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Cellularity of endomorphism algebras of Young permutation modules

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A B S T R A C T

Let E be an n-dimensional vector space. Then the symmetric group $\mathrm{Sym}(n)$ acts on E by permuting the elements of a basis and hence on the r-fold tensor product $E^{\otimes r}$. Bowman, Doty and Martin ask, in [1], whether the endomorphism algebra $\mathrm{End}_{\mathrm{Sym}(n)}(E^{\otimes r})$ is cellular. The module $E^{\otimes r}$ is the permutation module for a certain Young $\mathrm{Sym}(n)$-set. We shall show that the endomorphism algebra of the permutation module on an arbitrary Young $\mathrm{Sym}(n)$-set is a cellular algebra. We determine, in terms of the point stabilisers which appear, when the endomorphism algebra is quasi-hereditary.

1. Introduction

We fix a positive integer n. The symmetric group of degree n is denoted $\mathrm{Sym}(n)$. For a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ of n we have the Young subgroup, i.e., the group $\mathrm{Sym}(\lambda) = \mathrm{Sym}(\lambda_1) \times \mathrm{Sym}(\lambda_2) \times \cdots$, regarded as a subgroup of $\mathrm{Sym}(n)$ in the usual way. By a Young $\mathrm{Sym}(n)$-set we mean a finite $\mathrm{Sym}(n)$-set such that each point stabiliser is conjugate to a Young subgroup. Let R be a commutative ring. Our interest is in the endomorphism algebra $\mathrm{End}_{\mathrm{Sym}(n)}(R\Omega)$ of the permutation module $R\Omega$ on a Young $\mathrm{Sym}(n)$-set Ω. We

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shall show that $\mathrm{End}_{\mathrm{Sym}(n)}(\mathbb{Z}\,\Omega)$ has a cellular structure, Theorem 6.3, hence by base change so has $\mathrm{End}_{\mathrm{Sym}(n)}(R\,\Omega)$, for an arbitrary commutative ring R.

Taking the base ring now to be a field k of positive characteristic, we give a criterion for $\mathrm{End}_{\mathrm{Sym}(n)}(k\Omega)$ to be a quasi-hereditary algebra, in terms of the set of partitions λ of n for which $\mathrm{Sym}(\lambda)$ occurs as a point stabiliser, and the characteristic p of k, see Theorem 6.5. This is applied to the case $\Omega = I(n,r)$, the set of maps from $\{1,\dots,r\}$ to $\{1,\dots,n\}$, for a positive integer r, with $\mathrm{Sym}(n)$ acting by composition of maps. The permutation module $kI(n,r)$ may be regarded as the rth tensor power $E^{\otimes r}$ of an n-dimensional vector space E, and we thus determine when $\mathrm{End}_{\mathrm{Sym}(n)}(E^{\otimes r})$ is quasi-hereditary, see Proposition 7.3.

We conclude by addressing two points raised by the referee, to whom I am most grateful. The first point is to relate quasihereditary endomorphism algebras of Young permutation modules to generalised Schur algebras and the second is to note that the decomposition numbers for the endomorphism algebra of an arbitrary Young permutation module are decomposition numbers for general linear groups.

Our procedure is to analyse the endomorphism algebra of a Young permutation module in the spirit of the Schur algebra $S(n,r)$ (which is a special case). Of particular importance to us will be the fact that the Schur algebra is quasi-hereditary. There are several approaches to this (see e.g. [7, Section A5] and [20]) but for us the most convenient is that of Green, [11]. This has the advantage of being a purely combinatorial account carried out over an arbitrary commutative base ring. So we regard what follows as a modest generalisation of some aspects of [11]: we follow Green's approach and notation to a large extent.

2. Preliminaries

We write $\mathrm{mod}(S)$ for the category of finitely generated modules over a ring S.

Let G be a finite group and K a field of characteristic 0. Let X be a finitely generated KG-module. Suppose that all composition factors of X are absolutely irreducible. Let $U_1,\dots,U_d$ be a complete set of pairwise non-isomorphic composition factors of X. We write X as a direct sum of simple modules $X = X_1\oplus\cdots\oplus X_h$. For $1\le i\le d$ let m_i be the number of elements $r\in\{1,\dots,h\}$ such that X_r is isomorphic to U_i. Let $S = \mathrm{End}_G(X)$. Then S is isomorphic to the product of the matrix algebras $M_{m_1}(K),\dots,M_{m_d}(K)$. We have an exact functor from $f : \mathrm{mod}(KG)\to\mathrm{mod}(S)$, given on objects by $f(Z) = \mathrm{Hom}_{\mathrm{Sym}(n)}(Z,X)$. Moreover we have $S = f(X) = \bigoplus_{r=1}^{h}\mathrm{Hom}_G(X_r,X)$. It follows that the modules $L_i = fU_i = \mathrm{Hom}_G(U_i,X)$, $1\le i\le d$, form a complete set of pairwise non-isomorphic irreducible S-modules.

The situation in positive characteristic is similar, cf. [10, (3.4) Proposition]. Suppose now that F is any field which is a splitting field for G. Let Y be a finitely generated KG-module in which every indecomposable summand is absolutely indecomposable. Let $V_1,\dots,V_e$ be a complete set of pairwise non-isomorphic indecomposable summands of Y. We write Y as a direct sum of indecomposable modules $Y = Y_1\oplus\cdots\oplus Y_k$. For $1\le j\le e$ let n_j be the number of elements $r\in\{1,\dots,k\}$ such that Y_r is isomorphic to V_j. Let

$T = \mathrm{End}_G(Y)$. Then each $P_j = \mathrm{Hom}_G(V_j, Y)$ is naturally a T-module and the modules $P_1, \dots, P_e$ form a complete set of pairwise non-isomorphic projective T-modules. Let N_j be the head of P_j, $1 \le j \le e$. Then the modules $N_1, \dots, N_e$ form a complete set of pairwise non-isomorphic irreducible T-modules. The dimension of N_j over F is n_j.

We now fix a positive integer n. We write $\mathrm{Par}(n)$ for the set of partitions of n. By the support $\zeta(\Omega)$ of a Young $\mathrm{Sym}(n)$-set Ω we mean the set of $\lambda \in \mathrm{Par}(n)$ such that the Young subgroup $\mathrm{Sym}(\lambda)$ is a point stabiliser. Let R be a commutative ring. For a Young $\mathrm{Sym}(n)$-set Ω we write $S_{\Omega,R}$ for the endomorphism algebra $\mathrm{End}_{\mathrm{Sym}(n)}(R\Omega)$ of the permutation module $R\Omega$. For $\lambda \in \mathrm{Par}(n)$ we write $M(\lambda)_R$ for the permutation module $R\,\mathrm{Sym}(n)/\mathrm{Sym}(\lambda)$.

We have the usual dominance partial order $\trianglelefteq$ on $\mathrm{Par}(n)$. Thus, for $\lambda = (\lambda_1, \lambda_2, \dots)$, $\mu = (\mu_1 \mu_2, \dots) \in \mathrm{Par}(n)$, we write $\lambda \trianglelefteq \mu$ if $\lambda_1 + \cdots + \lambda_a \le \mu_1 + \cdots + \mu_a$ for all $1 \le a \le n$.

Recall that the Specht modules $\mathrm{Sp}(\lambda)_{\mathbb{Q}}$, $\lambda \in \mathrm{Par}(n)$, form a complete set of pairwise irreducible $\mathbb{Q}\mathrm{Sym}(n)$-modules. For $\lambda \in \mathrm{Par}(n)$ we have $M(\lambda)_{\mathbb{Q}} = \mathrm{Sp}(\lambda)_{\mathbb{Q}} \oplus C$, where C is a direct sum of modules of the form $\mathrm{Sp}(\mu)$ with $\lambda \triangleleft \mu$, and moreover every Specht module $\mathrm{Sp}(\mu)_{\mathbb{Q}}$ with $\lambda \triangleleft \mu$ occurs in C (see for example [14, 14.1]).

For a Young $\mathrm{Sym}(n)$-set Ω we define

$$\zeta^{\trianglerighteq}(\Omega) = \{\mu \in \mathrm{Par}(n) \mid \mu \trianglerighteq \lambda \text{ for some } \lambda \in \zeta(\Omega)\}.$$

Thus the set of composition factors of $\mathbb{Q}\Omega$ is $\{\mathrm{Sp}(\mu)_{\mathbb{Q}} \mid \mu \in \zeta^{\trianglerighteq}(\Omega)\}$ and, setting $\nabla_\Omega(\mu)_{\mathbb{Q}} = \mathrm{Hom}_{\mathrm{Sym}(n)}(\mathrm{Sp}(\mu)_{\mathbb{Q}}, \mathbb{Q}\Omega)$, we have the following.

Lemma 2.1. *The modules $\nabla_\Omega(\lambda)_{\mathbb{Q}}$, $\lambda \in \zeta^{\trianglerighteq}(\Omega)$, form a complete set of pairwise non-isomorphic irreducible $S_{\Omega,\mathbb{Q}}$-modules.*

Remark 2.2. Since $S_{\Omega,\mathbb{Q}}$ is a direct sum of matrix algebras over $\mathbb{Q}$ it is semisimple, all irreducible modules are absolutely irreducible and $\dim_{\mathbb{Q}} S_{\Omega,\mathbb{Q}} = \sum_{\lambda \in \zeta^{\trianglerighteq}(\Omega)} (\dim_{\mathbb{Q}} \nabla_\Omega(\lambda)_{\mathbb{Q}})^2$.

We now let k be a field of characteristic $p > 0$. For $\lambda \in \mathrm{Par}(n)$ we have the Young module $Y(\lambda)$ for $k\mathrm{Sym}(n)$, labelled by λ, as described in [7, Section 4.4] for example. Then we have $M(\lambda)_k = Y(\lambda) \oplus C$, where C is a direct sum of Young modules $Y(\mu)$, with $\lambda \triangleleft \mu$, see for example [7, Section 4.4 (1) (v)]. A partition $\lambda = (\lambda_1, \lambda_2, \dots)$ will be called p-restricted (also called column p-regular) if $\lambda_i - \lambda_{i+1} < p$ for all $i \ge 1$. A partition λ has a unique expression

$$\lambda = \sum_{i \ge 0} p^i \lambda(i)$$

where each $\lambda(i)$ is a p-restricted partition. This is called the base p (or p-adic) expansion of λ.

We write $\Lambda(n)$ for the set of all n-tuples of non-negative integers. An expression $\lambda = \sum_{i \ge 0} p^i \gamma(i)$, with all $\gamma(i) \in \Lambda(n)$ (but not necessarily restricted) will be called a weak p expansion.

For an n-tuple of non-negative integers γ we write $\overline{\gamma}$ for the partition obtained by arranging the entries in descending order.

Definition 2.3. For $\lambda, \mu \in \mathrm{Par}(n)$ we shall say that μ p-dominates λ, and write $\mu \trianglerighteq_p \lambda$ (or $\lambda \trianglelefteq_p \mu$) if there exists a weak p expansion $\lambda = \sum_{i\geq 0} p^i \gamma(i)$, such that $\mu(i) \trianglerighteq \overline{\gamma(i)}$ for all $i \geq 0$, where $\mu = \sum_{i\geq 0} p^i \mu(i)$ is the base p expansion of μ.

Note that $\lambda \trianglelefteq_p \mu$ implies $\lambda \trianglelefteq \mu$.

By [6, Section 3, Remark], for $\lambda, \mu \in \mathrm{Par}(n)$, the module $Y(\mu)$ appears as a component of $M(\lambda)_k$ if and only if $\lambda \trianglelefteq_p \mu$. For a Young $\mathrm{Sym}(n)$-set Ω we define

$$\zeta^{\trianglerighteq_p}(\Omega) = \{\mu \in \mathrm{Par}(n) \mid \mu \trianglerighteq_p \lambda \text{ for some } \lambda \in \zeta(\Omega)\}.$$

Writing $P_\Omega(\mu) = \mathrm{Hom}_{\mathrm{Sym}(n)}(Y(\mu), k\Omega)$ and writing $L_\Omega(\mu)$ for the head of $P_\Omega(\mu)$, for $\mu \in \zeta^{\trianglerighteq_p}(\Omega)$ we have the following.

Lemma 2.4. *The modules $L_\Omega(\lambda)$, $\lambda \in \zeta^{\trianglerighteq_p}(\Omega)$, form a complete set of pairwise non-isomorphic irreducible $S_{\Omega,k}$-modules.*

3. Basic constructions

We fix a positive integer n and a Young $\mathrm{Sym}(n)$-set Ω. Here we assume the base ring R is either the ring integers $\mathbb{Z}$ or the field of rational numbers $\mathbb{Q}$. We write $M_{\Omega,R}$, or just M_R for the permutation module $R\Omega$ over $R\mathrm{Sym}(n)$. We also just write M for $M_{\Omega,\mathbb{Z}}$. We shall sometimes write simply S_R for $S_{\Omega,R}$ and just S for $S_\mathbb{Z}$. We identify S with a subring or $S_\mathbb{Q}$ in the natural way.

Let $\{\mathcal{O}_\alpha \mid \alpha \in \Lambda_\Omega\}$ be a complete set of orbits in Ω. For $\lambda \in \zeta(\Omega)$ we pick $\alpha(\lambda) \in \Lambda_\Omega$ such that $\mathrm{Sym}(\lambda)$ is a point stabiliser of some element of $\mathcal{O}_{\alpha(\lambda)}$.

We put $M_{\alpha,R} = R\,\mathcal{O}_\alpha$, and sometimes write just M_α for $M_{\alpha,\mathbb{Z}}$, for $\alpha \in \Lambda_\Omega$. For $\beta \in \Lambda_\Omega$ we define the element ξ_β of S_R to be the projection onto $M_{\beta,R}$ coming from the decomposition $M_R = \bigoplus_{\alpha\in\Lambda_\Omega} M_{\alpha,R}$. Then each ξ_α is idempotent and we have the orthogonal decomposition:

$$1_S = \sum_{\alpha\in\Lambda_\Omega} \xi_\alpha.$$

For a left S_R-module V and $\beta \in \Lambda_\Omega$ we have the β weight space ${}^\beta V = \xi_\beta V$ and the weight space decomposition

$$V = \bigoplus_{\alpha\in\Lambda_\Omega} {}^\alpha V.$$

For $\lambda \in \mathrm{Par}(n)$ we define

$$^{\lambda}V = \begin{cases} \xi_{\alpha(\lambda)}V, & \text{if } \lambda \in \zeta(\Omega); \\ 0, & \text{otherwise.} \end{cases}$$

Similar remarks apply to weight spaces of right S_R-modules.

Lemma 3.1. *Let $\lambda \in \zeta^{\trianglerighteq}(\Omega)$. Then we have:*
(i) $\dim_{\mathbb{Q}} {}^{\lambda}\nabla_{\Omega}(\lambda)_{\mathbb{Q}} = 1$*; and*
(ii) if $\mu \in \mathrm{Par}(n)$ and ${}^{\mu}\nabla_{\Omega}(\lambda)_{\mathbb{Q}} \neq 0$ then $\mu \trianglelefteq \lambda$.

Proof. Let $\mu \in \mathrm{Par}(n)$ and suppose ${}^{\mu}\nabla_{\Omega}(\lambda)_{\mathbb{Q}} \neq 0$. Thus $\xi_{\mu}\mathrm{Hom}_{\mathrm{Sym}(n)}(\mathrm{Sp}(\lambda)_{\mathbb{Q}}, M_{\mathbb{Q}}) \neq 0$ i.e., $\mathrm{Hom}_{\mathrm{Sym}(n)}(\mathrm{Sp}(\lambda)_{\mathbb{Q}}, M(\mu)_{\mathbb{Q}}) \neq 0$ and so $\mu \trianglerighteq \lambda$, giving (ii). Moreover

$$\xi_{\lambda}\mathrm{Hom}_{\mathrm{Sym}(n)}(\mathrm{Sp}(\lambda)_{\mathbb{Q}}, M_{\mathbb{Q}}) = \mathrm{Hom}_{\mathrm{Sym}(n)}(\mathrm{Sp}(\lambda)_{\mathbb{Q}}, M(\lambda)_{\mathbb{Q}}) = \mathbb{Q}$$

giving (i). □

For $\lambda \in \mathrm{Par}(n)$ we set

$$\xi_{\lambda} = \begin{cases} \xi_{\alpha(\lambda)}, & \text{if } \lambda \in \zeta(\Omega): \\ 0, & \text{otherwise.} \end{cases}$$

For $\lambda \in \mathrm{Par}(n)$ we set $S_R(\lambda) = S_R\xi_{\lambda}S_R$ and for $\sigma \subseteq \mathrm{Par}(n)$ set

$$S_R(\sigma) = \sum_{\lambda \in \sigma} S_R(\lambda).$$

We also write simply $S(\lambda)$ for $S_{\mathbb{Z}}(\lambda)$ and $S(\sigma)$ for $S_{\mathbb{Z}}(\sigma)$.

Let $\leq$ be a partial order on $\mathrm{Par}(n)$ which is a refinement of the dominance partial order. For $\lambda \in \zeta(\Omega)$ we set $S_R(\geq \lambda) = S_R(\sigma)$, where $\sigma = \{\mu \in \mathrm{Par}(n) \,|\, \mu \geq \lambda\}$, and $S_R(> \lambda) = S_R(\tau)$, where $\tau = \{\mu \in \mathrm{Par}(n) \,|\, \mu > \lambda\}$. Thus

$$S_R(\geq \lambda) = S_R\xi_{\lambda}S_R + S_R(> \lambda).$$

We set $V_R(\lambda) = S_R(\geq \lambda)/S_R(> \lambda)$. So we have

$$V_R(\lambda)^{\lambda} = (S_R\xi_{\lambda} + S_R(> \lambda))/S_R(> \lambda),$$
$$^{\lambda}V_R(\lambda) = (\xi_{\lambda}S_R + S_R(> \lambda))/S_R(> \lambda)$$

and the multiplication map $S_R\xi_{\lambda} \times \xi_{\lambda}S_R \to S_R$ induces a surjective map

$$\phi_R(\lambda) : V_R(\lambda)^{\lambda} \otimes_R {}^{\lambda}V_R(\lambda) \to V_R(\lambda).$$

For left S_R-modules P, Q and $\lambda \in \mathrm{Par}(n)$ we define $\mathrm{Hom}^{\lambda}_{\mathrm{Sym}(n)}(P, Q)$ to be the R-submodule of $\mathrm{Hom}_{\mathrm{Sym}(n)}(P, Q)$ spanned by all composite maps $f \circ g$, with $f \in$

$\mathrm{Hom}_{\mathrm{Sym}(n)}(M(\lambda)_R, Q)$ and $g \in \mathrm{Hom}_{\mathrm{Sym}(n)}(P, M(\lambda)_R)$. For a subset σ of $\mathrm{Par}(n)$ we set

$$\mathrm{Hom}^{\sigma}_{\mathrm{Sym}(n)}(P,Q) = \sum_{\lambda\in\sigma} \mathrm{Hom}^{\lambda}_{\mathrm{Sym}(n)}(P,Q).$$

We note some similarity of our approach here via these groups of homomorphisms with the approach to Schur algebras due to Erdmann, [8], via stratification.

For $\lambda \in \mathrm{Par}(n)$ we define $\mathrm{Hom}^{\geq\lambda}_{\mathrm{Sym}(n)}(P,Q) = \mathrm{Hom}^{\sigma}_{\mathrm{Sym}(n)}(P,Q)$, where $\sigma = \{\mu \in \mathrm{Par}(n) \mid \mu \geq \lambda\}$, and $\mathrm{Hom}^{>\lambda}_{\mathrm{Sym}(n)}(P,Q) = \mathrm{Hom}^{\tau}_{\mathrm{Sym}(n)}(P,Q)$, where $\tau = \{\mu \in \mathrm{Par}(n) \mid \mu > \lambda\}$.

Note that if $\lambda \notin \zeta(\Omega)$ then $V_R(\lambda) = 0$. Suppose $\lambda \in \zeta(\Omega)$. Then we have

$$\begin{aligned} S_R\xi_\lambda S_R &= \sum_{\alpha,\beta,\gamma,\delta\in\Lambda_\Omega} \mathrm{Hom}_{\mathrm{Sym}(n)}(M_{\alpha,R}, M_{\beta,R})\xi_\lambda \mathrm{Hom}_{\mathrm{Sym}(n)}(M_{\gamma,R}, M_{\delta,R}) \\ &= \sum_{\alpha,\delta\in\Lambda_\Omega} \mathrm{Hom}_{\mathrm{Sym}(n)}(M_{\alpha,R}, M_{\alpha(\lambda)})\xi_\lambda \mathrm{Hom}_{\mathrm{Sym}(n)}(M_{\alpha(\lambda)}, M_{\delta,R}) \\ &= \bigoplus_{\alpha,\beta\in\Lambda_\Omega} \mathrm{Hom}^{\lambda}_{\mathrm{Sym}(n)}(M_{\alpha,R}, M_{\beta,R}) \end{aligned}$$

and hence

$$S_R(\sigma) = \bigoplus_{\alpha,\beta\in\Lambda_\Omega} \mathrm{Hom}^{\sigma}_{\mathrm{Sym}(n)}(M_{\alpha,R}, M_{\beta,R}) \tag{1}$$

for $\sigma \subseteq \mathrm{Par}(n)$. In particular we have

$$S_R(\geq \lambda) = \bigoplus_{\alpha,\beta\in\Lambda_\Omega} \mathrm{Hom}^{\geq\lambda}_{\mathrm{Sym}(n)}(M_{\alpha,R}, M_{\beta,R})$$

and

$$S_R(> \lambda) = \bigoplus_{\alpha,\beta\in\Lambda_\Omega} \mathrm{Hom}^{>\lambda}_{\mathrm{Sym}(n)}(M_{\alpha,R}, M_{\beta,R})$$

and hence

$$V_R(\lambda) = \bigoplus_{\alpha,\beta\in\Lambda_\Omega} \mathrm{Hom}^{\geq\lambda}_{\mathrm{Sym}(n)}(M_{\alpha,R}, M_{\beta,R})/\mathrm{Hom}^{>\lambda}_{\mathrm{Sym}(n)}(M_{\alpha,R}, M_{\beta,R}). \tag{2}$$

Example 3.2. Of crucial importance is the motivating example of the usual Schur algebra $S(n,r)$. Let R be a commutative ring and let E_R be a free R-module of rank n. Then $\mathrm{Sym}(r)$ acts on the r-fold tensor product $E_R^{\otimes r} = E_R \otimes \cdots \otimes_R E_R$ by place permutation, and the Schur algebra $S_R(n,r)$ may be realised as $\mathrm{End}_{\mathrm{Sym}(r)}(E_R^{\otimes r})$.

We choose an R-basis $e_1, \ldots, e_n$ of E_R. We write $I(n,r)$ for the set of maps from $\{1,\ldots,r\}$ to $\{1,\ldots,n\}$. We regard $i \in I(n,r)$ as an r-tuple of elements $(i_1,\ldots,i_r)$ with entries in $\{1,\ldots,n\}$ (where $i_a = i(a)$, $1 \le a \le r$). The group $\mathrm{Sym}(r)$ acts on $I(n,r)$ by composition of maps, i.e., by $w \cdot i = i \circ w^{-1}$, for $w \in \mathrm{Sym}(r)$, $i \in I(n,r)$. For $i \in I(n,r)$ we write e_i for $e_{i_1} \otimes \cdots \otimes e_{i_r}$. Then we have $w \cdot e_i = e_{i\circ w^{-1}}$, for $i \in I(n,r)$.

Thus we may regard $E_R^{\otimes r}$ as the $R\mathrm{Sym}(r)$ permutation module $R\Omega$ on $\Omega = I(n,r)$. Note that $\zeta(\Omega) = \Lambda^+(n,r)$, the set of partitions of r with at most n parts. We write $\Lambda(n,r)$ for the set of weights, i.e., the set of n-tuples of non-negative integers $\alpha = (\alpha_1 \ldots, \alpha_n)$ such that $\alpha_1 + \cdots + \alpha_n = r$. An element i of $I(n,r)$ has weight $\mathrm{wt}(i) = (\alpha_1,\ldots,\alpha_n) \in \Lambda(n,r)$, where $\alpha_a = |i^{-1}(a)|$, for $1 \le a \le n$. For $\alpha \in \Lambda(n,r)$ we have the orbit $\mathcal{O}_\alpha$ consisting or all $i \in I(n,r)$ such that $\mathrm{wt}(i) = \alpha$. Then $R\Omega = \bigoplus_{\alpha\in\Lambda(n,r)} R\mathcal{O}_\alpha$.

4. Groups of homomorphisms between Young permutation modules

In the situation of the Example 3.2 it follows from the quasi-hereditary structure of $S_{\mathbb{Z}}(n,r)$ that $V_{\mathbb{Z}}(\lambda)$ is a free abelian group - indeed an explicit basis is given by Green in [11, (7.3) Theorem, (ii), (iii)]. Thus, taking $r = n$, from Section 3, (2), we have the following.

Lemma 4.1. *For all $\lambda, \mu, \tau \in \mathrm{Par}(n)$ the quotient*

$$\mathrm{Hom}^{\ge\lambda}_{\mathrm{Sym}(n)}(M(\mu), M(\tau))/\mathrm{Hom}^{>\lambda}_{\mathrm{Sym}(n)}(M(\mu), M(\tau))$$

is torsion free.

We can improve on this somewhat. A subset σ of $\mathrm{Par}(n)$ will be called cosaturated (also said to be a coideal) if whenever $\lambda, \mu \in \sigma$, $\lambda \in \sigma$ and $\lambda \le \mu$ then $\mu \in \sigma$.

Proposition 4.2. *Let σ, τ be cosaturated subsets of $\mathrm{Par}(n)$ with the $\tau \subseteq \sigma$. Then, for all $\mu, \nu \in \mathrm{Par}(n)$, the quotient*

$$\mathrm{Hom}^{\sigma}_{\mathrm{Sym}(n)}(M(\mu), M(\nu))/\mathrm{Hom}^{\tau}_{\mathrm{Sym}(n)}(M(\mu), M(\nu))$$

is torsion free.

Proof. If there is a cosaturated subset θ with $\tau \subset \theta \subset \sigma$ (and $\theta \ne \sigma, \tau$) and if

$$\mathrm{Hom}^{\sigma}_{\mathrm{Sym}(n)}(M(\mu), M(\nu))/\mathrm{Hom}^{\theta}_{\mathrm{Sym}(n)}(M(\mu), M(\nu))$$

and

$$\mathrm{Hom}^{\theta}_{\mathrm{Sym}(n)}(M(\mu), M(\nu))/\mathrm{Hom}^{\tau}_{\mathrm{Sym}(n)}(M(\mu), M(\nu))$$

are torsion free then so is

$$\mathrm{Hom}^{\sigma}_{\mathrm{Sym}(n)}(M(\mu), M(\nu))/\mathrm{Hom}^{\tau}_{\mathrm{Sym}(n)}(M(\mu), M(\nu)).$$

Thus we are reduced to the case $\tau = \sigma \backslash \{\lambda\}$, where λ is a minimal element of σ. We choose a total order $\preceq$ on $\mathrm{Par}(n)$ refining $\leq$ such that, writing out the elements of $\mathrm{Par}(n)$ in descending order $\lambda^1 \succ \lambda^2 \cdots \succ \lambda^h$ we have $\tau = \{\lambda^1, \ldots, \lambda^k\}$, $\sigma = \{\lambda^1, \ldots, \lambda^{k+1}\}$ (so $\lambda = \lambda^{k+1}$) for some k. Then we have

$$\begin{aligned}\mathrm{Hom}^{\sigma}_{\mathrm{Sym}(n)}&(M(\mu), M(\nu))/\mathrm{Hom}^{\tau}_{\mathrm{Sym}(n)}(M(\mu), M(\nu)\\ &= \mathrm{Hom}^{\succeq\lambda}_{\mathrm{Sym}(n)}(M(\mu), M(\nu))/\mathrm{Hom}^{\succ\lambda}_{\mathrm{Sym}(n)}(M(\mu), M(\nu)\end{aligned}$$

which is torsion free by the Lemma. □

Returning to the general situation we have, by the Proposition and Section 3, (2), the following results.

Corollary 4.3. *The S-module $V(\lambda)$ is torsion free.*

Corollary 4.4. *Let σ be a cosaturated set (with respect to $\leq$). Then $S(\sigma)$ is a pure submodule of S.*

5. Cosaturated $\mathrm{Sym}(n)$-sets

From Corollary 4.4, if σ is any cosaturated subset of $\mathrm{Par}(n)$ then we may identify $\mathbb{Q} \otimes_{\mathbb{Z}} S(\sigma)$ with an $S_{\Omega,\mathbb{Q}}$-submodule of $S_{\Omega,\mathbb{Q}}$ via the natural map $\mathbb{Q} \otimes_{\mathbb{Z}} S(\sigma) \to S_{\mathbb{Q}}$.

We now suppose that Ω is cosaturated, by which we mean that $\zeta(\Omega)$ is a cosaturated subset of $\mathrm{Par}(n)$. We check that much of the structure, described by Green for the Schur algebras in [11], still stands in this more general case.

Let σ be a cosaturated subset of the support $\zeta(\Omega)$ of Ω. Let $\mu \in \zeta(\Omega)$. If $\nabla_\Omega(\mu)_{\mathbb{Q}}$ is a composition factor of $S_{\mathbb{Q}}(\sigma)$ then it is a composition factor of $S_{\mathbb{Q}}(\lambda)$ and hence of $S_{\mathbb{Q}}\xi_\lambda$, for some $\lambda \in \sigma$. Thus we have $\mathrm{Hom}_{\mathrm{Sym}(n)}(S_{\mathbb{Q}}\xi_\lambda, \nabla_\Omega(\mu)_{\mathbb{Q}}) \neq 0$ and so $\mu \geq \lambda$, Lemma 3.1(ii), and therefore $\mu \in \sigma$.

We fix $\lambda \in \zeta(\Omega)$. Then $\mathrm{Hom}_{\mathrm{Sym}(n)}(S_{\mathbb{Q}}\xi_\lambda, \nabla_\Omega(\lambda)_{\mathbb{Q}}) = {}^{\lambda}\nabla_\Omega(\lambda)_{\mathbb{Q}} = \mathbb{Q}$, by Lemma 3.1(i), so that $\nabla_\Omega(\lambda)_{\mathbb{Q}}$ is a composition factor of $S(\geq \lambda)_{\mathbb{Q}}$, but not of $S_{\mathbb{Q}}(> \lambda)$. Now we can write $S_{\mathbb{Q}}(\geq \lambda) = S_{\mathbb{Q}}(> \lambda) \oplus I$ for some ideal I which, as a left $S_{\mathbb{Q}}$-module, has only the composition factor $\nabla_\Omega(\lambda)_{\mathbb{Q}}$. Hence I is isomorphic to the matrix algebra $M_d(\mathbb{Q})$, where $d = \dim \nabla_\Omega(\lambda)_{\mathbb{Q}}$, and, as a left $S_{\mathbb{Q}}$-module $S_{\mathbb{Q}}(\geq \lambda)/S_{\mathbb{Q}}(> \lambda)$ is a direct sum of d copies of $\nabla_\Omega(\lambda)_{\mathbb{Q}}$. Hence

$$\begin{aligned}\dim_{\mathbb{Q}} {}^{\lambda}V_{\mathbb{Q}}(\lambda) &= \dim_{\mathbb{Q}} \mathrm{Hom}_{\mathrm{Sym}(n)}(S_{\mathbb{Q}}\xi_\lambda, V_{\mathbb{Q}}(\lambda))\\ &= d \dim_{\mathbb{Q}} \mathrm{Hom}_{\mathrm{Sym}(n)}(S_{\mathbb{Q}}\xi_\lambda, \nabla_\Omega(\lambda)_{\mathbb{Q}})\end{aligned}$$

$$= d \dim_{\mathbb{Q}} {}^{\lambda}\nabla_{\Omega}(\lambda)_{\mathbb{Q}} = d.$$

Thus $\dim V_{\mathbb{Q}}(\lambda)^{\lambda} \otimes_{\mathbb{Q}} {}^{\lambda}V_{\mathbb{Q}}(\lambda) = \dim V_{\mathbb{Q}}(\lambda)$ and we have:

$$\text{the natural map } V_{\mathbb{Q}}(\lambda)^{\lambda} \otimes_{\mathbb{Q}} {}^{\lambda}V_{\mathbb{Q}}(\lambda) \to V_{\mathbb{Q}}(\lambda) \text{ is an isomorphism.} \tag{1}$$

We now consider the integral version. We have the natural surjective map $V(\lambda)^{\lambda} \otimes_{\mathbb{Z}} {}^{\lambda}V(\lambda) \to V(\lambda)$. But the rank of $V(\lambda)^{\lambda}$ is the dimension of $V_{\mathbb{Q}}(\lambda)^{\lambda}$, the rank of ${}^{\lambda}V(\lambda)$ is the dimension of ${}^{\lambda}V_{\mathbb{Q}}(\lambda)$, and the rank of $V(\lambda)$ is the dimension of $V_{\mathbb{Q}}(\lambda)$ so that, by (1), $V(\lambda)^{\lambda} \otimes_{\mathbb{Z}} {}^{\lambda}V(\lambda)$ and $V(\lambda)$ have the same rank. Thus the surjective map $V(\lambda)^{\lambda} \otimes_{\mathbb{Z}} {}^{\lambda}V(\lambda) \to V(\lambda)$ is an isomorphism.

We have shown the following.

Proposition 5.1. *Assume Ω is cosaturated. Then, for each $\lambda \in \mathrm{Par}(n)$, the map*

$$V(\lambda)^{\lambda} \otimes_{\mathbb{Z}} {}^{\lambda}V(\lambda) \to V(\lambda)$$

induced by multiplication in S, is an isomorphism.

Remark 5.2. If k is a field then the corresponding algebras $S_{\Omega,k}$ over k are Morita equivalent to those considered by Mathas and Soriano in [17]. There they determined the blocks of such algebras (for the Schur algebras themselves this was done in [5], and for the quantised Schur algebras by Cox in [2]).

6. Cellularity of endomorphism algebras of Young permutation modules

We now establish our main result, namely that the endomorphism algebra of a Young permutation module has the structure of a cellular algebra. We first recall the notion of a cellular algebra due to Graham and Lehrer, [9]. (We have made some minor notational changes to be consistent with the notation above. The most serious of these is the reversal of the partial order from the definition given in [9].)

Definition 6.1. Let A be an algebra over a commutative ring R. A cell datum for $(\Lambda^+, N, C, {}^*)$ for A consists of the following.

(C1) A partially ordered set Λ^+ and for each $\lambda \in \Lambda^+$ a finite set $N(\lambda)$ and an injective map $C : \coprod_{\lambda\in\Lambda^+} N(\lambda) \times N(\lambda) \to A$ with image an R-basis of A.

(C2) For $\lambda \in \Lambda^+$ and $t, u \in N(\lambda)$ we write $C(t, u) = C^{\lambda}_{t,u} \in R$. Then * is an R-linear anti-involution of A such that $(C^{\lambda}_{t,u})^* = C^{\lambda}_{u,t}$.

(C3) If $\lambda \in \Lambda^+$ and $t, u \in N(\lambda)$ then for any element $a \in A$ we have

$$aC^{\lambda}_{t,u} \equiv \sum_{t'\in N(\lambda)} r_a(t', t)C^{\lambda}_{t',u} \pmod{A(> \lambda)}$$

where $r_a(t',t) \in R$ is independent of u and where $A(>\lambda)$ is the R-submodule of A generated by $\{C^{\mu}_{t'',u''} \mid \mu \in \Lambda^+, \mu > \lambda \text{ and } t'', u'' \in N(\mu)\}$.

We say that A is a cellular R-algebra if it admits a cell datum.

Let G be a finite group. Let Ω be a finite G-set and let R be a commutative ring. Now G acts on $\Omega \times \Omega$. If $\mathcal{A} \subseteq \Omega \times \Omega$ is G-stable then we have an element $a_{\mathcal{A}} \in \mathrm{End}_G(R\Omega)$ satisfying

$$a_{\mathcal{A}}(x) = \sum_y y$$

where the sum is over all $y \in \Omega$ such that $(y,x) \in \mathcal{A}$. We write $\mathrm{Orb}_G(\Omega \times \Omega)$ for the set of G-orbits in $\Omega \times \Omega$. Then $\mathrm{End}_{RG}(R\Omega)$ free over R on basis $a_{\mathcal{A}}$, $\mathcal{A} \in \mathrm{Orb}_G(\Omega \times \Omega)$. We have an involution on $\Omega \times \Omega$ defined by $(x,y)^* = (y,x)$, $x, y \in \Omega$. For a G-stable subset $\mathcal{A}$ of $\Omega \times \Omega$ we write $\mathcal{A}^*$ for the G-stable set $\{(x,y)^* \mid (x,y) \in \mathcal{A}\}$.

For $\mathcal{A}, \mathcal{B} \in \mathrm{Orb}_G(\Omega \times \Omega)$ we have

$$a_{\mathcal{A}} a_{\mathcal{B}} = \sum_{\mathcal{C} \in \mathrm{Orb}_G(\Omega \times \Omega)} n^{\mathcal{C}}_{\mathcal{A},\mathcal{B}} a_{\mathcal{C}}$$

where, for fixed $x \in \mathcal{A}$, $y \in \mathcal{B}$, the coefficient $n^{\mathcal{C}}_{\mathcal{A},\mathcal{B}}$ is the cardinality of the set $\{z \in \mathcal{C} \mid (x,z) \in \mathcal{A} \text{ and } (z,y) \in \mathcal{B}\}$. It follows that $\mathrm{End}_{RG}(R\Omega)$ has an involutory anti-automorphism satisfying $a^*_{\mathcal{D}} = a_{\mathcal{D}^*}$, for a G-stable subset $\mathcal{D}$ of $\Omega \times \Omega$.

The notion of cellularity has built into it an involutory anti-automorphism * and in the case of endomorphism algebras of permutation modules, we shall always use the one just defined.

We now restrict to the case $G = \mathrm{Sym}(n)$ with Ω a Young $\mathrm{Sym}(n)$-set as usual and label by $\mathcal{O}_\alpha$, $\alpha \in \Lambda_\Omega$, the G-orbits in Ω. Now, for $\alpha \in \Lambda_\Omega$ and $x \in \Omega$ we have

$$\xi_\alpha(x) = \begin{cases} x, & \text{if } x \in \mathcal{O}_\alpha; \\ 0, & \text{otherwise.} \end{cases}$$

Hence $\xi_\alpha = a_{\mathcal{A}}$, where $\mathcal{A} = \{(x,x) \mid x \in \mathcal{O}_\alpha\}$ and therefore $\xi^*_\alpha = \xi_\alpha$. In particular we have $\xi^*_\lambda = \xi_\lambda$ for $\lambda \in \zeta(\Omega)$. Thus we also have $S_{\Omega,R}(\sigma)^* = S_{\Omega,R}(\sigma)$, for $\sigma \subseteq \mathrm{Par}(n)$.

Note that if Γ is a G-stable subset of Ω then we have the idempotent $e_\Gamma \in S_{\Omega,R}$ given on elements of Ω by

$$e_\Gamma(x) = \begin{cases} x, & \text{if } x \in \Gamma\,; \\ 0, & \text{if } x \notin \Gamma. \end{cases}$$

Thus $e_\Gamma = a_{\mathcal{C}}$ where $\mathcal{C} = \{(y,y) \mid y \in \Gamma\}$ and $e^*_\Gamma = e_\Gamma$.

So now let Γ be a Young $\mathrm{Sym}(n)$-set and let Ω be a cosaturated Young $\mathrm{Sym}(n)$-set containing Γ. We have the idempotent $e = e_\Gamma \in S_{\Omega,R}$ as above and $S_{\Gamma,R} = \mathrm{End}_{\mathrm{Sym}(n)}(R\Gamma)$ is naturally identified with $eS_{\Omega,R}e$.

Lemma 6.2. *For $\lambda \in \zeta(\Omega)$ we have $e_\Gamma \nabla_\Omega(\lambda)_{\mathbb{Q}} \neq 0$ if and only if $\lambda \in \zeta^{\trianglerighteq}(\Gamma)$.*

Proof. We have $e_\Gamma = \sum_{\alpha\in\Lambda_\Gamma} \xi_\alpha$. Hence $e_\Gamma \nabla_\Omega(\lambda)_{\mathbb{Q}} \neq 0$ if and only if $\xi_\alpha \nabla_\Omega(\lambda)_{\mathbb{Q}} \neq 0$ i.e., $\sum_{\beta\in\Lambda_\Omega} \xi_\alpha \mathrm{Hom}_{\mathrm{Sym}(n)}(\mathrm{Sp}(\lambda)_{\mathbb{Q}}, M_{\beta,\mathbb{Q}}) \neq 0$, for some $\alpha \in \Lambda_\Gamma$. Hence $e_\Gamma \nabla_\Omega(\lambda)_{\mathbb{Q}} \neq 0$ if and only if $\mathrm{Hom}_{\mathrm{Sym}(n)}(\mathrm{Sp}(\lambda)_{\mathbb{Q}}, M_{\beta,\mathbb{Q}}) \neq 0$ for some $\beta \in \Lambda_\Gamma$, i.e., if and only if $\mathrm{Hom}_{\mathrm{Sym}(n)}(\mathrm{Sp}(\lambda), M(\mu)_{\mathbb{Q}}) \neq 0$ for some $\mu \in \zeta(\Gamma)$, i.e., if and only if there exists $\mu \in \zeta(\Gamma)$ such that $\mu \trianglelefteq \lambda$. $\square$

We fix a partial order $\leq$ on $\zeta(\Omega)$ refining the partial order $\trianglelefteq$.

Let $\lambda \in \zeta(\Omega)$. We have the section $V(\lambda) = S(\geq \lambda)/S(> \lambda)$ of $S = S_\Omega$.

We write J^{op} for the opposite ring of a ring J. We write S^{env} for the enveloping algebra $S \otimes_{\mathbb{Z}} S^{\mathrm{op}}$. We identify an (S,S)-bimodule with a left S^{env}-module in the usual way.

We have the idempotent $\tilde{e} = e\otimes e \in S^{\mathrm{env}}$ and hence the Schur functor $\tilde{f} : \mathrm{mod}(S^{\mathrm{env}}) \to \mathrm{mod}(\tilde{e}S^{\mathrm{env}}\tilde{e})$ as in [12, Chapter 6]. Moreover, $\tilde{e}S^{\mathrm{env}}\tilde{e} = eSe \otimes_{\mathbb{Z}} (eSe)^{\mathrm{op}}$. Now $\tilde{f}$ is exact so applying it to the isomorphism $V(\lambda)^\lambda \otimes_{\mathbb{Z}} {}^\lambda V(\lambda) \to V(\lambda)$ of Proposition 5.1 we obtain an isomorphism

$$e\,V(\lambda)^\lambda \otimes_{\mathbb{Z}} {}^\lambda V(\lambda)\,e \to eV(\lambda)e. \tag{1}$$

Now $\xi_\lambda S + S(> \lambda) = (S\xi_\lambda + S(> \lambda))^*$ so that $eV(\lambda)e \neq 0$ if and only if $eV(\lambda)^\lambda \neq 0$. Moreover, $V(\lambda)^\lambda$ is a $\mathbb{Z}$-form of $\nabla(\lambda)_{\mathbb{Q}}$ so that $eV(\lambda)e \neq 0$ if and only if $e\nabla_\Omega(\lambda)_{\mathbb{Q}} \neq 0$. Hence by, Lemma 6.2:

$$eV(\lambda)e \neq 0 \text{ if and only if } \lambda \in \zeta^{\trianglerighteq}(\Gamma). \tag{2}$$

We now assemble our cell data. We have the set $\Lambda^+ = \zeta^{\trianglerighteq}(\Gamma)$ with partial order induced from the partial order $\leq$ on $\zeta(\Omega)$ (and also denoted $\leq$). Let $\lambda \in \Lambda^+$. We let $n_\lambda = \dim_{\mathbb{Q}} e\nabla_\Omega(\lambda)_{\mathbb{Q}}$ and set $N(\lambda) = \{1, \ldots, n_\lambda\}$. The rank of $eV(\lambda)^\lambda$ is n_λ. We choose elements $d_{\lambda,1}, \ldots, d_{\lambda,n_\lambda}$ of $eS\xi_\lambda$ such that the elements $d_{\lambda,1} + S(> \lambda), \ldots, d_{\lambda,n_\lambda} + S(> \lambda)$ form a $\mathbb{Z}$-basis of $eV(\lambda)^\lambda = (eS\xi_\lambda + S(> \lambda))/S(> \lambda)$. Then $d^*_{\lambda,1}, \ldots, d^*_{\lambda,n_\lambda}$ belong to $(eS\xi_\lambda)^* = \xi_\lambda Se$ and the elements $d^*_{\lambda,1} + S(> \lambda), \ldots, d^*_{\lambda,n_\lambda} + S(> \lambda)$ form a $\mathbb{Z}$-basis of ${}^\lambda V(\lambda)e = (\xi_\lambda Se + S(> \lambda))/S(> \lambda)$. The product $d_{\lambda,t}d^*_{\lambda,u}$ belongs to $eS\xi_\lambda Se$. We define $C : \coprod_{\lambda\in\Lambda^+} N(\lambda) \times N(\lambda) \to eSe$ by $C(t,u) = C^\lambda_{t,u} = d_{\lambda,t}d^*_{\lambda,u}$, for $t, u \in N(\lambda)$.

Let M be the $\mathbb{Z}$-span of all $C^\lambda_{t,u}$, $\lambda \in \Lambda^+$, $t, u \in N(\lambda)$. We claim that $M = eSe$. We have $S = \sum_{\lambda\in\Lambda_\Omega} S\xi_\lambda S$ so that if the claim is false then there exists $\lambda \in \Lambda_\Omega$ such that $eS\xi_\lambda Se \nsubseteq M$. In that case we choose λ minimal with this property. First suppose that $\lambda \notin \zeta^{\trianglerighteq}(\Gamma)$. Then we have $eV(\lambda)e = 0$, by (2), i.e., $eS\xi_\lambda Se \subseteq S(> \lambda)$ and so

$eS\xi_\lambda Se \subseteq eS(>\lambda)e$. However, $eS(>\lambda)e = \sum_{\mu>\lambda} eS\xi_\mu Se \subseteq M$, by minimality of λ and so $eS\xi_\lambda Se \subseteq M$. Thus we have $\lambda \in \Lambda^+ = \zeta^{\trianglerighteq}(\Gamma)$.

Now by (1) the map

$$(eS\xi_\lambda + S(>\lambda)) \otimes_{\mathbb{Z}} (\xi_\lambda Se + S(>\lambda)) \to eS\xi_\lambda Se + S(>\lambda)$$

induced by multiplication is surjective. Moreover we have $eS\xi_\lambda + S(>\lambda) = \sum_{t=1}^{n_\lambda} \mathbb{Z}d_{\lambda,t} + S(>\lambda)$ and $\xi_\lambda Se + S(>\lambda) = \sum_{u=1}^{n_\lambda} \mathbb{Z}d^*_{\lambda,u} + S(>\lambda)$ so that

$$eS\xi_\lambda Se \subseteq \sum_{t,u=1}^{n_\lambda} \mathbb{Z}d_{\lambda,t}d^*_{\lambda,u} + S(>\lambda) = \sum_{t,u=1}^{n_\lambda} \mathbb{Z}C^\lambda_{t,u} + S(>\lambda)$$

and hence

$$eS\xi_\lambda Se \subseteq \sum_{t,u=1}^{n_\lambda} \mathbb{Z}C^\lambda_{t,u} + eS(>\lambda)e.$$

But now $\sum_{t,u=1}^{n_\lambda} \mathbb{Z}C^\lambda_{t,u} \subseteq M$ by definition and again $eS(>\lambda)e \subseteq M$ by the minimality of λ so that $eS\xi_\lambda Se \subseteq M$ and the claim is established.

The elements $C^\lambda_{t,u}$, $\lambda \in \Lambda^+$, $1 \leq t,u \leq n_\lambda$ form a spanning set of $eS_\Omega e = S_\Gamma$. But the rank of eSe is the $\mathbb{Q}$-dimension of $eS_{\mathbb{Q}}e$, i.e., the $\mathbb{Q}$-dimension of $S_{\Gamma,\mathbb{Q}}$ and this is $\sum_{\lambda\in\Lambda^+}(\dim e\nabla_\Omega(\lambda))^2$ by Remark 2.2. Hence the elements $C^\lambda_{t,u}$, with $\lambda \in \Lambda^+$, $t,u \in N(\lambda)$, form a $\mathbb{Z}$-basis of eSe.

We have now checked the defining properties (C1) and (C2) of cell structure and it remains to check (C3). We fix $\lambda \in \Lambda^+$ and let $1 \leq t,u \leq n_\lambda$. Let $a \in eSe$. Then we have

$$aC^\lambda_{t,u} = ad_{\lambda,t}d^*_{\lambda,u}.$$

Now we have $\sum_{i=1}^{n_\lambda} \mathbb{Z}d_{\lambda,i} + S(>\lambda) = eS\xi_\lambda + S(>\lambda)$ so we may write $ad_{\lambda,t} = \sum_{t'=1}^{n_\lambda} r_a(t',t)d_{\lambda,t'} + y$ for some integers $r_a(t',t)$ and an element y of $S(>\lambda)$. Thus we have

$$\begin{aligned} aC^\lambda_{t,u} =& ad_{\lambda,t}d^*_{\lambda,u} = \sum_{t'=1}^{n_\lambda} r_a(t',t)d_{\lambda,t'}d^*_{\lambda,u} + yd^*_{\lambda,u} \\ &= \sum_{t'=1}^{n_\lambda} r_a(t',t)C^\lambda_{t',u} + yd^*_{\lambda,u} \end{aligned}$$

and hence

$$aC^\lambda_{t,u} = \sum_{t'=1}^{n_\lambda} r_a(t',t)C^\lambda_{t',u} \pmod{S(>\lambda)}.$$

We have thus checked defining property (C3) and hence proved the following.

Theorem 6.3. *Let Γ be a Young* $\mathrm{Sym}(n)$*-set. Then* $(\Lambda^+, N, C, {}^*)$ *is a cell structure on* $S_{\Gamma,\mathbb{Z}} = eS_{\Omega,\mathbb{Z}}e = \mathrm{End}_{\mathrm{Sym}(n)}(\mathbb{Z}\Gamma)$.

One now obtains a cell structure on $\mathrm{End}_{\mathrm{Sym}(n)}(R\Gamma)$, for any commutative ring R by base change.

Remark 6.4. This implies that, over a field, any endomorphism algebra of a Young $\mathrm{Sym}(n)$-set is Schurian, i.e., every irreducible module is absolutely irreducible, by [9, (3.2) Proposition (ii) and (3.4) Theorem (i)]. One may also deduce this from eSe-theory as in [12, Section 6.2] and the result for Schur algebras (cf. [12, 3.5 Remarks (i)]).

There is also the question of when an endomorphism algebra over a field k is quasi-hereditary. If k has characteristic 0 then $\mathrm{End}_{\mathrm{Sym}(n)}(k\Gamma)$ is semisimple and there is nothing to consider. We assume now that the characteristic of k is $p > 0$. By [9, Remark 3.10] (see also [15], [16]) $\mathrm{End}_{\mathrm{Sym}(n)}(k\Gamma)$ is quasi-hereditary if and only if the number of irreducible $\mathrm{End}_{\mathrm{Sym}(n)}(k\Gamma)$-modules (up to isomorphism) is equal to the length of the cell chain, i.e., $|\zeta^{\trianglerighteq}(\Gamma)|$. By Lemma 2.4, the number of irreducible $\mathrm{End}_{\mathrm{Sym}(n)}(k\Gamma)$-modules is $|\zeta^{\trianglerighteq_p}(\Gamma)|$. Moreover, we have $\zeta^{\trianglerighteq_p}(\Gamma) \subseteq \zeta^{\trianglerighteq}(\Gamma)$ and so $\mathrm{End}_{\mathrm{Sym}(n)}(k\Gamma)$ is quasi-hereditary if and only if $\zeta^{\trianglerighteq}(\Gamma) \subseteq \zeta^{\trianglerighteq_p}(\Gamma)$. We spell this out in the following result.

Theorem 6.5. *Let k be a field of characteristic $p > 0$ and let Γ be a Young* $\mathrm{Sym}(n)$*-set. Then the endomorphism algebra* $\mathrm{End}_{\mathrm{Sym}(n)}(k\Gamma)$ *of the permutation module $k\Gamma$ is quasi-hereditary if and only if for every partition λ of n such that the Young subgroup* $\mathrm{Sym}(\lambda)$ *appears as the stabiliser of a point of Γ and every partition $\mu \trianglerighteq \lambda$ there exists a partition τ such that* $\mathrm{Sym}(\tau)$ *appears as a point stabiliser and such that μ p-dominates τ, i.e., there exists a weak p expansion $\tau = \sum_{i\geq 0} p^i\gamma(i)$, with $\gamma(i) \in \Lambda(n)$, and $\overline{\gamma(i)} \trianglelefteq \mu(i)$ for all i (where $\mu = \sum_{i\geq 0} p^i\mu(i)$ is the base p-expansion of μ and where $\overline{\gamma(i)}$ is the partition obtained by writing the parts of $\gamma(i)$ in descending order, for $i \geq 0$).*

Remark 6.6. We emphasise that the above gives a criterion for the endomorphism algebra $\mathrm{End}_{\mathrm{Sym}(n)}(k\Gamma)$ of the Young permutation module $k\Gamma$ to be quasi-hereditary with respect to any labelling of the simple modules by a partially ordered set (which may have nothing to do with those considered above) thanks to the result of König and Xi, [16, Theorem 3]. Thus if Γ does not satisfy the condition above then $S_{\Gamma,k}$ can not have finite global dimension by [16, Theorem 3] and hence is not quasi-hereditary.

7. Example: tensor powers

Let R be a commutative ring and let E_R be a free R-module on basis $e_{1,R}, \ldots, e_{n,R}$. Let r be a positive integer and let $I(n,r)$ be the set described in Example 3.2. Then the r-fold tensor product $E_R^{\otimes r} = E_R \otimes_R \otimes \cdots \otimes_R E_R$ has R-basis $e_{i,R} = e_{i_1,R} \otimes \cdots \otimes e_{i_r,R}$, $i \in I(n,r)$, and we thus identify $E_R^{\otimes r}$ with $RI(n,r)$, the free R-module on $I(n,r)$.

Remark 7.1. The symmetric group $\mathrm{Sym}(r)$ acts on $E_R^{\otimes r}$ by place permutations, i.e., $w \cdot e_{i,R} = e_{i\circ w^{-1},R}$, for $w \in \mathrm{Sym}(r)$, $i \in I(n,r)$. Thus we may regard $E_R^{\otimes r}$ as the permutation module $RI(n,r)$, with $\mathrm{Sym}(r)$, acting on $I(n,r)$ by $w \cdot i = i \circ w^{-1}$. The endomorphism algebra $\mathrm{End}_{\mathrm{Sym}(r)}(E_R^{\otimes r})$ is the Schur algebra $S_R(n,r)$.

The stabiliser of $i \in I(n,r)$ is the direct product of the symmetric groups on the fibres of i (regarded as a subgroup of $\mathrm{Sym}(r)$ in the usual way). Hence $I(n,r)$ is a Young $\mathrm{Sym}(r)$-set. Hence $E_R^{\otimes r}$ is a Young permutation module and hence $S_R(n,r)$ is cellular. Moreover, $\zeta(I(n,r))$ is the set $\Lambda^+(n,r)$ of all partitions of r with at most n parts. This is a cosaturated set and hence for a prime p we have $\zeta(I(n,r)) = \zeta^{\trianglerighteq}(I(n,r)) = \zeta^{\trianglerighteq_p}(I(n,r))$. Hence, for a field k of characteristic p the Schur algebra $S_k(n,r)$ is quasi-hereditary.

However, this is not a new proof since our treatment relies crucially on a detail from Green's analysis of $S_{\mathbb{Z}}(n,r)$ as in [11], at least in the case $n = r$. (See Example 3.2 above and the proofs of the results of Section 4.)

We now regard E_R as an $R\mathrm{Sym}(n)$-module with $\mathrm{Sym}(n)$ permuting the basis $e_{1,R}, \ldots, e_{n,R}$ in the natural way. This action induces an action on the tensor product $E_R^{\otimes r}$. Specifically, we have $w \cdot e_{i,R} = e_{w\circ i,R}$, for $w \in \mathrm{Sym}(n)$, $i \in I(n,r)$, and we thus regard $E_R^{\otimes r}$ as the permutation module $RI(n,r)$. For $w \in \mathrm{Sym}(n)$, $i \in I(n,r)$ we have $w \circ i = i$ if and only if w acts as the identity on the image of i, so that the stabiliser of i is the group of symmetries of the complement of the image of i in $\{1, \ldots, n\}$, identified with a subgroup of $\mathrm{Sym}(n)$ in the usual way. Thus $I(n,r)$ is a Young $\mathrm{Sym}(n)$-set so we have the following consequence of Theorem 6.3, answering a question raised in [1].

Proposition 7.2. *The endomorphism algebra* $\mathrm{End}_{\mathrm{Sym}(n)}(E_R^{\otimes r}) = \mathrm{End}_{\mathrm{Sym}(n)}(RI(n,r))$ *is a cellular algebra.*

The support of $I(n,r)$ consists of hook partitions, more precisely we have

$$\zeta(I(n,r)) = \{(a, 1^b) \mid a + b = n, 1 \leq b \leq r\}.$$

Hence we have

$$\zeta^{\trianglerighteq}(I(n,r)) = \{\lambda = (\lambda_1, \lambda_2, \ldots) \in \mathrm{Par}(n) \mid \lambda_1 \geq n - r\}.$$

Let k be a field of characteristic $p > 0$. Then $\mathrm{End}_{\mathrm{Sym}(n)}(E_k^{\otimes r})$ is quasi-hereditary if and only if $\zeta^{\trianglerighteq}(I(n,r)) \subseteq \zeta^{\trianglerighteq_p}(I(n,r)$, i.e., if and only if for every $\mu = (\mu_1, \mu_2, \ldots) \in \mathrm{Par}(n)$ with $\mu_1 \geq n - r$ there exists some $\lambda = (a, 1^b)$, $1 \leq b \leq r$, such that $\lambda \trianglelefteq_p \mu$.

We are able to give an explicit list of quasi-hereditary algebras arising in the above manner.

Proposition 7.3. *Let k be a field of characteristic $p > 0$. Let n be a positive integer and E an n-dimensional k-vector space with basis $e_1, \ldots, e_n$. We regard E as a $k\mathrm{Sym}(n)$-module with $\mathrm{Sym}(n)$ permuting the basis in the obvious way. For $r \geq 1$ we regard the* *rth tensor power* $E^{\otimes r}$ *as a* $k\mathrm{Sym}(n)$*-module via the usual tensor product action. Then* $\mathrm{End}_{\mathrm{Sym}(n)}(E^{\otimes r})$ *is quasi-hereditary if and only if:*

(i) p *does not divide* n*; and*

(ii) either $n < 2p$ *(and* r *is arbitrary) or* $n > 2p$ *and* $r < p$.

Proof. We see this in a number of steps. We regard $E^{\otimes r}$ as the permutation module $kI(n,r)$, as above, with $\mathrm{Sym}(n)$ acting by $w \cdot i = w \circ i$, for $w \in \mathrm{Sym}(n)$, $i \in I(n,r)$. We shall say that $I(n,r)$ is quasi-hereditary if $\mathrm{End}_{\mathrm{Sym}(n)}(E^{\otimes r})$ is.

Step 1. If p divides n then $I(n,r)$ is not quasi-hereditary.

We have $(n-1,1) \in \zeta(I(n,r))$ and $(n,0) \trianglerighteq (n-1,1)$ so that $(n,0) \in \zeta^{\trianglerighteq}(I(n,r))$. Now $n = pm$, for some positive integer m, so that $\mu = (n,0) = p(m,0)$ has base p expansion $(n,0) = \sum_{i\geq 0} p^i \mu(i)$, with restricted part $\mu(0) = 0$. Thus if $\tau = (a,1^b)$ has weak p-expansion $\tau = \sum_{i \geq 0} p^i \gamma(i)$ and $\overline{\gamma(i)} \trianglelefteq \mu(i)$, for all i, then $\gamma(0) = 0$ and τ is divisible by p. However, this is not the case so no such weak p-expansion exists and $\mu \in \zeta^{\trianglerighteq}(I(n,r)) \backslash \zeta^{\trianglerighteq_p}(I(n,r))$. Thus $\zeta^{\trianglerighteq}(I(n,r)) \neq \zeta^{\trianglerighteq_p}(I(n,r))$ and $I(n,r)$ is not quasi-hereditary.

Step 2. If p does not divide n then $I(n,1)$ is quasi-hereditary.

We have $\zeta(I(n,1)) = \{(n-1,1)\}$. If $\mu \in \zeta^{\trianglerighteq}(I(n,1)) \backslash \zeta^{\trianglerighteq_p}(I(n,r))$ then $\mu = (n,0)$. Now n has base p expansion $n = \sum_{i\geq 0} p^i n_i$, with $0 \leq n_i < p$ for all $i \geq 0$ and $n_0 \neq 0$ and μ has base p expansion $\mu = \sum_{i \geq 0} p^i \mu(i)$, with $\mu(i) = (n_i, 0)$, for all $i \geq 0$.

But now we write

$$\tau = (n-1,1) = (n_0 - 1, 1) + \sum_{i\geq 1} p^i (n_i, 0)$$

and τ has weak p-expansion $\tau = \sum_{i\geq 0} p^i \gamma(i)$, with $\gamma(0) = (n_0 - 1, 1)$, $\gamma(i) = (n_i, 0)$ for $i \geq 1$. Moreover $\overline{\gamma(i)} \leq \mu(i)$, for all i so that $(n,0) \in \zeta^{\trianglerighteq_p}(I(n,1))$. Thus $\zeta^{\trianglerighteq}(I(n,1)) = \zeta^{\trianglerighteq_p}(I(n,1))$ and $I(n,1)$ is quasi-hereditary.

Step 3. If $\mu \in \zeta^{\trianglerighteq}(I(n,r))$ is p-restricted then $\mu \in \zeta^{\trianglerighteq_p}(I(n,r))$

We have $\mu \trianglerighteq (a,1^b)$ for some $n = a + b$, $1 \leq b \leq r$. The partition μ has base p expansion $\mu = \sum_{i\geq 0} p^i \mu(i)$, with $\mu(i) = 0$ for all $i \geq 1$.

But now $\tau = (a,1^b)$ has weak p-expansion $\tau = \sum_{i \geq 0} p^i \gamma(i)$, with $\gamma(0) = (a,1^b)$ and $\gamma(i) = 0$ for all $i \geq 1$. Furthermore we have $\overline{\gamma(i)} \trianglelefteq \mu(i)$ for all $i \geq 0$ so $\mu \in \zeta^{\trianglerighteq_p}(I(n,r))$.

Step 4. If $n < p$ then $I(n,r)$ is quasi-hereditary.

This follows from Step 3 since all elements of $\mathrm{Par}(n)$ are restricted.

Step 5. If $p < n < 2p$ then $I(n,r)$ is quasi-hereditary.

For a contradiction suppose not and let $\mu = (\mu_1, \mu_2, \ldots) \in \zeta^{\trianglerighteq}(I(n,r)) \backslash \zeta^{\trianglerighteq_p}(I(n,r))$. We have $\mu \trianglerighteq (a,1^b)$ for some a,b with $n = a+b$, $1 \leq b \leq r$. Choose a,b with this property with $b \geq 1$ minimal. If $b = 1$ then $\mu \in \zeta^{\trianglerighteq}(I(n,1))$, which by Step 2 is $\zeta^{\trianglerighteq_p}(I(n,1))$. Thus we have $b \geq 2$.

We claim that $\mu_1 = a$. Since $\mu \trianglerighteq (a, 1^b)$ the length l, say, of μ is at most the length of $(a, 1^b)$, i.e., $b+1$. Put $\xi = (\xi_1, \xi_2, \dots) = (a+1, 1^{b-1})$. If $\mu_1 > a$ then $\mu_1 \geq \xi_1$ and, for $1 < i \leq l$, we have

$$\mu_1 + \cdots + \mu_i \geq a + 1 + (i-1) = a + i = \xi_1 + \cdots + \xi_i.$$

So $\mu \trianglerighteq \xi = (a+1, 1^{b-1})$, which is a contradiction, and the claim is established.

Note that μ is non-restricted, by Step 3, and, since μ is a partition of $n < 2p$ in the base p expansion $\mu = \sum_{i \geq 0} p^i \mu(i)$ of μ, we must have $\mu(1) = (1, 0)$ and $\mu(i) = 0$ for $i \geq 2$. Let $\tau = (a, 1^b)$. Then $\tau \trianglelefteq \mu$ implies that $\tau - (p, 0) \trianglelefteq \mu - (p, 0) = \mu(0)$. But now

$$\tau = (a, 1^b) = (a - p, 1^b) + p(1, 0)$$

so we have the weak p expansion $\tau = \sum_{i \geq 0} p^i \gamma(i)$ with $\gamma(0) = (a - p, 1^b)$, $\gamma(1) = (1, 0)$ and $\gamma(i) = 0$ for $i > 1$. Since $\overline{\gamma(i)} \trianglelefteq \mu(i)$ for all $i \geq 0$ we have $(a, 1^b) \trianglelefteq_p \mu$ and so $\mu \in \zeta^{\trianglerighteq_p}(I(n,r))$, a contradiction.

Step 6. If $n > 2p$ and $r \geq p$ then $I(n,r)$ is not quasi-hereditary.

Note that $\zeta(I(n,r))$ contains $(n-p, 1^p)$ and hence $\zeta^{\trianglerighteq}(I(n,r))$ contains $\mu = (n-p, p)$. Now we have $\mu = (n - 2p, 0) + p(1, 1)$ and so $\mu = \mu(0) + p\xi$, where $\mu(0)$ has at most one part and ξ has two parts. Hence in the base p expansion $\mu = \sum_{i \geq 0} p^i \mu(i)$, there is for some $j \geq 1$, such that $\mu(j)$ has two parts.

Now if $\mu \in \zeta^{\trianglerighteq_p}(I(n,r))$ then there exists some $\tau = (a, 1^b)$ with weak p expansion $\tau = \sum_{i \geq 0} p^i \gamma(i)$ such that $\overline{\gamma(i)} \trianglelefteq \mu(i)$ for all $i \geq 0$. But then $\gamma(j)$ must have at least two parts. Since $j \geq 1$, the partition $\tau = (a, 1^b)$ has two parts of size at least p. This is not the case so there is no such weak p expansion and $\mu \notin \zeta^{\trianglerighteq_p}(I(n,r))$. Thus $\zeta^{\trianglerighteq}(I(n,r)) \neq \zeta^{\trianglerighteq_p}(I(n,r))$ and $I(n,r)$ is not quasi-hereditary.

Step 7. If $n > 2p$, if p does not divide n and if $r < p$, then $I(n,r)$ is quasi-hereditary.

If not there exists $\mu = (\mu_1, \mu_2, \dots) \in \zeta^{\trianglerighteq}(I(n,r)) \backslash \zeta^{\trianglerighteq_p}(I(n,r))$. Thus $\mu \trianglerighteq (a, 1^b)$, for some $n = a + b$, $b \geq 1$ and, as in Step 5, we choose such $(a, 1^b)$ with b minimal. Again, by Step 2, we have $b \geq 2$.

We claim that $\mu_1 = a$. If not, we get $\mu \trianglerighteq (a+1, 1^{b-1})$ as in Step 5, contradicting the minimality of b.

Thus we have $\mu_2 + \cdots + \mu_n = n - \mu_1 = b < p$, in particular we have $\mu_i < p$ for all $i \geq 1$. Hence in the base p expansion $\mu = \sum_{i \geq 0} p^i \mu(i)$, for all $i \geq 1$ we have $\mu(i) = (c_i, 0, \dots, 0)$, for some $0 \leq c_i < p$. Also, $\mu(0) = (k, \mu_2, \dots, \mu_n)$, for some $k > 0$.

Now we have

$$\tau = (a, 1^b) = (k + \sum_{i \geq 1} p^i c_i, 1^b) = (k, 1^b) + \sum_{i \geq 1} p^i (c_i, 0, \dots, 0).$$

Thus we have the weak p-expansion $\tau = \sum_{i\geq 0} p^i\gamma(i)$, with $\gamma(0) = (k, 1^b)$ and $\gamma(i) = (c_i, 0, \ldots, 0)$, for $i \geq 1$. Furthermore, $\overline{\gamma(i)} \trianglelefteq \mu(i)$, for all $i \geq 0$ so that $\mu \in \zeta^{\trianglerighteq_p}(I(n,r))$ and therefore $\zeta^{\trianglerighteq}(I(n,r)) = \zeta^{\trianglerighteq_p}(I(n,r))$ and $I(n,r)$ is quasi-hereditary. $\square$

Let k be a field. Recall that, for $\delta \in k$, and r a positive integer we have the partition algebra $P_r(\delta)$ over k. One may find a detailed account of the construction and properties of $P_r(\delta)$ in for example the papers by Paul P. Martin, [18], [19], and [13], [1]. Suppose now that k has characteristic $p > 0$ and $\delta = n1_k$, for some positive integer n. Let E_n be an n-dimensional vector space with basis $e_1, \ldots, e_n$. Then $P_r(n) = P_r(n1_k)$ acts on $E_n^{\otimes r}$. By a result of Halverson-Ram, [13, Theorem 3.6] the image of the representation $P_r(n) \to \mathrm{End}_k(E_n^{\otimes r})$ is $\mathrm{End}_{\mathrm{Sym}(n)}(E^{\otimes r})$. Moreover, for $n \gg 0$ the action of $P_r(n)$ is faithful. Let $N = n + ps$, for s suitably large, so that $P_r(n) = P_r(N)$ acts faithfully on $E_N^{\otimes r}$. Thus $P_r(n)$ is quasi-hereditary if and only if $\mathrm{End}_{\mathrm{Sym}(N)}(E_N^{\otimes r})$ is. Hence from Proposition 7.3 we have the following, which is a special case of a result of König and Xi, [16, Theorem 1.4].

Corollary 7.4. *The partition algebra $P_r(n)$ is quasi-hereditary if and only if n is prime to p and $r < p$.*

8. Quasi-hereditary endomorphism algebras of Young permutation modules and generalised Schur algebras

This section and the next were prompted by two questions raised by the referee, to whom I am most grateful.

The first point is to relate quasi-hereditary endomorphism algebras of Young permutation modules to generalised Schur algebras, as in [3], [4], [6], and second is to note that the decomposition numbers for endomorphism algebras of Young permutation modules are decomposition numbers for general linear groups.

We fix a positive integer n and a field k of characteristic $p > 0$. We write k_s for the sign module for $\mathrm{Sym}(n)$ i.e., the field k regarded as the one dimensional $k\mathrm{Sym}(n)$-module on which a permutation acts according to sign. For a finite dimensional $k\mathrm{Sym}(n)$-module Y, the endomorphism algebra $\mathrm{End}_{\mathrm{Sym}(n)}(Y)$ is isomorphic to the endomorphism algebra $\mathrm{End}_{\mathrm{Sym}(n)}(k_s \otimes Y)$. Let σ be a subset of $\mathrm{Par}(n)$. We consider the endomorphism algebra $\mathrm{End}_{\mathrm{Sym}(n)}(Y)$, where Y is a $k\mathrm{Sym}(n)$-module of the form $Y = \bigoplus_{\lambda\in\sigma} Y(\lambda)^{(d_\lambda)}$, with $d_\lambda \neq 0$, for all $\lambda \in \sigma$.

We now bring into play the representation theory of the Schur algebra $S(n,r)$ over k, for $r \geq 1$. We write $\Lambda^+(n,r)$ for the set of partitions of r with at most n parts. For $\lambda \in \Lambda^+(n,r)$ we have an irreducible module $L_n(\lambda)$, as in [7]. The modules $L_n(\lambda)$, $\lambda \in \Lambda^+(n,r)$, form a complete set of pairwise non-isomorphic irreducible modules. The algebra $S(n,r)$ is quasi-hereditary with respect to the partial order $\trianglelefteq$ on $\Lambda^+(n,r)$ and this labelling of the irreducible modules. We have the costandard module $\nabla_n(\lambda)$ and tilting module $T_n(\lambda)$, for $\lambda \in \mathrm{Par}(n)$.

As in [12] we may realise the group algebra $k\mathrm{Sym}(n)$ as the algebra $eS(n,n)e$, for a certain idempotent $e \in S(n,n)$ and as in [12, Chapter 6] we have the Schur functor $f : \mathrm{mod}(S(n,n)) \to \mathrm{mod}(k\mathrm{Sym}(n))$, given on objects by $fV = eV$. For $\lambda \in \mathrm{Par}(n)$ we write λ' for the transpose partition. For a subset τ of $\mathrm{Par}(n)$ we set $\tau' = \{\lambda' \mid \lambda \in \tau\}$. We have $fT_n(\lambda) = k_s \otimes Y(\lambda')$, for $\lambda \in \mathrm{Par}(n)$, by [6, (3.6) Lemma (ii)]. Hence we have $k_s \otimes Y = \bigoplus_{\lambda\in\pi} fT_n(\lambda)^{(d_\lambda)}$, where $\pi = \sigma'$. Moreover, for $\lambda, \mu \in \mathrm{Par}(n)$ the Schur functor induces an isomorphism from $\mathrm{Hom}_{S(n,n)}(T_n(\lambda), T_n(\mu))$ to $\mathrm{Hom}_{\mathrm{Sym}(n)}(fT_n(\lambda), fT_n(\mu))$, by [7, 4.4(1)(ii)]. Thus $\mathrm{End}_{\mathrm{Sym}(n)}(Y)$ is isomorphic to $\mathrm{End}_{S(n,n)}(T)$, where $T = \bigoplus_{\lambda\in\pi} T_n(\lambda)^{(d_\lambda)}$.

We now assume that σ is cosaturated. Let $S(\pi)$ be the generalised Schur algebra, defined by the saturated set π, as in [3], [4]. We say that a finite dimensional $S(n,n)$-module V belongs to π if each composition factor of V has the form $L_n(\mu)$ for some $\mu \in \pi$. For $\lambda \in \pi$ the module $T_n(\lambda)$ belongs to π and so is naturally a module for $S(\pi)$, and indeed $T_n(\lambda)$ is the indecomposable tilting module for $S(\pi)$ labelled by λ. Thus T is a full tilting module for $S(\pi)$ and $\mathrm{End}_{\mathrm{Sym}(n)}(Y)$ is, up to Morita equivalence, the Ringel dual of $S(\pi)$. We have shown:

Proposition 8.1. *Let σ be a cosaturated subset of $\mathrm{Par}(n)$. Let Y be a module of the form $\bigoplus_{\lambda\in\sigma} Y(\lambda)^{d_\lambda)}$, with $d_\lambda \neq 0$ for all $\lambda \in \sigma$. Then $\mathrm{End}_{\mathrm{Sym}(n)}(Y)$ is (up to Morita equivalence) the Ringel dual of the generalised Schur algebra $S(\pi)$, where $\pi = \sigma'$.*

Corollary 8.2. *Let Ω be a Young $\mathrm{Sym}(n)$-set. If the algebra $\mathrm{End}_{\mathrm{Sym}(n)}(k\Omega)$ is quasihereditary then it is, up to Morita equivalence, the Ringel dual of the generalised Schur algebra $S(\pi)$, for $\mathrm{GL}_n(k)$, where $\pi = \sigma'$, with $\sigma = \zeta^{\trianglerighteq}(\Omega)$.*

Proof. The summands of $k\Omega$ are the Young modules $Y(\lambda)$, $\lambda \in \zeta^{\trianglerighteq_p}(\Omega)$, by Theorem 6.3, and the condition for $\mathrm{End}_{\mathrm{Sym}}(k\Omega)$ to be quasihereditary is that $\zeta^{\trianglerighteq_p}(\Omega)$ is the cosaturated set $\zeta^{\trianglerighteq}(\Omega)$, whence the result. $\square$

9. Decomposition numbers of endomorpism algebras of Young permutation modules

The point here is to check that the decomposition numbers for endomorphism algebras of Young permutation modules, in particular the partition algebras considered in Section 7, are decomposition numbers for general linear groups. One may see this as a generalisation of the Theorem of James for symmetric groups, see e.g., [12, (6.6g) Theorem].

Our first task is to show that our labelling of simple modules in the case $I = I(n,n)$, with action $w \cdot i = i \circ w^{-1}$, agrees with the usual labelling of simple modules for the Schur algebra $S = S(n,n)$, by highest weight. For $i \in I$, let e_i be the corresponding basis element of the permutation module kI. We have $\Lambda_I = \Lambda(n,n)$, the set of sequences of elements of $\{1,\dots,n\}$ of length n, whose sum is n. The content of $i = (i_1,\dots,i_n)$ is the element $\alpha = (\alpha_1,\dots,\alpha_n) \in \Lambda(n,n)$, where α_1 is the number of 1s in i, where α_2 is the number of 2s, etc. For $\alpha \in \Lambda(n,n)$ the corresponding orbit $\mathcal{O}_\alpha$ is the set of $i \in I(n,n)$ with content α. For $\alpha \in \Lambda(n,n)$ weight space $\xi_\alpha(kI)$ is the span of the elements e_i, with $i \in I(n,n)$ having content α. For $\lambda \in \mathrm{Par}(n)$ we have $\xi_\alpha \mathrm{Hom}_{\mathrm{Sym}(n)}(M(\lambda), kI) = \mathrm{Hom}_{\mathrm{Sym}(n)}(M(\lambda), \xi_\alpha(kI))$. By Frobenius reciprocity we have

$$\dim \xi_\alpha \mathrm{Hom}_{\mathrm{Sym}(n)}(M(\lambda), kI) = \dim \mathrm{Hom}_{\mathrm{Sym}(\lambda)}(k, \xi_\alpha(kI))$$

and $\xi_\alpha(kI)$ is the permutation module on the $\mathrm{Sym}(\lambda)$-set on the set of $i \in I(n,n)$ that have content α. Hence $\dim \xi_\alpha \mathrm{Hom}_{\mathrm{Sym}(n)}(M(\lambda), kI)$ is the number $N(\lambda, \alpha)$, say of $\mathrm{Sym}(\lambda)$-orbits of elements of $i \in I(n,n)$ which have content α.

Now the α weight space of $S\xi_\lambda$ is $\xi_\alpha S\xi_\lambda$. This has k-basis consisting of all ξ_{ij} such that i has content α and j has content λ. We recall from, [12, Section 2.3], that for $i, i', j, j' \in I(n,n)$ we have $\xi_{ij} = \xi_{i'j'}$ if and only if there exists $w \in \mathrm{Sym}(n)$ such that $i' = i \circ w$ and $j' = j \circ w$. We write j_λ for the element $(1, \ldots, 1, 2, \ldots, \ldots)$ of $I(n,n)$ whose entries are weakly increasing and in which 1 occurs λ_1 times, 2 occurs λ_2 times, and so on. Thus if $i, j \in I(n,n)$ have contents α, λ then we may write $\xi_{ij} = \xi_{hj_\lambda}$, for some $h \in I(n,n)$ with content α and, for $h, h' \in I(n,n)$ with content α we have $\xi_{ij_\lambda} = \xi_{h'j_\lambda}$ if and only if there exists $w \in \mathrm{Sym}(\lambda)$ such that $h' = h \circ w$. Hence the dimension of $\xi_\alpha S\xi_\lambda$ is equal $N(\alpha, \lambda)$. Thus the projective modules $\mathrm{Hom}_{\mathrm{Sym}(n)}(M(\lambda), kI)$ and $S\xi_\lambda$ have the same weight space multiplicities. It follows from the classification of irreducible $S(n,n)$-modules by highest weight that finite dimensional modules with the same weight space multiplicities have the same composition factors (counted according to multiplicity). Moreover, the Cartan matrix of $S(n,n)$ is non-singular (e.g., by [7, Proposition A 2.2 (iv)]) so that finite dimensional projective modules with the same composition factors (counted according to multiplicity) are isomorphic. Hence $\mathrm{Hom}_{\mathrm{Sym}(n)}(M(\lambda), kI)$ and $S\xi_\lambda$ are isomorphic.

For $\lambda \in \mathrm{Par}(n)$ we write $Q(\lambda)$ for the projective cover of $L_n(\lambda)$. Now, for $\lambda, \mu \in \mathrm{Par}(n)$, we have $\dim \mathrm{Hom}_S(S\xi_\lambda, L_n(\mu)) = \dim L_n(\mu)^\lambda$ so that

$$S\xi_\lambda = \bigoplus_{\mu \in \mathrm{Par}(n)} Q(\mu)^{(m_{\lambda\mu})}$$

where $m_{\lambda\mu} = \dim L_n(\mu)^\lambda$. Since $L_n(\lambda)$ has unique highest weight λ and this occurs with multiplicity one we have

$$\mathrm{Hom}_{\mathrm{Sym}(n)}(M(\lambda), kI) = Q(\lambda) \bigoplus (\bigoplus_{\mu \rhd \lambda} Q(\mu)^{(m_{\lambda\mu})})$$

But we also have

$$M(\lambda) = Y(\lambda) \bigoplus (\bigoplus_{\mu \rhd \lambda} Y(\mu)^{(m_{\lambda\mu})})$$

by [7, 4.4 (1)(v)], and hence

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{Sym}(n)}(M(\lambda), kI)\\ &\qquad = \mathrm{Hom}_{\mathrm{Sym}(n)}(Y(\lambda), kI) \bigoplus (\bigoplus_{\mu \rhd \lambda} \mathrm{Hom}_{\mathrm{Sym}(n)}(Y(\mu), kI)^{(m_{\lambda\mu})}).\end{aligned}$$

Thus we have

$$Q(\lambda) \bigoplus (\bigoplus_{\mu \rhd \lambda} Q(\mu)^{(m_{\lambda\mu})}) = P_I(\lambda) \bigoplus (\bigoplus_{\mu \rhd \lambda} P_I(\mu)^{(m_{\lambda\mu})}).$$

Now it follows by descending induction that $Q(\lambda) = P_I(\lambda)$, for all $\lambda \in \mathrm{Par}(n)$. Hence we have:

$$L_I(\lambda) = L_n(\lambda), \text{ for all } \lambda \in \mathrm{Par}(n). \tag{1}$$

We briefly consider the more general case of permutation modules over a finite group G. Let Ω be a finite G-set and J a commutative ring. We write $S_{\Omega,J}$, or just S_Ω for $\mathrm{End}_G(J\Omega)$. For a G-stable subset $\mathcal{O}$ of $\Omega \times \Omega$ we write $a_{\mathcal{O},\Omega}$ for the element of S_Ω given on an element $x \in \Omega$ by $a_{\mathcal{O},\Omega}(x) = \sum_y y$, with the sum over all $y \in \Omega$ such that $(y, x) \in \mathcal{O}$. Then, as in Section 6, S_Ω has J-basis $a_{\mathcal{O},\Omega}$, with $\mathcal{O}$ ranging over the G-orbits in $\Omega \times \Omega$.

Let Γ be a G-stable subset of Ω. We have the idempotent $e_\Gamma \in S_\Omega$, whose value on $x \in \Omega$ is x if $x \in \Gamma$ and 0 otherwise. For an orbit $\mathcal{O}$ in $\Omega \times \Omega$ we have

$$e_\Gamma a_{\mathcal{O},\Omega} e_\Gamma = \begin{cases} a_{\mathcal{O},\Omega}, & \text{if} \quad \mathcal{O} \subseteq \Gamma \times \Gamma \\ 0, & \text{otherwise.} \end{cases}$$

We identify the J-algebra S_Γ with $e_\Gamma S_\Omega e_\Gamma$ by identifying $a_{\mathcal{O},\Gamma}$ with $a_{\mathcal{O},\Omega}$, for a G-orbit $\mathcal{O}$ in $\Gamma \times \Gamma$. We have the Schur functor $f_{\Omega,\Gamma} : \mathrm{mod}(S_\Omega) \to \mathrm{mod}(S_\Gamma)$, as in [12, Chapter 6], given on objects by $f_{\Omega,\Gamma} Z = e_\Gamma Z$. (One should perhaps write $e_{\Gamma,J}$ and $f_{\Omega,\Gamma,J}$ for e_Γ and $f_{\Omega,\Gamma}$ but we hope that in what follows the base ring will be clear from the context.) Note that if $\Gamma \subseteq I \subseteq \Omega$ are G-sets then $f_{\Omega,\Gamma} = f_{I,\Gamma} \circ f_{\Omega,I}$.

Let F be a field. Let V be a finite dimensional FG-module. Then $\mathrm{Hom}_G(V, F\Omega)$ is naturally an S_Ω-module. We leave the details of the following to the reader.

We have an isomorphism of S_Γ-modules from $\mathrm{Hom}_G(V, F\Gamma)$ to $f_{\Omega,\Gamma}\mathrm{Hom}_G(V, F\Omega)$, taking $\theta \in \mathrm{Hom}_G(V, F\Gamma)$ to $i \circ \theta \in \mathrm{Hom}_G(V, F\Omega)$, where $i : F\Gamma \to F\Omega$ is inclusion. (2)

We now restrict attention to the case in which $G = \mathrm{Sym}(n)$ and Ω is a Young $\mathrm{Sym}(n)$-set. Let R be a principal ideal domain with characteristic 0 field of fractions K. Let $\mathfrak{m}$ be a maximal ideal of R and let k be a field containing the residue field $R/\mathfrak{m}$ and suppose that k has characteristic $p > 0$. The modules $P_\Omega(\lambda) = \mathrm{Hom}_G(Y(\lambda), k\Omega)$, $\lambda \in \zeta^{\trianglerighteq_p}(\Omega)$

form a complete set of pairwise non-isomorphic projective indecomposable S_Ω-modules and, writing $L_\Omega(\lambda)$ for the head of $P_\Omega(\lambda)$, the modules $L_\Omega(\lambda)$, $\lambda \in \zeta^{\trianglerighteq_p}(\Omega)$, form a complete set of pairwise non-isomorphic irreducible S_Ω-modules (see Section 1).

Lemma 9.1. *Let Γ be a Young $\mathrm{Sym}(n)$-subset of the Young $\mathrm{Sym}(n)$-set Ω. Then we have:*
(i) $P_\Gamma(\lambda) = f_{\Omega,\Gamma} P_\Omega(\lambda)$, for $\lambda \in \zeta^{\trianglerighteq_p}(\Gamma)$;
(ii) $\nabla_\Gamma(\lambda)_K = f_{\Omega,\Gamma}\nabla_\Omega(\lambda)_K$, for $\lambda \in \zeta^{\trianglerighteq}(\Gamma)$; and
(iii) $f_{\Omega,\Gamma} L_\Omega(\lambda) = L_\Gamma(\lambda)$ for $\lambda \in \zeta^{\trianglerighteq_p}(\Gamma)$.

Proof. (i) For $\lambda \in \zeta^{\trianglerighteq_p}(\Gamma)$, we have from (2) that $e_\Gamma \mathrm{Hom}_G(Y(\lambda), k\Omega)$ is isomorphic to $\mathrm{Hom}_G(Y(\lambda), k\Gamma)$ i.e., $P_\Gamma(\lambda) = f_{\Omega,\Gamma} P_\Omega(\lambda)$.

(ii) Similar.

(iii) We first note that in order to show that the pair (Ω, Γ) of Young $\mathrm{Sym}(n)$-sets has the desired property (i.e., $f_{\Omega,\Gamma} L_\Omega(\lambda) = L_\Gamma(\lambda)$, for all $\lambda \in \zeta^{\trianglerighteq_p}(\Gamma)$) it is enough to show that $e_\Gamma L_\Omega(\lambda) \neq 0$ for all $\lambda \in \zeta^{\trianglerighteq_p}(\Gamma)$. If that is the case then, for $\lambda \in \zeta^{\trianglerighteq_p}(\Gamma)$, since $P_\Omega(\lambda)$ maps onto $L_\Omega(\lambda)$ the module $f_{\Omega,\Gamma} P_\Omega(\lambda) = P_\Gamma(\lambda)$ maps onto the non-zero simple module $f_{\Omega,\Gamma} L_\Omega(\lambda)$. But $P_\Gamma(\lambda)$ has simple head $L_\Gamma(\lambda)$ and hence $f_{\Omega,\Gamma} L_\Omega(\lambda) = L_\Gamma(\lambda)$.

Suppose that $\Omega = I = I(n,n)$ and $\Gamma \subseteq I(n,n)$. We take $\Lambda_\Omega = \Lambda(n,n)$ and, for $\alpha \in \Lambda(n,n)$, take $\mathcal{O}_{\alpha,I}$ to the set of all $i \in I$ with content α. We take Λ_Γ to the set of all $\alpha \in I$ such that $\mathcal{O}_{\alpha,I} \subseteq \Gamma$ and $\mathcal{O}_{\alpha,\Gamma} = \mathcal{O}_{\alpha,\Omega}$, for $\alpha \in \Lambda_\Gamma$. Then, for $\lambda \in \mathrm{Par}(n)$ we have $e_\Gamma L_I(\lambda) \neq 0$ if and only if $\xi_\mu L_I(\lambda) \neq 0$, for some $\mu \in \zeta(\Gamma)$ and the condition for this is $\lambda \in \zeta^{\trianglerighteq_p}(\Gamma)$, by [6, Section 3, Remark]. Thus the pair (Ω, Γ) has the desired property.

Now suppose that $\zeta^{\trianglerighteq_p}(\Omega) = \zeta^{\trianglerighteq_p}(\Gamma)$. Then S_Ω and S_Γ have the same number of isomorphism classes of irreducible modules. Hence $f_\Omega L_\Omega(\lambda)$ is a non-zero irreducible S_Γ-module, for all $\lambda \in \zeta^{\trianglerighteq_p}(\Omega) = \zeta^{\trianglerighteq_p}(\Gamma)$, by [12, (6.2g) Theorem]. Thus the pair (Ω, Γ) has the desired property. Note that this applies in two important cases.

In the first we may take $\Gamma = I = I(n,n)$ and Ω any Young $\mathrm{Sym}(n)$-set containing I.

We shall say that a $\mathrm{Sym}(n)$ is basic if for each $\lambda \in \zeta(\Omega)$ there is only one orbit containing an element with point stabilizer $\mathrm{Sym}(\lambda)$. In the second case we take Ω arbitrary and $\Gamma \subseteq \Omega$ any basic $\mathrm{Sym}(n)$-subset.

We now consider the general case. We take $\Gamma_0 \subseteq \Gamma$ to be a basic $\mathrm{Sym}(n)$-subset. Suppose that the pair (Ω, Γ_0) has the desired property. Then, for $\lambda \in \zeta^{\trianglerighteq_p}(\Gamma) = \zeta^{\trianglerighteq_p}(\Gamma_0)$ we have $f_{\Gamma,\Gamma_0} f_{\Omega,\Gamma} L_\Omega(\lambda) \neq 0$ and hence $f_{\Omega,\Gamma} L_\Omega(\lambda) \neq 0$. Hence (Ω, Γ) has the desired property.

Thus we may assume that Γ is basic. Now choosing a basic $\mathrm{Sym}(n)$-subset Ω_0 of Ω containing Γ and repeating the argument we get that it is enough to know that (Ω_0, Γ) has the desired property. Thus we may assume that $\Gamma \subseteq \Omega \subseteq I$. But now, from the first case considered above we know that, for $\lambda \in \zeta^{\trianglerighteq_p}(\Gamma)$ we have $f_{I,\Omega} L_I(\lambda) = L_\Omega(\lambda)$ and $f_{\Omega,\Gamma} f_{I,\Omega} L_I(\lambda) \neq 0$. Hence $f_{\Omega,\Gamma} L_\Omega(\lambda) \neq 0$ for all $\lambda \in \zeta^{\trianglerighteq_p}(\Gamma)$, and we are done. □

We note in passing the following.

If α and $\beta \in \Lambda_\Omega$ are such that $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$ have a common point stabiliser then for a finite dimensional S_Ω-module V we have

$$\dim{}^\alpha V = \dim{}^\beta V. \tag{3}$$

Proof. (i) We set $X = \zeta^{\trianglerighteq_p}(\Omega)$. For $V \in \mathrm{mod}(S_\Omega)$ we define $g_\alpha(V) = \dim{}^\alpha V$ and $g_\beta(V) = \dim{}^\beta V$. By additivity of weight space multiplicities on short exact sequences, it is enough to prove that $g_\alpha(V) = g_\beta(V)$ for V irreducible. We set $g_{\alpha,\lambda} = g_\alpha(L(\lambda))$, $g_{\beta,\lambda} = g_\beta(L(\lambda))$ and $G_{\alpha,\lambda} = g_\alpha(P(\lambda))$, $G_{\beta,\lambda} = g_\beta(P(\lambda))$, for $\lambda \in X$. Now we have

$$G_{\alpha,\lambda} = \dim \xi_\alpha \mathrm{Hom}_{\mathrm{Sym}(n)}(Y(\lambda), k\Omega) = \dim \mathrm{Hom}_{\mathrm{Sym}(n)}(Y(\lambda), k\mathcal{O}_\alpha)$$

and for the same reason $G_{\beta,\lambda} = \dim \mathrm{Hom}_{\mathrm{Sym}(n)}(Y(\lambda), k\mathcal{O}_\beta)$. But $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$ are isomorphic $\mathrm{Sym}(n)$-set so that $k\mathcal{O}_\alpha$ and $k\mathcal{O}_\beta$ are isomorphic $\mathrm{Sym}(n)$-modules and so $G_{\alpha,\lambda} = G_{\beta,\lambda}$, for all $\lambda \in X$.

For $\lambda, \mu \in X$ we have the Cartan integer $c_{\lambda\mu}$, the multiplicity of $L_\Omega(\mu)$ are a composition factor of $P_\Omega(\lambda)$. Again by additivity we have

$$G_{\alpha,\lambda} = g_\alpha(P(\lambda)) = \sum_{\mu \in X} c_{\lambda\mu} g_\alpha(L_\Omega(\mu)) = \sum_{\mu \in X} c_{\lambda\mu} g_{\alpha,\mu}$$

and $G_{\beta,\lambda} = \sum_{\mu \in X} c_{\lambda\mu} g_{\beta,\mu}$. Hence we have

$$\sum_{\mu \in X} c_{\lambda\mu} g_{\alpha,\mu} = \sum_{\mu \in X} c_{\lambda\mu} g_{\beta,\mu}$$

for all $\lambda \in X$. Since S_Ω is cellular, its Cartan matrix $C = (c_{\lambda\mu})_{\lambda,\mu \in X}$ is non-singular by [16, Proposition 1.2] and hence $g_{\alpha,\lambda} = g_{\beta,\lambda}$ for all $\lambda \in X$, i.e., $g_\alpha(V) = g_\beta(V)$, for all irreducible S_Ω-modules, as required. □

Note that the same result works over a field of characteristic 0.

To $\lambda \in \zeta^{\trianglerighteq}(\Omega)$ and $\mu \in \zeta^{\trianglerighteq_p}(\Omega)$ we have attached the simple $S_{\Omega,K}$-module $\nabla_\Omega(\lambda)_K$ and simple $S_\Omega = S_{\Omega,k}$-module $L_\Omega(\lambda)$. We have the natural map $S_R(\Omega) \to S_{\Omega,k}$ giving rise to an isomorphism $k \otimes_R S_{\Omega,R} \to S_{\Omega,k}$ of k-algebras by which we identify $k \otimes_R S_{\Omega,R}$ and $S_{\Omega,k}$. Let $\nabla_\Omega(\lambda)_R$ be a R-form of $\nabla_\Omega(\lambda)_K$. Then the multiplicity of $L_\Omega(\mu)$, as a composition factor of $k \otimes_R \nabla_\Omega(\lambda)_R$ is independent of the choice of R-form and we denote it $[\nabla_\Omega(\lambda)_K : L_\Omega(\mu)]$.

For $\lambda, \mu \in \Lambda^+(n,r)$ we have the multiplicity $[\nabla_n(\lambda)_K : L_n(\mu)]_n$ of $L_n(\mu)$ as a composition factor of a modular reduction of $\nabla_n(\lambda)_K$. For $N \geq n$ we have $[\nabla_n(\lambda) : L_n(\mu)]_n = [\nabla_N(\lambda) : L_N(\mu)]_N$, by [12, (6.6e) Theorem (i)]. For partitions λ, μ of the same degree we write simply $[\lambda : \mu]$ for the decomposition number $[\nabla_n(\lambda) : L_n(\mu)]_n$, where n is at least the number of parts of λ and of μ.

We shall show that these numbers are also the decomposition numbers for the endomorphism algebra of an arbitrary $\mathrm{Sym}(n)$-set.

Proposition 9.2. *Let* Ω *be a Young* $\mathrm{Sym}(n)$*-set. Then for* $\lambda \in \zeta^{\trianglerighteq}(\Omega)$ *and* $\mu \in \zeta^{\trianglerighteq_p}(\Omega)$ *we have*

$$[\nabla_\Omega(\lambda)_K : L_\Omega(\mu)] = [\lambda : \mu].$$

The result holds for $\Omega = I = I(n,n)$ by definition. We now argue by repeated application of the theorem of T. Martins, [12, (6.6d) Theorem]. Suppose that Ω is a Young $\mathrm{Sym}(n)$-set containing $I = I(n,n)$. Then $\zeta(\Omega) = \zeta(I) = \zeta^{\trianglerighteq}(\Omega) = \zeta^{\trianglerighteq}(I) = \mathrm{Par}(n)$ and for $\lambda, \mu \in \mathrm{Par}(n)$ we have

$$[\nabla_\Omega(\lambda)_K : L_\Omega(\lambda)] = [f_{\Omega,I}\nabla(\lambda)_K : f_{\Omega,I}L_\Omega(\mu)] = [\nabla_I(\lambda)_K : L_I(\mu)]$$

by [12, (6.6d) Theorem], and this we already know to be $[\lambda : \mu]$. Now suppose that Ω is arbitrary and choose $\tilde{\Omega}$ a Young $\mathrm{Sym}(n)$-set containing Ω and I. Then we have shown $[\nabla_{\tilde{\Omega}}(\lambda)_K : L_{\tilde{\Omega}}(\mu)] = [\lambda : \mu]$, for $\lambda \in \zeta^{\trianglerighteq}(\Omega)$, $\mu \in \zeta^{\trianglerighteq_p}(\Omega)$, and hence

$$\begin{aligned}[\nabla_\Omega(\lambda)_K : L_\Omega(\mu)] &= [f_{\tilde{\Omega},\Omega}\nabla_{\tilde{\Omega}}(\lambda)_K : f_{\tilde{\Omega},\Omega}L_{\tilde{\Omega}}(\mu)] \\ &= [\nabla_{\tilde{\Omega}}(\lambda)_K : L_{\tilde{\Omega}}(\mu)] \\ &= [\lambda : \mu]\end{aligned}$$

by [12, (6.6d) Theorem] again.

References

[1] C. Bowman, S.R. Doty, S. Martin, Integral Schur-Weyl duality for partition algebras, arXiv:1906.00457v1, 2019.
[2] A.G. Cox, The blocks of the q-Schur algebra, J. Algebra 207 (1998) 306–325.
[3] S. Donkin, On Schur algebras and related algebras I, J. Algebra 104 (1986) 310–328.
[4] S. Donkin, On Schur algebras and related algebras II, J. Algebra 111 (1987) 354–364.
[5] S. Donkin, On Schur algebras and related algebras IV: the blocks of the Schur algebras, J. Algebra 168 (1994) 400–429.
[6] S. Donkin, On tilting modules for algebraic groups, Math. Z. 212 (1993) 39–60.
[7] S. Donkin, The q-Schur Algebra, LMS Lecture Notes, vol. 253, Cambridge University Press, 1998.
[8] Karin Erdmann, Stratifying systems, filtration multiplicities and symmetric groups, J. Algebra Appl. 5 (2005) 551–555.
[9] J.J. Graham, G.I. Lehrer, Cellular algebras, Invent. Math. 123 (1996) 1–34.
[10] J.A. Green, A theorem on modular endomorphism rings, Ill. J. Math. 32 (1988) 510–519.
[11] J.A. Green, Combinatorics and the Schur algebra, J. Pure Appl. Algebra 88 (1993) 89–106.
[12] J.A. Green, Polynomial Representations of GL_n, second edition, Lecture Notes in Mathematics, vol. 830, Springer, 2007, with an Appendix on Schenstead Correspondence and Littelmann Paths by K. Erdmann, J.A. Green and M. Schocker.
[13] T. Halverson, A. Ram, Partition algebras, Eur. J. Comb. 28 (2005) 869–921.
[14] G.D. James, The Representation Theory of the Symmetric Groups, Lecture Notes in Mathematics, vol. 682, Springer, 1970.
[15] Steffen König, Changchang Xi, On the structure of cellular algebras, in: I. Reiten, S. Smalø, Ø. Solberg (Eds.), Algebras and Modules II, in: Canadian Math. Society Conference Proceedings, vol. 24, 1998, pp. 365–386.
[16] Steffen König, Changchang Xi, When is a cellular algebra quasi-hereditary, Math. Ann. 315 (1999) 281–293.

[17] Andrew Mathas, Marcos Soriano, Blocks of truncated q-Schur algebras of type A, in: Vyjayanthi Chari, Jacob Greenstein, Kallash C. Misra, K.N. Raghavan, Sankaran Viswananth (Eds.), Recent Developments in Algebraic and Combinatorial Aspects of Representations Theory, in: Contemporary Mathematics, vol. 602, 2013, pp. 123–142.
[18] P. Martin, Potts Models and Related Problems in Statistical Mechanics, Series on Advances in Statistical Mechanics, vol. 5, World Scientific Publishing Co., Inc., Teaneck, NJ, 1991.
[19] P. Martin, Temperley-Lieb algebras for nonplanar statistical mechanics - the partition algebra construction, J. Knot Theory Ramif. 3 (1) (1994) 51–82.
[20] B. Parshall, Finite Dimensional Algebras and Algebraic Groups, Contemporary Math., vol. 82, 1989, pp. 97–114.