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Vertex operator algebras associated to type B affine Lie algebras on admissible half-integer levels ☆

Ozren Perše

Department of Mathematics, University of Zagreb, Bijenička 30, 10 000 Zagreb, Croatia

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Abstract

Let $L(n-l+\frac{1}{2},0)$ be the vertex operator algebra associated to an affine Lie algebra of type $B_l^{(1)}$ at level $n-l+\frac{1}{2}$, for a positive integer n. We classify irreducible $L(n-l+\frac{1}{2},0)$-modules and show that every $L(n-l+\frac{1}{2},0)$-module is completely reducible. In the special case $n=1$, we study a category of weak $L(-l+\frac{3}{2},0)$-modules which are in the category $\mathcal{O}$ as modules for the associated affine Lie algebra. We classify irreducible objects in that category and prove semisimplicity of that category.

Keywords: Vertex operator algebras; Affine Kac–Moody algebras; Admissible modules

1. Introduction

Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra and $\hat{\mathfrak{g}}$ the associated affine Lie algebra. For any complex number k, denote by $L(k,0)$ the irreducible highest weight $\hat{\mathfrak{g}}$-module with the highest weight $k\Lambda_0$. Then $L(k,0)$ has a natural vertex operator algebra structure for any $k \neq -h^\vee$. The representation theory of $L(k,0)$ heavily depends on the choice of level $k \in \mathbb{C}$. If k is a positive integer, $L(k,0)$ is a rational vertex operator algebra (cf. [10,21]), i.e. the category of $\mathbb{Z}_+$-graded weak $L(k,0)$-modules is semisimple. Irreducible objects in that category are integrable highest weight $\hat{\mathfrak{g}}$-modules of level k (cf. [10,18]). The corresponding associative algebra $A(L(k,0))$, defined in [21], is finite-dimensional (cf. [17]). In some cases such as $k \notin \mathbb{Q}$ or

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E-mail address: perse@math.hr.

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$k < -h^\vee$ (studied in [13,14]), categories of $L(k,0)$-modules have significantly different structure then categories of $L(k,0)$-modules for a positive integer k. But there are examples of rational levels k such that the category of weak $L(k,0)$-modules which are in the category $\mathcal{O}$ as $\hat{\mathfrak{g}}$-modules, has similar structure as the category of $\mathbb{Z}_+$-graded weak $L(k,0)$-modules for positive integer levels k. These are so-called admissible levels, defined by V. Kac and M. Wakimoto (cf. [15,16]). D. Adamović (cf. [1,2]) studied vertex operator algebras associated to affine Lie algebras of type $C_l^{(1)}$ on admissible half-integer levels. D. Adamović and A. Milas [4], and C. Dong, H. Li and G. Mason [6] studied vertex operator algebras associated to affine Lie algebras of type $A_1^{(1)}$ on all admissible levels. It is shown that in these cases vertex operator algebra $L(k,0)$ has finitely many irreducible weak modules from the category $\mathcal{O}$ and that every weak $L(k,0)$-module from the category $\mathcal{O}$ is completely reducible. One can say that these vertex operator algebras are rational in the category $\mathcal{O}$. In [4], authors gave a conjecture that vertex operator algebras $L(k,0)$, for all admissible levels k, are rational in the category $\mathcal{O}$. In this paper we give examples of a vertex operator algebras $L(k,0)$ on admissible levels k for which we prove some parts of the conjecture from [4]. Admissible modules for affine Lie algebras were also recently studied in [3,9,11,20].

We consider the case of an affine Lie algebra of type $B_l^{(1)}$ and the corresponding vertex operator algebra $L(n-l+\frac{1}{2},0)$, for any positive integer n. We show that $n-l+\frac{1}{2}$ is an admissible level for this affine Lie algebra. The results on admissible modules from [15] imply that $L(n-l+\frac{1}{2},0)$ is a quotient of the generalized Verma module by the maximal ideal generated by a singular vector. Using results from [10,21], we can identify the corresponding associative algebra $A(L(n-l+\frac{1}{2},0))$ with a certain quotient of $U(\mathfrak{g})$. Algebra $A(L(n-l+\frac{1}{2},0))$ is infinite-dimensional in this case. Using methods from [2,19] we get that irreducible $A(L(n-l+\frac{1}{2},0))$-modules from the category $\mathcal{O}$ are in one-to-one correspondence with zeros of the certain set of polynomials $\mathcal{P}_0$. By calculating certain polynomials from that set we obtain the classification of irreducible finite-dimensional $A(L(n-l+\frac{1}{2},0))$-modules, and in the special case $n=1$, the classification of irreducible $A(L(-l+\frac{3}{2},0))$-modules from the category $\mathcal{O}$. Using results from [21], we obtain the classification of irreducible $L(n-l+\frac{1}{2},0)$-modules, and in the special case $n=1$, the classification of irreducible weak $L(-l+\frac{3}{2},0)$-modules from the category $\mathcal{O}$. Using these classifications and results from [16], we show that every $L(n-l+\frac{1}{2},0)$-module is completely reducible, and in the case $n=1$, that every weak $L(-l+\frac{3}{2},0)$-module from the category $\mathcal{O}$ is completely reducible.

The method for classification of irreducible $L(k,0)$-modules used in this paper depends on a relatively simple formula for the singular vector in the generalized Verma module. For a general admissible level k, a more global method for classification is needed.

2. Vertex operator algebras associated to affine Lie algebras

This section is preliminary. We recall some necessary definitions and fix the notation. We review certain results about vertex operator algebras and corresponding modules. The emphasis is on the class of vertex operator algebras associated to affine Lie algebras, because we study a special case in that class in Sections 3 and 4.

2.1. Vertex operator algebras and modules

Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra (cf. [5,7,8]). An *ideal* in a vertex operator algebra V is a subspace I of V satisfying $Y(a,z)I \subseteq I[[z,z^{-1}]]$ for any $a \in V$. Given an ideal I in V, such that $\mathbf{1} \notin I$, $\omega \notin I$, the quotient V/I admits a natural vertex operator algebra structure.

Let (M, Y_M) be a weak module for a vertex operator algebra V (cf. [18]). A $\mathbb{Z}_+$-*graded weak* V-module (cf. [10]) is a weak V-module M together with a $\mathbb{Z}_+$-gradation $M = \bigoplus_{n=0}^{\infty} M(n)$ such that

$$a_m M(n) \subseteq M(n + r - m - 1) \quad \text{for } a \in V_{(r)},\ m, n, r \in \mathbb{Z},$$

where $M(n) = 0$ for $n < 0$ by definition.

A weak V-module M is called a V-*module* if $L(0)$ acts semisimply on M with the decomposition into $L(0)$-eigenspaces $M = \bigoplus_{\alpha \in \mathbb{C}} M_{(\alpha)}$ such that for any $\alpha \in \mathbb{C}$, $\dim M_{(\alpha)} < \infty$ and $M_{(\alpha+n)} = 0$ for $n \in \mathbb{Z}$ sufficiently small.

2.2. *Zhu's $A(V)$ theory*

Let V be a vertex operator algebra. Following [21], we define bilinear maps $*: V \times V \to V$ and $\circ: V \times V \to V$ as follows. For any homogeneous $a \in V$ and for any $b \in V$, let

$$a \circ b = \operatorname{Res}_z \frac{(1+z)^{\operatorname{wt} a}}{z^2} Y(a, z) b,$$
$$a * b = \operatorname{Res}_z \frac{(1+z)^{\operatorname{wt} a}}{z} Y(a, z) b$$

and extend to $V \times V \to V$ by linearity. Denote by $O(V)$ the linear span of elements of the form $a \circ b$, and by $A(V)$ the quotient space $V/O(V)$. For $a \in V$, denote by $[a]$ the image of a under the projection of V onto $A(V)$. The multiplication $*$ induces the multiplication on $A(V)$ and $A(V)$ has a structure of an associative algebra.

Proposition 1. [10, Proposition 1.4.2] *Let I be an ideal of V. Assume $\mathbf{1} \notin I$, $\omega \notin I$. Then the associative algebra $A(V/I)$ is isomorphic to $A(V)/A(I)$, where $A(I)$ is the image of I in $A(V)$.*

For any homogeneous $a \in V$ we define $o(a) = a_{\operatorname{wt} a - 1}$ and extend this map linearly to V.

Proposition 2. [21, Theorems 2.1.2 and 2.2.1]

(a) *Let $M = \bigoplus_{n=0}^{\infty} M(n)$ be a $\mathbb{Z}_+$-graded weak V-module. Then $M(0)$ is an $A(V)$-module defined as follows*:

$$[a].v = o(a)v,$$

for any $a \in V$ and $v \in M(0)$.

(b) *Let U be an $A(V)$-module. Then there exists a $\mathbb{Z}_+$-graded weak V-module M such that the $A(V)$-modules $M(0)$ and U are isomorphic.*

Proposition 3. [21, Theorem 2.2.2] *The equivalence classes of the irreducible $A(V)$-modules and the equivalence classes of the irreducible $\mathbb{Z}_+$-graded weak V-modules are in one-to-one correspondence.*

2.3. Modules for affine Lie algebras

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ with a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. Let Δ be the root system of $(\mathfrak{g}, \mathfrak{h})$, $\Delta_+ \subset \Delta$ the set of positive roots, θ the highest root and $(\cdot,\cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ the Killing form, normalized by the condition $(\theta, \theta) = 2$. Denote by $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ the set of simple roots of $\mathfrak{g}$, and by $\Pi^\vee = \{h_1, \ldots, h_l\}$ the set of simple coroots of $\mathfrak{g}$.

The affine Lie algebra $\hat{\mathfrak{g}}$ associated to $\mathfrak{g}$ is the vector space $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$ equipped with the usual bracket operation and the canonical central element c (cf. [12]). Let $h^\vee$ be the dual Coxeter number of $\hat{\mathfrak{g}}$. Let $\hat{\mathfrak{g}} = \hat{\mathfrak{n}}_- \oplus \hat{\mathfrak{h}} \oplus \hat{\mathfrak{n}}_+$ be the corresponding triangular decomposition of $\hat{\mathfrak{g}}$.

Denote by $\hat{\Delta}$ the set of roots of $\hat{\mathfrak{g}}$, by $\hat{\Delta}_+$ the set of positive roots of $\hat{\mathfrak{g}}$, and by $\hat{\Pi}$ the set of simple roots of $\hat{\mathfrak{g}}$. Denote by $\hat{\Delta}^{\mathrm{re}}$ the set of real roots of $\hat{\mathfrak{g}}$ and let $\hat{\Delta}^{\mathrm{re}}_+ = \hat{\Delta}^{\mathrm{re}} \cap \hat{\Delta}_+$. With $\alpha^\vee$ denote the coroot of a real root $\alpha \in \hat{\Delta}^{\mathrm{re}}$. For any $\lambda \in \hat{\mathfrak{h}}^*$ set $D(\lambda) = \{\lambda - \sum_{\alpha \in \hat{\Pi}} n_\alpha \alpha \mid n_\alpha \in \mathbb{Z}_+ \ \forall \alpha \in \hat{\Pi}\}$.

We say that a $\hat{\mathfrak{g}}$-module M is from the *category* $\mathcal{O}$ (cf. [12]) if Cartan subalgebra $\hat{\mathfrak{h}}$ acts semi-simply on M with finite-dimensional weight spaces and there exists a finite number of elements $\nu_1, \ldots, \nu_k \in \hat{\mathfrak{h}}^*$ such that for every weight ν of M holds $\nu \in \bigcup_{i=1}^k D(\nu_i)$.

For every weight $\lambda \in \hat{\mathfrak{h}}^*$, denote by $M(\lambda)$ the Verma module for $\hat{\mathfrak{g}}$ with highest weight λ, and by $L(\lambda)$ the irreducible $\hat{\mathfrak{g}}$-module with highest weight λ.

Let U be a $\mathfrak{g}$-module, and let $k \in \mathbb{C}$. Let $\hat{\mathfrak{g}}_+ = \mathfrak{g} \otimes t\mathbb{C}[t]$ act trivially on U and c as the scalar multiplication operator k. Considering U as a $\mathfrak{g} \oplus \mathbb{C}c \oplus \hat{\mathfrak{g}}_+$-module, we have the induced $\hat{\mathfrak{g}}$-module (so-called *generalized Verma module*)

$$N(k, U) = U(\hat{\mathfrak{g}}) \otimes_{U(\mathfrak{g} \oplus \mathbb{C}c \oplus \hat{\mathfrak{g}}_+)} U.$$

For a fixed $\mu \in \mathfrak{h}^*$, denote by $V(\mu)$ the irreducible highest-weight $\mathfrak{g}$-module with highest weight μ. Denote by P_+ the set of dominant integral weights of $\mathfrak{g}$, i.e. $P_+ = \{\mu \in \mathfrak{h}^* \mid \mu(h_i) \in \mathbb{Z}_+, \text{ for } i = 1, \ldots, l\}$. Denote by $\omega_1, \ldots, \omega_l \in P_+$ the fundamental weights of $\mathfrak{g}$, defined by $\omega_i(h_j) = \delta_{ij}$ for all $i, j = 1, \ldots, l$.

We shall use the notation $N(k, \mu)$ to denote the $\hat{\mathfrak{g}}$-module $N(k, V(\mu))$. Denote by $J(k, \mu)$ the maximal proper submodule of $N(k, \mu)$ and $L(k, \mu) = N(k, \mu)/J(k, \mu)$.

Denote by $\Lambda_0 \in \hat{\mathfrak{h}}^*$ the weight defined by $\Lambda_0(c) = 1$ and $\Lambda_0(h) = 0$ for any $h \in \mathfrak{h}$. $N(k, \mu)$ is a highest-weight module with highest weight $k\Lambda_0 + \mu$, and a quotient of the Verma module $M(k\Lambda_0 + \mu)$. $L(k, \mu)$ is the unique irreducible highest-weight module with highest weight $k\Lambda_0 + \mu$, i.e. $L(k, \mu) \cong L(k\Lambda_0 + \mu)$.

2.4. Admissible modules for affine Lie algebras

Let $\hat{\Delta}^{\vee\mathrm{re}}$ (respectively $\hat{\Delta}^{\vee\mathrm{re}}_+$) $\subset \hat{\mathfrak{h}}$ be the set of real (respectively positive real) coroots of $\hat{\mathfrak{g}}$. Fix $\lambda \in \hat{\mathfrak{h}}^*$. Let $\hat{\Delta}^{\vee\mathrm{re}}_\lambda = \{\alpha \in \hat{\Delta}^{\vee\mathrm{re}} \mid \langle \lambda, \alpha \rangle \in \mathbb{Z}\}$, $\hat{\Delta}^{\vee\mathrm{re}}_{\lambda+} = \hat{\Delta}^{\vee\mathrm{re}}_\lambda \cap \hat{\Delta}^{\vee\mathrm{re}}_+$, $\hat{\Pi}^\vee$ the set of simple coroots in $\hat{\Delta}^{\vee\mathrm{re}}$ and $\hat{\Pi}^\vee_\lambda = \{\alpha \in \hat{\Delta}^{\vee\mathrm{re}}_{\lambda+} \mid \alpha \text{ not equal to a sum of several coroots from } \hat{\Delta}^{\vee\mathrm{re}}_{\lambda+}\}$. Define ρ in the usual way, and denote by $w.\lambda$ the "shifted" action of an element w of the Weyl group of $\hat{\mathfrak{g}}$.

Recall that a weight $\lambda \in \hat{\mathfrak{h}}^*$ is called *admissible* (cf. [15,16,20]) if the following properties are satisfied:

$$\langle \lambda + \rho, \alpha \rangle \notin -\mathbb{Z}_+ \quad \text{for all } \alpha \in \hat{\Delta}^{\vee\mathrm{re}}_+,$$

$$\mathbb{Q}\hat{\Delta}_{\lambda}^{\vee\mathrm{re}} = \mathbb{Q}\hat{\Pi}^{\vee}.$$

The irreducible $\hat{\mathfrak{g}}$-module $L(\lambda)$ is called admissible if the weight $\lambda \in \hat{\mathfrak{h}}^*$ is admissible.

We shall use the following results of V. Kac and M. Wakimoto:

Proposition 4. [15, Corollary 2.1] *Let λ be an admissible weight. Then*

$$L(\lambda) = \frac{M(\lambda)}{\sum_{\alpha \in \hat{\Pi}_{\lambda}^{\vee}} U(\hat{\mathfrak{g}})v^{\alpha}},$$

where $v^{\alpha} \in M(\lambda)$ is a singular vector of weight $r_{\alpha}.\lambda$, the highest weight vector of $M(r_{\alpha}.\lambda) = U(\hat{\mathfrak{g}})v^{\alpha} \subset M(\lambda)$.

Proposition 5. [16, Theorem 4.1] *Let M be a $\hat{\mathfrak{g}}$-module from the category $\mathcal{O}$ such that for any irreducible subquotient $L(\nu)$ the weight ν is admissible. Then M is completely reducible.*

2.5. Vertex operator algebras $N(k,0)$ and $L(k,0)$, for $k \neq -h^{\vee}$

Since $V(0)$ is the one-dimensional trivial $\mathfrak{g}$-module, it can be identified with $\mathbb{C}$. Denote by $\mathbf{1} = 1 \otimes 1 \in N(k,0)$. We note that $N(k,0)$ is spanned by the elements of the form $x_1(-n_1-1)\cdots x_m(-n_m-1)\mathbf{1}$, where $x_1,\dots,x_m \in \mathfrak{g}$ and $n_1,\dots,n_m \in \mathbb{Z}_+$, with $x(n)$ denoting the representation image of $x \otimes t^n$ for $x \in \mathfrak{g}$ and $n \in \mathbb{Z}$. Vertex operator map $Y(\cdot,z): N(k,0) \to (\operatorname{End} N(k,0))[[z,z^{-1}]]$ is uniquely determined by defining $Y(\mathbf{1},z)$ to be the identity operator on $N(k,0)$ and

$$Y\big(x(-1)\mathbf{1},z\big) = \sum_{n\in\mathbb{Z}} x(n)z^{-n-1},$$

for $x \in \mathfrak{g}$. In the case that $k \neq -h^{\vee}$, $N(k,0)$ has a Virasoro element

$$\omega = \frac{1}{2(k+h^{\vee})}\sum_{i=1}^{\dim\mathfrak{g}} x^i(-1)^2\mathbf{1},$$

where $\{x^i\}_{i=1,\dots,\dim\mathfrak{g}}$ is an arbitrary orthonormal basis of $\mathfrak{g}$ with respect to the form $(\cdot,\cdot)$. In [10], the following results are proved:

Proposition 6. [10, Theorem 2.4.1] *If $k \neq -h^{\vee}$, the quadruple $(N(k,0), Y, \mathbf{1}, \omega)$ defined above is a vertex operator algebra.*

Proposition 7. [10, Theorem 3.1.1] *The associative algebra $A(N(k,0))$ is canonically isomorphic to $U(\mathfrak{g})$. The isomorphism is given by $F: A(N(k,0)) \to U(\mathfrak{g})$*

$$F\big(\big[x_1(-n_1-1)\cdots x_m(-n_m-1)\mathbf{1}\big]\big) = (-1)^{n_1+\cdots+n_m}x_m\cdots x_1,$$

for any $x_1,\dots,x_m \in \mathfrak{g}$ and any $n_1,\dots,n_m \in \mathbb{Z}_+$.

Since every $\hat{\mathfrak{g}}$-submodule of $N(k,0)$ is also an ideal in the vertex operator algebra $N(k,0)$, it follows that $L(k,0)$ is a vertex operator algebra, for any $k \neq -h^\vee$. The associative algebra $A(L(k,0))$ is identified in the next proposition, in the case when the maximal $\hat{\mathfrak{g}}$-submodule of $N(k,0)$ is generated by one singular vector.

Proposition 8. *Assume that the maximal $\hat{\mathfrak{g}}$-submodule of $N(k,0)$ is generated by a singular vector, i.e. $J(k,0) = U(\hat{\mathfrak{g}})v_{sing}$. Then*

$$A\big(L(k,0)\big) \cong \frac{U(\mathfrak{g})}{\langle Q\rangle},$$

where $\langle Q\rangle$ is the two-sided ideal of $U(\mathfrak{g})$ generated by $Q = F([v_{sing}])$.

Let U be a $\mathfrak{g}$-module. Then U is an $A(L(k,0))$-module if and only if $QU = 0$.

3. Modules for vertex operator algebra $L(n-l+\frac{1}{2},0)$ associated to affine Lie algebra of type $B_l^{(1)}$

Let $\mathfrak{g}$ be the simple Lie algebra of type B_l, and $\hat{\mathfrak{g}}$ the affine Lie algebra associated to $\mathfrak{g}$. In this section we study the vertex operator algebra $L(n-l+\frac{1}{2},0)$ associated to $\hat{\mathfrak{g}}$, for a positive integer n. Using results from [10,21], we identify the corresponding associative algebra $A(L(n-l+\frac{1}{2},0))$ with a quotient of $U(\mathfrak{g})$ by the ideal generated by a certain vector. Using methods from [2,19], we show that the highest weights $\mu \in P_+$ of irreducible finite-dimensional $A(L(n-l+\frac{1}{2},0))$-modules are characterized by the condition

$$(\mu,\epsilon_1) \leqslant n-\frac{1}{2},$$

where ϵ_1 is the maximal short root for $\mathfrak{g}$. Using Zhu's theory we obtain the classification of irreducible $L(n-l+\frac{1}{2},0)$-modules. From this classification and results from [16], we obtain the semisimplicity of the category of $L(n-l+\frac{1}{2},0)$-modules. Using similar techniques, we also show that there are finitely many irreducible weak $L(n-l+\frac{1}{2},0)$-modules from the category $\mathcal{O}$.

3.1. Simple Lie algebra of type B_l

Let $\Delta = \{\pm\epsilon_i \mid i=1,\dots,l\} \cup \{\pm(\epsilon_i \pm \epsilon_j) \mid i,j=1,\dots,l,\ i\neq j\}$ be the root system of type B_l. Fix the set of positive roots $\Delta_+ = \{\epsilon_i \mid i=1,\dots,l\} \cup \{\epsilon_i-\epsilon_j \mid i<j\} \cup \{\epsilon_i+\epsilon_j \mid i\neq j\}$. Then the simple roots are $\alpha_1=\epsilon_1-\epsilon_2, \alpha_2=\epsilon_2-\epsilon_3,\dots,\alpha_{l-1}=\epsilon_{l-1}-\epsilon_l, \alpha_l=\epsilon_l$. The highest root is $\theta=\epsilon_1+\epsilon_2=\alpha_1+2\alpha_2+2\alpha_3+\cdots+2\alpha_l$.

Let $\mathfrak{g}$ be the simple Lie algebra associated to the root system of type B_l. Let e_i, f_i, h_i, $i=1,\dots,l$ be the Chevalley generators of $\mathfrak{g}$. Fix the root vectors:

$$\begin{aligned}
e_{\epsilon_i-\epsilon_j} &= \big[e_i,\big[e_{i+1},\big[\dots[e_{j-2},e_{j-1}]\dots\big]\big]\big], \quad i<j,\\
e_{\epsilon_i} &= \big[e_i,\big[e_{i+1},\big[\dots[e_{l-1},e_l]\dots\big]\big]\big],\\
e_{\epsilon_i+\epsilon_j} &= \frac{1}{2}[e_{\epsilon_i},e_{\epsilon_j}], \quad i<j,
\end{aligned}$$

$$f_{\epsilon_i-\epsilon_j}=\big[f_{j-1},\big[f_{j-2},\big[\ldots[f_{i+1},f_i]\ldots\big]\big]\big],\quad i<j,$$
$$f_{\epsilon_i}=\big[f_l,\big[f_{l-1},\big[\ldots[f_{i+1},f_i]\ldots\big]\big]\big],$$
$$f_{\epsilon_i+\epsilon_j}=\frac{1}{2}[f_{\epsilon_j},f_{\epsilon_i}],\quad i<j.$$

Denote by $h_\alpha=\alpha^\vee=[e_\alpha,f_\alpha]$ coroots, for any positive root $\alpha\in\Delta_+$. It is clear that $h_{\alpha_i}=h_i$. Let $\mathfrak{g}=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+$ be the corresponding triangular decomposition of $\mathfrak{g}$.

3.2. *Vertex operator algebra $L(n-l+\frac{1}{2},0)$ associated to affine Lie algebra of type $B_l^{(1)}$*

Let $\hat{\mathfrak{g}}$ be the affine Lie algebra associated to simple Lie algebra of type B_l. We want to show that the maximal $\hat{\mathfrak{g}}$-submodule of $N(n-l+\frac{1}{2},0)$ for $n\in\mathbb{N}$ is generated by a singular vector. We need two lemmas to prove that.

Denote by λ_n the weight $(n-l+\frac{1}{2})\Lambda_0$ for $n\in\mathbb{N}$. Then $N(n-l+\frac{1}{2},0)$ is a quotient of $M(\lambda_n)$ and $L(n-l+\frac{1}{2},0)\cong L(\lambda_n)$.

Lemma 9. *The weight $\lambda_n=(n-l+\frac{1}{2})\Lambda_0$ is admissible and*

$$\hat{\Pi}^\vee_{\lambda_n}=\big\{(\delta-\epsilon_1)^\vee,\alpha_1^\vee,\alpha_2^\vee,\ldots,\alpha_l^\vee\big\},$$

for any $n\in\mathbb{N}$. Furthermore

$$\big\langle\lambda_n+\rho,\alpha_i^\vee\big\rangle=1\quad\textit{for } i=1,\ldots,l,$$
$$\big\langle\lambda_n+\rho,(\delta-\epsilon_1)^\vee\big\rangle=2n.$$

Proof. We have to show

$$\big\langle\lambda_n+\rho,\tilde{\alpha}^\vee\big\rangle\notin-\mathbb{Z}_+\quad\text{for any }\tilde{\alpha}\in\hat{\Delta}_+^{\mathrm{re}},$$
$$\mathbb{Q}\hat{\Delta}^{\vee\mathrm{re}}_{\lambda_n}=\mathbb{Q}\hat{\Pi}^\vee.$$

Any positive real root $\tilde{\alpha}\in\hat{\Delta}_+^{\mathrm{re}}$ of $\hat{\mathfrak{g}}$ is of the form $\tilde{\alpha}=\alpha+m\delta$, for $m>0$ and $\alpha\in\Delta$ or $m=0$ and $\alpha\in\Delta_+$. Denote by $\bar{\rho}$ the sum of fundamental weights of $\mathfrak{g}$. Then $\rho=h^\vee\Lambda_0+\bar{\rho}=(2l-1)\Lambda_0+\bar{\rho}$, and $\lambda_n+\rho=(n+l-\frac{1}{2})\Lambda_0+\bar{\rho}$. We have

$$\begin{aligned}\big\langle\lambda_n+\rho,\tilde{\alpha}^\vee\big\rangle&=\left\langle\left(n+l-\frac{1}{2}\right)\Lambda_0+\bar{\rho},(\alpha+m\delta)^\vee\right\rangle\\&=\frac{2}{(\alpha,\alpha)}\left(m\left(n+l-\frac{1}{2}\right)+(\bar{\rho},\alpha)\right).\end{aligned}$$

If $m=0$ and $\alpha\in\Delta_+$, then $(\bar{\rho},\alpha)>0$, and it is clear that $\langle\lambda_n+\rho,\tilde{\alpha}^\vee\rangle\notin-\mathbb{Z}_+$.

Let $m\geqslant1$. We have two cases: $(\alpha,\alpha)=2$ and $(\alpha,\alpha)=1$.

If $(\alpha,\alpha)=2$ and m is odd, then $\langle\lambda_n+\rho,\tilde{\alpha}^\vee\rangle\notin\mathbb{Z}$, which implies $\langle\lambda_n+\rho,\tilde{\alpha}^\vee\rangle\notin-\mathbb{Z}_+$.

If $(\alpha,\alpha)=2$ and m is even, then $m\geqslant 2$, and since $(\bar\rho,\alpha)\geqslant -(2l-1)$ for any $\alpha\in\Delta$, we have

$$\left\langle\lambda_n+\rho,\tilde\alpha^\vee\right\rangle=m\left(n+l-\frac{1}{2}\right)+(\bar\rho,\alpha)\geqslant 2\left(n+l-\frac{1}{2}\right)-(2l-1)=2n\geqslant 2,$$

which implies $\langle\lambda_n+\rho,\tilde\alpha^\vee\rangle\notin -\mathbb{Z}_+$.

If $(\alpha,\alpha)=1$, then $\alpha=\pm\epsilon_i$ for some $i=1,\dots,l$, which implies $(\bar\rho,\alpha)\geqslant -l$. Then

$$\left\langle\lambda_n+\rho,\tilde\alpha^\vee\right\rangle=2\left(m\left(n+l-\frac{1}{2}\right)+(\bar\rho,\alpha)\right)\geqslant 2\left(n+l-\frac{1}{2}-l\right)=2n-1\geqslant 1,$$

which implies $\langle\lambda_n+\rho,\tilde\alpha^\vee\rangle\notin -\mathbb{Z}_+$.

Thus, $\langle\lambda_n+\rho,\tilde\alpha^\vee\rangle\notin -\mathbb{Z}_+$ for any $\tilde\alpha\in\hat\Delta_+^{\mathrm{re}}$. Furthermore,

$$\begin{aligned}&\left\langle\lambda_n+\rho,\alpha_i^\vee\right\rangle=1\quad\text{for } i=1,\dots,l,\\&\left\langle\lambda_n+\rho,(\delta-\epsilon_1)^\vee\right\rangle=2n\in\mathbb{N},\end{aligned}$$

which implies

$$\hat\Pi^\vee_{\lambda_n}=\left\{(\delta-\epsilon_1)^\vee,\alpha_1^\vee,\alpha_2^\vee,\dots,\alpha_l^\vee\right\}.$$

It follows that $\mathbb{Q}\hat\Delta^{\vee\mathrm{re}}_{\lambda_n}=\mathbb{Q}\hat\Pi^\vee_{\lambda_n}=\mathbb{Q}\hat\Pi^\vee$. $\square$

Lemma 10. *Vector*

$$v_n=\left(-\frac{1}{4}e_{\epsilon_1}(-1)^2+e_{\epsilon_1-\epsilon_2}(-1)e_{\epsilon_1+\epsilon_2}(-1)+\cdots+e_{\epsilon_1-\epsilon_l}(-1)e_{\epsilon_1+\epsilon_l}(-1)\right)^n\mathbf{1}$$

is a singular vector in $N(n-l+\frac{1}{2},0)$ for any $n\in\mathbb{N}$.

Proof. It is sufficient to show

$$\begin{aligned}&e_i(0).v_n=0,\quad i=1,\dots,l,\\&f_\theta(1).v_n=0.\end{aligned}$$

We introduce the notation

$$u=-\frac{1}{4}e_{\epsilon_1}(-1)^2+e_{\epsilon_1-\epsilon_2}(-1)e_{\epsilon_1+\epsilon_2}(-1)+\cdots+e_{\epsilon_1-\epsilon_l}(-1)e_{\epsilon_1+\epsilon_l}(-1)\in U(\hat{\mathfrak g}).$$

It can easily be checked that vectors $e_i(0)$ commute with u in $U(\hat{\mathfrak g})$ for any $i=1,\dots,l$, which implies

$$e_i(0).v_n=e_i(0).u^n\mathbf{1}=\left[e_i(0),u^n\right]\mathbf{1}=0,\quad i=1,\dots,l.$$

Similarly, we can show

$$\begin{aligned}
\left[f_\theta(1), u\right] = {} & \left(k+l-\frac{3}{2}\right)e_1(-1) - e_1(-1)h_{\epsilon_1+\epsilon_2}(0) - \frac{1}{2}e_{\epsilon_1}(-1)f_{\epsilon_2}(0) \\
& - e_{\epsilon_1+\epsilon_3}(-1)f_{\epsilon_2+\epsilon_3}(0) - e_{\epsilon_1-\epsilon_3}(-1)f_{\epsilon_2-\epsilon_3}(0) - \cdots \\
& - e_{\epsilon_1+\epsilon_l}(-1)f_{\epsilon_2+\epsilon_l}(0) - e_{\epsilon_1-\epsilon_l}(-1)f_{\epsilon_2-\epsilon_l}(0).
\end{aligned}$$

It can easily be checked that vectors $f_{\epsilon_2}(0), f_{\epsilon_2+\epsilon_3}(0), f_{\epsilon_2-\epsilon_3}(0), \ldots, f_{\epsilon_2+\epsilon_l}(0), f_{\epsilon_2-\epsilon_l}(0)$ commute with u in $U(\hat{\mathfrak{g}})$ and that

$$\left[h_{\epsilon_1+\epsilon_2}(0), u^m\right] = 2mu^m, \quad \text{for } m \in \mathbb{N}.$$

We have:

$$\begin{aligned}
f_\theta(1).v_n &= f_\theta(1).u^n\mathbf{1} = \left[f_\theta(1), u^n\right]\mathbf{1} \\
&= \left[f_\theta(1), u\right]u^{n-1}\mathbf{1} + u\left[f_\theta(1), u\right]u^{n-2}\mathbf{1} + \cdots + u^{n-1}\left[f_\theta(1), u\right]\mathbf{1} \\
&= \left(k+l-\frac{3}{2}\right)e_1(-1)u^{n-1}\mathbf{1} - 2(n-1)e_1(-1)u^{n-1}\mathbf{1} \\
&\quad + \left(u\left(k+l-\frac{3}{2}\right)e_1(-1)u^{n-2} - ue_1(-1)\cdot 2(n-2)u^{n-2}\right)\mathbf{1} + \cdots \\
&\quad + u^{n-1}\left(k+l-\frac{3}{2}\right)e_1(-1)\mathbf{1} \\
&= n\left(k+l-\frac{3}{2}\right)e_1(-1)u^{n-1}\mathbf{1} - 2\big((n-1)+(n-2)+\cdots+1\big)e_1(-1)u^{n-1}\mathbf{1} \\
&= \left(n\left(k+l-\frac{3}{2}\right) - 2\cdot\frac{(n-1)n}{2}\right)e_1(-1)u^{n-1}\mathbf{1} \\
&= n\left(k+l-n-\frac{1}{2}\right)e_1(-1)u^{n-1}\mathbf{1} = 0,
\end{aligned}$$

because $k = n-l+\frac{1}{2}$. Thus $\hat{\mathfrak{n}}_+.v_n = 0$, and v_n is a singular vector in $N(n-l+\frac{1}{2}, 0)$. □

Theorem 11. *The maximal $\hat{\mathfrak{g}}$-submodule of $N(n-l+\frac{1}{2}, 0)$ is $J(n-l+\frac{1}{2}, 0) = U(\hat{\mathfrak{g}})v_n$, where*

$$v_n = \left(-\frac{1}{4}e_{\epsilon_1}(-1)^2 + e_{\epsilon_1-\epsilon_2}(-1)e_{\epsilon_1+\epsilon_2}(-1) + \cdots + e_{\epsilon_1-\epsilon_l}(-1)e_{\epsilon_1+\epsilon_l}(-1)\right)^n\mathbf{1}, \quad n \in \mathbb{N}.$$

Proof. It follows from Proposition 4 and Lemma 9 that the maximal submodule of the Verma module $M(\lambda_n)$ is generated by $l+1$ singular vectors with weights

$$r_{\delta-\epsilon_1}.\lambda_n, r_{\alpha_1}.\lambda_n, \ldots, r_{\alpha_l}.\lambda_n.$$

It follows from Lemma 10 that v_n is a singular vector of weight $\lambda_n - 2n\delta + 2n\epsilon_1 = r_{\delta-\epsilon_1}.\lambda_n$. Other singular vectors have weights

$$r_{\alpha_i}.\lambda_n = \lambda_n - \langle\lambda_n+\rho, \alpha_i^\vee\rangle\alpha_i = \lambda_n - \alpha_i, \quad i = 1, \ldots, l,$$

so the images of these vectors under the projection of $M(\lambda_n)$ onto $N(n-l+\frac{1}{2},0)$ are 0. Therefore, the maximal submodule of $N(n-l+\frac{1}{2},0)$ is generated by the vector v_n, i.e. $J(n-l+\frac{1}{2},0)=U(\hat{\mathfrak{g}})v_n$. $\square$

Using Theorem 11 and Proposition 8 we can identify the associative algebra $A(L(n-l+\frac{1}{2},0))$:

Proposition 12. *The associative algebra $A(L(n-l+\frac{1}{2},0))$ is isomorphic to the algebra $U(\mathfrak{g})/I_n$, where I_n is the two-sided ideal of $U(\mathfrak{g})$ generated by*

$$v_n' = \left(-\frac{1}{4}e_{\epsilon_1}^2 + e_{\epsilon_1-\epsilon_2}e_{\epsilon_1+\epsilon_2} + \cdots + e_{\epsilon_1-\epsilon_l}e_{\epsilon_1+\epsilon_l}\right)^n.$$

3.3. Modules for associative algebra $A(L(n-l+\frac{1}{2},0))$

In this subsection we present the method from [2,4,19] for classification of irreducible $A(L(n-l+\frac{1}{2},0))$-modules from the category $\mathcal{O}$ by solving certain systems of polynomial equations.

Denote by $_L$ the adjoint action of $U(\mathfrak{g})$ on $U(\mathfrak{g})$ defined by $X_L f=[X,f]$ for $X\in\mathfrak{g}$ and $f\in U(\mathfrak{g})$. Let R be a $U(\mathfrak{g})$-submodule of $U(\mathfrak{g})$ generated by the vector v_n'. Clearly, R is an irreducible finite-dimensional $U(\mathfrak{g})$-module isomorphic to $V(2n\epsilon_1)$. Let R_0 be the zero-weight subspace of R.

Proposition 13. ([2, Proposition 2.4.1], [4, Lemma 3.4.3]) *Let $V(\mu)$ be an irreducible highest weight $U(\mathfrak{g})$-module with the highest weight vector v_μ, for $\mu\in\mathfrak{h}^*$. The following statements are equivalent*:

(1) *$V(\mu)$ is an $A(L(n-l+\frac{1}{2},0))$-module,*
(2) *$RV(\mu)=0$,*
(3) *$R_0v_\mu=0$.*

Proof. The equivalence of (1) and (2) follows from the fact that R generates the ideal I_n. Clearly, (2) implies (3). To prove the converse, suppose that $R_0v_\mu=0$. We claim that $RV(\mu)$ is a $\mathfrak{g}$-submodule of $V(\mu)$. Let $x\in\mathfrak{g}$, $r\in R$ and $v\in V(\mu)$. We get

$$x(rv)=[x,r]v+r(xv).$$

Since R is a $\mathfrak{g}$-module, $[x,r]\in R$, and that implies $x(rv)\in RV(\mu)$. Since $V(\mu)$ is irreducible, $RV(\mu)=0$ or $RV(\mu)=V(\mu)$. To prove that $RV(\mu)=0$, it is enough to show that $v_\mu\notin RV(\mu)$. Clearly,

$$RV(\mu)=RU(\mathfrak{n}_-)v_\mu=U(\mathfrak{n}_-)Rv_\mu,$$

since R is a $\mathfrak{g}$-module under the adjoint action. Since $R\subseteq U(\mathfrak{g})$, the Poincaré–Birkhoff–Witt theorem implies that every element of $r\in R$ can be written as a linear combination of elements of the form $r_0r_-r_+$, where $r_0\in S(\mathfrak{h})$, $r_-\in U(\mathfrak{n}_-)$ and $r_+\in U(\mathfrak{n}_+)$. It follows that, if the weight of r is positive, then $rv_\mu=0$, and if the weight of r is negative, then the weight of rv_μ is

$\mu + wt(r) < \mu$. From this we obtain that $v_\mu \in RV(\mu)$ if and only if $R_0 v_\mu \neq 0$. Thus, $R_0 v_\mu = 0$ implies $v_\mu \notin RV(\mu)$, which implies $RV(\mu) = 0$. $\square$

Let $r \in R_0$. Clearly there exists the unique polynomial $p_r \in S(\mathfrak{h})$ such that

$$r v_\mu = p_r(\mu) v_\mu.$$

Set $\mathcal{P}_0 = \{p_r \mid r \in R_0\}$. We have:

Corollary 14. *There is one-to-one correspondence between*

(1) *irreducible $A(L(n-l+\frac{1}{2}, 0))$-modules from the category $\mathcal{O}$,*
(2) *weights $\mu \in \mathfrak{h}^*$ such that $p(\mu) = 0$ for all $p \in \mathcal{P}_0$.*

3.4. Construction of some polynomials in $\mathcal{P}_0$

The following lemmas are obtained by direct calculations in $U(\mathfrak{g})$:

Lemma 15. *Let $X \in \mathfrak{g}$ and $Y_1, \ldots, Y_m \in U(\mathfrak{g})$. Then*

$$(X^n)_L(Y_1 \cdot \ldots \cdot Y_m) = \sum_{\substack{(k_1,\ldots,k_m) \in \mathbb{Z}_+^m \\ \sum k_i = n}} \binom{n}{k_1, \ldots, k_m} (X^{k_1})_L Y_1 \cdot \ldots \cdot (X^{k_m})_L Y_m.$$

Lemma 16.

(1) $(e_\alpha^m)_L(f_\alpha^m) \in m! \cdot h_\alpha \cdot \ldots \cdot (h_\alpha - m + 1) + U(\mathfrak{g}) e_\alpha$, $\forall \alpha \in \Delta_+$;
(2) $(e_\alpha^k)_L(f_\alpha^m) \in U(\mathfrak{g}) e_\alpha$, *for $k > m$ and $\forall \alpha \in \Delta_+$*;
(3) $(e_{\epsilon_1}^{2k})_L(f_{\epsilon_1+\epsilon_i}^k) = (-1)^k (2k)! \cdot e_{\epsilon_1-\epsilon_i}^k$, $i = 2, \ldots, l$;
(4) $(e_{\epsilon_1}^{2k+j})_L(f_{\epsilon_1+\epsilon_i}^k) = 0$, *for $j > 0$ and $i = 2, \ldots, l$*;
(5) $(e_{\epsilon_1}^r)_L(f_{\epsilon_1-\epsilon_i}^k) \in U(\mathfrak{g}) \mathfrak{n}_+$, *for $r > 0$ and $i = 2, \ldots, l$*;
(6) $e_\alpha^k p(h) = p(h - k\alpha(h)) e_\alpha^k$, $\forall p \in S(\mathfrak{h})$;
(7) $(e_{\epsilon_1+\epsilon_i}^k)_L(f_{\epsilon_1+\epsilon_2}^m) = m(m-1)\cdots(m-k+1) f_{\epsilon_1+\epsilon_2}^{m-k} f_{\epsilon_2-\epsilon_i}^k$, *for $i = 3, \ldots, l$ and $k \leqslant m$*;
(8) $(e_{\epsilon_1+\epsilon_i}^k)_L(f_{\epsilon_1-\epsilon_2}^m) \in U(\mathfrak{g}) \mathfrak{n}_+$, *for $i = 3, \ldots, l$ and $k > 0$*;
(9) $(e_{\epsilon_1+\epsilon_2}^k)_L(f_{\epsilon_2-\epsilon_i}^m) \in U(\mathfrak{g}) e_{\epsilon_1+\epsilon_i}$, *for $i = 3, \ldots, l$ and $k > 0$*;
(10) $(e_\alpha^k)_L(f_\alpha^m) \in m(m-1)\cdots(m-k+1) f_\alpha^{m-k} \cdot (h_\alpha - m + k) \cdot \ldots \cdot (h_\alpha - m + 1) + U(\mathfrak{g}) e_\alpha$, $\forall \alpha \in \Delta_+$ *and $k \leqslant m$*;
(11) $(e_{\epsilon_1-\epsilon_i}^k)_L(f_{\epsilon_2-\epsilon_i}^k) = k! e_{\epsilon_1-\epsilon_2}^k$ *for $i = 3, \ldots, l$*;
(12) $(e_{\epsilon_1-\epsilon_i}^k)_L(f_{\epsilon_1-\epsilon_2}^m) \in U(\mathfrak{g}) \mathfrak{n}_+$, *for $i = 3, \ldots, l$ and $k > 0$.*

Lemma 17. *Let $\beta_1, \ldots, \beta_k, \gamma_1, \ldots, \gamma_m \in \Delta_+$ such that $\sum_{i=1}^k \beta_i = \sum_{i=1}^m \gamma_i$. Let $Y_1, Y_2 \in U(\mathfrak{g})$ such that*

$$Y_1 = e_{\beta_1} \cdots e_{\beta_k}, \qquad [e_{\beta_i}, e_{\beta_j}] = 0, \quad \text{for all } i, j,$$

$$Y_2 = f_{\gamma_1} \cdots f_{\gamma_m}, \qquad [f_{\gamma_i}, f_{\gamma_j}] = 0, \quad \textit{for all } i, j.$$

Then

$$Y_1\,{}_L Y_2 \in e_{\beta_1} \cdots e_{\beta_k} f_{\gamma_1} \cdots f_{\gamma_m} + U(\mathfrak{g})\mathfrak{n}_+,$$
$$Y_2\,{}_L Y_1 \in (-1)^m e_{\beta_1} \cdots e_{\beta_k} f_{\gamma_1} \cdots f_{\gamma_m} + U(\mathfrak{g})\mathfrak{n}_+.$$

In the following lemma we calculate $l+1$ polynomials in the set $\mathcal{P}_0$. This is a crucial technical result needed for classification of irreducible $L(n-l+\frac{1}{2}, 0)$-modules.

Lemma 18. *Let*

(1) $$q(h) = \sum_{\substack{(k_1,\dots,k_l)\in\mathbb{Z}_+^l \\ \sum k_i = n}} \frac{1}{k_1!4^{k_1}} \cdot (h_{\epsilon_1} - 2k_2 - \cdots - 2k_l) \cdot \ldots \cdot (h_{\epsilon_1} - 2n + 1)$$
$$\cdot (h_{\epsilon_1-\epsilon_l} - k_{l-1} - \cdots - k_2) \cdot \ldots \cdot (h_{\epsilon_1-\epsilon_l} - k_{l-1} - \cdots - k_2 - k_l + 1) \cdot \ldots$$
$$\cdot h_{\epsilon_1-\epsilon_2} \cdot \ldots \cdot (h_{\epsilon_1-\epsilon_2} - k_2 + 1),$$

(2) $$p_i(h) = h_i(h_i - 1) \cdot \ldots \cdot (h_i - n + 1)\left(h_{\epsilon_i+\epsilon_{i+1}} + l - i - \frac{1}{2}\right)$$
$$\cdot \left(h_{\epsilon_i+\epsilon_{i+1}} + l - i - \frac{3}{2}\right) \cdot \ldots \cdot \left(h_{\epsilon_i+\epsilon_{i+1}} + l - n - i + \frac{1}{2}\right),$$

for $i = 1, \dots, l-1$,

(3) $$p_l(h) = h_l(h_l - 1) \cdot \ldots \cdot (h_l - 2n + 1).$$

Then $p_1, \dots, p_l, q \in \mathcal{P}_0$.

Proof. (1) We claim that

$$\left(e_{\epsilon_1}^{2n} f_{\epsilon_1}^{4n}\right)_L \left(-\frac{1}{4}e_{\epsilon_1}^2 + e_{\epsilon_1-\epsilon_2}e_{\epsilon_1+\epsilon_2} + \cdots + e_{\epsilon_1-\epsilon_l}e_{\epsilon_1+\epsilon_l}\right)^n \in cq(h) + U(\mathfrak{g})\mathfrak{n}_+,$$

for some $c \neq 0$.

We introduce the notation:

$$\bar{u} = -\frac{1}{4}e_{\epsilon_1}^2 + e_{\epsilon_1-\epsilon_2}e_{\epsilon_1+\epsilon_2} + \cdots + e_{\epsilon_1-\epsilon_l}e_{\epsilon_1+\epsilon_l} \in U(\mathfrak{g}).$$

It can easily be checked that

$$\left(f_{\epsilon_1}^4\right)_L \bar{u} = 24\left(-\frac{1}{4}f_{\epsilon_1}^2 + f_{\epsilon_1-\epsilon_2}f_{\epsilon_1+\epsilon_2} + \cdots + f_{\epsilon_1-\epsilon_l}f_{\epsilon_1+\epsilon_l}\right)$$

and $(f_{\epsilon_1}^5)_L \bar{u} = 0$, which implies

$$\begin{aligned}
&\left(f_{\epsilon_1}^{4n}\right)_L\left(-\frac{1}{4}e_{\epsilon_1}^2+e_{\epsilon_1-\epsilon_2}e_{\epsilon_1+\epsilon_2}+\cdots+e_{\epsilon_1-\epsilon_l}e_{\epsilon_1+\epsilon_l}\right)^n\\
&\quad=\left(f_{\epsilon_1}^{4n}\right)_L\left(\bar{u}^n\right)=\frac{(4n)!}{(4!)^n}\left(\left(f_{\epsilon_1}^{4}\right)_L\bar{u}\right)^n\\
&\quad=(4n)!\left(-\frac{1}{4}f_{\epsilon_1}^2+f_{\epsilon_1-\epsilon_2}f_{\epsilon_1+\epsilon_2}+\cdots+f_{\epsilon_1-\epsilon_l}f_{\epsilon_1+\epsilon_l}\right)^n.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left(e_{\epsilon_1}^{2n}f_{\epsilon_1}^{4n}\right)_L\left(-\frac{1}{4}e_{\epsilon_1}^2+e_{\epsilon_1-\epsilon_2}e_{\epsilon_1+\epsilon_2}+\cdots+e_{\epsilon_1-\epsilon_l}e_{\epsilon_1+\epsilon_l}\right)^n\\
&\quad=c'\left(e_{\epsilon_1}^{2n}\right)_L\left(-\frac{1}{4}f_{\epsilon_1}^2+f_{\epsilon_1-\epsilon_2}f_{\epsilon_1+\epsilon_2}+\cdots+f_{\epsilon_1-\epsilon_l}f_{\epsilon_1+\epsilon_l}\right)^n,
\end{aligned}\tag{1}$$

where $c'\neq 0$. Since all root vectors $f_{\epsilon_1}, f_{\epsilon_1-\epsilon_2}, f_{\epsilon_1+\epsilon_2},\ldots,f_{\epsilon_1-\epsilon_l}, f_{\epsilon_1+\epsilon_l}\in\mathfrak{g}$ commute, we have

$$\begin{aligned}
&\left(-\frac{1}{4}f_{\epsilon_1}^2+f_{\epsilon_1-\epsilon_2}f_{\epsilon_1+\epsilon_2}+\cdots+f_{\epsilon_1-\epsilon_l}f_{\epsilon_1+\epsilon_l}\right)^n\\
&\quad=\sum_{\substack{(k_1,\ldots,k_l)\in\mathbb{Z}_+^l\\ \sum k_i=n}}\binom{n}{k_1,\ldots,k_l}(-1)^{k_1}\frac{1}{4^{k_1}}f_{\epsilon_1}^{2k_1}f_{\epsilon_1-\epsilon_2}^{k_2}f_{\epsilon_1+\epsilon_2}^{k_2}\cdots f_{\epsilon_1-\epsilon_l}^{k_l}f_{\epsilon_1+\epsilon_l}^{k_l}.
\end{aligned}\tag{2}$$

We calculate the action of $e_{\epsilon_1}^{2n}$ on every summand above. By using claim (5) from Lemma 16 we obtain

$$\left(e_{\epsilon_1}^{r}\right)_L\left(f_{\epsilon_1-\epsilon_2}^{k_2}\cdots f_{\epsilon_1-\epsilon_l}^{k_l}\right)\in U(\mathfrak{g})\mathfrak{n}_+,\quad r>0,$$

which implies

$$\begin{aligned}
&\left(e_{\epsilon_1}^{2n}\right)_L\left(f_{\epsilon_1}^{2k_1}f_{\epsilon_1-\epsilon_2}^{k_2}f_{\epsilon_1+\epsilon_2}^{k_2}\cdots f_{\epsilon_1-\epsilon_l}^{k_l}f_{\epsilon_1+\epsilon_l}^{k_l}\right)\\
&\quad=\left(e_{\epsilon_1}^{2n}\right)_L\left(f_{\epsilon_1}^{2k_1}f_{\epsilon_1+\epsilon_2}^{k_2}\cdots f_{\epsilon_1+\epsilon_l}^{k_l}f_{\epsilon_1-\epsilon_2}^{k_2}\cdots f_{\epsilon_1-\epsilon_l}^{k_l}\right)\\
&\quad\in\left[\left(e_{\epsilon_1}^{2n}\right)_L\left(f_{\epsilon_1}^{2k_1}f_{\epsilon_1+\epsilon_2}^{k_2}\cdots f_{\epsilon_1+\epsilon_l}^{k_l}\right)\right]\left(f_{\epsilon_1-\epsilon_2}^{k_2}\cdots f_{\epsilon_1-\epsilon_l}^{k_l}\right)+U(\mathfrak{g})\mathfrak{n}_+.
\end{aligned}\tag{3}$$

Using Lemma 15 we can calculate:

$$\begin{aligned}
&\left(e_{\epsilon_1}^{2n}\right)_L\left(f_{\epsilon_1}^{2k_1}f_{\epsilon_1+\epsilon_2}^{k_2}\cdots f_{\epsilon_1+\epsilon_l}^{k_l}\right)\\
&\quad=\left(e_{\epsilon_1}^{2n}\right)_L\left(f_{\epsilon_1+\epsilon_2}^{k_2}\cdots f_{\epsilon_1+\epsilon_l}^{k_l}f_{\epsilon_1}^{2k_1}\right)\\
&\quad=\sum_{\substack{(m_1,\ldots,m_l)\in\mathbb{Z}_+^l\\ \sum m_i=2n}}\binom{2n}{m_1,\ldots,m_l}\left[\left(e_{\epsilon_1}^{m_2}\right)_L\left(f_{\epsilon_1+\epsilon_2}^{k_2}\right)\cdots\left(e_{\epsilon_1}^{m_l}\right)_L\left(f_{\epsilon_1+\epsilon_l}^{k_l}\right)\left(e_{\epsilon_1}^{m_1}\right)_L\left(f_{\epsilon_1}^{2k_1}\right)\right].
\end{aligned}$$

It follows from claims (2) and (4) from Lemma 16 that only nontrivial summand above is for $m_1 = 2k_1, m_2 = 2k_2, \dots, m_l = 2k_l$. By using claims (1) and (3) from Lemma 16 we obtain:

$$\begin{aligned}
&\big(e_{\epsilon_1}^{2n}\big)_L\big(f_{\epsilon_1}^{2k_1} f_{\epsilon_1+\epsilon_2}^{k_2} \cdots f_{\epsilon_1+\epsilon_l}^{k_l}\big)\\
&\quad\in \frac{(2n)!}{(2k_1)!\cdots(2k_l)!}\big(e_{\epsilon_1}^{2k_2}\big)_L\big(f_{\epsilon_1+\epsilon_2}^{k_2}\big)\cdots\big(e_{\epsilon_1}^{2k_l}\big)_L\big(f_{\epsilon_1+\epsilon_l}^{k_l}\big)\big(e_{\epsilon_1}^{2k_1}\big)_L\big(f_{\epsilon_1}^{2k_1}\big) + U(\mathfrak{g})e_{\epsilon_1}\\
&\quad\in (2n)!(-1)^{n-k_1} e_{\epsilon_1-\epsilon_2}^{k_2}\cdots e_{\epsilon_1-\epsilon_l}^{k_l}\cdot h_{\epsilon_1}\cdots(h_{\epsilon_1}-2k_1+1) + U(\mathfrak{g})e_{\epsilon_1}.
\end{aligned}$$

By putting the expression above in relation (3), we obtain:

$$\begin{aligned}
&\big(e_{\epsilon_1}^{2n}\big)_L\big(f_{\epsilon_1}^{2k_1} f_{\epsilon_1-\epsilon_2}^{k_2} f_{\epsilon_1+\epsilon_2}^{k_2} \cdots f_{\epsilon_1-\epsilon_l}^{k_l} f_{\epsilon_1+\epsilon_l}^{k_l}\big)\\
&\quad\in (2n)!(-1)^{n-k_1} e_{\epsilon_1-\epsilon_2}^{k_2}\cdots e_{\epsilon_1-\epsilon_l}^{k_l}\cdot h_{\epsilon_1}\cdots(h_{\epsilon_1}-2k_1+1) f_{\epsilon_1-\epsilon_2}^{k_2}\cdots f_{\epsilon_1-\epsilon_l}^{k_l} + U(\mathfrak{g})\mathfrak{n}_+.
\end{aligned}$$

By using claims (6) and (1) from Lemma 16, we have:

$$\begin{aligned}
&\big(e_{\epsilon_1}^{2n}\big)_L\big(f_{\epsilon_1}^{2k_1} f_{\epsilon_1-\epsilon_2}^{k_2} f_{\epsilon_1+\epsilon_2}^{k_2} \cdots f_{\epsilon_1-\epsilon_l}^{k_l} f_{\epsilon_1+\epsilon_l}^{k_l}\big)\\
&\quad\in (2n)!(-1)^{n-k_1}(h_{\epsilon_1}-2k_2-\cdots-2k_l)\cdots(h_{\epsilon_1}-2n+1)\\
&\qquad\cdot e_{\epsilon_1-\epsilon_2}^{k_2}\cdots e_{\epsilon_1-\epsilon_l}^{k_l} f_{\epsilon_1-\epsilon_l}^{k_l}\cdots f_{\epsilon_1-\epsilon_2}^{k_2} + U(\mathfrak{g})\mathfrak{n}_+\\
&\quad\in (2n)!(-1)^{n-k_1}(h_{\epsilon_1}-2k_2-\cdots-2k_l)\cdots(h_{\epsilon_1}-2n+1)\\
&\qquad\cdot e_{\epsilon_1-\epsilon_2}^{k_2}\cdots e_{\epsilon_1-\epsilon_{l-1}}^{k_{l-1}}\cdot\big(k_l! h_{\epsilon_1-\epsilon_l}\cdots(h_{\epsilon_1-\epsilon_l}-k_l+1)\big) f_{\epsilon_1-\epsilon_{l-1}}^{k_{l-1}}\cdots f_{\epsilon_1-\epsilon_2}^{k_2} + U(\mathfrak{g})\mathfrak{n}_+\\
&\quad\in (2n)!(-1)^{n-k_1} k_l!(h_{\epsilon_1}-2k_2-\cdots-2k_l)\cdots(h_{\epsilon_1}-2n+1)\\
&\qquad\cdot(h_{\epsilon_1-\epsilon_l}-k_{l-1}-\cdots-k_2)\cdots(h_{\epsilon_1-\epsilon_l}-k_{l-1}-\cdots-k_2-k_l+1)\\
&\qquad\cdot e_{\epsilon_1-\epsilon_2}^{k_2}\cdots e_{\epsilon_1-\epsilon_{l-1}}^{k_{l-1}}\cdot f_{\epsilon_1-\epsilon_{l-1}}^{k_{l-1}}\cdots f_{\epsilon_1-\epsilon_2}^{k_2} + U(\mathfrak{g})\mathfrak{n}_+\cdots\\
&\quad\in (2n)!(-1)^{n-k_1} k_l!\cdots k_2!(h_{\epsilon_1}-2k_2-\cdots-2k_l)\cdot\ldots\cdot(h_{\epsilon_1}-2n+1)\\
&\qquad\cdot(h_{\epsilon_1-\epsilon_l}-k_{l-1}-\cdots-k_2)\cdot\ldots\cdot(h_{\epsilon_1-\epsilon_l}-k_{l-1}-\cdots-k_2-k_l-1)\cdot\ldots\\
&\qquad\cdot h_{\epsilon_1-\epsilon_2}\cdot\ldots\cdot(h_{\epsilon_1-\epsilon_2}-k_2+1) + U(\mathfrak{g})\mathfrak{n}_+.
\end{aligned}$$

It follows from relation (2) that

$$\begin{aligned}
&\big(e_{\epsilon_1}^{2n}\big)_L\bigg(-\frac{1}{4}f_{\epsilon_1}^2 + f_{\epsilon_1-\epsilon_2}f_{\epsilon_1+\epsilon_2} + \cdots + f_{\epsilon_1-\epsilon_l}f_{\epsilon_1+\epsilon_l}\bigg)^n\\
&\quad\in (-1)^n(2n)!n! \sum_{\substack{(k_1,\dots,k_l)\in\mathbb{Z}_+^l\\ \sum k_i=n}} \frac{1}{k_1!4^{k_1}}\cdot(h_{\epsilon_1}-2k_2-\cdots-2k_l)\cdot\ldots\cdot(h_{\epsilon_1}-2n+1)\\
&\qquad\cdot(h_{\epsilon_1-\epsilon_l}-k_{l-1}-\cdots-k_2)\cdot\ldots\cdot(h_{\epsilon_1-\epsilon_l}-k_{l-1}-\cdots-k_2-k_l-1)\cdot\ldots\\
&\qquad\cdot h_{\epsilon_1-\epsilon_2}\cdot\ldots\cdot(h_{\epsilon_1-\epsilon_2}-k_2+1) + U(\mathfrak{g})\mathfrak{n}_+.
\end{aligned}$$

Finally, relation (1) implies

$$
\begin{aligned}
&\left(e_{\epsilon_1}^{2n} f_{\epsilon_1}^{4n}\right)_L\left(-\frac{1}{4}e_{\epsilon_1}^2+e_{\epsilon_1-\epsilon_2}e_{\epsilon_1+\epsilon_2}+\cdots+e_{\epsilon_1-\epsilon_l}e_{\epsilon_1+\epsilon_l}\right)^n\\
&\quad\in c\sum_{\substack{(k_1,\dots,k_l)\in\mathbb{Z}_+^l\\ \sum k_i=n}}\frac{1}{k_1!4^{k_1}}\cdot(h_{\epsilon_1}-2k_2-\cdots-2k_l)\cdot\ldots\cdot(h_{\epsilon_1}-2n+1)\\
&\qquad\cdot(h_{\epsilon_1-\epsilon_l}-k_{l-1}-\cdots-k_2)\cdot\ldots\cdot(h_{\epsilon_1-\epsilon_l}-k_{l-1}-\cdots-k_2-k_l-1)\cdot\ldots\\
&\qquad\cdot h_{\epsilon_1-\epsilon_2}\cdot\ldots\cdot(h_{\epsilon_1-\epsilon_2}-k_2+1)+U(\mathfrak{g})\mathfrak{n}_+,
\end{aligned}
$$

for some $c\neq 0$, and the proof is complete.

(2) First notice that

$$
\left(f_{\epsilon_1-\epsilon_2}^n f_{\epsilon_1+\epsilon_2}^n\right)_L\left(-\frac{1}{4}e_{\epsilon_1}^2+e_{\epsilon_1-\epsilon_2}e_{\epsilon_1+\epsilon_2}+\cdots+e_{\epsilon_1-\epsilon_l}e_{\epsilon_1+\epsilon_l}\right)^n\in R_0.
$$

Lemma 17 implies that we can calculate the corresponding polynomial from

$$
\begin{aligned}
&\left(-\frac{1}{4}e_{\epsilon_1}^2+e_{\epsilon_1-\epsilon_2}e_{\epsilon_1+\epsilon_2}+\cdots+e_{\epsilon_1-\epsilon_l}e_{\epsilon_1+\epsilon_l}\right)_L^n\left(f_{\epsilon_1-\epsilon_2}^n f_{\epsilon_1+\epsilon_2}^n\right)\\
&\quad=\sum_{\substack{(k_1,\dots,k_l)\in\mathbb{Z}_+^l\\ \sum k_i=n}}\binom{n}{k_1,\dots,k_l}(-1)^{k_1}\frac{1}{4^{k_1}}\left(e_{\epsilon_1}^{2k_1}e_{\epsilon_1-\epsilon_2}^{k_2}e_{\epsilon_1+\epsilon_2}^{k_2}\cdots e_{\epsilon_1-\epsilon_l}^{k_l}e_{\epsilon_1+\epsilon_l}^{k_l}\right)_L\left(f_{\epsilon_1-\epsilon_2}^n f_{\epsilon_1+\epsilon_2}^n\right).
\end{aligned}
\tag{4}
$$

By using claims (7) and (8) from Lemma 16, we have

$$
\begin{aligned}
&\left(e_{\epsilon_1}^{2k_1}e_{\epsilon_1-\epsilon_2}^{k_2}e_{\epsilon_1+\epsilon_2}^{k_2}\cdots e_{\epsilon_1-\epsilon_l}^{k_l}e_{\epsilon_1+\epsilon_l}^{k_l}\right)_L\left(f_{\epsilon_1-\epsilon_2}^n f_{\epsilon_1+\epsilon_2}^n\right)\\
&\quad=\left(e_{\epsilon_1-\epsilon_2}^{k_2}\cdots e_{\epsilon_1-\epsilon_l}^{k_l}e_{\epsilon_1}^{2k_1}e_{\epsilon_1+\epsilon_2}^{k_2}\cdots e_{\epsilon_1+\epsilon_{l-1}}^{k_{l-1}}e_{\epsilon_1+\epsilon_l}^{k_l}\right)_L\left(f_{\epsilon_1+\epsilon_2}^n f_{\epsilon_1-\epsilon_2}^n\right)\\
&\quad\in\left(e_{\epsilon_1-\epsilon_2}^{k_2}\cdots e_{\epsilon_1-\epsilon_l}^{k_l}e_{\epsilon_1}^{2k_1}e_{\epsilon_1+\epsilon_2}^{k_2}\cdots e_{\epsilon_1+\epsilon_{l-1}}^{k_{l-1}}\right)_L\big(n(n-1)\cdots(n-k_l+1)\\
&\qquad\cdot f_{\epsilon_1+\epsilon_2}^{n-k_l}f_{\epsilon_2-\epsilon_l}^{k_l}f_{\epsilon_1-\epsilon_2}^n\big)+U(\mathfrak{g})\mathfrak{n}_+\\
&\quad=\left(e_{\epsilon_1-\epsilon_2}^{k_2}\cdots e_{\epsilon_1-\epsilon_l}^{k_l}e_{\epsilon_1}^{2k_1}e_{\epsilon_1+\epsilon_2}^{k_2}\cdots e_{\epsilon_1+\epsilon_{l-2}}^{k_{l-2}}\right)_L\big(n(n-1)\cdots(n-k_l+1)\\
&\qquad\cdot(n-k_l)\cdots(n-k_l-k_{l-1}+1)f_{\epsilon_1+\epsilon_2}^{n-k_l-k_{l-1}}f_{\epsilon_2-\epsilon_{l-1}}^{k_{l-1}}f_{\epsilon_2-\epsilon_l}^{k_l}f_{\epsilon_1-\epsilon_2}^n\big)+U(\mathfrak{g})\mathfrak{n}_+\cdots\\
&\quad=\left(e_{\epsilon_1-\epsilon_2}^{k_2}\cdots e_{\epsilon_1-\epsilon_l}^{k_l}e_{\epsilon_1}^{2k_1}e_{\epsilon_1+\epsilon_2}^{k_2}\right)_L\big(n(n-1)\cdots(n-k_l-\cdots-k_3+1)\\
&\qquad\cdot f_{\epsilon_1+\epsilon_2}^{n-k_l-\cdots-k_3}f_{\epsilon_2-\epsilon_3}^{k_3}\cdots f_{\epsilon_2-\epsilon_{l-1}}^{k_{l-1}}f_{\epsilon_2-\epsilon_l}^{k_l}f_{\epsilon_1-\epsilon_2}^n\big)+U(\mathfrak{g})\mathfrak{n}_+\\
&\quad=\left(e_{\epsilon_1-\epsilon_2}^{k_2}\cdots e_{\epsilon_1-\epsilon_l}^{k_l}e_{\epsilon_1}^{2k_1}e_{\epsilon_1+\epsilon_2}^{k_2}\right)_L\big(n(n-1)\cdots(k_1+k_2+1)\\
&\qquad\cdot f_{\epsilon_1+\epsilon_2}^{k_1+k_2}f_{\epsilon_2-\epsilon_3}^{k_3}\cdots f_{\epsilon_2-\epsilon_{l-1}}^{k_{l-1}}f_{\epsilon_2-\epsilon_l}^{k_l}f_{\epsilon_1-\epsilon_2}^n\big)+U(\mathfrak{g})\mathfrak{n}_+.
\end{aligned}
$$

Claims (9), (8), (10) and (6) from Lemma 16 imply

$$\begin{aligned}
&\big(e_{\epsilon_1-\epsilon_2}^{k_2}\cdots e_{\epsilon_1-\epsilon_l}^{k_l}e_{\epsilon_1}^{2k_1}e_{\epsilon_1+\epsilon_2}^{k_2}\cdots e_{\epsilon_1+\epsilon_{l-1}}^{k_{l-1}}e_{\epsilon_1+\epsilon_l}^{k_l}\big)_L\big(f_{\epsilon_1+\epsilon_2}^{n}f_{\epsilon_1-\epsilon_2}^{n}\big)\\
&\quad\in\big(e_{\epsilon_1-\epsilon_2}^{k_2}\cdots e_{\epsilon_1-\epsilon_l}^{k_l}e_{\epsilon_1}^{2k_1}\big)_L\big(n(n-1)\cdots(k_1+k_2+1)\\
&\qquad\cdot\big(e_{\epsilon_1+\epsilon_2}^{k_2}\big)_L\big(f_{\epsilon_1+\epsilon_2}^{k_1+k_2}\big)\cdot f_{\epsilon_2-\epsilon_3}^{k_3}\cdots f_{\epsilon_2-\epsilon_{l-1}}^{k_{l-1}}f_{\epsilon_2-\epsilon_l}^{k_l}f_{\epsilon_1-\epsilon_2}^{n}\big)+U(\mathfrak{g})\mathfrak{n}_+\\
&\quad=\big(e_{\epsilon_1-\epsilon_2}^{k_2}\cdots e_{\epsilon_1-\epsilon_l}^{k_l}e_{\epsilon_1}^{2k_1}\big)_L\big(n\cdots(k_1+k_2+1)\cdot(k_1+k_2)\cdots(k_1+1)f_{\epsilon_1+\epsilon_2}^{k_1}\\
&\qquad\cdot(h_{\epsilon_1+\epsilon_2}-k_1)\cdots(h_{\epsilon_1+\epsilon_2}-k_1-k_2+1)f_{\epsilon_2-\epsilon_3}^{k_3}\cdots f_{\epsilon_2-\epsilon_{l-1}}^{k_{l-1}}f_{\epsilon_2-\epsilon_l}^{k_l}f_{\epsilon_1-\epsilon_2}^{n}\big)+U(\mathfrak{g})\mathfrak{n}_+\\
&\quad=\big(e_{\epsilon_1-\epsilon_2}^{k_2}\cdots e_{\epsilon_1-\epsilon_l}^{k_l}e_{\epsilon_1}^{2k_1}\big)_L\big(n(n-1)\cdots(k_1+1)f_{\epsilon_1+\epsilon_2}^{k_1}f_{\epsilon_2-\epsilon_3}^{k_3}\cdots f_{\epsilon_2-\epsilon_{l-1}}^{k_{l-1}}f_{\epsilon_2-\epsilon_l}^{k_l}f_{\epsilon_1-\epsilon_2}^{n}\\
&\qquad\cdot(h_{\epsilon_1+\epsilon_2}-k_1-k_3-\cdots-k_l)\cdots(h_{\epsilon_1+\epsilon_2}-k_1-k_2-k_3-\cdots-k_l+1)\big)+U(\mathfrak{g})\mathfrak{n}_+\\
&\quad=\big(e_{\epsilon_1-\epsilon_2}^{k_2}\cdots e_{\epsilon_1-\epsilon_l}^{k_l}e_{\epsilon_1}^{2k_1}\big)_L\big(n(n-1)\cdots(k_1+1)f_{\epsilon_1+\epsilon_2}^{k_1}f_{\epsilon_2-\epsilon_3}^{k_3}\cdots f_{\epsilon_2-\epsilon_{l-1}}^{k_{l-1}}f_{\epsilon_2-\epsilon_l}^{k_l}f_{\epsilon_1-\epsilon_2}^{n}\\
&\qquad\cdot(h_{\epsilon_1+\epsilon_2}-n+k_2)\cdots(h_{\epsilon_1+\epsilon_2}-n+1)\big)+U(\mathfrak{g})\mathfrak{n}_+.
\end{aligned}$$

It follows from claims (3) and (5) from Lemma 16 that

$$\begin{aligned}
&\big(e_{\epsilon_1-\epsilon_2}^{k_2}\cdots e_{\epsilon_1-\epsilon_l}^{k_l}e_{\epsilon_1}^{2k_1}e_{\epsilon_1+\epsilon_2}^{k_2}\cdots e_{\epsilon_1+\epsilon_{l-1}}^{k_{l-1}}e_{\epsilon_1+\epsilon_l}^{k_l}\big)_L\big(f_{\epsilon_1+\epsilon_2}^{n}f_{\epsilon_1-\epsilon_2}^{n}\big)\\
&\quad\in\big(e_{\epsilon_1-\epsilon_2}^{k_2}\cdots e_{\epsilon_1-\epsilon_l}^{k_l}\big)_L\big(n(n-1)\cdots(k_1+1)(-1)^{k_1}(2k_1)!\,e_{\epsilon_1-\epsilon_2}^{k_1}f_{\epsilon_2-\epsilon_3}^{k_3}\cdots f_{\epsilon_2-\epsilon_l}^{k_l}f_{\epsilon_1-\epsilon_2}^{n}\\
&\qquad\cdot(h_{\epsilon_1+\epsilon_2}-n+k_2)\cdots(h_{\epsilon_1+\epsilon_2}-n+1)\big)+U(\mathfrak{g})\mathfrak{n}_+.
\end{aligned}$$

By using claims (11), (12), (10) and (1) from Lemma 16, we obtain

$$\begin{aligned}
&\big(e_{\epsilon_1-\epsilon_2}^{k_2}\cdots e_{\epsilon_1-\epsilon_l}^{k_l}e_{\epsilon_1}^{2k_1}e_{\epsilon_1+\epsilon_2}^{k_2}\cdots e_{\epsilon_1+\epsilon_{l-1}}^{k_{l-1}}e_{\epsilon_1+\epsilon_l}^{k_l}\big)_L\big(f_{\epsilon_1+\epsilon_2}^{n}f_{\epsilon_1-\epsilon_2}^{n}\big)\\
&\quad\in\big(e_{\epsilon_1-\epsilon_2}^{k_2}\cdots e_{\epsilon_1-\epsilon_{l-1}}^{k_{l-1}}\big)_L\big(n(n-1)\cdots(k_1+1)(-1)^{k_1}(2k_1)!\,e_{\epsilon_1-\epsilon_2}^{k_1}f_{\epsilon_2-\epsilon_3}^{k_3}\cdots f_{\epsilon_2-\epsilon_{l-1}}^{k_{l-1}}\\
&\qquad\cdot k_l!\,e_{\epsilon_1-\epsilon_2}^{k_l}f_{\epsilon_1-\epsilon_2}^{n}(h_{\epsilon_1+\epsilon_2}-n+k_2)\cdots(h_{\epsilon_1+\epsilon_2}-n+1)\big)+U(\mathfrak{g})\mathfrak{n}_+\\
&\quad=\big(e_{\epsilon_1-\epsilon_2}^{k_2}\cdots e_{\epsilon_1-\epsilon_{l-2}}^{k_{l-2}}\big)_L\big(n(n-1)\cdots(k_1+1)(-1)^{k_1}(2k_1)!\,e_{\epsilon_1-\epsilon_2}^{k_1}f_{\epsilon_2-\epsilon_3}^{k_3}\cdots f_{\epsilon_2-\epsilon_{l-2}}^{k_{l-2}}\\
&\qquad\cdot k_{l-1}!\,e_{\epsilon_1-\epsilon_2}^{k_{l-1}}k_l!\,e_{\epsilon_1-\epsilon_2}^{k_l}f_{\epsilon_1-\epsilon_2}^{n}(h_{\epsilon_1+\epsilon_2}-n+k_2)\cdots(h_{\epsilon_1+\epsilon_2}-n+1)\big)+U(\mathfrak{g})\mathfrak{n}_+=\cdots\\
&\quad=\big(e_{\epsilon_1-\epsilon_2}^{k_2}\big)_L\big(n(n-1)\cdots(k_1+1)(-1)^{k_1}(2k_1)!\,k_3!\cdots k_l!e_{\epsilon_1-\epsilon_2}^{k_1}e_{\epsilon_1-\epsilon_2}^{k_3}\cdots e_{\epsilon_1-\epsilon_2}^{k_l}\\
&\qquad\cdot f_{\epsilon_1-\epsilon_2}^{n}(h_{\epsilon_1+\epsilon_2}-n+k_2)\cdots(h_{\epsilon_1+\epsilon_2}-n+1)\big)+U(\mathfrak{g})\mathfrak{n}_+\\
&\quad=n(n-1)\cdots(k_1+1)(-1)^{k_1}(2k_1)!\,k_3!\cdots k_l!\,e_{\epsilon_1-\epsilon_2}^{n-k_2}\big(e_{\epsilon_1-\epsilon_2}^{k_2}\big)_L\big(f_{\epsilon_1-\epsilon_2}^{n}\big)\\
&\qquad\cdot(h_{\epsilon_1+\epsilon_2}-n+k_2)\cdots(h_{\epsilon_1+\epsilon_2}-n+1)+U(\mathfrak{g})\mathfrak{n}_+\\
&\quad=n(n-1)\cdots(k_1+1)(-1)^{k_1}(2k_1)!k_3!\cdots k_l!e_{\epsilon_1-\epsilon_2}^{n-k_2}n(n-1)\cdots(n-k_2+1)f_{\epsilon_1-\epsilon_2}^{n-k_2}\\
&\qquad\cdot(h_{\epsilon_1-\epsilon_2}-n+k_2)\cdots(h_{\epsilon_1-\epsilon_2}-n+1)(h_{\epsilon_1+\epsilon_2}-n+k_2)\cdots(h_{\epsilon_1+\epsilon_2}-n+1)+U(\mathfrak{g})\mathfrak{n}_+\\
&\quad=\frac{n!}{k_1!}(-1)^{k_1}(2k_1)!k_3!\cdots k_l!n!h_{\epsilon_1-\epsilon_2}\cdots(h_{\epsilon_1-\epsilon_2}-n+1)\\
&\qquad\cdot(h_{\epsilon_1+\epsilon_2}-n+k_2)\cdots(h_{\epsilon_1+\epsilon_2}-n+1)+U(\mathfrak{g})\mathfrak{n}_+.
\end{aligned}$$

It follows from relation (4) that

$$\begin{aligned}
&\left(-\frac{1}{4}e_{\epsilon_1}^2 + e_{\epsilon_1-\epsilon_2}e_{\epsilon_1+\epsilon_2} + \cdots + e_{\epsilon_1-\epsilon_l}e_{\epsilon_1+\epsilon_l}\right)^n{}_L\left(f_{\epsilon_1-\epsilon_2}^n f_{\epsilon_1+\epsilon_2}^n\right) \\
&\quad \in (n!)^3 h_{\epsilon_1-\epsilon_2}\cdots(h_{\epsilon_1-\epsilon_2}-n+1) \\
&\quad\quad \cdot \sum_{\substack{(k_1,\ldots,k_l)\in\mathbb{Z}_+^l \\ \sum k_i=n}} \binom{2k_1}{k_1}\frac{1}{4^{k_1}}\frac{1}{k_2!}(h_{\epsilon_1+\epsilon_2}-n+k_2)\cdots(h_{\epsilon_1+\epsilon_2}-n+1) + U(\mathfrak{g})\mathfrak{n}_+.
\end{aligned}$$

Since

$$\begin{aligned}
&\sum_{\substack{(k_1,\ldots,k_l)\in\mathbb{Z}_+^l \\ \sum k_i=n}} \binom{2k_1}{k_1}\frac{1}{4^{k_1}}\frac{1}{k_2!}(h_{\epsilon_1+\epsilon_2}-n+k_2)\cdots(h_{\epsilon_1+\epsilon_2}-n+1) \\
&\quad = \sum_{\substack{(k_1,k_2)\in\mathbb{Z}_+^2 \\ k_1+k_2\leqslant n}} \binom{n-k_1-k_2+l-3}{n-k_1-k_2}\binom{2k_1}{k_1}\frac{1}{4^{k_1}}\binom{h_{\epsilon_1+\epsilon_2}-n+k_2}{k_2} \\
&\quad = \sum_{k_2=0}^{n}\binom{h_{\epsilon_1+\epsilon_2}-n+k_2}{k_2}\sum_{k_1=0}^{n-k_2}\binom{n-k_1-k_2+l-3}{n-k_1-k_2}\binom{2k_1}{k_1}\frac{1}{4^{k_1}} \\
&\quad = \sum_{k_2=0}^{n}\binom{h_{\epsilon_1+\epsilon_2}-n+k_2}{k_2}\binom{n-k_2+l-\frac{5}{2}}{n-k_2} = \binom{h_{\epsilon_1+\epsilon_2}+l-\frac{3}{2}}{n},
\end{aligned}$$

we get

$$\begin{aligned}
&\left(f_{\epsilon_1-\epsilon_2}^n f_{\epsilon_1+\epsilon_2}^n\right)_L\left(-\frac{1}{4}e_{\epsilon_1}^2 + e_{\epsilon_1-\epsilon_2}e_{\epsilon_1+\epsilon_2} + \cdots + e_{\epsilon_1-\epsilon_l}e_{\epsilon_1+\epsilon_l}\right)^n \\
&\quad \in c_1 h_{\epsilon_1-\epsilon_2}\cdot\ldots\cdot(h_{\epsilon_1-\epsilon_2}-n+1)\left(h_{\epsilon_1+\epsilon_2}+l-\frac{3}{2}\right)\cdot\ldots\cdot\left(h_{\epsilon_1+\epsilon_2}+l-n-\frac{1}{2}\right) + U(\mathfrak{g})\mathfrak{n}_+ \\
&\quad = c_1 p_1(h) + U(\mathfrak{g})\mathfrak{n}_+, \qquad (5)
\end{aligned}$$

for some $c_1 \neq 0$.

Thus, $p_1 \in \mathcal{P}_0$. Let us prove $p_i \in \mathcal{P}_0$, for $1 = 2, \ldots, l-1$. Using notation

$$\bar{u} = -\frac{1}{4}e_{\epsilon_1}^2 + e_{\epsilon_1-\epsilon_2}e_{\epsilon_1+\epsilon_2} + \cdots + e_{\epsilon_1-\epsilon_l}e_{\epsilon_1+\epsilon_l},$$

as in (1), we have

$$\begin{aligned}
\left(f_{\epsilon_1+\epsilon_i}^2\right)_L\bar{u} = 2\Big(&-\frac{1}{4}f_{\epsilon_i}^2 - e_{\epsilon_1-\epsilon_i}f_{\epsilon_1+\epsilon_i} - \cdots - e_{\epsilon_{i-1}-\epsilon_i}f_{\epsilon_{i-1}+\epsilon_i} \\
&+ f_{\epsilon_i-\epsilon_{i+1}}f_{\epsilon_i+\epsilon_{i+1}} + \cdots + f_{\epsilon_i-\epsilon_l}f_{\epsilon_i+\epsilon_l}\Big)
\end{aligned}$$

and $(f^3_{\epsilon_1+\epsilon_i})_L\bar{u}=0$, for $1=2,\dots,l-1$, which implies

$$\begin{aligned}
&\big(f^{2n}_{\epsilon_1+\epsilon_i}\big)_L\bigg(-\frac{1}{4}e^2_{\epsilon_1}+e_{\epsilon_1-\epsilon_2}e_{\epsilon_1+\epsilon_2}+\cdots+e_{\epsilon_1-\epsilon_l}e_{\epsilon_1+\epsilon_l}\bigg)^n\\
&\quad=\big(f^{2n}_{\epsilon_1+\epsilon_i}\big)_L\big(\bar{u}^n\big)=\frac{(2n)!}{(2!)^n}\big(\big(f^2_{\epsilon_1+\epsilon_i}\big)_L\bar{u}\big)^n\\
&\quad=(2n)!\bigg(-\frac{1}{4}f^2_{\epsilon_i}-e_{\epsilon_1-\epsilon_i}f_{\epsilon_1+\epsilon_i}-\cdots-e_{\epsilon_{i-1}-\epsilon_i}f_{\epsilon_{i-1}+\epsilon_i}\\
&\qquad+f_{\epsilon_i-\epsilon_{i+1}}f_{\epsilon_i+\epsilon_{i+1}}+\cdots+f_{\epsilon_i-\epsilon_l}f_{\epsilon_i+\epsilon_l}\bigg)^n.
\end{aligned}$$

Since all root vectors $f_{\epsilon_i}, e_{\epsilon_1-\epsilon_i}, f_{\epsilon_1+\epsilon_i},\dots,e_{\epsilon_{i-1}-\epsilon_i}, f_{\epsilon_{i-1}+\epsilon_i}, f_{\epsilon_i-\epsilon_{i+1}}, f_{\epsilon_i+\epsilon_{i+1}},\dots,f_{\epsilon_i-\epsilon_l}, f_{\epsilon_i+\epsilon_l}\in\mathfrak{g}$ commute, we get

$$\begin{aligned}
&\bigg(-\frac{1}{4}f^2_{\epsilon_i}-e_{\epsilon_1-\epsilon_i}f_{\epsilon_1+\epsilon_i}-\cdots-e_{\epsilon_{i-1}-\epsilon_i}f_{\epsilon_{i-1}+\epsilon_i}+f_{\epsilon_i-\epsilon_{i+1}}f_{\epsilon_i+\epsilon_{i+1}}+\cdots+f_{\epsilon_i-\epsilon_l}f_{\epsilon_i+\epsilon_l}\bigg)^n\\
&\quad=\sum_{\substack{(k_1,\dots,k_l)\in\mathbb{Z}^l_+\\ \sum k_j=n}}\binom{n}{k_1,\dots,k_l}(-1)^{k_1+\cdots+k_i}\frac{1}{4^{k_i}}f^{2k_i}_{\epsilon_i}e^{k_1}_{\epsilon_1-\epsilon_i}f^{k_1}_{\epsilon_1+\epsilon_i}\cdots e^{k_{i-1}}_{\epsilon_{i-1}-\epsilon_i}f^{k_{i-1}}_{\epsilon_{i-1}+\epsilon_i}\\
&\qquad\cdot f^{k_{i+1}}_{\epsilon_i-\epsilon_{i+1}}f^{k_{i+1}}_{\epsilon_i+\epsilon_{i+1}}\cdots f^{k_l}_{\epsilon_i-\epsilon_l}f^{k_l}_{\epsilon_i+\epsilon_l}.
\end{aligned}\tag{6}$$

If any of indices $k_1,\dots,k_{i-1}$ is nonzero, then the corresponding summand in (6) is an element of $U(\mathfrak{g})\mathfrak{n}_+$. Thus we obtain

$$\begin{aligned}
&\big(f^{2n}_{\epsilon_1+\epsilon_i}\big)_L\bigg(-\frac{1}{4}e^2_{\epsilon_1}+e_{\epsilon_1-\epsilon_2}e_{\epsilon_1+\epsilon_2}+\cdots+e_{\epsilon_1-\epsilon_l}e_{\epsilon_1+\epsilon_l}\bigg)^n\\
&\quad\in(2n)!\sum_{\substack{(k_i,\dots,k_l)\in\mathbb{Z}^l_+\\ \sum k_j=n}}\binom{n}{k_i,\dots,k_l}(-1)^{k_i}\frac{1}{4^{k_i}}f^{2k_i}_{\epsilon_i}f^{k_{i+1}}_{\epsilon_i-\epsilon_{i+1}}f^{k_{i+1}}_{\epsilon_i+\epsilon_{i+1}}\cdots f^{k_l}_{\epsilon_i-\epsilon_l}f^{k_l}_{\epsilon_i+\epsilon_l}+U(\mathfrak{g})\mathfrak{n}_+\\
&\quad=(2n)!\bigg(-\frac{1}{4}f^2_{\epsilon_i}+f_{\epsilon_i-\epsilon_{i+1}}f_{\epsilon_i+\epsilon_{i+1}}+\cdots+f_{\epsilon_i-\epsilon_l}f_{\epsilon_i+\epsilon_l}\bigg)^n+U(\mathfrak{g})\mathfrak{n}_+.
\end{aligned}$$

Let $\mathfrak{g}'$ be the subalgebra of $\mathfrak{g}$ associated to roots $\alpha_i,\dots,\alpha_l$. Then $\mathfrak{g}'$ is a simple Lie algebra of type B_{l-i+1}. Let $\mathfrak{g}'=\mathfrak{n}'_-\oplus\mathfrak{h}'\oplus\mathfrak{n}'_+$ be the corresponding triangular decomposition of $\mathfrak{g}'$. Universal enveloping algebra $U(\mathfrak{g}')$ is embedded in $U(\mathfrak{g})$ in the natural way. Vector

$$\bigg(-\frac{1}{4}f^2_{\epsilon_i}+f_{\epsilon_i-\epsilon_{i+1}}f_{\epsilon_i+\epsilon_{i+1}}+\cdots+f_{\epsilon_i-\epsilon_l}f_{\epsilon_i+\epsilon_l}\bigg)^n\in U(\mathfrak{g}')$$

is the lowest weight vector for $\mathfrak{g}'$. Let R' be a $\mathfrak{g}'$-module generated with this vector, and R'_0 zero-weight subspace of R'. R' is then a $\mathfrak{g}'$-module with highest weight $2\epsilon_i$ and highest weight vector

$$\left(-\frac{1}{4}e_{\epsilon_i}^2 + e_{\epsilon_i-\epsilon_{i+1}}e_{\epsilon_i+\epsilon_{i+1}} + \cdots + e_{\epsilon_i-\epsilon_l}e_{\epsilon_i+\epsilon_l}\right)^n.$$

Clearly,

$$\left(f_{\epsilon_i-\epsilon_{i+1}}^n f_{\epsilon_i+\epsilon_{i+1}}^n\right)_L \left(-\frac{1}{4}e_{\epsilon_i}^2 + e_{\epsilon_i-\epsilon_{i+1}}e_{\epsilon_i+\epsilon_{i+1}} + \cdots + e_{\epsilon_i-\epsilon_l}e_{\epsilon_i+\epsilon_l}\right)^n \in R'_0.$$

If we apply relation (5) to subalgebra $\mathfrak{g}'$ of type B_{l-i+1}, we get

$$\begin{aligned}
&\left(f_{\epsilon_i-\epsilon_{i+1}}^n f_{\epsilon_i+\epsilon_{i+1}}^n\right)_L \left(-\frac{1}{4}e_{\epsilon_i}^2 + e_{\epsilon_i-\epsilon_{i+1}}e_{\epsilon_i+\epsilon_{i+1}} + \cdots + e_{\epsilon_i-\epsilon_l}e_{\epsilon_i+\epsilon_l}\right)^n \\
&\quad \in c_1 h_{\epsilon_i-\epsilon_{i+1}} \cdot \ldots \cdot (h_{\epsilon_i-\epsilon_{i+1}} - n + 1)\left(h_{\epsilon_i+\epsilon_{i+1}} + l - i - \frac{1}{2}\right) \cdot \ldots \\
&\qquad \cdot \left(h_{\epsilon_i+\epsilon_{i+1}} + l - n - i + \frac{1}{2}\right) + U(\mathfrak{g}')\mathfrak{n}'_+ \\
&\quad = c_1\, p_i(h) + U(\mathfrak{g}')\mathfrak{n}'_+.
\end{aligned}$$

Clearly, there exists $Y \in U(\mathfrak{n}'_+)$ such that

$$\begin{aligned}
&Y_L \left(-\frac{1}{4}f_{\epsilon_i}^2 + f_{\epsilon_i-\epsilon_{i+1}}f_{\epsilon_i+\epsilon_{i+1}} + \cdots + f_{\epsilon_i-\epsilon_l}f_{\epsilon_i+\epsilon_l}\right)^n \\
&\quad = \left(f_{\epsilon_i-\epsilon_{i+1}}^n f_{\epsilon_i+\epsilon_{i+1}}^n\right)_L \left(-\frac{1}{4}e_{\epsilon_i}^2 + e_{\epsilon_i-\epsilon_{i+1}}e_{\epsilon_i+\epsilon_{i+1}} + \cdots + e_{\epsilon_i-\epsilon_l}e_{\epsilon_i+\epsilon_l}\right)^n,
\end{aligned}$$

which implies

$$\begin{aligned}
&\left(Y f_{\epsilon_1+\epsilon_i}^{2n}\right)_L \left(-\frac{1}{4}e_{\epsilon_1}^2 + e_{\epsilon_1-\epsilon_2}e_{\epsilon_1+\epsilon_2} + \cdots + e_{\epsilon_1-\epsilon_l}e_{\epsilon_1+\epsilon_l}\right)^n \\
&\quad \in (2n)!\, Y_L \left(-\frac{1}{4}f_{\epsilon_i}^2 + f_{\epsilon_i-\epsilon_{i+1}}f_{\epsilon_i+\epsilon_{i+1}} + \cdots + f_{\epsilon_i-\epsilon_l}f_{\epsilon_i+\epsilon_l}\right)^n + U(\mathfrak{g})\mathfrak{n}_+ \\
&\quad = c_i\, p_i(h) + U(\mathfrak{g}')\mathfrak{n}'_+ + U(\mathfrak{g})\mathfrak{n}_+ \\
&\quad = c_i\, p_i(h) + U(\mathfrak{g})\mathfrak{n}_+,
\end{aligned} \tag{7}$$

for some $c_i \neq 0$. Thus $p_i \in \mathcal{P}_0$, for $i = 2, \ldots, l-1$.

(3) We claim that

$$\left(e_{\epsilon_l}^{2n} f_{\epsilon_1+\epsilon_l}^{2n}\right)_L \left(-\frac{1}{4}e_{\epsilon_1}^2 + e_{\epsilon_1-\epsilon_2}e_{\epsilon_1+\epsilon_2} + \cdots + e_{\epsilon_1-\epsilon_l}e_{\epsilon_1+\epsilon_l}\right)^n \in c_l\, p_l(h) + U(\mathfrak{g})\mathfrak{n}_+,$$

for some $c_l \neq 0$.

Using notation

$$\bar{u} = -\frac{1}{4}e_{\epsilon_1}^2 + e_{\epsilon_1-\epsilon_2}e_{\epsilon_1+\epsilon_2} + \cdots + e_{\epsilon_1-\epsilon_l}e_{\epsilon_1+\epsilon_l} \in U(\mathfrak{g}),$$

as in (1), we have

$$\left(f_{\epsilon_1+\epsilon_l}^2\right)_L \bar{u} = -2\left(\frac{1}{4}f_{\epsilon_l}^2 + f_{\epsilon_1+\epsilon_l}e_{\epsilon_1-\epsilon_l} + f_{\epsilon_2+\epsilon_l}e_{\epsilon_2-\epsilon_l} + \cdots + f_{\epsilon_{l-1}+\epsilon_l}e_{\epsilon_{l-1}-\epsilon_l}\right)$$

and $(f_{\epsilon_1+\epsilon_l}^3)_L \bar{u} = 0$, which implies

$$\begin{aligned}
&\left(f_{\epsilon_1+\epsilon_l}^{2n}\right)_L \left(-\frac{1}{4}e_{\epsilon_1}^2 + e_{\epsilon_1-\epsilon_2}e_{\epsilon_1+\epsilon_2} + \cdots + e_{\epsilon_1-\epsilon_l}e_{\epsilon_1+\epsilon_l}\right)^n \\
&\quad = \left(f_{\epsilon_1+\epsilon_l}^{2n}\right)_L \left(\bar{u}^n\right) = \frac{(2n)!}{(2!)^n}\left(\left(f_{\epsilon_1+\epsilon_l}^2\right)_L \bar{u}\right)^n \\
&\quad = (-1)^n (2n)! \left(\frac{1}{4}f_{\epsilon_l}^2 + f_{\epsilon_1+\epsilon_l}e_{\epsilon_1-\epsilon_l} + \cdots + f_{\epsilon_{l-1}+\epsilon_l}e_{\epsilon_{l-1}-\epsilon_l}\right)^n.
\end{aligned}$$

We obtain

$$\begin{aligned}
&\left(e_{\epsilon_l}^{2n} f_{\epsilon_1+\epsilon_l}^{2n}\right)_L \left(-\frac{1}{4}e_{\epsilon_1}^2 + e_{\epsilon_1-\epsilon_2}e_{\epsilon_1+\epsilon_2} + \cdots + e_{\epsilon_1-\epsilon_l}e_{\epsilon_1+\epsilon_l}\right)^n \\
&\quad = (-1)^n (2n)! \left(e_{\epsilon_l}^{2n}\right)_L \left(\frac{1}{4}f_{\epsilon_l}^2 + f_{\epsilon_1+\epsilon_l}e_{\epsilon_1-\epsilon_l} + f_{\epsilon_2+\epsilon_l}e_{\epsilon_2-\epsilon_l} + \cdots + f_{\epsilon_{l-1}+\epsilon_l}e_{\epsilon_{l-1}-\epsilon_l}\right)^n.
\end{aligned}$$

Clearly,

$$\begin{aligned}
&\left(\frac{1}{4}f_{\epsilon_l}^2 + f_{\epsilon_1+\epsilon_l}e_{\epsilon_1-\epsilon_l} + f_{\epsilon_2+\epsilon_l}e_{\epsilon_2-\epsilon_l} + \cdots + f_{\epsilon_{l-1}+\epsilon_l}e_{\epsilon_{l-1}-\epsilon_l}\right)^n \\
&\quad = \sum_{\substack{(k_1,\ldots,k_l)\in\mathbb{Z}_+^l \\ \sum k_i = n}} \binom{n}{k_1,\ldots,k_l} \frac{1}{4^{k_l}} f_{\epsilon_l}^{2k_l} f_{\epsilon_1+\epsilon_l}^{k_1} e_{\epsilon_1-\epsilon_l}^{k_1} \cdots f_{\epsilon_{l-1}+\epsilon_l}^{k_{l-1}} e_{\epsilon_{l-1}-\epsilon_l}^{k_{l-1}} \\
&\quad = \sum_{\substack{(k_1,\ldots,k_l)\in\mathbb{Z}_+^l \\ \sum k_i = n}} \binom{n}{k_1,\ldots,k_l} \frac{1}{4^{k_l}} f_{\epsilon_l}^{2k_l} f_{\epsilon_1+\epsilon_l}^{k_1} \cdots f_{\epsilon_{l-1}+\epsilon_l}^{k_{l-1}} e_{\epsilon_1-\epsilon_l}^{k_1} \cdots e_{\epsilon_{l-1}-\epsilon_l}^{k_{l-1}}.
\end{aligned} \tag{8}$$

If any of indices $k_1 \ldots k_{l-1}$ is nonzero, then the corresponding summand above is an element of $U(\mathfrak{g})\mathfrak{n}_+$. It follows that

$$\begin{aligned}
&\left(e_{\epsilon_l}^{2n} f_{\epsilon_1+\epsilon_l}^{2n}\right)_L \left(-\frac{1}{4}e_{\epsilon_1}^2 + e_{\epsilon_1-\epsilon_2}e_{\epsilon_1+\epsilon_2} + \cdots + e_{\epsilon_1-\epsilon_l}e_{\epsilon_1+\epsilon_l}\right)^n \\
&\quad \in (-1)^n (2n)! \frac{1}{4^n}\left(e_{\epsilon_l}^{2n}\right)_L \left(f_{\epsilon_l}^{2n}\right) + U(\mathfrak{g})\mathfrak{n}_+.
\end{aligned}$$

By using claim (1) from Lemma 16, we obtain

$$\begin{aligned}&\left(e_{\epsilon_l}^{2n} f_{\epsilon_1+\epsilon_l}^{2n}\right)_L \left(-\frac{1}{4}e_{\epsilon_1}^2 + e_{\epsilon_1-\epsilon_2}e_{\epsilon_1+\epsilon_2} + \cdots + e_{\epsilon_1-\epsilon_l}e_{\epsilon_1+\epsilon_l}\right)^n \\ &\quad \in c_l h_l(h_l-1)\cdot\ldots\cdot(h_l-2n+1) + U(\mathfrak{g})\mathfrak{n}_+,\end{aligned}$$

for some $c_l \neq 0$, and the proof is complete. □

Proposition 19. *There are finitely many irreducible $A(L(n-l+\frac{1}{2},0))$-modules from the category $\mathcal{O}$.*

Proof. It follows from Corollary 14 that highest weights $\mu \in \mathfrak{h}^*$ of irreducible $A(L(n-l+\frac{1}{2},0))$-modules $V(\mu)$ satisfy $p(\mu)=0$ for all $p \in \mathcal{P}_0$. Lemma 18 implies that $p_1(\mu)=p_2(\mu)=\cdots=p_l(\mu)=0$ for such weights μ. Every weight $\mu \in \mathfrak{h}^*$ is uniquely determined by its values $\mu_i=\mu(h_i)$, for $i=1,\ldots,l$. The equation $p_l(\mu)=0$ is

$$\mu_l(\mu_l-1)\cdot\ldots\cdot(\mu_l-2n+1)=0,$$

which implies that there are $2n$ distinct values for μ_l. The equation $p_{l-1}(\mu)=0$ is

$$\mu_{l-1}\cdot\ldots\cdot(\mu_{l-1}-n+1)\left(\mu_{l-1}+\mu_l+\frac{1}{2}\right)\cdot\ldots\cdot\left(\mu_{l-1}+\mu_l-n+\frac{3}{2}\right)=0,$$

which implies that for every fixed μ_l, there are $2n$ distinct values for μ_{l-1}. If we continue this procedure, at the end we get that equation $p_1(\mu)=0$ is

$$\begin{aligned}&\mu_1\cdot\ldots\cdot(\mu_1-n+1)\left(\mu_1+2\mu_2+\cdots+2\mu_{l-1}+\mu_l+l-\frac{3}{2}\right)\cdot\ldots \\ &\quad\cdot\left(\mu_1+2\mu_2+\cdots+2\mu_{l-1}+\mu_l+l-n-\frac{1}{2}\right)=0,\end{aligned}$$

which implies that for fixed $\mu_2,\ldots,\mu_l$, there are $2n$ distinct values for μ_1.

Thus, there are at most $(2n)^l$ weights $\mu \in \mathfrak{h}^*$ such that $p_1(\mu)=p_2(\mu)=\cdots=p_l(\mu)=0$, which implies the claim of the theorem. □

It follows from Zhu's theory that:

Theorem 20. *There are finitely many irreducible weak $L(n-l+\frac{1}{2},0)$-modules from the category $\mathcal{O}$.*

3.5. Classification of irreducible $L(n-l+\frac{1}{2},0)$-modules

In this section we classify irreducible $L(n-l+\frac{1}{2},0)$-modules. It follows from Proposition 3 that irreducible $\mathbb{Z}_+$-graded weak $L(n-l+\frac{1}{2},0)$-modules are in one-to-one correspondence with irreducible $A(L(n-l+\frac{1}{2},0))$-modules. Specially, irreducible $L(n-l+\frac{1}{2},0)$-modules are in one-to-one correspondence with finite-dimensional irreducible $A(L(n-l+\frac{1}{2},0))$-modules.

It follows from Proposition 12 that $A(L(n-l+\frac{1}{2},0)) \cong \frac{U(\mathfrak{g})}{I_n}$, which implies that every finite-dimensional irreducible $A(L(n-l+\frac{1}{2},0))$-module is a finite-dimensional irreducible $\mathfrak{g}$-module, and therefore is of the form $V(\mu)$, where $\mu \in P_+$ is a dominant integral weight of $\mathfrak{g}$.

Lemma 21. *Assume that $V(\mu)$, $\mu \in P_+$ is an $A(L(n-l+\frac{1}{2},0))$-module. Then $(\mu,\epsilon_1) \leqslant n-\frac{1}{2}$.*

Proof. If $V(\mu)$, $\mu \in P_+$ is an $A(L(n-l+\frac{1}{2},0))$-module, then Corollary 14 and Lemma 18 imply that $q(\mu)=0$. Thus,

$$\begin{aligned} 0 = \sum_{\substack{(k_1,\dots,k_l)\in\mathbb{Z}_+^l \\ \sum k_i = n}} & \frac{1}{k_1! 4^{k_1}} \cdot \bigl(\mu(h_{\epsilon_1}) - 2k_2 - \cdots - 2k_l\bigr) \cdot \ldots \cdot \bigl(\mu(h_{\epsilon_1}) - 2n + 1\bigr) \\ & \cdot \bigl(\mu(h_{\epsilon_1-\epsilon_l}) - k_{l-1} - \cdots - k_2\bigr) \cdot \ldots \cdot \bigl(\mu(h_{\epsilon_1-\epsilon_l}) - k_{l-1} - \cdots - k_2 - k_l - 1\bigr) \cdot \ldots \\ & \cdot \mu(h_{\epsilon_1-\epsilon_2}) \cdot \ldots \cdot \bigl(\mu(h_{\epsilon_1-\epsilon_2}) - k_2 + 1\bigr). \end{aligned}$$

We claim that this relation implies $(\mu,\epsilon_1) \leqslant n-\frac{1}{2}$, or equivalently $\mu(h_{\epsilon_1}) \leqslant 2n-1$. Suppose that $\mu(h_{\epsilon_1}) \geqslant 2n$. We claim that, under that assumption, all the summands above are nonnegative. Let $(k_1,\dots,k_l) \in \mathbb{Z}_+^l$ be any l-tuple, such that $\sum_{i=1}^l k_i = n$. It is clear that

$$\bigl(\mu(h_{\epsilon_1}) - 2k_2 - \cdots - 2k_l\bigr) \cdot \ldots \cdot \bigl(\mu(h_{\epsilon_1}) - 2n + 1\bigr) \geqslant 0.$$

Assume that

$$\begin{aligned} & \bigl(\mu(h_{\epsilon_1-\epsilon_l}) - k_{l-1} - \cdots - k_2\bigr) \cdot \ldots \cdot \bigl(\mu(h_{\epsilon_1-\epsilon_l}) - k_{l-1} - \cdots - k_2 - k_l - 1\bigr) \cdot \ldots \\ & \quad \cdot \mu(h_{\epsilon_1-\epsilon_2}) \cdot \ldots \cdot \bigl(\mu(h_{\epsilon_1-\epsilon_2}) - k_2 + 1\bigr) \neq 0. \end{aligned}$$

Then, from $\mu(h_{\epsilon_1-\epsilon_2}) \cdot \ldots \cdot (\mu(h_{\epsilon_1-\epsilon_2}) - k_2 + 1) \neq 0$, and from $\mu(h_{\epsilon_1-\epsilon_2}) \in \mathbb{Z}_+$ follows $\mu(h_{\epsilon_1-\epsilon_2}) \geqslant k_2$, which implies $\mu(h_{\epsilon_1-\epsilon_2}) \cdot \ldots \cdot (\mu(h_{\epsilon_1-\epsilon_2}) - k_2 + 1) > 0$. Since $h_{\epsilon_1-\epsilon_3} = h_{\epsilon_1-\epsilon_2} + h_{\epsilon_2-\epsilon_3}$, we have $\mu(h_{\epsilon_1-\epsilon_3}) \geqslant \mu(h_{\epsilon_1-\epsilon_2}) \geqslant k_2$. From $\mu(h_{\epsilon_1-\epsilon_3} - k_2) \cdot \ldots \cdot (\mu(h_{\epsilon_1-\epsilon_3}) - k_2 - k_3 + 1) \neq 0$, and from $\mu(h_{\epsilon_1-\epsilon_3}) \geqslant k_2$ follows $\mu(h_{\epsilon_1-\epsilon_3}) \geqslant k_2 + k_3$, which implies $\mu(h_{\epsilon_1-\epsilon_3} - k_2) \cdot \ldots \cdot (\mu(h_{\epsilon_1-\epsilon_3}) - k_2 - k_3 + 1) > 0$. Inductively, we obtain

$$\begin{aligned} & \bigl(\mu(h_{\epsilon_1-\epsilon_l}) - k_{l-1} - \cdots - k_2\bigr) \cdot \ldots \cdot \bigl(\mu(h_{\epsilon_1-\epsilon_l}) - k_{l-1} - \cdots - k_2 - k_l - 1\bigr) \cdot \ldots \\ & \quad \cdot \mu(h_{\epsilon_1-\epsilon_2}) \cdot \ldots \cdot \bigl(\mu(h_{\epsilon_1-\epsilon_2}) - k_2 + 1\bigr) > 0. \end{aligned}$$

Thus, we have shown that all the summands above are nonnegative.

On the other hand, for $k_1 = n$, $k_2 = k_3 = \cdots = k_l = 0$, we have the summand

$$\frac{1}{n! 4^n} \mu(h_{\epsilon_1}) \cdot \ldots \cdot \bigl(\mu(h_{\epsilon_1}) - 2n + 1\bigr) > 0.$$

Therefore

$$0 < \sum_{\substack{(k_1,\ldots,k_l)\in\mathbb{Z}_+^l \\ \sum k_i = n}} \frac{1}{k_1! 4^{k_1}} \cdot \bigl(\mu(h_{\epsilon_1}) - 2k_2 - \cdots - 2k_l\bigr) \cdot \ldots \cdot \bigl(\mu(h_{\epsilon_1}) - 2n + 1\bigr)$$
$$\cdot \bigl(\mu(h_{\epsilon_1-\epsilon_l}) - k_{l-1} - \cdots - k_2\bigr) \cdot \ldots \cdot \bigl(\mu(h_{\epsilon_1-\epsilon_l}) - k_{l-1} - \cdots - k_2 - k_l - 1\bigr) \cdot \ldots$$
$$\cdot \mu(h_{\epsilon_1-\epsilon_2}) \cdot \ldots \cdot \bigl(\mu(h_{\epsilon_1-\epsilon_2}) - k_2 + 1\bigr),$$

which is a contradiction. Thus, $(\mu, \epsilon_1) \leqslant n - \frac{1}{2}$. □

The converse of Lemma 21 also holds:

Lemma 22. *Let $\mu \in P_+$, such that $(\mu, \epsilon_1) \leqslant n - \frac{1}{2}$. Then $V(\mu)$ is an $A(L(n-l+\frac{1}{2}, 0))$-module.*

Proof. Since $A(L(n-l+\frac{1}{2}, 0)) \cong \frac{U(\mathfrak{g})}{I_n}$, we have to show $I_n.V(\mu) = 0$. Since the ideal I_n is generated by the vector v_n', it is sufficient to show that v_n' annihilates $V(\mu)$.

Suppose that there exists a vector $u \in V(\mu)$, such that $v_n'.u \neq 0$. From the structure of the root system of type B_l follows that the lowest weight in the module $V(\mu)$ is $-\mu$. Then the weight of the vector u is of the form $-\mu + \sum_{i=1}^{l} k_i\alpha_i$, where $k_i \in \mathbb{Z}_+$ for $i = 1, \ldots, l$, and the weight of the vector $v_n'.u$ has to be of the form $\mu - \sum_{i=1}^{l} t_i\alpha_i$, where $t_i \in \mathbb{Z}_+$ for $i = 1, \ldots, l$. Since v_n' has the weight $2n\epsilon_1$, we obtain the equation: $-\mu + \sum_{i=1}^{l} k_i\alpha_i + 2n\epsilon_1 = \mu - \sum_{i=1}^{l} t_i\alpha_i$, or equivalently

$$2\mu - 2n\epsilon_1 = \sum_{i=1}^{l} m_i\alpha_i,$$

where $m_i = k_i + t_i \in \mathbb{Z}_+$ for $i = 1, \ldots, l$. It follows from the equation above that $(2\mu - 2n\epsilon_1, \epsilon_1) = (\sum_{i=1}^{l} m_i\alpha_i, \epsilon_1)$. Since $(\alpha_1, \epsilon_1) = 1$ and $(\alpha_i, \epsilon_1) = 0$ for $i = 2, \ldots, l$, we obtain $2(\mu, \epsilon_1) - 2n = m_1$. From $(\mu, \epsilon_1) \leqslant n - \frac{1}{2}$ we obtain

$$m_1 = 2(\mu, \epsilon_1) - 2n \leqslant -1,$$

which is a contradiction with $m_i \in \mathbb{Z}_+$ for $i = 1, \ldots, l$.

Thus, v_n' annihilates $V(\mu)$, and $V(\mu)$ is an $A(L(n-l+\frac{1}{2}, 0))$-module. □

Proposition 23. *The set*

$$\left\{ V(\mu) \;\middle|\; \mu \in P_+,\ (\mu, \epsilon_1) \leqslant n - \frac{1}{2} \right\}$$

provides the complete list of irreducible finite-dimensional $A(L(n-l+\frac{1}{2}, 0))$-modules.

It follows from Zhu's theory that:

Theorem 24. *The set*

$$\left\{ L\left(n - l + \frac{1}{2}, \mu\right) \;\middle|\; \mu \in P_+,\ (\mu, \epsilon_1) \leqslant n - \frac{1}{2} \right\}$$

provides the complete list of irreducible $L(n-l+\frac{1}{2}, 0)$-modules.

3.6. Complete reducibility in category of $L(n-l+\frac{1}{2},0)$-modules

In this subsection we show that every $L(n-l+\frac{1}{2},0)$-module is completely reducible. The following lemma is crucial for proving complete reducibility.

Lemma 25. *Let $L(\lambda)$ be a $L(n-l+\frac{1}{2},0)$-module. Then the weight λ is admissible.*

Proof. If $L(\lambda)$ is a $L(n-l+\frac{1}{2},0)$-module, then Theorem 24 implies that $\lambda=(n-l+\frac{1}{2})\Lambda_0+\mu$, for some weight $\mu\in P_+$, such that $(\mu,\epsilon_1)\leqslant n-\frac{1}{2}$. It follows that

$$\langle\lambda+\rho,\alpha_i^\vee\rangle=\langle\mu,\alpha_i^\vee\rangle+1\in\mathbb{N}\quad\text{for } i=1,\dots,l,$$
$$\langle\lambda+\rho,(\delta-\epsilon_1)^\vee\rangle=2n-2(\mu,\epsilon_1)\in\mathbb{N},$$

which implies that λ is admissible weight and that $\hat{\Pi}_\lambda^\vee=\{(\delta-\epsilon_1)^\vee,\alpha_1^\vee,\alpha_2^\vee,\dots,\alpha_l^\vee\}$. □

Lemma 26. *Let M be a $L(n-l+\frac{1}{2},0)$-module. Then M is from the category $\mathcal{O}$ as a $\hat{\mathfrak{g}}$-module.*

Proof. Let $M=\bigoplus_{\alpha\in\mathbb{C}}M_{(\alpha)}$, where $L(0)u=\alpha u$ for any $u\in M_{(\alpha)}$, $\dim M_{(\alpha)}<\infty$ for any $\alpha\in\mathbb{C}$ and $M_{(\alpha+n)}=0$ for $n\in\mathbb{Z}$ sufficiently small. It follows from $a_mM_{(\alpha)}\subset M_{(\alpha+\mathrm{wt}\,a-m-1)}$ for $a\in V$, that $M_{(\alpha)}$ is a $\mathfrak{g}$-module, for any $\alpha\in\mathbb{C}$. Since $M_{(\alpha)}$ is finite-dimensional, $\mathfrak{h}$ acts semisimply on $M_{(\alpha)}$, which implies that $\hat{\mathfrak{h}}$ acts semisimply on M with finite-dimensional weight spaces. Let $v\in M$ be a singular vector of weight $\lambda\in\hat{\mathfrak{h}}^*$. Then $L(\lambda)$ is an irreducible subquotient of M, which implies that $L(\lambda)$ is a $L(n-l+\frac{1}{2},0)$-module. It follows from Theorem 24 that there are finitely many irreducible $L(n-l+\frac{1}{2},0)$-modules, which implies that there exists a finite number of weights $\nu_1,\dots,\nu_k\in\hat{\mathfrak{h}}^*$ such that for every weight ν of M holds $\nu\in\bigcup_{i=1}^kD(\nu_i)$. Thus $\hat{\mathfrak{g}}$-module M is from the category $\mathcal{O}$. □

Theorem 27. *Let M be a $L(n-l+\frac{1}{2},0)$-module. Then M is completely reducible.*

Proof. Let $L(\lambda)$ be some irreducible subquotient of M. Then $L(\lambda)$ is a $L(n-l+\frac{1}{2},0)$-module, and Lemma 25 implies that λ is admissible weight. It follows from Lemma 26 that M is from the category $\mathcal{O}$, and then Proposition 5 implies that M is completely reducible. □

4. Weak $L(-l+\frac{3}{2},0)$-modules from category $\mathcal{O}$

In this section we study the special case $n=1$, i.e. the smallest admissible half-integer level $-l+\frac{3}{2}$. In this case we find a basis for the vector space $\mathcal{P}_0$, defined in Section 3.3, from which we get the classification of irreducible weak $L(-l+\frac{3}{2},0)$-modules from the category $\mathcal{O}$. We also show that every weak $L(-l+\frac{3}{2},0)$-module from the category $\mathcal{O}$ is completely reducible.

4.1. Classification of irreducible weak $L(-l+\frac{3}{2},0)$-modules from category $\mathcal{O}$

It follows from Corollary 14 that irreducible $A(L(-l+\frac{3}{2},0))$-modules are in one-to-one correspondence with weights $\mu\in\mathfrak{h}^*$ such that $p(\mu)=0$ for all $p\in\mathcal{P}_0$, where $\mathcal{P}_0=\{p_r\mid r\in R_0\}$.

In this case, R is a highest weight $U(\mathfrak{g})$-module with the highest weight $2\epsilon_1 = 2\omega_1$, and R_0 is zero-weight subspace of R.

Lemma 28.

$$\dim R_0 \leqslant l.$$

Proof. In this proof we use induction on l. We use the notation $V_l(\mu)$ for the highest weight module for simple Lie algebra of type B_l, with the highest weight $\mu \in \mathfrak{h}^*$.

For $l = 2$ it is easily checked that $\dim R = 14$ and $\dim R_0 = 2$. Assume that the claim of this lemma holds for simple Lie algebra of type B_{l-1}, $l-1 \geqslant 2$. Let $\mathfrak{g}$ be simple Lie algebra of type B_l. Let $\mathfrak{g}'$ be the subalgebra of $\mathfrak{g}$ associated to roots $\alpha_2, \dots, \alpha_l$. $\mathfrak{g}'$ is then a simple Lie algebra of type B_{l-1}. We can decompose $\mathfrak{g}$-module $V_l(2\omega_1)$ into the direct sum of irreducible $\mathfrak{g}'$-modules. If we denote by v the highest weight vector of $\mathfrak{g}$-module $V_l(2\omega_1)$, then it can be easily checked that $f_{\epsilon_1-\epsilon_2}^2.v$, $f_{\epsilon_1-\epsilon_2}.v$, $f_{\epsilon_1}^2 f_{\epsilon_1-\epsilon_2}.v$, v, $f_{\epsilon_1}^4.v$ and $(-\frac{1}{4} f_{\epsilon_1}^2 + f_{\epsilon_1-\epsilon_2} f_{\epsilon_1+\epsilon_2} + \cdots + f_{\epsilon_1-\epsilon_l} f_{\epsilon_1+\epsilon_l}).v$ are highest weight vectors for $\mathfrak{g}'$, which generate $\mathfrak{g}'$-modules isomorphic to $V_{l-1}(2\omega_1)$, $V_{l-1}(\omega_1)$, $V_{l-1}(\omega_1)$, $V_{l-1}(0)$, $V_{l-1}(0)$ and $V_{l-1}(0)$, respectively. It follows from Weyl's formula for the dimension of irreducible module that $\dim V_l(2\omega_1) = 2l^2 + 3l$ and $\dim V_l(\omega_1) = 2l + 1$, which implies that the direct sum of $\mathfrak{g}'$-modules above is $V_l(2\omega_1)$. Clearly, there are no zero-weight vectors for $\mathfrak{g}$ in $\mathfrak{g}'$-modules generated by highest weight vectors $f_{\epsilon_1-\epsilon_2}.v$, $f_{\epsilon_1}^2 f_{\epsilon_1-\epsilon_2}.v$, v and $f_{\epsilon_1}^4.v$. The inductive assumption implies that there are at most $l-1$ linearly independent zero-weight vectors for $\mathfrak{g}$ in $\mathfrak{g}'$-module $V_{l-1}(2\omega_1)$, which implies $\dim R_0 \leqslant (l-1) + 1 = l$. □

In the next lemma we find a basis for the vector space $\mathcal{P}_0$.

Lemma 29.

$$\mathcal{P}_0 = \mathrm{span}_{\mathbb{C}}\{p_1, \dots, p_l\},$$

where

$$p_i(h) = h_i\left(h_{\epsilon_i+\epsilon_{i+1}} + l - i - \frac{1}{2}\right), \quad \textit{for } i = 1, \dots, l-1,$$
$$p_l(h) = h_l(h_l - 1).$$

Proof. Lemma 18 implies that $p_1, \dots, p_l$ are linearly independent polynomials in the set $\mathcal{P}_0$. It follows from the definition of the set $\mathcal{P}_0$ that $\dim \mathcal{P}_0 \leqslant \dim R_0$, and using Lemma 28 we get $\dim \mathcal{P}_0 \leqslant l$. Thus, polynomials $p_1, \dots, p_l$ form a basis for $\mathcal{P}_0$. □

Proposition 30. *For every subset $S = \{i_1, \dots, i_k\} \subseteq \{1, 2, \dots, l-1\}$, $i_1 < \cdots < i_k$, we define weights*

$$\mu_S = \sum_{j=1}^{k}\left(i_j + 2\sum_{s=j+1}^{k}(-1)^{s-j} i_s + (-1)^{k-j+1}\left(l - \frac{1}{2}\right)\right)\omega_{i_j},$$

$$\mu'_S = \sum_{j=1}^{k} \left(i_j + 2 \sum_{s=j+1}^{k} (-1)^{s-j} i_s + (-1)^{k-j+1} \left(l + \frac{1}{2} \right) \right) \omega_{i_j} + \omega_l,$$

where $\omega_1, \ldots, \omega_l$ are fundamental weights for $\mathfrak{g}$. Then the set

$$\left\{ V(\mu_S), V\left(\mu'_S\right) \,\middle|\, S \subseteq \{1, 2, \ldots, l-1\} \right\}$$

provides the complete list of irreducible $A(L(-l + \frac{3}{2}, 0))$-modules from the category $\mathcal{O}$.

Proof. Since

$$h_{\epsilon_i + \epsilon_{i+1}} = h_i + 2h_{i+1} + \cdots + 2h_{l-1} + h_l,$$

it follows from Corollary 14 and Lemma 29 that highest weights $\mu \in \mathfrak{h}^*$ of irreducible $A(L(-l + \frac{3}{2}, 0))$-modules $V(\mu)$ are in one-to-one correspondence with solutions of the system of polynomial equations

$$\begin{aligned}
&h_1 \left(h_1 + 2h_2 + \cdots + 2h_{l-1} + h_l + l - \frac{3}{2} \right) = 0, \\
&h_2 \left(h_2 + 2h_3 + \cdots + 2h_{l-1} + h_l + l - \frac{5}{2} \right) = 0, \\
&\quad \vdots \\
&h_{l-1} \left(h_{l-1} + h_l + \frac{1}{2} \right) = 0, \\
&h_l (h_l - 1) = 0.
\end{aligned}$$

Clearly, $h_l \in \{0, 1\}$. Let $S = \{i_1, \ldots, i_k\}$, $i_1 < \cdots < i_k$ be the subset of $\{1, 2, \ldots, l-1\}$ such that $h_i = 0$ for $i \notin S$ and $h_i \neq 0$ for $i \in S$.

First consider the case $h_l = 0$. Then we have the system

$$\begin{aligned}
&h_{i_1} + 2h_{i_2} + \cdots + 2h_{i_k} + l - i_1 - \frac{1}{2} = 0, \\
&h_{i_2} + 2h_{i_3} + \cdots + 2h_{i_k} + l - i_2 - \frac{1}{2} = 0, \\
&\quad \vdots \\
&h_{i_{k-1}} + 2h_{i_k} + l - i_{k-1} - \frac{1}{2} = 0, \\
&h_{i_k} + l - i_k - \frac{1}{2} = 0.
\end{aligned} \tag{9}$$

The solution of this triangular system is

$$h_{i_j} = i_j + 2 \sum_{s=j+1}^{k} (-1)^{s-j} i_s + (-1)^{k-j+1} \left(l - \frac{1}{2} \right), \quad \text{for } j = 1, \ldots, k.$$

It follows that $V(\mu_S)$ is irreducible $A(L(-l+\frac{3}{2},0))$-module.

In the case when $h_l = 1$, we have the system

$$\begin{aligned}
&h_{i_1} + 2h_{i_2} + \cdots + 2h_{i_k} + l - i_1 + \frac{1}{2} = 0,\\
&h_{i_2} + 2h_{i_3} + \cdots + 2h_{i_k} + l - i_2 + \frac{1}{2} = 0,\\
&\vdots\\
&h_{i_{k-1}} + 2h_{i_k} + l - i_{k-1} + \frac{1}{2} = 0,\\
&h_{i_k} + l - i_k + \frac{1}{2} = 0,
\end{aligned}$$

whose solution is

$$h_{i_j} = i_j + 2\sum_{s=j+1}^{k} (-1)^{s-j} i_s + (-1)^{k-j+1}\left(l + \frac{1}{2}\right), \quad \text{for } j = 1, \ldots, k.$$

It follows that $V(\mu'_S)$ is irreducible $A(L(-l+\frac{3}{2},0))$-module, which completes the proof. □

It follows from Zhu's theory that:

Theorem 31. *The set*

$$\left\{L\left(-l+\frac{3}{2},\mu_S\right), L\left(-l+\frac{3}{2},\mu'_S\right) \,\middle|\, S \subseteq \{1,2,\ldots,l-1\}\right\}$$

provides the complete list of irreducible weak $L(-l+\frac{3}{2},0)$-modules from the category $\mathcal{O}$.

Theorem 31 implies that there are 2^l irreducible weak $L(-l+\frac{3}{2},0)$-modules from the category $\mathcal{O}$.

4.2. Complete reducibility of weak $L(-l+\frac{3}{2},0)$-modules from category $\mathcal{O}$

We introduce the notation $\lambda_S = (-l+\frac{3}{2})\Lambda_0 + \mu_S$ and $\lambda'_S = (-l+\frac{3}{2})\Lambda_0 + \mu'_S$, for every $S \subseteq \{1,2,\ldots,l-1\}$. The following lemma is crucial for proving complete reducibility.

Lemma 32. *Weights $\lambda_S, \lambda'_S \in \hat{\mathfrak{h}}^*$ are admissible, for every $S \subseteq \{1,2,\ldots,l-1\}$.*

Proof. We will prove that the weight λ_S is admissible for every $S \subseteq \{1,2,\ldots,l-1\}$. The proof of admissibility of weights λ'_S for every $S \subseteq \{1,2,\ldots,l-1\}$ is similar. We have to show

$$\langle \lambda_S + \rho, \tilde{\alpha}^\vee \rangle \notin -\mathbb{Z}_+ \quad \text{for any } \tilde{\alpha} \in \hat{\Delta}_+^{\mathrm{re}}, \tag{10}$$

$$\mathbb{Q}\hat{\Delta}_{\lambda_S}^{\vee\mathrm{re}} = \mathbb{Q}\hat{\Pi}^\vee. \tag{11}$$

Let us prove first the relation (10). Any positive real root $\tilde{\alpha} \in \hat{\Delta}_+^{\mathrm{re}}$ of $\hat{\mathfrak{g}}$ is of the form $\tilde{\alpha} = \alpha + m\delta$, for $m > 0$ and $\alpha \in \Delta$ or $m = 0$ and $\alpha \in \Delta_+$. It can be easily checked, as in Lemma 9, that

$$\langle \lambda_S + \rho, \tilde{\alpha}^\vee \rangle = \frac{2}{(\alpha, \alpha)} \left(m\left(l + \frac{1}{2}\right) + (\bar{\rho}, \alpha) + (\mu_S, \alpha) \right), \tag{12}$$

where $\bar{\rho}$ is the sum of fundamental weights of $\mathfrak{g}$. Let $S = \{i_1, \ldots, i_k\} \subseteq \{1, 2, \ldots, l-1\}$, $i_1 < \cdots < i_k$. Proposition 30 implies that $\mu_S = \sum_{j=1}^{k} h_{i_j} \omega_{i_j}$, where

$$h_{i_j} = i_j + 2 \sum_{s=j+1}^{k} (-1)^{s-j} i_s + (-1)^{k-j+1} \left(l - \frac{1}{2} \right), \quad \text{for } j = 1, \ldots, k.$$

From the system (9) easily follows

$$h_{i_j} + h_{i_{j+1}} = -(i_{j+1} - i_j) \quad \text{for } j = 1, \ldots, k-1. \tag{13}$$

We will consider three cases in this proof.

Case 1. The root α is of the form $\alpha = \pm(\epsilon_i - \epsilon_j)$, for $i, j = 1, \ldots, l$, $i < j$.

Then $(\bar{\rho}, \epsilon_i - \epsilon_j) = j - i$. Let $s, t \in \{1, \ldots, k\}$ be indices such that $S \cap \{i, i+1, \ldots, j-1\} = \{i_s, \ldots, i_t\}$. Clearly, $i_s \geqslant i$ and $i_t \leqslant j-1$. Furthermore, $(\mu_S, \epsilon_i - \epsilon_j) = h_{i_s} + \cdots + h_{i_t}$.

First consider the case $\alpha = \epsilon_i - \epsilon_j$, $i < j$ and $m \geqslant 0$.

If $t - s + 1$ is even, then using relation (13) we obtain

$$(\mu_S, \epsilon_i - \epsilon_j) = -(i_{s+1} - i_s) - \cdots - (i_t - i_{t-1}) \geqslant -(i_t - i_s),$$

and relation (12) implies

$$\begin{aligned} \langle \lambda_S + \rho, \tilde{\alpha}^\vee \rangle &\geqslant (\bar{\rho}, \epsilon_i - \epsilon_j) + (\mu_S, \epsilon_i - \epsilon_j) \geqslant (j - i) - (i_t - i_s) \\ &= (j - i_t) + (i_s - i) > 0. \end{aligned}$$

Suppose now that $t - s + 1$ is odd. Then $(\mu_S, \epsilon_i - \epsilon_j) \notin \mathbb{Z}$, and if $m = 0$, then $\langle \lambda_S + \rho, \tilde{\alpha}^\vee \rangle \notin \mathbb{Z}$. Let $m \geqslant 1$. Then

$$\begin{aligned} (\mu_S, \epsilon_i - \epsilon_j) &= h_{i_s} + \cdots + h_{i_{t-1}} + h_{i_t} = -(i_{s+1} - i_s) - \cdots - (i_{t-1} - i_{t-2}) + h_{i_t} \\ &\geqslant -(i_{t-1} - i_s) + h_{i_t}. \end{aligned}$$

If $k - t$ is even, we get

$$h_{i_t} = i_t + 2(i_{t+2} - i_{t+1}) + \cdots + 2(i_k - i_{k-1}) - \left(l - \frac{1}{2} \right) \geqslant -\left(l - \frac{1}{2} \right),$$

which implies

$$(\mu_S, \epsilon_i - \epsilon_j) \geqslant -(i_{t-1} - i_s) - \left(l - \frac{1}{2}\right).$$

We obtain

$$\begin{aligned}
\langle \lambda_S + \rho, \tilde{\alpha}^\vee \rangle &\geqslant l + \frac{1}{2} + (\bar{\rho}, \epsilon_i - \epsilon_j) + (\mu_S, \epsilon_i - \epsilon_j) \\
&\geqslant l + \frac{1}{2} + (j - i) - (i_{t-1} - i_s) - \left(l - \frac{1}{2}\right) \\
&= (j - i_{t-1}) + (i_s - i) + 1 > 0.
\end{aligned}$$

If $k - t$ is odd, then

$$h_{i_t} = i_t + 2(i_{t+2} - i_{t+1}) + \cdots + 2(i_{k-1} - i_{k-2}) - 2i_k + \left(l - \frac{1}{2}\right) \geqslant l - \frac{1}{2} - 2i_k,$$

which implies

$$(\mu_S, \epsilon_i - \epsilon_j) \geqslant -(i_{t-1} - i_s) + l - \frac{1}{2} - 2i_k.$$

We obtain

$$\begin{aligned}
\langle \lambda_S + \rho, \tilde{\alpha}^\vee \rangle &\geqslant l + \frac{1}{2} + (\bar{\rho}, \epsilon_i - \epsilon_j) + (\mu_S, \epsilon_i - \epsilon_j) \\
&\geqslant l + \frac{1}{2} + (j - i) - (i_{t-1} - i_s) + l - \frac{1}{2} - 2i_k \\
&= 2(l - i_k) + (j - i_{t-1}) + (i_s - i) > 0.
\end{aligned}$$

Thus, we have proved that, if $\alpha = \epsilon_i - \epsilon_j$, $i < j$ and $m \geqslant 0$, then $\langle \lambda_S + \rho, \tilde{\alpha}^\vee \rangle \notin -\mathbb{Z}_+$.

Now, let us consider the case $\alpha = -(\epsilon_i - \epsilon_j)$, $i < j$ and $m \geqslant 1$.

Then

$$\langle \lambda_S + \rho, \tilde{\alpha}^\vee \rangle = m\left(l + \frac{1}{2}\right) - (\bar{\rho}, \epsilon_i - \epsilon_j) - (\mu_S, \epsilon_i - \epsilon_j).$$

If $t - s + 1$ is even, then $(\mu_S, \epsilon_i - \epsilon_j)$ is an integer and $(\mu_S, \epsilon_i - \epsilon_j) \leqslant 0$, so if m is odd, then $\langle \lambda_S + \rho, \tilde{\alpha}^\vee \rangle \notin \mathbb{Z}$. If m is even, then $m \geqslant 2$, and we get

$$\langle \lambda_S + \rho, \tilde{\alpha}^\vee \rangle \geqslant 2\left(l + \frac{1}{2}\right) - (j - i) = (l - j) + l + i + 1 > 0.$$

If $t - s + 1$ is odd, then $(\mu_S, \epsilon_i - \epsilon_j) = h_{i_s} + \cdots + h_{i_{t-1}} + h_{i_t} \leqslant h_{i_t}$. If $k - t$ is even, then

$$\begin{aligned}h_{i_t} &= i_t + 2(i_{t+2} - i_{t+1}) + \cdots + 2(i_k - i_{k-1}) - \left(l - \frac{1}{2}\right)\\ &\leqslant i_t + 2(i_k - i_{t+1}) - \left(l - \frac{1}{2}\right),\end{aligned}$$

which implies

$$(\mu_S, \epsilon_i - \epsilon_j) \leqslant i_t + 2(i_k - i_{t+1}) - \left(l - \frac{1}{2}\right).$$

It follows

$$\begin{aligned}\langle \lambda_S + \rho, \tilde{\alpha}^\vee \rangle &\geqslant l + \frac{1}{2} - (j - i) - \left(i_t + 2(i_k - i_{t+1}) - \left(l - \frac{1}{2}\right)\right)\\ &= 2(l - i_k) + (i_{t+1} - i_t) + (i_{t+1} - j) + i > 0.\end{aligned}$$

If $k - t$ is odd, then

$$\begin{aligned}h_{i_t} &= i_t + 2(i_{t+2} - i_{t+1}) + \cdots + 2(i_{k-1} - i_{k-2}) - 2i_k + \left(l - \frac{1}{2}\right)\\ &\leqslant i_t + 2(i_{k-1} - i_{t+1}) - 2i_k + \left(l - \frac{1}{2}\right),\end{aligned}$$

which implies

$$(\mu_S, \epsilon_i - \epsilon_j) \leqslant i_t + 2(i_{k-1} - i_{t+1}) - 2i_k + \left(l - \frac{1}{2}\right).$$

It follows

$$\begin{aligned}\langle \lambda_S + \rho, \tilde{\alpha}^\vee \rangle &\geqslant l + \frac{1}{2} - (j - i) - \left(i_t + 2(i_{k-1} - i_{t+1}) - 2i_k + \left(l - \frac{1}{2}\right)\right)\\ &= 2(i_k - i_{k-1}) + (i_{t+1} - i_t) + \big(i_{t+1} - (j-1)\big) + i > 0.\end{aligned}$$

We have proved that, if $\alpha = -(\epsilon_i - \epsilon_j)$, $i < j$ and $m \geqslant 1$, then $\langle \lambda_S + \rho, \tilde{\alpha}^\vee \rangle \notin -\mathbb{Z}_+$.

Case 2. The root α is of the form $\alpha = \pm\epsilon_i$, for $i = 1, \ldots, l$.

Then $(\bar{\rho}, \epsilon_i) = l - i + \frac{1}{2}$. Let $s \in \{1, \ldots, k\}$ be the index such that $S \cap \{i, i+1, \ldots, l-1\} = \{i_s, \ldots, i_k\}$. Clearly, $i_{s-1} < i \leqslant i_s$. Furthermore, $(\mu_S, \epsilon_i) = h_{i_s} + \cdots + h_{i_k}$.

First consider the case $\alpha = \epsilon_i$ and $m \geqslant 0$.

Then

$$\langle \lambda_S + \rho, \tilde{\alpha}^\vee \rangle = 2\left(m\left(l + \frac{1}{2}\right) + (\bar{\rho}, \epsilon_i) + (\mu_S, \epsilon_i)\right). \tag{14}$$

If $k - s + 1$ is even, then using relation (13) we obtain

$$(\mu_S, \epsilon_i) = -(i_{s+1} - i_s) - \cdots - (i_k - i_{k-1}) \geqslant -(i_k - i_s),$$

which implies

$$\begin{aligned}\langle \lambda_S + \rho, \tilde{\alpha}^\vee \rangle &\geqslant 2(\bar{\rho}, \epsilon_i) + 2(\mu_S, \epsilon_i) \geqslant (2l - 2i + 1) - 2(i_k - i_s) \\ &= 2(l - i_k) + 2(i_s - i) + 1 > 0.\end{aligned}$$

If $k - s + 1$ is odd, then

$$\begin{aligned}(\mu_S, \epsilon_i) &= h_{i_s} + \cdots + h_{i_{k-1}} + h_{i_k} = -(i_{s+1} - i_s) - \cdots - (i_{k-1} - i_{k-2}) + h_{i_k} \\ &\geqslant -(i_{k-1} - i_s) + \left(-l + i_k + \frac{1}{2}\right) = -l + \frac{1}{2} + i_k - i_{k-1} + i_s,\end{aligned}$$

which implies

$$\begin{aligned}\langle \lambda_S + \rho, \tilde{\alpha}^\vee \rangle &\geqslant 2(\bar{\rho}, \epsilon_i) + 2(\mu_S, \epsilon_i) \geqslant (2l - 2i + 1) - l + \frac{1}{2} + i_k - i_{k-1} + i_s \\ &= (l - i) + (i_k - i_{k-1}) + (i_s - i) + \frac{3}{2} > 0.\end{aligned}$$

Thus, we have proved that, if $\alpha = \epsilon_i$ and $m \geqslant 0$, then $\langle \lambda_S + \rho, \tilde{\alpha}^\vee \rangle \notin -\mathbb{Z}_+$.

Now, let us consider the case $\alpha = -\epsilon_i$ and $m \geqslant 1$. Since $h_{i_k} < 0$, using relation (13) we get $(\mu_S, \epsilon_i) = h_{i_s} + \cdots + h_{i_k} \leqslant 0$. It follows

$$\begin{aligned}\langle \lambda_S + \rho, \tilde{\alpha}^\vee \rangle &= 2\left(m\left(l + \frac{1}{2}\right) - (\bar{\rho}, \epsilon_i) - (\mu_S, \epsilon_i)\right) \\ &\geqslant 2l + 1 - (2l - 2i + 1) = 2i > 0,\end{aligned}$$

which implies that $\langle \lambda_S + \rho, \tilde{\alpha}^\vee \rangle \notin -\mathbb{Z}_+$ holds in this case.

Case 3. The root α is of the form $\alpha = \pm(\epsilon_i + \epsilon_j)$, for $i, j = 1, \ldots, l$, $i < j$.

Then $(\bar{\rho}, \epsilon_i + \epsilon_j) = 2l - j - i + 1$. Let $s, t \in \{1, \ldots, k\}$ be indices such that $S \cap \{i, i+1, \ldots, j-1\} = \{i_s, \ldots, i_t\}$. Clearly, $i_s \geqslant i$ and $i_t \leqslant j - 1$. Furthermore, $(\mu_S, \epsilon_i + \epsilon_j) = h_{i_s} + \cdots + h_{i_t} + 2(h_{i_{t+1}} + \cdots + h_{i_k})$.

First consider the case $\alpha = \epsilon_i + \epsilon_j$, $i < j$ and $m \geqslant 0$.

Then

$$\begin{aligned}\langle \lambda_S + \rho, \tilde{\alpha}^\vee \rangle &= m\left(l + \frac{1}{2}\right) + (\bar{\rho}, \epsilon_i + \epsilon_j) + (\mu_S, \epsilon_i + \epsilon_j) \\ &\geqslant 2l - j - i + 1 + (\mu_S, \epsilon_i + \epsilon_j).\end{aligned}$$

Suppose that $t - s + 1$ is even. If $k - t$ is also even, then using relation (13) we obtain

$$(\mu_S, \epsilon_i + \epsilon_j) \geqslant -(i_t - i_s) - 2(i_k - i_{t+1}),$$

which implies

$$\begin{aligned}\langle\lambda_S+\rho,\tilde{\alpha}^\vee\rangle &\geqslant 2l-j-i+1-(i_t-i_s)-2(i_k-i_{t+1})\\ &=2(l-i_k)+(i_s-i)+(i_{t+1}-i_t)+(i_{t+1}-j)+1>0.\end{aligned}$$

If $k-t$ is odd, then using relation (13) we obtain

$$\begin{aligned}(\mu_S,\epsilon_i+\epsilon_j) &\geqslant -(i_t-i_s)-2(i_{k-1}-i_{t+1})+2h_{i_k},\\ &=-(i_t-i_s)-2(i_{k-1}-i_{t+1})+2\left(-l+i_k+\frac{1}{2}\right),\end{aligned}$$

which implies

$$\begin{aligned}\langle\lambda_S+\rho,\tilde{\alpha}^\vee\rangle &\geqslant 2l-j-i+1-(i_t-i_s)-2(i_{k-1}-i_{t+1})-2l+2i_k+1\\ &=2(i_k-i_{k-1})+(i_{t+1}-i_t)+(i_{t+1}-j)+(i_s-i)+2>0.\end{aligned}$$

Suppose now that $t-s+1$ is odd. Then

$$(\mu_S,\epsilon_i+\epsilon_j)=h_{i_s}+\cdots+h_{i_{t-1}}+(h_{i_t}+2h_{i_{t+1}}+\cdots+2h_{i_k}).$$

By using relation (13) and system (9) we get

$$\begin{aligned}(\mu_S,\epsilon_i+\epsilon_j) &\geqslant -(i_{t-1}-i_s)+(h_{i_t}+2h_{i_{t+1}}+\cdots+2h_{i_k})\\ &=-(i_{t-1}-i_s)+\left(-l+i_t+\frac{1}{2}\right).\end{aligned}$$

It follows

$$\begin{aligned}\langle\lambda_S+\rho,\tilde{\alpha}^\vee\rangle &\geqslant 2l-j-i+1-(i_{t-1}-i_s)-l+i_t+\frac{1}{2}\\ &=(l-j)+(i_t-i_{t-1})+(i_s-i)+\frac{3}{2}>0.\end{aligned}$$

Thus, we have proved that, if $\alpha=\epsilon_i+\epsilon_j$, $i<j$ and $m\geqslant 0$, then $\langle\lambda_S+\rho,\tilde{\alpha}^\vee\rangle\notin -\mathbb{Z}_+$.

The only case left is $\alpha=-(\epsilon_i+\epsilon_j)$, $i<j$ and $m\geqslant 1$. Then

$$\langle\lambda_S+\rho,\tilde{\alpha}^\vee\rangle=m\left(l+\frac{1}{2}\right)-(\bar{\rho},\epsilon_i+\epsilon_j)-(\mu_S,\epsilon_i+\epsilon_j).$$

If $t-s+1$ is even, then $(\mu_S,\epsilon_i+\epsilon_j)\in\mathbb{Z}$, so if m is odd, then $\langle\lambda_S+\rho,\tilde{\alpha}^\vee\rangle\notin\mathbb{Z}$. Let m be even. Then $m\geqslant 2$. Clearly

$$(\mu_S,\epsilon_i+\epsilon_j)=(h_{i_s}+\cdots+h_{i_k})+(h_{i_{t+1}}+\cdots+h_{i_k})\leqslant 0,$$

which implies

$$\langle\lambda_S+\rho,\tilde{\alpha}^\vee\rangle \geqslant 2\left(l+\frac{1}{2}\right)-(2l-j-i+1)=i+j>0.$$

If $t-s+1$ is odd, then using relation (13) and system (9) we obtain

$$(\mu_S,\epsilon_i+\epsilon_j)=(h_{i_s}+\cdots+h_{i_{t-1}})+(h_{i_t}+2h_{i_{t+1}}+\cdots+2h_{i_k})\leqslant -l+i_t+\frac{1}{2}.$$

It follows

$$\begin{aligned}\langle\lambda_S+\rho,\tilde{\alpha}^\vee\rangle &\geqslant \left(l+\frac{1}{2}\right)-(2l-j-i+1)-\left(-l+i_t+\frac{1}{2}\right)\\ &=(j-1-i_t)+i>0.\end{aligned}$$

We have proved that, if $\alpha=-(\epsilon_i+\epsilon_j)$, $i<j$ and $m\geqslant 1$, then $\langle\lambda_S+\rho,\tilde{\alpha}^\vee\rangle\notin -\mathbb{Z}_+$.

Thus, we have proved the relation (10). Moreover, it can be easily checked that coroots

$$\begin{aligned}&(\delta-\alpha_{i_j})^\vee,\quad j=1,\ldots,k,\\ &\alpha_{i_j}^\vee+\alpha_{i_j+1}^\vee+\cdots+\alpha_{i_{j+1}}^\vee,\quad j=1,\ldots,k-1,\\ &\alpha_i^\vee,\quad i\notin S,\ i\in\{1,2,\ldots,l\}\end{aligned}$$

are elements of the set $\hat{\Delta}_{\lambda_S}^{\vee\mathrm{re}}$ which implies $\mathbb{Q}\hat{\Delta}_{\lambda_S}^{\vee\mathrm{re}}=\mathbb{Q}\hat{\Pi}^\vee$, and the relation (11) is proved. □

Theorem 33. *Let M be a weak $L(-l+\frac{3}{2},0)$-module from the category $\mathcal{O}$. Then M is completely reducible.*

Proof. Let $L(\lambda)$ be some irreducible subquotient of M. Then $L(\lambda)$ is a $L(-l+\frac{3}{2},0)$-module, and Theorem 31 implies that there exists $S\subseteq\{1,2,\ldots,l-1\}$ such that $\lambda=(-l+\frac{3}{2})\Lambda_0+\mu_S$ or $\lambda=(-l+\frac{3}{2})\Lambda_0+\mu'_S$. It follows from Lemma 32 that such λ is admissible. Proposition 5 now implies that M is completely reducible. □

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