the source algebra of the block c is $\mathcal{O}(P \rtimes \tilde{E})e_\theta$. Here Z acts trivially on P and $e_\theta \in \mathcal{O}Z$ is the central idempotent corresponding to an irreducible ordinary character θ of Z . Set $N = P \rtimes \tilde{E}$. Note that e_θ is still a block of $C_N(R)$ for any $R \leq P$. Since $l_{N_G(P)}(c) = 1$, there is a unique irreducible character v of \tilde{E} covering θ . Then by Clifford's Theorem, we have $v(1)^2 = |\tilde{E} : Z|$. Recall that a finite group K is said to be of *central type* if K has an irreducible character χ satisfying $\chi(1)^2 = |K : Z(K)|$, where $Z(K)$ is the centre of K (see [5]). Denote by $Z(\tilde{E})$ the centre of \tilde{E} . It is clear that $v(1)^2 \leq |\tilde{E} : Z(\tilde{E})|$. Since $Z \leq Z(\tilde{E})$, we have $Z = Z(\tilde{E})$ and $v(1)^2 = |\tilde{E} : Z(\tilde{E})|$ which means that \tilde{E} is of central type. Therefore by [4, Lemma 2] and Lemma 2.1, E is a direct product of two isomorphic groups. Next, we will show that E acts diagonally on P . This can be deduced from the following general fact.

Lemma 2.2. *Let D be an abelian p -group of rank 2 and $F \leq \text{Aut}(P)$ an abelian p' -group which is a direct product of two isomorphic subgroups. Then we have the decompositions $F = F_1 \times F_2$ and $D = D_1 \times D_2$ such that F_1 acts faithfully on D_1 and centralises D_2 and F_2 acts faithfully on D_2 and centralises D_1 and $F_1 \cong F_2$. In particular, F_1 and F_2 are cyclic groups of order dividing $(p-1)$.*

Proof. We will exhibit it by induction on $|D|$. When D is elementary abelian, it is actually done in [6, Proposition 5.3]. We may assume that $n \geq 2$ or $m \geq 2$. Let $\Phi(D)$ be the Frattini subgroup of D . So $D/\Phi(D)$ is $C_p \times C_p$. Let π be the canonical map from F to $\text{Aut}(D/\Phi(D))$. For any subset X of F , \bar{X} denotes the image of X under π . It is clear that π is injective. So there exist two subgroups F_1 and F_2 of F and two subgroups D_1 and D_2 of D containing $\Phi(D)$ satisfying the properties $\bar{F} = \bar{F}_1 \times \bar{F}_2$ and $D/\Phi(D) = D_1/\Phi(D) \times D_2/\Phi(D)$ and \bar{F}_1 acts faithfully on $D_1/\Phi(D)$ and centralises $D_2/\Phi(D)$ and \bar{F}_2 acts faithfully on $D_2/\Phi(D)$ and centralises $D_1/\Phi(D)$ and $\bar{F}_1 \cong \bar{F}_2$. Hence, D_1 and D_2 are F -stable and they fulfil

- (i) $D_1 = [D_1, F_1] \cdot \Phi(D)$ and $[D_1, F_2] \subseteq \Phi(D)$ and F_1 acts faithfully on D_1 ;
- (ii) $D_2 = [D_2, F_2] \cdot \Phi(D)$ and $[D_2, F_1] \subseteq \Phi(D)$ and F_2 acts faithfully on D_2 ;
- (iii) $D_1 \cap D_2 = \Phi(D)$ and $D_1/\Phi(D) \cong C_p \cong D_2/\Phi(D)$ and $D = D_1 \cdot D_2$.

Suppose that $\Phi(D)$ is cyclic. Then $D = C_p \times C_{p^m}$ with $m \geq 2$ and $\Phi(D) = C_{p^{m-1}}$. Since $D_2 = [D_2, F_1] \times C_{D_2}(F_1)$ and $[D_2, F_1] \subseteq \Phi(D)$, $\Phi(D) = [D_2, F_1] \times C_{\Phi(D)}(F_1)$. Then either $[D_2, F_1] = 1$ or $C_{\Phi(D)}(F_1) = 1$ by the assumption that $\Phi(D)$ is cyclic. If $[D_2, F_1] = 1$, then $\Phi(D) \leq D_2 \leq C_D(F_1)$. Clearly, $D = [D, F_1] \times C_D(F_1)$ and D_2 is a maximal subgroup of D . Thus, $D_2 = C_D(F_1)$ and $[D, F_1] = [D_1, F_1]$. Since F_1 and F_2 commute with each other and $[D_1, F_2] \subseteq \Phi(D) \subseteq C_D(F_1)$, $[[D_1, F_1], F_2] = 1$. So $[D_1, F_1] \leq C_P(F_2)$. Since F_1 acts faithfully on D_1 and $D_1 = [D_1, F_1] \times C_{D_1}(F_1)$, F_1 acts faithfully on $[D_1, F_1]$. Thus, the decompositions $F = F_1 \times F_2$ and $D = [D_1, F_1] \times D_2$ are what we want. We may assume that $C_{\Phi(D)}(F_1) = 1$. Then $\Phi(D) = [D_2, F_1]$ and $D_2 = \Phi(D) \times C_{D_2}(F_1)$. If $C_{\Phi(D)}(F_2) = 1$, we can get $\Phi(D) = [D_1, F_2]$ and $D_1 = \Phi(D) \times C_{D_1}(F_2)$ similarly. Then $D = C_{D_1}(F_2) \times C_{D_2}(F_1)$ which is impossible. So $C_{\Phi(D)}(F_2) \neq 1$. Then

replacing D_2 by D_1 in the previous argument, we can obtain the decompositions that we need.

Suppose $\Phi(D)$ is of rank 2. Then both D_1 and D_2 are of rank 2. Let K be subgroup of F consisting of automorphisms acting trivially on D_1 . Then $D = [D, K] \times C_D(K)$ and $D_1 \leq C_D(K)$. Hence, K has to be trivial since D_1 has rank 2. This means F acts faithfully on D_1 . By induction, we have $D_1 = D_{11} \times D_{12}$ and $F = F_{11} \times F_{12}$ such that F_{11} acts faithfully on D_{11} and centralises D_{12} and F_{12} acts faithfully on D_{12} and centralises D_{11} and $F_{11} \cong F_{12}$. Then $D = [D, F_{11}] \times C_D(F_{11})$ and $D_{11} = [D_{11}, F_{11}] \leq [D, F_{11}]$ and $D_{11} \leq C_D(F_{12})$. In particular, $C_{[D, F_{11}]}(F_{12}) \neq 1$. But $[D, F_{11}]$ is cyclic. Then $[D, F_{11}] \leq C_D(F_{12})$ and moreover $C_D(F_{12}) = [D, F_{11}] \times (C_D(F_{11}) \cap C_D(F_{12}))$. But $C_D(F_{12})$ is also cyclic. We have $[D, F_{11}] = C_D(F_{12})$. Similarly, we can prove that $[D, F_{12}] = C_D(F_{11})$. Then the decompositions $D = [D, F_{11}] \times [D, F_{12}]$ and $F = F_{11} \times F_{12}$ are what we want. We are done. \square

Hence, by Lemma 2.2, we have $E = E_1 \times E_2$ and $P = P_1 \times P_2$ such that

- (i) E_1 acts faithfully on P_1 and centralises P_2 ;
- (ii) E_2 acts faithfully on P_1 and centralises P_1 ;
- (iii) $E_1 \cong E_2$ are cyclic groups of order l , which l is a positive integer dividing $(p-1)$.

Let's borrow the notation in the paragraph under Lemma 2.1. The following lemma gives some information about the degrees and number of irreducible ordinary characters of $\mathcal{O}Ne_\theta$, which is similar with [6, Proposition 5.3]. We will skip the proof.

Lemma 2.3. *Set A to be $\mathcal{O}Ne_\theta$. Remind that $|E| = l^2$. Denote $\frac{p^n-1}{l}$ and $\frac{p^m-1}{l}$ by L_n and L_m , respectively. Then the degree of an element of $\text{Irr}_K(A)$ is either l or l^2 and $\text{Irr}_K(A)$ has $p^n + p^m - 1$ elements of degree l and $L_n \cdot L_m$ elements of degree l^2 .*

3. The extension of local system

Keep the notation as above. In this section, we will use the so-called (G, b) -local system introduced by Puig and Usami in [12] to prove the main theorem.

First, let us recall some notation and state the definition of (G, b) -local system under our setting (see [12]).

Let $\mathcal{CF}_K(G)$ be the vector space of K -valued class functions of G and $\mathcal{BCF}_K(G)$ be the vector space of K -valued class functions on the set $G_{p'}$ of p' -elements of G . It is clear that the set of irreducible ordinary characters of G is a K -basis of $\mathcal{CF}_K(G)$ and the set of irreducible Brauer characters of G is a K -basis of $\mathcal{BCF}_K(G)$. For $\chi, \chi' \in \mathcal{CF}_K(G)$, we denote by $\langle \chi, \chi' \rangle$ the inner product of χ and χ' .

Let u be a p -element of G . We have the well-known surjective K -linear map $d_G^u : \mathcal{CF}_K(G) \rightarrow \mathcal{BCF}_K(C_G(u))$ defined by $d_G^u(\chi)(s) = \chi(us)$ for any $\chi \in \mathcal{CF}_K(G)$ and $s \in C_G(u)_{p'}$. It has a section $e_G^u : \mathcal{BCF}_K(C_G(u)) \rightarrow \mathcal{CF}_K(G)$ such that for $\varphi \in \mathcal{BCF}_K(C_G(u))$, $e_G^u(\varphi)(g) = 0$ if the p -part of g is not conjugate to u in G .

For the block b , let $\mathcal{CF}_{\mathcal{K}}(G, b)$ be the subspace of $\mathcal{CF}_{\mathcal{K}}(G)$ generated by the elements in $\text{Irr}_{\mathcal{K}}(G, b)$ and $\mathcal{L}_{\mathcal{K}}(G, b)$ the group of generalized characters in b . Also, let $\mathcal{CF}_{\mathcal{K}}^{\circ}(G, b) = \mathcal{CF}_{\mathcal{K}}(G, b) \cap \text{Ker}(d_G^1)$ and $\mathcal{L}_{\mathcal{K}}^{\circ}(G, b) = \mathcal{L}_{\mathcal{K}}(G, b) \cap \text{Ker}(d_G^1)$.

Definition 3.1. (Puig-Usami [12, 3.2]) With the above notation and assumption. Let X be an E -stable non-empty set of subgroups of P and assume that X contains any subgroup of P containing an element of X . Let Γ be a map over X sending $Q \in X$ to a bijective isometry

$$\Gamma_Q : \mathcal{BCF}_{\mathcal{K}}(C_N(Q), e_{\theta}) \longrightarrow \mathcal{BCF}_{\mathcal{K}}(C_G(Q), b_Q).$$

If Γ satisfies the following conditions, then Γ is called a (G, b) -local system over X .

(i) For any $Q \in X$, any $\eta \in \mathcal{BCF}_{\mathcal{K}}(C_N(Q), e_{\theta})$ and any $s \in E$, we have $\Gamma_Q(\eta)^s = \Gamma_{Q^s}(\eta^s)$.

(ii) For any $Q \in X$ and any $\eta \in \mathcal{L}_{\mathcal{K}}(C_N(Q), e_{\theta})$, the sum

$$\sum_u e_{C_G(Q)}^u (\Gamma_{Q \cdot \langle u \rangle} (d_{C_N(Q)}^u(\eta)))$$

where u runs over a set of representatives U_Q for the orbits of $C_E(Q)$ in P , is a generalized character of $C_G(Q)$.

Let Γ be a (G, b) -local system over X . Such Γ always exists by [12, 3.4.2]. For any $Q \in X$, we have a map $\Delta_Q : \mathcal{CF}_{\mathcal{K}}(C_N(Q), e_{\theta}) \longrightarrow \mathcal{CF}_{\mathcal{K}}(C_G(Q), b_Q)$ defined by

$$\Delta_Q(\eta) = \sum_{u \in U_Q} e_{C_G(Q)}^u (\Gamma_{Q \cdot \langle u \rangle} (d_{C_N(Q)}^u(\eta))).$$

Then by [12, 3.3 and 3.4] Δ_Q gives a perfect isometry between the block e_{θ} of $C_N(Q)$ and the block b_Q of $C_G(Q)$ and $\Delta_Q(\lambda * \eta) = \lambda * \Delta_Q(\eta)$ for any $\lambda \in \mathcal{CF}_{\mathcal{K}}(P)^{C_E(Q)}$ and $\eta \in \mathcal{CF}_{\mathcal{K}}(C_N(Q))$. Here, $\mathcal{CF}_{\mathcal{K}}(P)^{C_E(Q)}$ denotes the set of $C_E(Q)$ -stable elements of $\mathcal{CF}_{\mathcal{K}}(P)$ and $*$ denotes the $*$ -construction of characters due to Broué and Puig (see [2]). Hence, if X contains the trivial subgroup 1 of P , then Δ_1 induces a perfect isometry between the block e_{θ} of N and the block b of G . Moreover, this is an isotypy in the sense of [1] by [17, Proposition 2.7].

In [12], Puig and Usami developed a criterion for the extendibility of the (G, b) -local system. With the notation above. Suppose that $1 \notin X$ and let Q be a maximal subgroup of P such that $Q \notin X$. Denote by X' the union of X and the E -orbit of Q . For any subset Y of $\mathcal{OC}_N(Q)$, denote by \bar{Y} the image of Y under the canonical map from $\mathcal{OC}_N(Q)$ to $\mathcal{OC}_N(Q)/Q$. We have the similar notation for $\mathcal{OC}_G(Q)$. So \bar{e}_{θ} and \bar{b}_Q are the blocks of $\bar{C}_N(Q)$ and $\bar{C}_G(Q)$ respectively. Set $\Delta_Q^{\circ} = \sum_{u \in U_{Q-Q}} e_{C_G(Q)}^u \circ \Gamma_{Q \cdot \langle u \rangle} \circ d_{C_N(Q)}^u$ (see [12, 3.6.2]). By [12, Proposition 3.7 and Remark 3.8], Δ_Q° induces a bijective isometry

$$\bar{\Delta}_Q^\circ : \mathcal{CF}_K^\circ(\bar{C}_N(Q), \bar{e}_\theta) \cong \mathcal{CF}_K^\circ(\bar{C}_G(Q), \bar{b}_Q)$$

such that $\bar{\Delta}_Q^\circ(\mathcal{L}_K^\circ(\bar{C}_N(Q), \bar{e}_\theta)) = \mathcal{L}_K^\circ(\bar{C}_G(Q), \bar{b}_Q)$. Clearly, $\bar{\Delta}_Q^\circ(\lambda * \eta) = \lambda * \bar{\Delta}_Q^\circ(\eta)$ for $\lambda \in \text{Irr}_K(\bar{P})^{C_E(Q)}$ and $\eta \in \mathcal{L}_K^\circ(\bar{C}_N(Q), \bar{e}_\theta)$ (see [15, Case 2.2]) and $\bar{\Delta}_Q^\circ$ is $N_E(Q)$ -stable. The following is the key criterion of extendibility.

Proposition 3.2. ([12, Proposition 3.11]) *With the notation above, the (G, b) -local system Γ over X can be extended to a (G, b) -local system Γ' over X' if and only if $\bar{\Delta}_Q^\circ$ can be extended to an $N_E(Q)$ -stable bijective isometry*

$$\bar{\Delta}_Q : \mathcal{CF}_K(\bar{C}_N(Q), \bar{e}_\theta) \cong \mathcal{CF}_K(\bar{C}_G(Q), \bar{b}_Q)$$

such that $\bar{\Delta}_Q(\mathcal{L}_K(\bar{C}_N(Q), \bar{e}_\theta)) = \mathcal{L}_K(\bar{C}_G(Q), \bar{b}_Q)$.

In order to prove Theorem 1.1, it suffices to show that there is a (G, b) -local system over the set of all the subgroups P . Hence, by Proposition 3.2, we can assume that there is a (G, b) -local system Γ over X such that $1 \notin X$ and Q is a maximal subgroup of P such that $Q \notin X$.

Theorem 3.3. *With the notation above and the assumptions of Section 2. Then $\bar{\Delta}_Q^\circ$ can be extended to an $N_E(Q)$ -stable bijective isometry*

$$\bar{\Delta}_Q : \mathcal{CF}_K(\bar{C}_N(Q), \bar{e}_\theta) \cong \mathcal{CF}_K(\bar{C}_G(Q), \bar{b}_Q)$$

such that $\bar{\Delta}_Q(\mathcal{L}_K(\bar{C}_N(Q), \bar{e}_\theta)) = \mathcal{L}_K(\bar{C}_G(Q), \bar{b}_Q)$.

Proof. By the structure of E and P , $C_E(Q)$ has only three possibilities: 1, E and E_1 or E_2 . So we will divided the proof into 3 cases.

Case 1 Assume that $C_E(Q) = 1$.

Then the blocks e_θ of $C_N(Q)$ and b_Q of $C_G(Q)$ are nilpotent. By the same argument as in [12, 4.4], $\bar{\Delta}_Q^\circ$ can be extended to an $N_E(Q)$ -stable bijective isometry $\bar{\Delta}_Q$.

Case 2 Assume that $C_E(Q) = E$.

Then Q has to be trivial subgroup of P and $N_E(Q) = E$. So $\bar{C}_N(Q) = N$ and $\bar{C}_G(Q) = G$ and we have a bijective isometry

$$\bar{\Delta}^\circ : \mathcal{CF}_K^\circ(N, e_\theta) \longrightarrow \mathcal{CF}_K^\circ(G, b)$$

such that $\bar{\Delta}^\circ(\mathcal{L}_K^\circ(N, e_\theta)) = \mathcal{L}_K^\circ(G, b)$.

The following technique we adopt to extend $\bar{\Delta}^\circ$ is essentially due to Kessar and Linckelmann (see [6, Theorem 4.1]).

By Lemma 2.3, we have the following disjoint union

$$\text{Irr}_{\mathcal{K}}(N, e_{\theta}) = \Lambda_1 \sqcup \Lambda_2,$$

where Λ_1 consists of irreducible ordinary characters of dimension l and Λ_2 consists of irreducible ordinary characters of dimension l^2 . Hence, $|\Lambda_1| = p^n + p^m - 1$ and $|\Lambda_2| = \frac{p^n-1}{l} \cdot \frac{p^m-1}{l}$. We can assume that $n \geq 2$. Then $|\Lambda_1| > 2$ and $|\Lambda_2| > 2$. Choose an element $\psi_i \in \Lambda_i$ and set $\Lambda'_i = \Lambda_i - \{\psi_i\}$ for $i = 1, 2$. Since $l_N(e_{\theta}) = 1$, it is easy to see

$$\mathcal{B} = \{\psi_1 - \psi'_1 \mid \psi'_1 \in \Lambda'_1\} \sqcup \{\psi_2 - \psi'_2 \mid \psi'_2 \in \Lambda'_2\} \sqcup \{\psi_2 - l\psi_1\}$$

is a \mathbb{Z} -basis of $\mathcal{L}_{\mathcal{K}}^{\circ}(N, e_{\theta})$. Since p is odd, $|\Lambda'_i| \geq 3$ for $i = 1, 2$. So by the same argument in [12, 4.4], for any $i = 1, 2$, there exists a subset $\Omega_i = \{\chi_{\psi_i}, \chi_{\psi'_i} \mid \psi'_i \in \Lambda'_i\}$ of $\text{Irr}_{\mathcal{K}}(G, b)$ and $\delta_i \in \{\pm 1\}$ such that $\bar{\Delta}^{\circ}(\psi_i - \psi'_i) = \delta_i(\chi_{\psi_i} - \chi_{\psi'_i})$. Since $\langle \psi_1 - \psi'_1, \psi_2 - \psi'_2 \rangle = 0$ for any $\psi'_1 \in \Lambda'_1$ and $\psi'_2 \in \Lambda'_2$, $\{\chi_{\psi_1}, \chi_{\psi'_1} \mid \psi'_1 \in \Lambda'_1\}$ and $\{\chi_{\psi_2}, \chi_{\psi'_2} \mid \psi'_2 \in \Lambda'_2\}$ have trivial intersection. Denote $\psi_2 - l\psi_1$ by μ . Then $\langle \mu, \psi_1 - \psi'_1 \rangle = -l$ for all $\psi'_1 \in \Lambda'_1$. Thus

$$\bar{\Delta}^{\circ}(\mu) = \delta_1(a - l)\chi_{\psi_1} + \delta_1 a \sum_{\psi'_1 \in \Lambda'_1} \chi_{\psi'_1} + \Xi \quad (3.1)$$

for some integer a and some element $\Xi \in \mathcal{L}_{\mathcal{K}}(N, e_{\theta})$ not involving any of elements in Ω_1 . Since $\langle \mu, \psi_2 - \psi'_2 \rangle = 1$ and $\bar{\Delta}^{\circ}(\psi_2 - \psi'_2) = \delta_2(\chi_{\psi_2} - \chi_{\psi'_2})$, Ξ must involve one of the two characters occurring in $\bar{\Delta}^{\circ}(\psi_2 - \psi'_2)$ for any $\psi'_2 \in \Lambda'_2$. Taking norms on both sides in equation (3.1), we have

$$\begin{aligned} 1 + l^2 &\geq (a - l)^2 + (p^n + p^m - 2)a^2 = (p^n + p^m - 1)a^2 - 2la + l^2 \\ \iff 1 &\geq (p^n + p^m - 1)a^2 - 2la \end{aligned} \quad (3.2)$$

Suppose that $a \leq 0$. Since a is integer and $p^n + p^m - 1, l$ are positive integers, a has to be 0.

Suppose that $a > 0$. Since $p^n + p^m - 1 > 2l$, $(p^n + p^m - 1)a^2 - 2la > (p^n + p^m - 1)(a^2 - a)$. This forces $a = 1$. Hence, $a = 0$ or 1. Notice that $\Xi \neq 0$. This implies (3.2) is a proper inequality. So a must be 0. Then equation (3.1) becomes

$$\bar{\Delta}^{\circ}(\mu) = -\delta_1 l \chi_{\psi_1} + \Xi.$$

Comparing norms, we have $\langle \Xi, \Xi \rangle = 1$.

For any $\psi'_2, \psi''_2 \in \Lambda'_2$,

$$\langle \bar{\Delta}^{\circ}(\mu), \delta_2(\chi_{\psi_2} - \chi_{\psi'_2}) \rangle = \langle \mu, \psi_2 - \psi'_2 \rangle = 1$$

and

$$\langle \bar{\Delta}^{\circ}(\mu), \delta_2(\chi_{\psi'_2} - \chi_{\psi''_2}) \rangle = \langle \mu, \psi'_2 - \psi''_2 \rangle = 0.$$

Then $\Xi = \delta_2 \chi_{\psi_2}$. But $\bar{\Delta}^\circ(\mu)(1) = 0$. This forces $\delta_1 = \delta_2$. Since \mathcal{B} is a \mathbb{Z} -basis of $\mathcal{L}_K^\circ(N, e_\theta)$, $\text{Irr}_K(G, b) = \Omega_1 \sqcup \Omega_2$. Hence, we get a bijective isometry $\bar{\Delta}$ from $\mathcal{L}_K(N, e_\theta)$ to $\mathcal{L}_K(G, b)$ mapping ψ_i and ψ'_i to χ_{ψ_i} and $\chi_{\psi'_i}$ respectively, where $i = 1, 2$. In particular, $l_G(b) = l_N(e_\theta) = 1$. Clearly, it is an extension of $\bar{\Delta}^\circ$. Since $\bar{\Delta}^\circ$ is E -stable and $l_G(b) = l_N(e_\theta) = 1$, $\bar{\Delta}$ is also E -stable.

Case 3 Assume that $C_E(Q) = E_i$ for some $i = 1, 2$.

We can assume that $C_E(Q) = E_1$ and then $1 \neq Q \leq P_2$ and $N_E(Q) = E$. It suffices to prove that $\bar{\Delta}_Q^\circ$ can extend to an E_2 -stable bijective isometry $\bar{\Delta}_Q : \mathcal{L}_K(\bar{C}_N(Q), \bar{e}_\theta) \rightarrow \mathcal{L}_K(\bar{C}_G(Q), \bar{b}_Q)$.

By [16, Theorem 1], $|\text{Irr}_K(\bar{C}_N(Q), \bar{e}_\theta)| = |\text{Irr}_K(\bar{C}_G(Q), \bar{b}_Q)|$ and $l_{\bar{C}_N(Q)}(\bar{e}_\theta) = l_{\bar{C}_G(Q)}(\bar{b}_Q)$ since the block \bar{b}_Q of $\bar{C}_G(Q)$ has a cyclic hyperfocal subgroup. It is clear $C_N(Q) = (P_1 \rtimes \bar{E}_1) \times P_2$ and $C_N(Q) \trianglelefteq N$, where \bar{E}_1 is the preimage of E_1 in \bar{E} . Hence, E_1 is the inertial quotient of the block e_θ of $C_N(Q)$ and P_1 is a hyperfocal subgroup with respect to E_1 . By [16, Theorem 1], $l_{C_N(Q)}(e_\theta) = l$. We will claim that N acts transitively on $\text{IBr}(C_N(Q), e_\theta)$. Indeed, this holds because $l_N(e_\theta) = 1$ by the assumption and $N/C_N(Q) \cong E_2$ is a cyclic group of order l .

Denote by $\text{Irr}_K(\bar{E}_1)_\theta$ the subset of $\text{Irr}_K(\bar{E}_1)$ consisting of characters covering θ . Then $|\text{Irr}_K(\bar{E}_1)_\theta| = l$ and we set $\text{Irr}_K(\bar{E}_1)_\theta = \{\tau_i \mid i = 1, 2, \dots, l\}$, which is transitively acted by N . Hence, we can write $\text{Irr}_K(\bar{E}_1)_\theta$ as $\{\tau^a \mid a \in E_2\}$ for any $\tau \in \text{Irr}_K(\bar{E}_1)_\theta$. By Clifford theorem, we have $\text{Res}_Z^{\bar{E}_1}(\tau_i) = \theta$ for any i and $\text{Ind}_Z^{\bar{E}_1}(\theta) = \sum_{i=1}^l \tau_i$. Let M be a representative of \bar{E}_1 -orbit of $\text{Irr}_K(P_1) - \{1_{P_1}\}$, where 1_{P_1} is the trivial character of P_1 . Then

$$\begin{aligned} & \text{Irr}_K(\bar{C}_N(Q), \bar{e}_\theta) \\ &= \{\tau_i \bar{\zeta}_j \mid \bar{\zeta}_j \in \text{Irr}_K(\bar{P}_2), i = 1, 2, \dots, l\} \cup \{\text{Ind}_{P_1 \times Z}^{P_1 \rtimes \bar{E}_1}(\xi \theta) \bar{\zeta}_j \mid \xi \in M, \bar{\zeta}_j \in \text{Irr}_K(\bar{P}_2)\}. \end{aligned}$$

We will write $\text{Ind}_{P_1 \times Z}^{P_1 \rtimes \bar{E}_1}(\xi \theta)$ and $\bar{\chi} \cdot 1_{\bar{P}_2}$ as $\text{Ind}(\xi)$ and $\bar{\chi}$ respectively for simplicity. Here, $\bar{\chi}$ is an element of $\mathcal{CF}_K(P_1 \times \bar{E}_1)$. Clearly, $\text{Ind}(\xi)$ is N and E_2 -stable for any $\xi \in M$. Similar to the argument of [15, Case 2],

$$\{(\sum_{i=1}^l \tau_i - \text{Ind}(\xi)) \bar{\zeta} \mid \xi \in M, \bar{\zeta} \in \text{Irr}_K(\bar{P}_2)\} \cup \{\tau_i - \tau_i \bar{\zeta} \mid i = 1, 2, \dots, l, 1_{\bar{P}_2} \neq \bar{\zeta} \in \text{Irr}_K(\bar{P}_2)\}$$

is a \mathbb{Z} -basis of $\mathcal{L}_K^\circ(\bar{C}_N(Q), \bar{e}_\theta)$.

Case 3.1 Assume that $\bar{P}_2 = 1$, i.e., $Q = P_2$.

Set $H = N_G(Q, b_Q)$. Then $H = C_G(Q)N_G(P, b_P)$ and b_Q is still a block of H . Let d be the Brauer correspondent of the block b_Q of H in $N_H(P)$. Then $l_{N_H(P)}(d) = 1$ by the assumption. We claim that $l_H(b_Q) = 1$.

Indeed, considering the canonical map from $\mathcal{O}H$ to $\mathcal{O}(H/Q)$, denote by \bar{X} the image of X under this canonical map for any subset X of $\mathcal{O}H$. Then \bar{b}_Q is still a block of $\bar{C}_G(Q)$

and $\bar{H}/\bar{C}_G(Q)$ is a cyclic group of order l . By [7, Lemma 3.5], $\text{Br}_{\bar{P}}(\bar{b}_Q) = \overline{\text{Br}_P(b_Q)}$. Since $l_{N_H(P)}(d) = 1$, \bar{d} is still a block of $\bar{N}_H(P)$. Therefore, $\text{Br}_{\bar{P}}(\bar{b}_Q) = \bar{d}$ is a block of $\bar{N}_H(P)$. Since $\bar{H}/\bar{C}_G(Q)$ is a p' -group, the blocks of \bar{H} covering the block \bar{b}_Q of $\bar{C}_G(Q)$ have the same defect group \bar{P} . Let i be a block of \bar{H} covering the block \bar{b}_Q of $\bar{C}_G(Q)$. Then $i = i\bar{b}_Q$ and \bar{P} is a defect group of the block i . Therefore, $\text{Br}_{\bar{P}}(i) \neq 0$ and $\text{Br}_{\bar{P}}(i)\text{Br}_{\bar{P}}(\bar{b}_Q) = \text{Br}_{\bar{P}}(i)$. However, $\text{Br}_{\bar{P}}(\bar{b}_Q)$ is a block of $\bar{N}_H(P)$ and $N_{\bar{H}}(\bar{P}) = \bar{N}_H(P)$. This means that there is only one block of \bar{H} covering the block \bar{b}_Q of $\bar{C}_G(Q)$ and then it must be \bar{b}_Q . Hence, the block \bar{b}_Q of \bar{H} has a defect group \bar{P} which is cyclic by our assumption. In particular, we have $l_H(b_Q) = l_{\bar{H}}(\bar{b}_Q) = l_{N_H(P)}(d) = 1$ since $N_{\bar{H}}(\bar{P}) = \bar{N}_H(P)$.

Moreover, we claim that there is a regular E_2 -orbit of $\text{Irr}_{\mathcal{K}}(\bar{C}_G(Q), \bar{b}_Q)$, namely, H acts transitively on it.

Indeed, since the block \bar{b}_Q of \bar{H} has a cyclic defect group, it must be nilpotent. By [11, Theorem 3.13], the block \bar{b}_Q of $\bar{C}_G(Q)$ is basic Morita equivalent to its Brauer correspondent. Note that the block \bar{b}_Q of $\bar{C}_G(Q)$ is not nilpotent since $l > 1$. This implies that every irreducible Brauer character of the block \bar{b}_Q of $\bar{C}_G(Q)$ can be uniquely lifted to an irreducible ordinary character by the theory of cyclic blocks.

On the other hand, since $l_{C_G(Q)}(b_Q) = l$ and $l_H(b_Q) = 1$ and $H/C_G(Q) \cong E_2$ has order l , H acts transitively on $\text{IBr}(C_G(Q), b_Q)$. Combining this with the argument above, there exists a regular H -orbit of $\text{Irr}_{\mathcal{K}}(\bar{C}_G(Q), \bar{b}_Q)$. We are done.

Case 3.1.1 Assume that $|M| = 1$.

Then $\text{rank}_{\mathcal{O}}(\mathcal{L}_{\mathcal{K}}^{\circ}(\bar{C}_N(Q), \bar{e}_{\theta})) = 1$ and $\mathcal{L}_{\mathcal{K}}^{\circ}(\bar{C}_N(Q), \bar{e}_{\theta}) = \mathbb{Z}(\text{Ind}(\xi) - \sum_{i=1}^l \tau_i)$. Since there is a regular E_2 -orbit of $\text{Irr}_{\mathcal{K}}(\bar{C}_G(Q), \bar{b}_Q)$, $\text{Irr}_{\mathcal{K}}(\bar{C}_G(Q), \bar{b}_Q) = \{\chi_0\} \cap \{\chi_1, \chi_2, \dots, \chi_l\}$ such that χ_0 is E_2 -stable and E_2 acts regularly on $\{\chi_1, \chi_2, \dots, \chi_l\}$. Then we have

$$\bar{\Delta}_Q^{\circ}(\text{Ind}(\xi) - \sum_{i=1}^l \tau_i) = \delta_0 \chi_0 - \sum_{i=1}^l \delta_i \chi_i$$

for some $\delta_0, \delta_i \in \{\pm 1\}$, $i = 1, 2, \dots, l$. Since $\bar{\Delta}_Q^{\circ}$ is E_2 -stable, we have $\delta_1 = \delta_2 = \dots = \delta_l = \delta_0$. If we write $\{\tau_1, \tau_2, \dots, \tau_l\}$ and $\{\chi_1, \chi_2, \dots, \chi_l\}$ as $\{\tau^a \mid a \in E_2\}$ and $\{\chi^a \mid a \in E_2\}$ respectively, then we can define a bijective isometry as below

$$\begin{aligned} \bar{\Delta}_Q : \mathcal{L}_{\mathcal{K}}(\bar{C}_N(Q), \bar{e}_{\theta}) &\longrightarrow \mathcal{L}_{\mathcal{K}}(\bar{C}_G(Q), \bar{b}_Q) \\ \text{Ind}(\xi) &\mapsto \delta_0 \chi_0 \\ \tau^a &\mapsto \delta_0 \chi^a. \end{aligned}$$

It is evident that it is an extension of $\bar{\Delta}_Q^{\circ}$ and E_2 -stable. We are done for this case.

Case 3.1.2 Assume that $|M| \geq 2$.

Then there are at least two different $\xi_1, \xi_2 \in M$. So $\text{Ind}(\xi_1) - \text{Ind}(\xi_2) \in \mathcal{L}_{\mathcal{K}}^{\circ}(\bar{C}_N(Q), \bar{e}_{\theta})$ and $\langle \text{Ind}(\xi_1) - \text{Ind}(\xi_2), \text{Ind}(\xi_1) - \text{Ind}(\xi_2) \rangle = 2$. Then there exist $\chi_1 \neq \chi_2 \in \text{Irr}_{\mathcal{K}}(\bar{C}_G(Q), \bar{b}_Q)$ such that

$$\bar{\Delta}_Q^{\circ}(\text{Ind}(\xi_1) - \text{Ind}(\xi_2)) = \delta(\chi_1 - \chi_2)$$

for some $\delta \in \{\pm 1\}$. Since $\bar{\Delta}_Q^{\circ}$ is E_2 -stable, we have ${}^a(\delta\chi_1 - \delta\chi_2) = \delta(\chi_1 - \chi_2)$ for any $a \in E_2$. This means that χ_1 and χ_2 are both E_2 -stable.

If there is a $\xi_3 \in M$ different from ξ_1 and ξ_2 , then there is a $\chi_3 \in \text{Irr}_{\mathcal{K}}(\bar{C}_G(Q), \bar{b}_Q)$ different from χ_1 and χ_2 such that

$$\bar{\Delta}_Q^{\circ}(\text{Ind}(\xi_1) - \text{Ind}(\xi_3)) = \delta\chi_1 - \delta\chi_3 \text{ or } -\delta\chi_2 + \delta\chi_3$$

and χ_3 is E_2 -stable; then we may choose the notation in such a way that

$$\bar{\Delta}_Q^{\circ}(\text{Ind}(\xi_1) - \text{Ind}(\xi_2)) = \delta(\chi_1 - \chi_2) \text{ and } \bar{\Delta}_Q^{\circ}(\text{Ind}(\xi_1) - \text{Ind}(\xi_3)) = \delta(\chi_1 - \chi_3)$$

for some E_2 -stable elements χ_1, χ_2, χ_3 of $\text{Irr}_{\mathcal{K}}(\bar{C}_G(Q), \bar{b}_Q)$.

If $|M| \geq 4$, then for any $\xi \in M - \{\xi_1, \xi_2, \xi_3\}$, there is a unique $\chi \in \text{Irr}_{\mathcal{K}}(\bar{C}_G(Q), \bar{b}_Q) - \{\chi_1, \chi_2, \chi_3\}$ such that

$$\bar{\Delta}_Q^{\circ}(\text{Ind}(\xi_1) - \text{Ind}(\xi)) = \delta(\chi_1 - \chi)$$

and χ is E_2 -stable.

In conclusion, we have an injective isometry

$$\Phi : \mathbb{Z}\{\text{Ind}(\xi) \mid \xi \in M\} \longrightarrow \mathcal{L}_{\mathcal{K}}(\bar{C}_G(Q), \bar{b}_Q)$$

mapping $\text{Ind}(\xi)$ to $\delta\chi_{\xi}$ such that

$$\Phi(\text{Ind}(\xi) - \text{Ind}(\xi')) = \bar{\Delta}_Q^{\circ}(\text{Ind}(\xi) - \text{Ind}(\xi'))$$

and χ_{ξ} is E_2 -stable for any $\xi, \xi' \in M$.

Denote $\text{Irr}_{\mathcal{K}}(\bar{C}_G(Q), \bar{b}_Q) - \{\chi_{\xi} \mid \xi \in M\}$ by Ω . Then $|\Omega| = l$ and E_2 acts on Ω . Since there is a regular E_2 -orbit of $\text{Irr}_{\mathcal{K}}(\bar{C}_G(Q), \bar{b}_Q)$, E_2 acts regularly on Ω . This means that Ω can be represented as $\{\chi^a \mid a \in E_2\}$ for some $\chi \in \Omega$.

Now we fix an element ξ of M . Suppose that χ does not get involved in $\bar{\Delta}_Q^{\circ}(\text{Ind}(\xi) - \sum_{i=1}^l \tau_i)$. Then there is $\xi' \in M$ such that $\langle \chi, \bar{\Delta}_Q^{\circ}(\text{Ind}(\xi') - \sum_{i=1}^l \tau_i) \rangle \neq 0$ since $\{\text{Ind}(\xi) - \sum_{i=1}^l \tau_i \mid \xi \in M\}$ is a \mathbb{Z} -basis of $\mathcal{L}_{\mathcal{K}}^{\circ}(\bar{C}_N(Q), \bar{e}_{\theta})$. Hence, χ has to get involved in $\bar{\Delta}_Q^{\circ}(\text{Ind}(\xi) - \sum_{i=1}^l \tau_i) - \bar{\Delta}_Q^{\circ}(\text{Ind}(\xi') - \sum_{i=1}^l \tau_i)$ which is $\delta(\chi_{\xi} - \chi_{\xi'})$. This is impossible. So χ must get

involved in $\bar{\Delta}_Q^\circ(\text{Ind}(\xi) - \sum_{i=1}^l \tau_i)$ for any $\xi \in M$. Since $\bar{\Delta}_Q^\circ$ and $\text{Ind}(\xi) - \sum_{i=1}^l \tau_i$ are E_2 -stable, χ^a has to get involved in $\bar{\Delta}_Q^\circ(\text{Ind}(\xi) - \sum_{i=1}^l \tau_i)$ for any $a \in E_2$ and $\xi \in M$. Since $\langle \text{Ind}(\xi) - \sum_{i=1}^l \tau_i, \text{Ind}(\xi) - \sum_{i=1}^l \tau_i \rangle = 1 + l$ and $\langle \text{Ind}(\xi) - \sum_{i=1}^l \tau_i, \text{Ind}(\xi) - \text{Ind}(\xi') \rangle = 1$, $\bar{\Delta}_Q^\circ(\text{Ind}(\xi) - \sum_{i=1}^l \tau_i) = \delta\chi_\xi - \sum_{a \in E_2} \delta_a \chi^a$ or $-\delta\chi_{\xi'} - \sum_{a \in E_2} \delta_a \chi^a$, where $\delta_a \in \{\pm 1\}$ for any $a \in E_2$. Note that the last situation can happen if and only if $|M| = 2$. By switching χ_ξ and $\chi_{\xi'}$ if necessary, we can assume that $\bar{\Delta}_Q^\circ(\text{Ind}(\xi) - \sum_{i=1}^l \tau_i) = \delta\chi_\xi - \sum_{a \in E_2} \delta_a \chi^a$. Since $\bar{\Delta}_Q^\circ$ is E_2 -stable and E_2 acts regularly on Ω , δ_a is equal to δ for any $a \in E_2$. Then we can define an E_2 -stable bijective isometry as follows

$$\begin{aligned} \bar{\Delta}_Q : \mathcal{L}_K(\bar{C}_N(Q), \bar{e}_\theta) &\longrightarrow \mathcal{L}_K(\bar{C}_G(Q), \bar{b}_Q) \\ \text{Ind}(\xi) &\mapsto \delta\chi_\xi \\ \tau^a &\mapsto \delta\chi^a. \end{aligned}$$

It is clear that $\bar{\Delta}_Q$ is an extension of $\bar{\Delta}_Q^\circ$.

Case 3.2 $\bar{P}_2 > 1$, namely, Q is a non-trivial proper subgroup of P_2 .

Then $\text{Ind}(\xi) - \text{Ind}(\xi)\bar{\zeta} \in \mathcal{L}_K^\circ(\bar{C}_N(Q), \bar{e}_\theta)$ for any $\xi \in M$ and $1_{\bar{P}_2} \neq \bar{\zeta} \in \text{Irr}_K(\bar{P}_2)$.

Now we fix an element $\xi \in M$. Since p is odd, $|\bar{P}_2| \geq 3$. Then there are at least two elements $\bar{\zeta}$ and $\bar{\zeta}'$ of $\text{Irr}_K(\bar{P}_2)$ different from $1_{\bar{P}_2}$. With the same argument in the first three paragraphs in Case 3.1.2, we can get a subset $\{\chi_\xi, \chi_{\bar{\zeta}} \mid 1_{\bar{P}_2} \neq \bar{\zeta} \in \text{Irr}_K(\bar{P}_2)\}$ of $\text{Irr}_K(\bar{C}_G(Q), \bar{b}_Q)$ such that

$$\bar{\Delta}_Q^\circ(\text{Ind}(\xi) - \text{Ind}(\xi)\bar{\zeta}) = \delta(\chi_\xi - \chi_{\bar{\zeta}})$$

for any $1_{\bar{P}_2} \neq \bar{\zeta} \in \text{Irr}_K(\bar{P}_2)$, where $\delta \in \{\pm 1\}$.

Given any $1 \neq a \in E_2$ and $1_{\bar{P}_2} \neq \bar{\zeta} \in \text{Irr}_K(\bar{P}_2)$, ${}^a(\text{Ind}(\xi)\bar{\zeta}) = \text{Ind}(\xi)({}^a\bar{\zeta})$. Since $\bar{\Delta}_Q^\circ$ is E_2 -stable, this means ${}^a\chi_\xi - {}^a\chi_{\bar{\zeta}} = \chi_\xi - \chi_{a\bar{\zeta}}$. Hence, we have χ_ξ is E_2 -stable and ${}^a\chi_{\bar{\zeta}} = \chi_{a\bar{\zeta}}$. On the other hand,

$$(\text{Ind}(\xi) - \text{Ind}(\xi)\bar{\zeta})\bar{\zeta} = (\text{Ind}(\xi) - \text{Ind}(\xi)\bar{\zeta}^2) - (\text{Ind}(\xi) - \text{Ind}(\xi)\bar{\zeta}).$$

Since $\bar{\Delta}_Q^\circ$ is compatible with $*$ -structure, using $\bar{\Delta}_Q^\circ$ on both sides in the above equality, we can get

$$\delta(\chi_\xi - \chi_{\bar{\zeta}}) * \bar{\zeta} = \delta(\chi_{\bar{\zeta}} - \chi_{\bar{\zeta}^2}).$$

Therefore, $\chi_{\bar{\zeta}} = \chi_\xi * \bar{\zeta}$ for any $1_{\bar{P}_2} \neq \bar{\zeta} \in \text{Irr}_K(\bar{P}_2)$.

Suppose that there is another element ξ' of M different from ξ . Similarly, we can get a subset $\{\chi_{\xi'} * \bar{\zeta} \mid \bar{\zeta} \in \text{Irr}_K(\bar{P}_2)\}$ of $\text{Irr}_K(\bar{C}_G(Q), \bar{b}_Q)$ such that $\bar{\Delta}_Q^\circ(\text{Ind}(\xi') - \text{Ind}(\xi')\bar{\zeta}) =$

$\delta'(\chi_{\xi'} - \chi_{\xi'} * \bar{\zeta})$ for any $1_{\bar{P}_2} \neq \bar{\zeta} \in \text{Irr}_{\mathcal{K}}(\bar{P}_2)$ and $\chi_{\xi'}$ is E_2 -stable, where $\delta' \in \{\pm 1\}$. Assume that $\{\chi_{\xi} * \bar{\zeta} \mid \bar{\zeta} \in \text{Irr}_{\mathcal{K}}(\bar{P}_2)\} \cap \{\chi_{\xi'} * \bar{\zeta} \mid \bar{\zeta} \in \text{Irr}_{\mathcal{K}}(\bar{P}_2)\} \neq \emptyset$. Then there is $\bar{\zeta}_0 \in \text{Irr}_{\mathcal{K}}(\bar{P}_2)$ such that $\chi_{\xi} = \chi_{\xi'} * \bar{\zeta}_0$. If $\bar{\zeta}_0 = 1_{\bar{P}_2}$, then $\chi_{\xi} = \chi_{\xi'}$. This implies that $\text{Ind}(\xi) - \text{Ind}(\xi')\bar{\zeta} = \pm(\text{Ind}(\xi') - \text{Ind}(\xi')\bar{\zeta})$ for any $1_{\bar{P}_2} \neq \bar{\zeta} \in \text{Irr}_{\mathcal{K}}(\bar{P}_2)$. This is impossible. Then $\bar{\zeta}_0$ is non-trivial. But it implies that $\chi_{\xi'} * \bar{\zeta}_0^2 = \chi_{\xi'}$ since $\langle \text{Ind}(\xi) - \text{Ind}(\xi')\bar{\zeta}_0, \text{Ind}(\xi') - \text{Ind}(\xi')\bar{\zeta}_0 \rangle = 0$. By [8, Theorem 8.10.8], $\text{Irr}_{\mathcal{K}}(\bar{P}_2)$ acts freely on irreducible ordinary characters of height zero in the block \bar{b}_Q of $\bar{C}_G(Q)$. Hence, $\bar{\zeta}_0^2 = 1_{\bar{P}_2}$ since the defect group of the block \bar{b}_Q of $\bar{C}_G(Q)$ is cyclic. But it is impossible because p is odd. Then

$$\{\chi_{\xi} * \bar{\zeta} \mid \bar{\zeta} \in \text{Irr}_{\mathcal{K}}(\bar{P}_2)\} \cap \{\chi_{\xi'} * \bar{\zeta} \mid \bar{\zeta} \in \text{Irr}_{\mathcal{K}}(\bar{P}_2)\} = \emptyset$$

for any different $\xi, \xi' \in M$. It is clear that $\chi_{\xi} * \bar{\zeta}$ is an irreducible ordinary character in the block \bar{b}_Q of $\bar{C}_G(Q)$ by [2, Corollary]. Then we get an injective isometry

$$\Psi : \mathbb{Z}\{\text{Ind}(\xi)\bar{\zeta} \mid \xi \in M, \bar{\zeta} \in \text{Irr}_{\mathcal{K}}(\bar{P}_2)\} \longrightarrow \mathcal{L}_{\mathcal{K}}(\bar{C}_G(Q), \bar{b}_Q)$$

mapping $\text{Ind}(\xi)\bar{\zeta}$ to $\delta_{\xi}(\chi_{\xi} * \bar{\zeta})$ such that $\Psi(\text{Ind}(\xi) - \text{Ind}(\xi')\bar{\zeta}) = \bar{\Delta}_Q^{\circ}(\text{Ind}(\xi) - \text{Ind}(\xi')\bar{\zeta})$ and χ_{ξ} is E_2 -stable for any $\xi \in M$ and $1_{\bar{P}_2} \neq \bar{\zeta} \in \text{Irr}_{\mathcal{K}}(\bar{P}_2)$, where $\delta_{\xi} \in \{\pm 1\}$.

At the same time, $\tau - \tau\bar{\zeta} \in \mathcal{L}_{\mathcal{K}}^{\circ}(\bar{C}_N(Q), \bar{e}_{\theta})$ for any $\tau \in \text{Irr}_{\mathcal{K}}(\bar{E}_1)_{\theta}$ and $1_{\bar{P}_2} \neq \bar{\zeta} \in \text{Irr}_{\mathcal{K}}(\bar{P}_2)$. Take an element τ of $\text{Irr}_{\mathcal{K}}(\bar{E}_1)_{\theta}$. With the same arguments as above, we can get an element χ_{τ} of $\text{Irr}_{\mathcal{K}}(\bar{C}_G(Q), \bar{b}_Q)$ and $\delta_{\tau} \in \{\pm 1\}$ such that $\bar{\Delta}_Q^{\circ}(\tau - \tau\bar{\zeta}) = \delta_{\tau}(\chi_{\tau} - \chi_{\tau} * \bar{\zeta})$ for any $1_{\bar{P}_2} \neq \bar{\zeta} \in \text{Irr}_{\mathcal{K}}(\bar{P}_2)$. Choosing any $1 \neq a \in E_2$ and $1_{\bar{P}_2} \neq \bar{\zeta} \in \text{Irr}_{\mathcal{K}}(\bar{P}_2)$, since $\bar{\Delta}_Q^{\circ}$ is E_2 -stable, we have

$$\delta_{a\tau}(\chi_{a\tau} - \chi_{a\tau} * {}^a\bar{\zeta}) = \bar{\Delta}_Q^{\circ}(a\tau - {}^a\tau({}^a\bar{\zeta})) = {}^a(\bar{\Delta}_Q^{\circ}(\tau - \tau\bar{\zeta})) = \delta_{\tau}({}^a\chi_{\tau} - {}^a\chi_{\tau} * {}^a\bar{\zeta}).$$

Then $\chi_{a\tau} = {}^a\chi_{\tau}$ or $\chi_{a\tau} = {}^a\chi_{\tau} * {}^a\bar{\zeta}$. If $\chi_{a\tau} = {}^a\chi_{\tau} * {}^a\bar{\zeta}$, then ${}^a\chi_{\tau} = \chi_{a\tau} * {}^a\bar{\zeta}$. Therefore, $\chi_{a\tau} = \chi_{a\tau} * {}^a(\bar{\zeta}^2)$, which is impossible. Hence, $\chi_{a\tau} = {}^a\chi_{\tau}$ and $\delta_{a\tau} = \delta_{\tau}$ for any $a \in E_2$ since E_2 acts transitively on $\text{Irr}_{\mathcal{K}}(\bar{E}_1)_{\theta}$. And we denote δ_{τ} by δ . By the facts that $\langle \tau - \tau\bar{\zeta}, \tau' - \tau'\bar{\zeta}' \rangle = 0$ and $\langle \text{Ind}(\xi) - \text{Ind}(\xi')\bar{\zeta}, \tau - \tau\bar{\zeta}' \rangle = 0$ for any $\tau \neq \tau' \in \text{Irr}_{\mathcal{K}}(\bar{E}_1)_{\theta}$ and $\xi \in M$ and $\bar{\zeta}, \bar{\zeta}' \in \text{Irr}_{\mathcal{K}}(\bar{P}_2) - \{1_{\bar{P}_2}\}$, we can get

$$\{\chi_{\tau} * \bar{\zeta} \mid \bar{\zeta} \in \text{Irr}_{\mathcal{K}}(\bar{P}_2)\} \cap \{\chi_{\tau'} * \bar{\zeta} \mid \bar{\zeta} \in \text{Irr}_{\mathcal{K}}(\bar{P}_2)\} = \emptyset$$

and

$$\{\chi_{\xi} * \bar{\zeta} \mid \xi \in M, \bar{\zeta} \in \text{Irr}_{\mathcal{K}}(\bar{P}_2)\} \cap \{\chi_{\tau} * \bar{\zeta} \mid \tau \in \text{Irr}_{\mathcal{K}}(\bar{E}_1)_{\theta}, \bar{\zeta} \in \text{Irr}_{\mathcal{K}}(\bar{P}_2)\} = \emptyset.$$

Hence, we have a well-defined E_2 -stable bijective isometry as below

$$\begin{aligned} \bar{\Delta}_Q : \mathcal{L}_{\mathcal{K}}(\bar{C}_N(Q), \bar{e}_{\theta}) &\longrightarrow \mathcal{L}_{\mathcal{K}}(\bar{C}_G(Q), \bar{b}_Q) \\ \text{Ind}(\xi)\bar{\zeta} &\mapsto \delta_{\xi}\chi_{\xi} * \bar{\zeta} \end{aligned}$$

$${}^a\tau\bar{\zeta} \mapsto \delta^a\chi_\tau * \bar{\zeta}.$$

It suffices to show that $\bar{\Delta}_Q$ is an extension of $\bar{\Delta}_Q^\circ$, namely,

$$\bar{\Delta}_Q^\circ(\text{Ind}(\xi) - \sum_{i=1}^l \tau_i) = \delta_\xi \chi_\xi - \delta \sum_{i=1}^l \chi_{\tau_i}$$

for any $\xi \in M$.

Choose an element ξ of M . Since $\langle \text{Ind}(\xi) - \sum_{i=1}^l \tau_i, \tau - \tau\bar{\zeta} \rangle = -1$, then at least χ_τ and $\chi_\tau * \bar{\zeta}$ must get involved in $\bar{\Delta}_Q^\circ(\text{Ind}(\xi) - \sum_{i=1}^l \tau_i)$ for any $\tau \in \text{Irr}_{\mathcal{K}}(\tilde{E}_1)_\theta$ and $\bar{\zeta} \in \text{Irr}_{\mathcal{K}}(\bar{P}_2) - \{1_{\bar{P}_2}\}$.

Keep the notation as above. Suppose that there are τ and $\bar{\zeta}$ such that $\chi_\tau * \bar{\zeta}$ gets involved in $\bar{\Delta}_Q^\circ(\text{Ind}(\xi) - \sum_{i=1}^l \tau_i)$. Since $\langle \text{Ind}(\xi) - \sum_{i=1}^l \tau_i, \tau\bar{\zeta} - \tau\bar{\zeta}' \rangle = 0$ for any $\bar{\zeta}' \in \text{Irr}_{\mathcal{K}}(\bar{P}_2)$ different from $\bar{\zeta}$ and $1_{\bar{P}_2}$, $\chi_\tau * \bar{\zeta}$ must get involved in $\bar{\Delta}_Q^\circ(\text{Ind}(\xi) - \sum_{i=1}^l \tau_i)$ for any $\bar{\zeta}$. At the same time, since $\bar{\Delta}_Q^\circ$ is E_2 -stable and $\text{Ind}(\xi) - \sum_{i=1}^l \tau_i$ is E_2 -stable, we have ${}^a(\chi_\tau * \bar{\zeta})$ must get involved in $\bar{\Delta}_Q^\circ(\text{Ind}(\xi) - \sum_{i=1}^l \tau_i)$ for any $a \in E_2$ and $\bar{\zeta}$. Then there are at least $l \cdot (|\bar{P}_2| - 1)$ different irreducible characters involved in $\bar{\Delta}_Q^\circ(\text{Ind}(\xi) - \sum_{i=1}^l \tau_i)$. This is impossible since $\langle \text{Ind}(\xi) - \sum_{i=1}^l \tau_i, \text{Ind}(\xi) - \sum_{i=1}^l \tau_i \rangle = 1 + l$ and $|\bar{P}_2| - 1 \geq 2$ and $l > 1$.

So for any $\xi \in M$, $\bar{\Delta}_Q^\circ(\text{Ind}(\xi) - \sum_{i=1}^l \tau_i) = a_\chi \chi - \delta \sum_{\tau} \chi_\tau$. Here, $a_\chi \in \{\pm 1\}$ and χ is an element of $\text{Irr}_{\mathcal{K}}(\bar{C}_G(Q), \bar{b}_Q) - \{\chi_\tau \mid \tau \in \text{Irr}_{\mathcal{K}}(\tilde{E}_1)_\theta\}$. Since $\langle \text{Ind}(\xi) - \sum_{i=1}^l \tau_i, \text{Ind}(\xi) - \text{Ind}(\xi)\bar{\zeta} \rangle = 1$ for any $\bar{\zeta} \neq 1_{\bar{P}_2}$, we have $a_\chi = \delta_\xi$ and $\chi = \chi_\xi$. We are done. \square

Then the proof of Theorem 1.1 will follow by Theorem 3.3 and [12, 3.4.2].

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