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Extensions of modules for $SL(2, K)$

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Abstract

In this paper, we consider the induced modules ∇ and the Weyl modules Δ for the algebraic group $G = SL(2, K)$ where K is an algebraically closed field of characteristic $p > 0$. We determine the G-modules $H^i(G_1, \nabla(s) \otimes \nabla(t))$ for all $i \geqslant 0$, where G_1 is the first Frobenius kernel of G. We then use it to find the Ext^1-spaces between twisted tensor products of Weyl modules and induced modules for G. Moreover, we describe explicitly the non-split extensions corresponding to ∇'s.

Keywords: Special linear group; Symmetric powers; Decomposition matrix

Introduction

In the theory of highest weight categories, the classes of modules ∇ and Δ are of central interest. In particular, twisted tensor products of these modules occur as important subquotients of ∇ and Δ (see [11,12]).

Here we consider these modules for the group $G = SL(2, K)$, the special linear group of dimension 2 over an algebraically closed field K of characteristic $p > 0$. Suppose that $F : G \to G$ is the corresponding Frobenius morphism and let G_1 denote the first Frobenius kernel of G. If V is a G-module then we denote by V^F its Frobenius twist. Considered as a G_1-module, V^F is trivial. Conversely, if W is a G-module on which G_1 acts trivially then $W \cong V^F$ for a unique G-module V and we write $W^{(-1)} := V$.

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Consider the Borel subgroup B of G consisting of lower triangular matrices and for $\lambda \in N$, let K_λ denote the 1-dimensional B-module of weight λ. Define the *induced* G-module $\nabla(\lambda)$ by

$$\nabla(\lambda) := \mathrm{Ind}_B^G(K_\lambda).$$

This is isomorphic to the symmetric power $S^\lambda E$ where E is the natural 2-dimensional G-module. The *Weyl* G-modules, $\Delta(\lambda)$, are defined by

$$\Delta(\lambda) := \nabla(\lambda)^*.$$

Note that $\operatorname{soc} \nabla(\lambda) = \operatorname{top} \Delta(\lambda) = L(\lambda)$ is simple and $\{L(\lambda)\colon \lambda \in N\}$ form a complete set of non-isomorphic simple G-modules. For $0 \leqslant \lambda \leqslant p-1$ we have $L(\lambda) = \nabla(\lambda) = \Delta(\lambda)$ and in general Steinberg's tensor product theorem tells us that if $\lambda = \sum_{i \geqslant 0} \lambda_i p^i$ is the p-adic expansion of λ then $L(\lambda)$ is given by

$$L(\lambda) = \bigotimes_{i \geqslant 0} L(\lambda_i)^{F^i}.$$

The simple G-modules are thus self-dual.

The modules $\nabla(\lambda)$ and $\Delta(\lambda)$ have highest weight λ occurring with multiplicity 1 and all their other weights μ satisfy $\mu < \lambda$.

In order to prove our results, we use the Lyndon–Hochschild–Serre 5-term exact sequence relating the Ext^1-spaces of G and G_1. For a rational G-module V, we have the exact sequence (see [3])

$$\begin{aligned} 0 \to\ & H^1\big(G, \big(V^{G_1}\big)^{(-1)}\big) \to H^1(G, V) \to H^1(G_1, V)^G \\ \to\ & H^2\big(G, \big(V^{G_1}\big)^{(-1)}\big) \to H^2(G, V). \end{aligned}$$

In Section 1, we describe properties of G_1-modules and we compute $\mathrm{Ext}^i_{G_1}(\Delta, \nabla)$ for $i \geqslant 0$ as G-modules. In Section 2, we use the 5-term exact sequence above and the results of Section 1 to compute $\mathrm{Ext}^1_G(\nabla(r)^{F^n} \otimes \Delta(s), \nabla(k)^{F^n} \otimes \nabla(t))$ for $0 \leqslant k, r$ and $0 \leqslant s, t \leqslant p^n - 1$. In particular, we show that it has at most dimension 1. We also find explicitly the non-split extensions corresponding to ∇. This filtration of ∇ by twisted tensor product of ∇'s and Δ's explains the symmetries observed in the decomposition matrix of G.

1. Computing $\mathrm{Ext}^i_{G_1}(\Delta, \nabla)$

The category of G_1-modules is equivalent to the category of U-modules where U is the restricted enveloping algebra of the Lie algebra of G. In particular, U is a self-injective algebra (see [14]). This category is very well understood [8,13]. The simple U-modules are the restriction of the $L(i)$ for $0 \leqslant i \leqslant p-1$ and the corresponding projective U-modules $P(i)$ have the following structure: for $0 \leqslant i \leqslant p-2$, $\operatorname{soc} P(i) = \operatorname{top} P(i) = L(i)$ and $\operatorname{rad} P(i)/\operatorname{soc} P(i) = L(j) \oplus L(j)$

where $i+j=p-2$ and for $i=p-1$ the projective module $P(p-1)=L(p-1)$ is simple. Thus the projective module $P(p-1)$ is alone in its block and $P(i)$ and $P(j)$ belong to the same block if and only if $i=j$ or $i+j=p-2$.

For an indecomposable non-projective U-module M, we denote by $\Omega(M)$ the kernel of the projective cover of M (and we define inductively $\Omega^k(M)=\Omega(\Omega^{k-1}(M))$. Similarly, we define $\Omega^{-1}(M)$ to be the cokernel of the injective hull of M (and we define inductively $\Omega^{-k}(M)$). The projective (injective) G_1-modules are restrictions of G-modules and for $n\geqslant 0$, we have an exact sequence of G-modules [4,16]

$$0\to\nabla(np+i)\to P(i)\otimes\nabla(n)^F\to\nabla\big((n+1)p+j\big)\to 0.$$

The restriction of this sequence to G_1 gives the projective cover of $\nabla((n+1)p+j)$ and the injective hull of $\nabla(np+i)$. The G_1-module $\nabla(np+i)$ has Loewy length 2 for $n\geqslant 1$. We have a sequence of G-modules [11,16]

$$0\to\nabla(n)^F\otimes\nabla(i)\to\nabla(np+i)\to\nabla(n-1)^F\otimes\Delta(j)\to 0 \tag{1}$$

and its restriction to G_1 gives the Loewy series of $\nabla(np+i)$ as a G_1-module.

Note finally that if V, W, and X are G-modules and $n\geqslant 0$ then $\mathrm{Ext}^n_{G_1}(V,W)$ has a natural structure of G-module and

$$\mathrm{Ext}^n_{G_1}\big(V,W\otimes X^F\big)\cong\mathrm{Ext}^n_{G_1}(V,W)\otimes X^F$$

as G-modules.

W. van der Kallen proved in [15] that if V is a G-module with a good filtration (that is a filtration with quotients isomorphic to some ∇'s) then $H^0(G_1,V)^{(-1)}$ has a good filtration and hence, by dimension shifting (see [6]), $H^i(G_1,V)^{(-1)}$ has a good filtration for all $i\geqslant 0$. Note that the module $V=\nabla\otimes\nabla$ has a good filtration and the next two propositions give the G-modules $H^i(G_1,V)=\mathrm{Ext}^i_{G_1}(\Delta,\nabla)$ for $i\geqslant 0$.

Write $t=t_1p+t_0$ and $s=s_1p+s_0$ where $0\leqslant s_0,t_0\leqslant p-1$.

Proposition 1.1. *For $i\geqslant 1$ we have*

$$\mathrm{Ext}^i_{G_1}\big(\Delta(s),\nabla(t)\big)\cong\begin{cases}\nabla(s_1+t_1+i)^F & \textit{if } s_0+t_0=p-2 \textit{ and } i \textit{ odd}\\ & \textit{or } s_0=t_0\leqslant p-2 \textit{ and } i \textit{ even,}\\ 0 & \textit{otherwise.}\end{cases}$$

Proof. From the block structure of G_1 we only need to consider the cases $s_0=t_0$ and $s_0+t_0=p-2$. Note that if $s_0=t_0=p-1$ then $\Delta(s)$ and $\nabla(t)$ are projective and so there is no non-split extension. Now suppose $s_0,t_0\leqslant p-2$.

$$\begin{aligned}&\mathrm{Ext}^i_{G_1}\big(\Delta(s_1p+s_0),\nabla(t_1p+t_0)\big)\\ &\quad\cong\mathrm{Ext}^i_{G_1}\big(\Omega^{-s_1}\big(\Delta(s_1p+s_0)\big),\Omega^{-s_1}\big(\nabla(t_1p+t_0)\big)\big)\\ &\quad\cong\begin{cases}\mathrm{Ext}^i_{G_1}\big(\Delta(s_0),\nabla\big((s_1+t_1)p+t_0\big)\big) & \text{if } s_1 \text{ even,}\\ \mathrm{Ext}^i_{G_1}\big(\Delta(p-2-s_0),\nabla\big((s_1+t_1)p+p-2-t_0\big)\big) & \text{if } s_1 \text{ odd.}\end{cases}\end{aligned}$$

Now consider the exact sequence

$$\begin{aligned}0 &\to \nabla\big((s_1+t_1)p+t_0\big) \to P(t_0)\otimes\nabla(s_1+t_1)^F\\ &\to \nabla\big((s_1+t_1+1)p+p-2-t_0\big)\to 0\end{aligned}$$

and apply $\mathrm{Hom}_{G_1}(\Delta(s_0),-)$ to get

$$\begin{aligned}0 &\to \mathrm{Hom}_{G_1}\big(\Delta(s_0),\nabla\big((s_1+t_1)p+t_0\big)\big)\\ &\to \mathrm{Hom}_{G_1}\big(\Delta(s_0),P(t_0)\otimes\nabla(s_1+t_1)^F\big)\\ &\to \mathrm{Hom}_{G_1}\big(\Delta(s_0),\nabla\big((s_1+t_1+1)p+p-2-t_0\big)\big)\\ &\to \mathrm{Ext}^1_{G_1}\big(\Delta(s_0),\nabla\big((s_1+t_1)p+t_0\big)\big)\to 0\end{aligned}\tag{2}$$

and

$$\begin{aligned}&\mathrm{Ext}^{i+1}_{G_1}\big(\Delta(s_0),\nabla\big((s_1+t_1)p+t_0\big)\big)\\ &\quad\cong \mathrm{Ext}^{i}_{G_1}\big(\Delta(s_0),\nabla\big((s_1+t_1+1)p+p_2-t_0\big)\big).\end{aligned}$$

Thus, if we prove the case $i=1$ then the result follows by induction. Now, observe that in the exact sequence (2) the first two terms are isomorphic ($\Delta(s_0)$ is simple and $P(t_0)\otimes\nabla(s_1+t_1)^F$ is the injective hull of $\nabla((s_1+t_1)p+t_0)$), hence the last two terms are isomorphic too and we get

$$\begin{aligned}&\mathrm{Ext}^1_{G_1}\big(\Delta(s_0),\nabla\big((s_1+t_1)p+t_0\big)\big)\\ &\quad\cong \mathrm{Hom}_{G_1}\big(\Delta(s_0),\nabla\big((s_1+t_1+1)p+p-2-t_0\big)\big)\\ &\quad\cong \mathrm{Hom}_{G_1}\big(\Delta(s_0),P(p-2-t_0)\otimes\nabla(s_1+t_1+1)^F\big)\\ &\quad\cong \mathrm{Hom}_{G_1}\big(\Delta(s_0),P(p-2-t_0)\big)\otimes\nabla(s_1+t_1+1)^F\\ &\quad\cong \begin{cases}\nabla(s_1+t_1+1)^F & \text{if } s_0+t_0=p-2,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

The proposition then follows by induction on i. $\square$

Proposition 1.2.

$$\mathrm{Hom}_{G_1}\big(\Delta(s_1p+s_0),\nabla(t_1p+t_0)\big)\cong\begin{cases}\big(\nabla(s_1)\otimes\nabla(t_1)\big)^F & \textit{if } s_0=t_0,\\ 0 & \textit{otherwise.}\end{cases}$$

Proof. Note that by the decomposition into blocks of G_1, we only need to consider the cases $s_0+t_0=p-2$ and $s_0=t_0$. Suppose for a start that $s_0,t_0\leqslant p-2$. Consider the exact sequence

$$0\to\nabla(t_1)^F\otimes\nabla(t_0)\to\nabla(t_1p+t_0)\to\nabla(t_1-1)^F\otimes\Delta(p-2-t_0)\to 0.$$

Apply $\mathrm{Hom}_{G_1}(\nabla(s_1p+s_0),-)$ to get the exact sequence

$$\begin{aligned}
0 \to\ & \mathrm{Hom}_{G_1}\big(\Delta(s_1p+s_0), \nabla(t_1)^F \otimes \nabla(t_0)\big) \\
\to\ & \mathrm{Hom}_{G_1}\big(\Delta(s_1p+s_0), \nabla(t_1p+t_0)\big) \\
\to\ & \mathrm{Hom}_{G_1}\big(\Delta(s_1p+s_0), \nabla(t_1-1)^F \otimes \Delta(p-2-t_0)\big) \\
\to\ & \mathrm{Ext}^1_{G_1}\big(\Delta(s_1p+s_0), \nabla(t_1)^F \otimes \nabla(t_0)\big) \\
\to\ & \mathrm{Ext}^1_{G_1}\big(\Delta(s_1p+s_0), \nabla(t_1p+t_0)\big).
\end{aligned} \tag{3}$$

Now,

$$\begin{aligned}
&\mathrm{Hom}_{G_1}\big(\Delta(s_1p+s_0), \nabla(t_1)^F \otimes \nabla(t_0)\big) \\
&\quad\cong \mathrm{Hom}_{G_1}\big(\nabla(t_0), \nabla(s_1p+s_0)\big) \otimes \nabla(t_1)^F \\
&\quad\cong \mathrm{Hom}_{G_1}\big(\nabla(t_0), P(s_0)\big) \otimes \nabla(s_1)^F \otimes \nabla(t_1)^F \\
&\quad\cong \begin{cases} \big(\nabla(s_1) \otimes \nabla(t_1)\big)^F & \text{if } s_0 = t_0, \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{Hom}_{G_1}\big(\Delta(s_1p+s_0), \nabla(t_1-1)^F \otimes \Delta(p-2-t_0)\big) \\
&\quad\cong \mathrm{Hom}_{G_1}\big(\nabla(p-2-t_0), \nabla(s_1p+s_0)\big) \otimes \nabla(t_1-1)^F \\
&\quad\cong \mathrm{Hom}_{G_1}\big(\nabla(p-2-t_0), P(s_0)\big) \otimes \nabla(s_1)^F \otimes \nabla(t_1-1)^F \\
&\quad\cong \begin{cases} \big(\nabla(s_1) \otimes \nabla(t_1-1)\big)^F & \text{if } s_0 + t_0 = p-2, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Using Proposition 1.1, we get

$$\begin{aligned}
&\mathrm{Ext}^1_{G_1}\big(\Delta(s_1p+s_0), \nabla(t_1)^F \otimes \nabla(t_0)\big) \\
&\quad\cong \mathrm{Ext}^1_{G_1}\big(\nabla(t_0), \nabla(s_1p+s_0)\big) \otimes \nabla(t_1)^F \\
&\quad\cong \begin{cases} \big(\nabla(s_1+1) \otimes \nabla(t_1)\big)^F & \text{if } s_0 + t_0 = p-2, \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

and

$$\mathrm{Ext}^1_{G_1}\big(\Delta(s_1p+s_0), \nabla(t_1p+t_0)\big) \cong \begin{cases} \nabla(s_1+t_1+1)^F & \text{if } s_0+t_0 = p-2, \\ 0 & \text{otherwise.} \end{cases}$$

So if $s_0 + t_0 = p-2$ and $p > 2$ (i.e., $s_0 \neq t_0$), then the exact sequence (3) becomes

$$\begin{aligned}
0 \to\ & \mathrm{Hom}_{G_1}\big(\Delta(s_1p+s_0), \nabla(t_1p+t_0)\big) \to \big(\nabla(s_1) \otimes \nabla(t_1-1)\big)^F \\
\to\ & \big(\nabla(s_1+1) \otimes \nabla(t_1)\big)^F \to \nabla(s_1+t_1+1)^F.
\end{aligned}$$

As

$$\begin{aligned}
\dim\big(\nabla(s_1+1) \otimes \nabla(t_1)\big)^F &= \dim\big(\nabla(s_1) \otimes \nabla(t_1-1)\big)^F \\
&\quad + \dim \nabla(s_1+t_1+1)^F,
\end{aligned}$$

we deduce that

$$\mathrm{Hom}_{G_1}\big(\Delta(s_1p+s_0),\nabla(t_1p+t_0)\big)=0.$$

If $s_0=t_0$ and $p=2$, the exact sequence (3) has the form

$$\begin{aligned}0 &\to \big(\nabla(s_1)\otimes\nabla(t_1)\big)^F \to \mathrm{Hom}_{G_1}\big(\Delta(s_1 2+s_0),\nabla(t_1 2+t_0)\big)\\ &\to \big(\nabla(s_1)\otimes\nabla(t_1-1)\big)^F \to \big(\nabla(s_1+1)\otimes\nabla(t_1)\big)^F \to \nabla(s_1+t_1+1)^F.\end{aligned}$$

Hence,

$$\mathrm{Hom}_{G_1}\big(\Delta(s_1 2+s_0),\nabla(t_1 2-t_0)\big)\cong\big(\nabla(s_1)\otimes\nabla(t_1)\big)^F.$$

Finally if $s_0=t_0$ and $p>2$ then clearly

$$\mathrm{Hom}_{G_1}\big(\Delta(s_1p+s_0),\nabla(t_1p+t_0)\big)\cong\big(\nabla(s_1)\otimes\nabla(t_1)\big)^F.$$

In the case where $s_0=t_0=p-1$, we have the following:

$$\begin{aligned}&\Delta(s_1p+s_0)\cong\Delta(s_1)^F\otimes\Delta(p-1),\\ &\nabla(t_1p+t_0)\cong\nabla(t_1)^F\otimes\nabla(p-1),\end{aligned}$$

and so

$$\begin{aligned}&\mathrm{Hom}_{G_1}\big(\Delta\big(s_1p+(p-1)\big),\nabla\big(t_1p+(p-1)\big)\big)\\ &\quad\cong\mathrm{Hom}_{G_1}\big(\Delta(p-1),\nabla(p-1)\big)\otimes\big(\nabla(s_1)\otimes\nabla(t_1)\big)^F\\ &\quad\cong\big(\nabla(s_1)\otimes\nabla(t_1)\big)^F.\end{aligned}$$

This completes the proof. □

2. Extensions of G-modules

In [5,7], Cox and Erdmann determined the Ext^1 and the Hom spaces between $\nabla(\lambda)$ and $\nabla(\mu)$ for arbitrary weights λ and μ. For completeness and to fix our notation, we state their result here.

For $0\leqslant a\leqslant p-1$ denote by $\hat a$, the integer such that $a+\hat a=p-1$. For a weight μ, define

$$\psi^0(\mu)=\left\{\sum_{i=0}^{u-1}\hat\mu_i p^i\colon\ u\geqslant 0\right\}$$

and

$$\begin{aligned}\psi^1(\mu)\ =\ &\left\{\sum_{i=0}^{u-1}\hat\mu_i p^i+p^{u+a}\colon\ \hat\mu_u\neq 0,\ a\geqslant 1,\ u\geqslant 0\right\}\\ &\cup\left\{\sum_{i=0}^{u}\hat\mu_i p^i\colon\ \hat\mu_u\neq 0,\ u\geqslant 0\right\}.\end{aligned}$$

With this notation we have,

$$\operatorname{Hom}_G\big(\nabla(\lambda),\nabla(\mu)\big)\cong\begin{cases}K & \text{if } \lambda=\mu+2d,\ d\in\psi^0(\mu),\\ 0 & \text{otherwise}\end{cases}\tag{4}$$

and

$$\operatorname{Ext}^1_G\big(\nabla(\lambda),\nabla(\mu)\big)\cong\begin{cases}K & \text{if } \lambda=\mu+2e,\ e\in\psi^1(\mu),\\ 0 & \text{otherwise.}\end{cases}\tag{5}$$

In [2], Cline determined all the Ext^1-spaces between simple G-modules. In particular, for simple modules $\nabla(r)^F\otimes\nabla(s)$ and $\nabla(k)^F\otimes\nabla(t)$, he proved that

$$\operatorname{Ext}^1_G\big(\nabla(r)^F\otimes\nabla(s),\nabla(k)^F\otimes\nabla(t)\big)\cong\begin{cases}K & \text{if } r=k\pm1,\ s+t=p-2,\\ 0 & \text{otherwise.}\end{cases}$$

The following theorem extends this result.

Theorem 2.1. *Let* $0\leqslant k,r$ *and* $0\leqslant s,t\leqslant p^n-1$ *then we have*

$$\begin{aligned}&\operatorname{Ext}^1_G\big(\nabla(r)^{F^n}\otimes\Delta(s),\nabla(k)^{F^n}\otimes\nabla(t)\big)\\ &\quad\cong\begin{cases}K & \textit{if } r=k+2e,\ e\in\psi^1(k),\ s=t\\ & \textit{or } r=k\pm1+2d,\ d\in\psi^0(k),\ t=t_0+t_1p^i,\ 0\leqslant t_0\leqslant p^i-1,\\ & s=t_0+(p^{n-i}-2-t_1)p^i,\\ 0 & \textit{otherwise.}\end{cases}\end{aligned}$$

Proof. In order to prove this theorem, we use the five terms exact sequence:

$$\begin{aligned}0\to\ & H^1\big(G,\big(V^{G_1}\big)^{(-1)}\big)\to H^1(G,V)\to H^1(G_1,V)^G\\ \to\ & H^2\big(G,\big(V^{G_1}\big)^{(-1)}\big)\to H^2(G,V),\end{aligned}$$

with $V=\Delta(r)^{F^n}\otimes\nabla(k)^{F^n}\otimes\nabla(s)\otimes\nabla(t)$.

Write $s=s_1p+s_0$ and $t=t_1p+t_0$. Let us first compute $H^1(G,(V_1^G)^{(-1)})$. Using Proposition 1.2, we have

$$\begin{aligned}V^{G_1}&=\operatorname{Hom}_{G_1}\big(\Delta(s),\nabla(t)\big)\otimes\Delta(r)^{F^n}\otimes\nabla(k)^{F^n}\\ &\cong\begin{cases}\big(\nabla(s_1)\otimes\nabla(t_1)\big)^F\otimes\Delta(r)^{F^n}\otimes\nabla(k)^{F^n} & \text{if } s_0=t_0,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Now,

$$\big(V^{G_1}\big)^{(-1)}\cong\begin{cases}\nabla(s_1)\otimes\nabla(t_1)\otimes\Delta(r)^{F^{n-1}}\otimes\nabla(k)^{F^{n-1}} & \text{if } s_0=t_0,\\ 0 & \text{otherwise.}\end{cases}$$

Hence for $s_0=t_0$ we have

$$H^1\big(G,\big(V^{G_1}\big)^{(-1)}\big)\cong\operatorname{Ext}^1_G\big(\nabla(r)^{F^{n-1}}\otimes\Delta(s_1),\nabla(k)^{F^{n-1}}\otimes\nabla(t_1)\big),$$

and is zero in all other cases.

Let us now compute $H^1(G_1, V)^G$. Using Proposition 1.1, we have

$$\begin{aligned} H^1(G_1, V) &= \operatorname{Ext}^1_{G_1}\big(\nabla(r)^{F^n} \otimes \Delta(s), \nabla(k)^{F^n} \otimes \nabla(t)\big) \\ &\cong \operatorname{Ext}^1_{G_1}\big(\Delta(s), \nabla(t)\big) \otimes \Delta(r)^{F^n} \otimes \nabla(k)^{F^n} \\ &\cong \begin{cases} \nabla(s_1 + t_1 + 1)^F \otimes \Delta(r)^{F^n} \otimes \nabla(k)^{F^n} & \text{if } s_0 + t_0 = p - 2, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Thus,

$$\begin{aligned} &H^1(G_1, V)^G \\ &\quad \cong \begin{cases} \operatorname{Hom}_G\big(\Delta(s_1 + t_1 + 1)^F, \Delta(r)^{F^n} \otimes \nabla(k)^{F^n}\big) & \text{if } s_0 + t_0 = p - 2, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Note that all the weights of $\Delta(r)^{F^n} \otimes \nabla(k)^{F^n}$ are multiples of p^n, so to get non-zero homomorphisms, we must have $s_1 + t_1 + 1 = cp^{n-1}$ for some c. But $s, t \leqslant p^n - 1$ implies that $s_1 + t_1 \leqslant 2p^{n-1} - 2$, thus $c = 1$ and $s_1 + t_1 + 1 = p^{n-1}$. Observe that

$$\begin{aligned} &\operatorname{Hom}_G\big(\Delta\big(p^{n-1}\big)^F, \Delta(r)^{F^n} \otimes \nabla(k)^{F^n}\big) \\ &\quad \cong \operatorname{Hom}_G\big(\nabla(r)^{F^n}, \nabla\big(p^{n-1}\big)^F \otimes \nabla(k)^{F^n}\big) \end{aligned}$$

and that all the weights of $\nabla(r)^{F^n}$ are multiple of p^n so the image of a homomorphism from $\nabla(r)^{F^n}$ to $\nabla(p^{n-1})^F \otimes \nabla(k)^{F^n}$ lies in the submodule $\nabla(1)^{F^n} \otimes \nabla(k)^{F^n} \leqslant \nabla(p^{n-1})^F \otimes \nabla(k)^{F^n}$. Hence,

$$\begin{aligned} &\operatorname{Hom}_G\big(\Delta\big(p^{n-1}\big)^F, \Delta(r)^{F^n} \otimes \nabla(k)^{F^n}\big) \\ &\quad \cong \operatorname{Hom}_G\big(\nabla(r)^{F^n}, \nabla(1)^{F^n} \otimes \nabla(k)^{F^n}\big) \cong \operatorname{Hom}_G\big(\nabla(r), \nabla(1) \otimes \nabla(k)\big). \end{aligned}$$

We claim that $\operatorname{Hom}_G(\nabla(r), \nabla(1) \otimes \nabla(k)) \cong K$ if $r = k \pm 1 + 2d$ where $d \in \psi^0(k)$ and zero otherwise. Consider the exact sequence

$$0 \to \nabla(z - 1) \to \nabla(1) \otimes \nabla(z) \to \nabla(z + 1) \to 0. \tag{6}$$

This sequence splits if and only if $z \neq -1 \pmod p$. Note that for $\operatorname{Hom}_G(\nabla(r), \nabla(1) \otimes \nabla(k))$ to be non-zero, we must have $r + k = 1 \pmod 2$. Now suppose $k = -1 \pmod p$ then we can assume $r \neq -1 \pmod p$ and so using (6) with $z = r$ we have

$$\begin{aligned} \operatorname{Hom}_G\big(\nabla(r), \nabla(1) \otimes \nabla(k)\big) &\cong \operatorname{Hom}_G\big(\nabla(1) \otimes \nabla(r), \nabla(k)\big) \\ &\cong \operatorname{Hom}_G\big(\nabla(r - 1) \oplus \nabla(r + 1), \nabla(k)\big). \end{aligned}$$

Now, using (4) we deduce that $\operatorname{Hom}_G(\nabla(r - 1), \nabla(k)) \cong K$ if and only if $r - 1 = k + 2d$ where $d \in \psi^0(k)$ and it is zero otherwise, and $\operatorname{Hom}_G(\nabla(r + 1), \nabla(k)) \cong K$ if and only if $r + 1 = k + 2d'$ where $d' \in \psi^0(k)$ and zero otherwise. Suppose they are both non-zero then $k + 1 + 2d = k - 1 + 2d'$. But this can only happen when

$d = 0$, $d' = 1$, and $r = k + 1$. This means that $k = p - 2 \pmod p$ and $r = -1 \pmod p$ contradicting our assumption. Now if $k \neq -1 \pmod p$ we use (6) with $z = k$ and the claim follows by a similar argument.

Hence, we have proved the following:

$$H^1(G, V)^G \cong \begin{cases} K & \text{if } s_0 + t_0 = p - 2,\ s_1 + t_1 = p^{n-1} - 1, \\ & r = k \pm 1 + 2d \text{ where } d \in \psi^0(k), \\ 0 & \text{otherwise.} \end{cases}$$

Let us now use the five term sequence to determine $H^1(G, V)$. We shall do this by induction on n. For $n = 1$ we have $s, t \leqslant p - 1$ and

$$H^1\bigl(G, \bigl(V^{G_1}\bigr)^{(-1)}\bigr) \cong \begin{cases} K & \text{if } r = k + 2e,\ e \in \psi^1(k), \text{ and } s = t, \\ 0 & \text{otherwise} \end{cases}$$

and

$$H^1(G_1, V)^G \cong \begin{cases} K & \text{if } r = k \pm 1 + 2d,\ d \in \psi^0(k), \text{ and } s + t = p - 2, \\ 0 & \text{otherwise;} \end{cases}$$

thus,

$$H^1(G, V) \cong \begin{cases} K & \text{if } r = k + 2e,\ e \in \psi^1(k), \text{ and } s = t, \\ & \text{or } r = k \pm 1 + 2d,\ d \in \psi^0(k), \text{ and } s + t = p - 2, \\ 0 & \text{otherwise.} \end{cases}$$

Now we use induction. Note that if $p = 2$ and $s_0 = t_0 = 0$ and $s_1 + t_1 = 2^{n-1} - 1$ then $\Delta(s_1)$ and $\nabla(t_1)$ are in different blocks of G_1 and so

$$\mathrm{Ext}^i_G\bigl(\nabla(r)^{F^{n-1}} \otimes \Delta(s_1), \nabla(k)^{F^{n-1}} \otimes \nabla(t_1)\bigr) = 0 \quad \text{for all } i.$$

So for all prime p we get

$$H^1(G, V) \cong \begin{cases} K & \text{if } r = k + 2e,\ e \in \psi^1(k), \text{ and } s = t, \\ & \text{or } r = k \pm 1 + 2d,\ d \in \psi^0(k),\ t = t_0 + t_1 p^i, \\ & 0 \leqslant t_0 \leqslant p^i - 1,\ s = t_0 + (p^{n-i} - 2 - t_1)p^i, \\ 0 & \text{otherwise.} \end{cases}$$

This completes the proof of our theorem. □

Note that if we set $n = 0$ and $s = t = 0$ in Theorem 2.1 we get Erdmann and Cox's result given by Eq. (5).

The following proposition shows that when $r = k - 1$ and $s = p^n - 2 - t$, the extension is given by $\nabla(kp^n + t)$. By considering weights, it is easy to see that no other extension described in Theorem 2.1 can be isomorphic to an induced module $\nabla(\lambda)$.

Proposition 2.1. *For $k \in N$ and $0 \leqslant t \leqslant p^n - 2$, there is an exact sequence of G-modules*

$$0 \to \nabla(k)^{F^n} \oplus \nabla(t) \to \nabla\bigl(kp^n + t\bigr) \to \nabla(k-1)^{F^n} \otimes \Delta\bigl(p^n - 2 - t\bigr) \to 0.$$

Moreover, $\nabla(kp^n + t)$ *is the only non-split extension, up to isomorphism, of* $\nabla(k-1)^{F^n} \otimes \Delta(p^n - t - 2)$ *by* $\nabla(k)^{F^n} \otimes \nabla(t)$.

Dually, the only non-split extension, up to isomorphism, of $\Delta(k)^{F^n} \otimes \Delta(t)$ *by* $\Delta(k-1)^{F^n} \otimes \nabla(p^n - t - 2)$ *is given by* $\Delta(kp^n + t)$.

Remark 1. For $k \in N$ we have an isomorphism between $\nabla(k-1)^{F^n} \otimes St_n$ and $\nabla(kp^n - 1)$ given by multiplication of polynomials. It is known that there is an isomorphism between these modules more generally, see, for example, [10, (II.3)].

Proof of Proposition 2.1. If $n = 1$ then we are done by (1) (Section 1). Suppose $n > 1$ and write $t = ap^{n-1} + d$ for $0 \leqslant a \leqslant p-1$ and $0 \leqslant d \leqslant p^{n-1} - 1$. Using induction we have an exact sequence

$$\begin{aligned} 0 \;\to\; &\nabla(kp+a)^{F^{n-1}} \otimes \nabla(d) \to \nabla\big((kp+a)p^{n-1} + d\big) \\ &\to\; \nabla\big(kp + (a-1)\big)^{F^{n-1}} \otimes \Delta\big(p^{n-1} - d - 2\big) \to 0. \end{aligned}$$

Using the exact sequences (1) for $\nabla(kp+a)^{F^{n-1}}$ and $\nabla(kp+(a-1))^{F^{n-1}}$ we get a filtration of $\nabla(kp^n + ap^{n-1} + d)$ with quotients

$$\begin{aligned} &\nabla(k-1)^{F^n} \otimes \Delta(p-a-1)^{F^{n-1}} \otimes \Delta\big(p^{n-1} - d - 2\big), \\ &\nabla(k)^{F^n} \otimes \nabla(a-1)^{F^{n-1}} \otimes \Delta\big(p^{n-1} - d - 2\big), \\ &\nabla(k-1)^{F^n} \otimes \Delta(p-a-2)^{F^{n-1}} \otimes \nabla(d), \\ &\nabla(k)^{F^n} \otimes \nabla(a)^{F^{n-1}} \otimes \nabla(d). \end{aligned}$$

Observe that the module $\nabla(kp^n + ap^{n-1} + d)$ is multiplicity-free, so that the four quotients have disjoint sets of weights. Hence, $\nabla(kp^n + t)/\nabla(k)^{F^n} \otimes \nabla(t)$ has a filtration with quotients

$$\begin{aligned} &\nabla(k-1)^{F^n} \otimes \Delta(p-a-1)^{F^{n-1}} \otimes \Delta\big(p^{n-1} - d - 2\big), \\ &\nabla(k-1)^{F^n} \otimes \Delta(p-a-2)^{F^{n-1}} \otimes \nabla(d). \end{aligned}$$

Note that for $a = p-1$ or $d = p^{n-1} - 1$, we only have one factor appearing and so we are done by Remark 1 above. So suppose $a \leqslant p-2$ and $d < p^{n-1} - 2$. Using a very similar argument to the proof of Theorem 2.1 we can show that

$$\begin{aligned} &\mathrm{Ext}^1_G\big(\nabla(k-1)^{F^n} \otimes \Delta(p-a-1)^{F^{n-1}} \otimes \Delta\big(p^{n-1} - d - 2\big), \\ &\qquad \nabla(k-1)^{F^n} \otimes \Delta(p-a-2)^{F^{n-1}} \otimes \nabla(d) \cong K. \end{aligned}$$

Now as $\nabla(kp^n + t)$ has simple top (see [1]), $\nabla(kp^n + t)/\nabla(k)^{F^n} \otimes \nabla(t)$ cannot be a direct sum of non-zero modules. By induction, we know that $\Delta(p^n - ap^{n-1} - d - 2)$ has a filtration with quotients

$$\Delta(p-a-1)^{F^{n-1}} \otimes \Delta\big(p^{n-1}-d-2\big),$$

$$\Delta(p-a-2)^{F^{n-1}} \otimes \nabla(d).$$

We deduce that the quotient $\nabla(kp^n+t)/\nabla(k)^{F^n} \otimes \nabla(t)$ is isomorphic to

$$\nabla(k-1)^{F^n} \otimes \Delta\big(p^n-ap^{n-1}-d-2\big) = \nabla(k-1)^{F^n} \otimes \Delta\big(p^n-2-t\big).$$

This completes the proof. □

Remark 2. S. Donkin suggested an alternative proof of Proposition 2.1. I shall sketch his argument here. Let us start with the exact sequence of B-modules

$$0 \to \nabla(s-1) \otimes K_{-1} \to \nabla(s) \to K_s \to 0 \tag{7}$$

for any positive integer s. Apply the Frobenius morphism F^n to the sequence (7) and tensor it with K_r for some $0 \leqslant r \leqslant p^n-1$. Then applying the induction functor from B-modules to G-modules and using the duality of induction (see [10, II.4]) gives the required sequence.

Remark 3. The composition factors of the ∇'s are known for $SL(2,K)$ (use, for example, Eq. (1) repeatedly) but Proposition 2.1 gives a direct explanation of the symmetries observed by A. Henke in the decomposition matrix of $SL(2,K)$ (see [9]). More precisely, if we write $\lambda = kp^n + t$ with $k \leqslant p-1$ then our proposition tells us that

$$\big[\nabla\big(kp^n+t\big) : L\big(kp^n+a\big)\big] = \big[\nabla(t) : L(a)\big],$$
$$\big[\nabla\big(kp^n+t\big) : L\big((k-1)p^n+b\big)\big] = \big[\nabla\big(p^n-2-t\big) : L(b)\big].$$

Let us write the decomposition matrix of G with the ∇'s on the horizontal axis and the L's on the vertical axis (see Figs. 1 and 2). Then for each $n \geqslant 1$ and each $1 \leqslant k \leqslant p-1$, the columns corresponding to $\nabla(kp^n+1)$ for $0 \leqslant t \leqslant p^n-1$ are obtained from the left bottom $p^n \times p^n$ block by

(1) translation of length k along the diagonal;
(2) translation of length $k-1$ along the diagonal and then reflection through the column corresponding to $\nabla(kp^n-1)$.

Hence, we can construct the decomposition matrix inductively starting with the left bottom $p \times p$ block which is just a diagonal matrix, as for $0 \leqslant r \leqslant p-1$ we have $\nabla(r) = L(r)$.

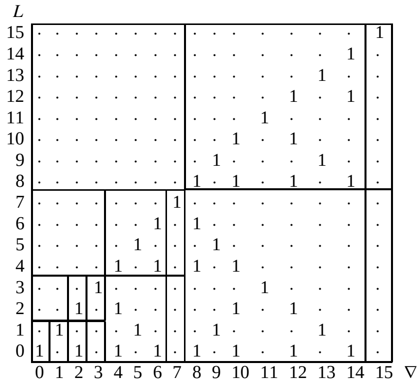

Fig. 1. Decomposition matrix for $p = 2$.

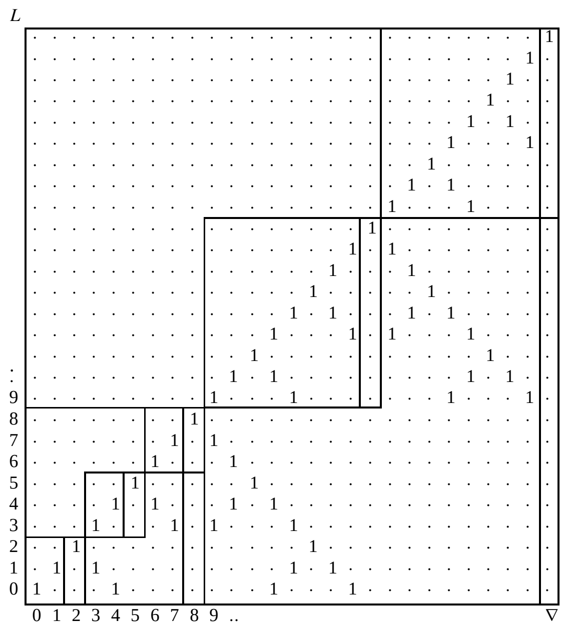

Fig. 2. Decomposition matrix for $p = 3$.

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