

The circuit ideal of a vector configuration [☆]

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Abstract

Given a configuration $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_n\} \subset \mathbb{Z}^d$, a basis ideal of \mathcal{A} is an ideal $J_{\mathcal{B}} = \langle \mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} : \mathbf{u} \in \mathcal{B} \rangle \subset \mathbf{k}[x_1, \dots, x_n]$ where \mathcal{B} spans the lattice $\mathcal{L}_{\mathcal{A}} = \{\mathbf{u} \in \mathbb{Z}^n : \sum \mathbf{a}_i u_i = \mathbf{0}\}$. Our main interest is to understand when the toric ideal, $I_{\mathcal{A}}$, of \mathcal{A} equals a basis ideal $J_{\mathcal{B}}$ with radical $I_{\mathcal{A}}$. The circuit ideal, $J_{\mathcal{C}_{\mathcal{A}}}$, of \mathcal{A} is an example of such a basis ideal.

We study such a $J_{\mathcal{B}}$ in relation to $I_{\mathcal{A}}$ from various algebraic and combinatorial perspectives with a special focus on $J_{\mathcal{C}_{\mathcal{A}}}$. We prove that the obstruction to equality of the ideals is the existence of certain polytopes. This result is based on a complete characterization of the standard pairs/associated primes of a monomial initial ideal of $J_{\mathcal{B}}$ and their differences from those for the corresponding toric initial ideal. Eisenbud and Sturmfels proved that the embedded primes of $J_{\mathcal{B}}$ are indexed by certain faces of the cone spanned by \mathcal{A} . We provide a necessary condition for a particular face to index an embedded prime and a partial converse. Finally, we compare various polyhedral fans associated to $I_{\mathcal{A}}$ and $J_{\mathcal{C}_{\mathcal{A}}}$. The Gröbner fan of $J_{\mathcal{C}_{\mathcal{A}}}$ is shown to refine that of $I_{\mathcal{A}}$ when the codimension of the ideals is at most two.

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1. Introduction

In this paper, we fix an ordered vector configuration $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_n\} \subset \mathbb{Z}^d$. Assume that the $d \times n$ integer matrix $A = [\mathbf{a}_1 \dots \mathbf{a}_n]$ whose columns are the elements of \mathcal{A} has rank d . Let $\mathcal{L}_{\mathcal{A}}$ be the $(n - d)$ -dimensional saturated lattice $\{\mathbf{u} \in \mathbb{Z}^n : A\mathbf{u} = \mathbf{0}\}$. We assume that $\mathcal{L}_{\mathcal{A}} \cap \mathbb{N}^n = \{\mathbf{0}\}$.

The *support* of a vector $\mathbf{u} \in \mathbb{Z}^n$ is defined to be $\text{supp}(\mathbf{u}) := \{i : u_i \neq 0\}$ and \mathbf{u} is *primitive* if the greatest common divisor of its components is one.

Definition 1.1. A vector $\mathbf{c} \in \mathcal{L}_{\mathcal{A}}$ is a *circuit* of \mathcal{A} if (1) \mathbf{c} is a non-zero primitive vector and (2) there does not exist a non-zero $\mathbf{d} \in \mathcal{L}_{\mathcal{A}}$ with $\text{supp}(\mathbf{d}) \subsetneq \text{supp}(\mathbf{c})$.

Let $\mathcal{C}_{\mathcal{A}}$ denote the set of all circuits of \mathcal{A} . Write $\mathbf{c} = \mathbf{c}^+ - \mathbf{c}^-$ where $c_j^+ = c_j$ if $c_j > 0$ and 0 otherwise, and $c_j^- = -c_j$ if $c_j < 0$ and 0 otherwise. Identify $\mathbf{c} \in \mathcal{C}_{\mathcal{A}}$ with the binomial $\mathbf{x}^{\mathbf{c}^+} - \mathbf{x}^{\mathbf{c}^-} \in \mathbf{k}[x_1, \dots, x_n] =: \mathbf{k}[\mathbf{x}]$ where \mathbf{k} is an algebraically closed field and $\mathbf{x}^{\mathbf{u}} := x_1^{u_1} x_2^{u_2} \dots x_n^{u_n}$. We refer to both \mathbf{c} and $\mathbf{x}^{\mathbf{c}^+} - \mathbf{x}^{\mathbf{c}^-}$ as a circuit of \mathcal{A} and denote both lists by $\mathcal{C}_{\mathcal{A}}$.

Definition 1.2. The *circuit ideal* of \mathcal{A} is the binomial ideal $J_{\mathcal{C}_{\mathcal{A}}} := \langle \mathcal{C}_{\mathcal{A}} \rangle \subseteq \mathbf{k}[\mathbf{x}]$.

The circuit ideal $J_{\mathcal{C}_{\mathcal{A}}}$ is a subideal of the binomial prime *toric ideal* of \mathcal{A}

$$I_{\mathcal{A}} := \langle \mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} : \mathbf{u} \in \mathcal{L}_{\mathcal{A}} \rangle.$$

Toric ideals are the defining ideals of *toric varieties* [7] and have numerous applications in combinatorics, optimization, algebra and algebraic geometry [16]. These connections make the computability of $I_{\mathcal{A}}$ an important practical concern.

Proposition 1.3. [16, Lemma 12.2] *Given a finite subset \mathcal{B} of $\mathcal{L}_{\mathcal{A}}$, define the ideal $J_{\mathcal{B}} := \langle \mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} : \mathbf{u} \in \mathcal{B} \rangle \subseteq \mathbf{k}[\mathbf{x}]$. A set \mathcal{B} spans $\mathcal{L}_{\mathcal{A}}$ if and only if $(J_{\mathcal{B}} : (x_1 x_2 \dots x_n)^{\infty}) = I_{\mathcal{A}}$.*

When \mathcal{B} spans $\mathcal{L}_{\mathcal{A}}$, $J_{\mathcal{B}}$ is called a *basis ideal* of \mathcal{A} . Proposition 1.3 is the starting point of the best algorithms to compute $I_{\mathcal{A}}$ since a spanning set \mathcal{B} of $\mathcal{L}_{\mathcal{A}}$ can be computed easily and each saturation in

$$(J_{\mathcal{B}} : (x_1 x_2 \dots x_n)^{\infty}) = (((J_{\mathcal{B}} : x_1^{\infty}) : x_2^{\infty}) \dots) : x_n^{\infty})$$

can be achieved by a Gröbner basis calculation ([9], [16, Chapter 12]). It can be checked that $\mathcal{C}_{\mathcal{A}}$ spans $\mathcal{L}_{\mathcal{A}}$ and hence $J_{\mathcal{C}_{\mathcal{A}}}$ is a basis ideal of \mathcal{A} and $I_{\mathcal{A}} = (J_{\mathcal{C}_{\mathcal{A}}} : (x_1 x_2 \dots x_n)^{\infty})$. Further, $I_{\mathcal{A}}$ is the radical of $J_{\mathcal{C}_{\mathcal{A}}}$. The main motivation behind this paper was to understand how close the circuit ideal is to the toric ideal, and in particular, when they are equal. The majority of our theorems hold for basis ideals $J_{\mathcal{B}}$ with the property that $\sqrt{J_{\mathcal{B}}} = I_{\mathcal{A}}$ and we use $J_{\mathcal{C}_{\mathcal{A}}}$ as our running example. The main question we address is the following.

Problem 1.4. When does a basis ideal $J_{\mathcal{B}}$ such that $\sqrt{J_{\mathcal{B}}} = I_{\mathcal{A}}$ equal $I_{\mathcal{A}}$?

In this paper, we investigate Problem 1.4 from several different angles. Let $\mathbb{N}\mathcal{A}$ denote the semigroup $\{A\mathbf{u} : \mathbf{u} \in \mathbb{N}^n\} \subset \mathbb{Z}^d$. Both $I_{\mathcal{A}}$ and $J_{\mathcal{B}}$ are homogeneous under multi-grading by $\mathbb{N}\mathcal{A}$

with $\mathbf{k}[\mathbf{x}]/I_{\mathcal{A}}$ having Hilbert function value one for all $\mathbf{b} \in \mathbb{N}\mathcal{A}$. In Section 2 we recall conditions for the equality of $I_{\mathcal{A}}$ and a basis ideal $J_{\mathcal{B}}$ and then exhibit various properties of $J_{\mathcal{B}}$ that contrast those of toric ideals. We interpret the multi-graded Hilbert function values of $\mathbf{k}[\mathbf{x}]/J_{\mathcal{B}}$.

From the point of view of Gröbner basis theory, it is natural to investigate $I_{\mathcal{A}}$ and $J_{\mathcal{B}}$ by examining the difference between their initial ideals with respect to a fixed weight vector ω . In Section 3, we give a complete characterization of the associated primes of a monomial initial ideal of a basis ideal $J_{\mathcal{B}}$ with $\sqrt{J_{\mathcal{B}}} = I_{\mathcal{A}}$ (Theorem 3.10) extending previously known characterizations of the associated primes of a monomial initial ideal of $I_{\mathcal{A}}$ [10]. The associated primes and the difference between the two monomial initial ideals are described in terms of certain polytopes that depend on \mathcal{A} and ω . Using this we answer Problem 1.4 by showing that the obstruction to equality of the ideals is the existence of certain polytopes of the above type (Theorem 3.17).

A second natural measure of the difference between the two ideals in Problem 1.4 is an understanding of the embedded primes of $J_{\mathcal{B}}$. Let $\text{cone}(\mathcal{A})$ denote the d -dimensional cone spanned by \mathcal{A} . Record a face σ of $\text{cone}(\mathcal{A})$ as the set of indices, j , of all \mathbf{a}_j that lie on σ . Eisenbud and Sturmfels [5] proved that the associated primes of $J_{\mathcal{B}}$ are all of the form $P_{\sigma} + I_{\mathcal{A}}$ where σ is some face of $\text{cone}(\mathcal{A})$ and $P_{\sigma} := \langle x_j : j \notin \sigma \rangle$. In particular, $I_{\mathcal{A}} = P_{[n]} + I_{\mathcal{A}}$ is indexed by the full face $[n] := \{1, 2, \dots, n\}$ of $\text{cone}(\mathcal{A})$. However, not all faces of $\text{cone}(\mathcal{A})$ need index an associated prime of $J_{\mathcal{B}}$ and Eisenbud and Sturmfels raise the following problem for the special case of $J_{\mathcal{C}_{\mathcal{A}}}$.

Problem 1.5. [5, Section 7] “It remains an interesting combinatorial problem to characterize the embedded primary components of the circuit ideal $J_{\mathcal{C}_{\mathcal{A}}}$. In particular, which faces of $\text{cone}(\mathcal{A})$ support an associated prime of $J_{\mathcal{C}_{\mathcal{A}}}$? An answer to this question might be valuable for the applications of binomial ideals to integer programming and statistics.”

In Section 4, we give a necessary condition for a prime $P_{\sigma} + I_{\mathcal{A}}$ to be an embedded prime of a basis ideal $J_{\mathcal{B}}$ with $\sqrt{J_{\mathcal{B}}} = I_{\mathcal{A}}$ (Theorem 4.3) using the results in Section 3. We also provide a partial converse to Theorem 4.3. As an application, we derive connections between the smoothness of the toric variety defined by a face σ of $\text{cone}(\mathcal{A})$ and $P_{\sigma} + I_{\mathcal{A}}$ being an embedded prime of $J_{\mathcal{C}_{\mathcal{A}}}$ when \mathcal{A} is a graded vector configuration.

Given a homogeneous ideal I and a weight vector $\omega \in \mathbb{R}^n$, let $\text{in}_{\omega}(I)$ be the initial ideal of I with respect to ω , $\sqrt{\text{in}_{\omega}(I)}$ the radical of $\text{in}_{\omega}(I)$, and $\text{top}(\text{in}_{\omega}(I))$ the intersection of the top-dimensional primary components of $\text{in}_{\omega}(I)$. These entities define three equivalence relations on \mathbb{R}^n as follows:

- (1) the *initial ideal* equivalence relation: $\mathbf{u} \sim \mathbf{v} \Leftrightarrow \text{in}_{\mathbf{u}}(I) = \text{in}_{\mathbf{v}}(I)$,
- (2) the *top* equivalence relation: $\mathbf{u} \sim \mathbf{v} \Leftrightarrow \text{top}(\text{in}_{\mathbf{u}}(I)) = \text{top}(\text{in}_{\mathbf{v}}(I))$, and
- (3) the *radical* equivalence relation: $\mathbf{u} \sim \mathbf{v} \Leftrightarrow \sqrt{\text{in}_{\mathbf{u}}(I)} = \sqrt{\text{in}_{\mathbf{v}}(I)}$.

For any homogeneous ideal I , the initial ideal equivalence classes form the cells of the *Gröbner fan* of I [13], [16, Chapter 2]. For $I_{\mathcal{A}}$ it is well known that the other two equivalence classes also form polyhedral fans—the radical equivalence relation gives the *secondary fan* of \mathcal{A} [2], [16, Chapter 8], and the top equivalence relation gives the *hypergeometric fan* of \mathcal{A} [14]. In Section 5 we prove that for $J_{\mathcal{C}_{\mathcal{A}}}$, the equivalence classes of the radical and top equivalence relations coincide with those for $I_{\mathcal{A}}$ (Theorem 5.15 and Proposition 5.16). However, the Gröbner fans of $I_{\mathcal{A}}$ and $J_{\mathcal{C}_{\mathcal{A}}}$ do not coincide in general. Corollary 5.18 proves that when the codimension of the ideals is at most two, the Gröbner fan of $J_{\mathcal{C}_{\mathcal{A}}}$ refines that of $I_{\mathcal{A}}$. Again, the theorems hold in greater generality than for circuit ideals.

2. Properties of $I_{\mathcal{A}}$ versus $J_{\mathcal{B}}$

Consider $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_n\} \subset \mathbb{Z}^d$ and the lattice $\mathcal{L}_{\mathcal{A}}$ as in the introduction. Let $\mathcal{B} \subseteq \mathbb{Z}^n$ be a spanning set of $\mathcal{L}_{\mathcal{A}}$ and consider the basis ideal

$$J_{\mathcal{B}} := \langle \mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} : \mathbf{u} \in \mathcal{B} \rangle \subseteq \mathbf{k}[\mathbf{x}].$$

The toric ideal $I_{\mathcal{A}}$ and the circuit ideal $J_{\mathcal{C}_{\mathcal{A}}}$ are of the form $J_{\mathcal{B}}$. In particular, since $I_{\mathcal{A}}$ is $J_{\mathcal{B}}$ for $\mathcal{B} = \mathcal{L}_{\mathcal{A}}$, every basis ideal $J_{\mathcal{B}}$ is contained in $I_{\mathcal{A}}$.

In this section we first collect conditions equivalent to the equality of $I_{\mathcal{A}}$ and $J_{\mathcal{B}}$. Many of these stem from combinatorics and optimization and most are well known [3,16]. We then contrast $J_{\mathcal{B}}$ with $I_{\mathcal{A}}$ in light of these conditions, using $J_{\mathcal{B}} = J_{\mathcal{C}_{\mathcal{A}}}$ in our examples.

Consider the semigroup homomorphism $\pi : \mathbb{N}^n \rightarrow \mathbb{N}\mathcal{A}$ such that $\mathbf{u} \mapsto \mathbf{A}\mathbf{u}$. The ideal $J_{\mathcal{B}}$ is homogeneous under the \mathcal{A} -grading of $\mathbf{k}[\mathbf{x}]$ by $\deg(x_i) = \mathbf{a}_i$ for $i = 1, \dots, n$ since every binomial of the form $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$ in $J_{\mathcal{B}}$ is \mathcal{A} -homogeneous with \mathcal{A} -degree $\pi(\mathbf{u}) = \mathbf{A}\mathbf{u} = \mathbf{A}\mathbf{v} = \pi(\mathbf{v})$. Let $\mathbf{H}_{J_{\mathcal{B}}} : \mathbb{N}\mathcal{A} \rightarrow \mathbb{N}$ be the \mathcal{A} -graded Hilbert function of $\mathbf{k}[\mathbf{x}]/J_{\mathcal{B}}$ given by $\mathbf{b} \mapsto \dim_{\mathbf{k}}(\mathbf{k}[\mathbf{x}]/J_{\mathcal{B}})_{\mathbf{b}}$. Let $\mathbf{H}_{I_{\mathcal{A}}}$ be the same for $I_{\mathcal{A}}$.

Since $\mathcal{L}_{\mathcal{A}} \cap \mathbb{N}^n = \{\mathbf{0}\}$, for each $\mathbf{b} \in \mathbb{N}\mathcal{A}$, the polyhedron $P_{\mathbf{b}} := \{\mathbf{x} \in \mathbb{R}_{\geq 0}^n : \mathbf{A}\mathbf{x} = \mathbf{b}\}$ is bounded [15] which implies that $\pi^{-1}(\mathbf{b}) := \{\mathbf{x} \in \mathbb{N}^n : \mathbf{A}\mathbf{x} = \mathbf{b}\} = P_{\mathbf{b}} \cap \mathbb{N}^n$ is finite for all $\mathbf{b} \in \mathbb{N}\mathcal{A}$. For a fixed $\mathbf{b} \in \mathbb{N}\mathcal{A}$, the set $\pi^{-1}(\mathbf{b})$ admits two natural graphs as follows. First, choose a binomial generating set $G(J_{\mathcal{B}})$ of $J_{\mathcal{B}}$ and let $\mathcal{F}^{\mathcal{B}}(\mathbf{b})$ be the graph on $\pi^{-1}(\mathbf{b})$ such that \mathbf{u} is adjacent to \mathbf{v} in $\mathcal{F}^{\mathcal{B}}(\mathbf{b})$ if $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$ is a monomial multiple of a binomial in $G(J_{\mathcal{B}})$. Next, fix a generic weight vector $\omega \in \mathbb{R}^n$ in the sense that the initial ideal $\text{in}_{\omega}(J_{\mathcal{B}})$ is a monomial ideal. Let $G_{\omega}(J_{\mathcal{B}})$ be the marked reduced Gröbner basis of $J_{\mathcal{B}}$ with respect to ω . Elements of $G_{\omega}(J_{\mathcal{B}})$ are \mathcal{A} -homogeneous binomials and the Gröbner basis being marked means that the first term in each binomial f is its initial term $\text{in}_{\omega}(f)$. Construct the directed graph $\mathcal{F}_{\omega}^{\mathcal{B}}(\mathbf{b})$ on $\pi^{-1}(\mathbf{b})$ by drawing an arrow from \mathbf{u} to \mathbf{v} in $\mathcal{F}_{\omega}^{\mathcal{B}}(\mathbf{b})$ if and only if $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$ is a monomial multiple of some marked binomial in $G_{\omega}(J_{\mathcal{B}})$. In the special case of $J_{\mathcal{B}} = I_{\mathcal{A}}$, we denote $\mathcal{F}^{\mathcal{B}}(\mathbf{b})$ by just $\mathcal{F}(\mathbf{b})$ and $\mathcal{F}_{\omega}^{\mathcal{B}}(\mathbf{b})$ by $\mathcal{F}_{\omega}(\mathbf{b})$.

Lemma 2.1. [3, Theorem 1.1] *Vectors $\mathbf{u}, \mathbf{v} \in \pi^{-1}(\mathbf{b})$ are in the same component of $\mathcal{F}^{\mathcal{B}}(\mathbf{b})$ if and only if $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$ lies in $J_{\mathcal{B}}$.*

Lemma 2.1 shows that while the edges in $\mathcal{F}^{\mathcal{B}}(\mathbf{b})$ depend on the choice of generating set $G(J_{\mathcal{B}})$, the components, and in particular the number of components, do not depend on this choice. Further, $\mathcal{F}^{\mathcal{B}}(\mathbf{b})$ and $\mathcal{F}_{\omega}^{\mathcal{B}}(\mathbf{b})$ partition $\pi^{-1}(\mathbf{b})$ identically into components. The following theorem collects results from [3,16].

Theorem 2.2. *The following statements are equivalent.*

- (1) *The ideals $I_{\mathcal{A}}$ and $J_{\mathcal{B}}$ are equal.*
- (2) *For every $\mathbf{b} \in \mathbb{N}\mathcal{A}$, the graph $\mathcal{F}^{\mathcal{B}}(\mathbf{b})$ is connected.*
- (3) *For every $\mathbf{b} \in \mathbb{N}\mathcal{A}$, the digraph $\mathcal{F}_{\omega}^{\mathcal{B}}(\mathbf{b})$ has a unique sink. (In this case, the unique sink \mathbf{u} in $\mathcal{F}_{\omega}^{\mathcal{B}}(\mathbf{b})$ is the optimal solution of the integer program $\text{minimize}\{\omega \cdot \mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \in \mathbb{N}^n\}$.)*
- (4) *For every $\mathbf{b} \in \mathbb{N}\mathcal{A}$ and generic weight vector $\omega \in \mathbb{R}^n$, $\text{in}_{\omega}(J_{\mathcal{B}})$ has a unique standard monomial of \mathcal{A} -degree \mathbf{b} . (In this case, the standard monomial of \mathcal{A} -degree \mathbf{b} is $\mathbf{x}^{\mathbf{u}}$ where \mathbf{u} is the unique sink in $\mathcal{F}_{\omega}^{\mathcal{B}}(\mathbf{b})$.)*
- (5) *For every $\mathbf{b} \in \mathbb{N}\mathcal{A}$, the Hilbert function value $\mathbf{H}_{J_{\mathcal{B}}}(\mathbf{b})$ is one.*

Proof. Statements (2)–(5) are all true if J_B equals I_A ; see [16, Chapters 4,5,10]. Further, $I_A = J_B$ if and only if $G_\omega(I_A) = G_\omega(J_B)$ if and only if for each \mathbf{b} , $\mathcal{F}_\omega^B(\mathbf{b})$ equals $\mathcal{F}_\omega(\mathbf{b})$ and hence if and only if $\mathcal{F}^B(\mathbf{b})$ and $\mathcal{F}(\mathbf{b})$ have the same components. Hence (1) is equivalent to (2) and (3). Since $J_B \subseteq I_A$, the two ideals are equal if and only if (5). The equivalence of (3) and (4) follows from Lemma 2.5 below. \square

Remark 2.3.

- (1) The connectivity of $\mathcal{F}(\mathbf{b})$ was used in [4], in the context of statistical sampling, to devise random walks on $\pi^{-1}(\mathbf{b})$. Of particular interest was the case where $I_A = J_{C_A}$ which allows $\pi^{-1}(\mathbf{b})$ to be connected using circuits of \mathcal{A} . In Section 3 we will see that under further assumptions on J_B , for most $\mathbf{b} \in \mathbb{N}A$, $\mathcal{F}^B(\mathbf{b})$ is in fact connected (Theorem 3.14), and that the set of $\mathbf{b} \in \mathbb{N}A$ for which $\mathcal{F}^B(\mathbf{b})$ is disconnected can be described precisely. See also [3].
- (2) The equality of I_A and J_{C_A} will allow all integer programs of the form

$$\text{minimize } \{\omega \cdot \mathbf{x} : A\mathbf{x} = \mathbf{b}, \mathbf{x} \in \mathbb{N}^n\}$$

as \mathbf{b} and ω vary to be solved by reduced Gröbner bases of J_{C_A} . The significance of this is that the circuits of \mathcal{A} are precisely the primitive edge directions of the polyhedra $P_{\mathbf{b}}$ as \mathbf{b} varies in $\mathbb{N}A$ and hence the directions taken by the simplex algorithm in solving linear programs of the form

$$\text{minimize } \{\omega \cdot \mathbf{x} : A\mathbf{x} = \mathbf{b}, \mathbf{x} \in \mathbb{R}_{\geq 0}^n\}.$$

Using circuit ideals, we now contrast various properties of J_B with those of I_A .

Proposition 2.4.

- (1) The graph $\mathcal{F}^B(\mathbf{b})$ may have arbitrarily many components, even if we restrict to the case of $A \in \mathbb{Z}^{1 \times 3}$.
- (2) The standard monomials of $\text{in}_\omega(J_B)$ of \mathcal{A} -degree \mathbf{b} are not necessarily the cheapest monomials of that degree with respect to ω .

Proof. (1) For any natural number $k \geq 2$, let $A_k = (k \ 2k+1 \ 3k+1)$ and let \mathcal{B}_k be the set of circuits of A_k . Since the three entries of A_k are pairwise relatively prime, the circuits are $x^{2k+1} - y^k$, $x^{3k+1} - z^k$, and $y^{3k+1} - z^{2k+1}$ with \mathcal{A} -degrees $2k^2 + k$, $3k^2 + k$, and $6k^2 + 5k + 1$, respectively. Thus the graph $\mathcal{F}^{\mathcal{B}_k}(b)$ has no edges when $b < 2k^2 + k$. In particular, this holds if we take $b = m(3k+1)$ for $m = \lfloor k/2 \rfloor$. This particular $\mathcal{F}^{\mathcal{B}_k}(b)$ has at least $m+1$ vertices $\{(j, j, m-j) : 0 \leq j \leq m\}$, so it has at least $m+1 = \lfloor k/2 \rfloor + 1$ components.

(2) Consider $\mathcal{A} = \{3, 4, 5\}$. The graded reverse lexicographic (grevlex) Gröbner basis of J_{C_A} with $a > b > c$ is

$$\{a^4 - b^3, ab^3 - c^3, b^5 - c^4, b^2c^3 - ac^4, a^3c^3 - bc^4, a^2bc^4 - c^6\}.$$

The monomials of degree 17 are a^4c , a^3b^2 , abc^2 and b^3c of which the last three are standard monomials of the above grevlex initial ideal of J_{C_A} . However, we see that the non-standard monomial a^4c is cheaper than the standard monomial a^3b^2 . \square

We now prove that $\mathbf{H}_{J_{\mathcal{B}}}(\mathbf{b})$ equals the number of components of $\mathcal{F}_{\omega}^{\mathcal{B}}(\mathbf{b})$, or equivalently, of $\mathcal{F}^{\mathcal{B}}(\mathbf{b})$. Proposition 2.4(1) then shows that the values of $\mathbf{H}_{J_{\mathcal{B}}}$ can be arbitrarily large even for d and n fixed. In contrast, $\mathbf{H}_{I_{\mathcal{A}}}(\mathbf{b}) = 1$ for all $\mathbf{b} \in \mathbb{N}A$.

Lemma 2.5. *Each component of $\mathcal{F}_{\omega}^{\mathcal{B}}(\mathbf{b})$ has a unique sink \mathbf{u} and $\mathbf{x}^{\mathbf{u}}$ is the unique standard monomial of $\text{in}_{\omega}(J_{\mathcal{B}})$ among all monomials $\mathbf{x}^{\mathbf{v}}$ such that \mathbf{v} is in the same component as \mathbf{u} . In particular, a monomial $\mathbf{x}^{\mathbf{u}}$ of \mathcal{A} -degree \mathbf{b} is a standard monomial of $\text{in}_{\omega}(J_{\mathcal{B}})$ if and only if for all $\mathbf{v} \neq \mathbf{u}$ in the same component of $\mathcal{F}_{\omega}^{\mathcal{B}}(\mathbf{b})$ as \mathbf{u} , $\omega \cdot \mathbf{u} < \omega \cdot \mathbf{v}$.*

Proof. Let D be an arbitrary component of $\mathcal{F}_{\omega}^{\mathcal{B}}(\mathbf{b})$, \mathbf{v} be an arbitrary vertex in D , and $\mathbf{x}^{\mathbf{u}}$ be the normal form of $\mathbf{x}^{\mathbf{v}}$ with respect to $G_{\omega}(J_{\mathcal{B}})$. Then $\mathbf{x}^{\mathbf{u}}$ is a standard monomial of $\text{in}_{\omega}(J_{\mathcal{B}})$ and by Lemma 2.1, \mathbf{u} is in D . If $\mathbf{x}^{\mathbf{u}'}$ is another standard monomial of $\text{in}_{\omega}(J_{\mathcal{B}})$ with \mathbf{u}' in D , then by Lemma 2.1, $f := \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{u}'} \in J_{\mathcal{B}}$ with $\text{in}_{\omega}(f)$ equal to either $\mathbf{x}^{\mathbf{u}}$ or $\mathbf{x}^{\mathbf{u}'}$, a contradiction. This implies that $\mathbf{x}^{\mathbf{u}}$ is the unique normal form of all $\mathbf{x}^{\mathbf{v}}$, $\mathbf{v} \in D$ and hence it is the unique sink in D . The remaining assertions now follow. \square

Proposition 2.6. *The Hilbert function value $\mathbf{H}_{J_{\mathcal{B}}}(\mathbf{b})$ equals the number of components of $\mathcal{F}_{\omega}^{\mathcal{B}}(\mathbf{b})$.*

Proof. By Lemma 2.5(2), each component of $\mathcal{F}_{\omega}^{\mathcal{B}}(\mathbf{b})$ contributes precisely one standard monomial of $\text{in}_{\omega}(J_{\mathcal{B}})$. The number of standard monomials of $\text{in}_{\omega}(J_{\mathcal{B}})$ of degree \mathbf{b} equals $\dim_{\mathbf{k}}(\mathbf{k}[\mathbf{x}]/\text{in}_{\omega}(J_{\mathcal{B}}))_{\mathbf{b}} = \dim_{\mathbf{k}}(\mathbf{k}[\mathbf{x}]/J_{\mathcal{B}})_{\mathbf{b}} = \mathbf{H}_{J_{\mathcal{B}}}(\mathbf{b})$. \square

Example 2.7. When $I_{\mathcal{A}} \neq J_{\mathcal{B}}$, the distribution of values of $\mathbf{H}_{J_{\mathcal{B}}}$ can be quite complicated. In Fig. 1, we plot these values for $\mathcal{B} = \mathcal{C}_{\mathcal{A}}$ of

$$A = \begin{pmatrix} 1 & 3 & 2 & 4 \\ 1 & 4 & 5 & 2 \end{pmatrix}.$$

The boundary of $\text{cone}(\mathcal{A})$ is shown by dashed lines. Notice that deep in the interior of the cone, all of the values are one. Theorem 3.14 proves this fact.

3. Monomial initial ideals of the circuit ideal

In this section let \mathcal{A} and \mathcal{B} be as in Section 2 with the further assumption that $\sqrt{J_{\mathcal{B}}} = I_{\mathcal{A}}$. This assumption always holds when \mathcal{B} is the set of circuits of \mathcal{A} [5, Proposition 7.10]. Fix a generic weight vector $\omega \in \mathbb{R}^n$ such that $\text{in}_{\omega}(I_{\mathcal{A}})$ and $\text{in}_{\omega}(J_{\mathcal{B}})$ are both monomial ideals. The main result of this section is Theorem 3.10 which characterizes the associated primes of $\text{in}_{\omega}(J_{\mathcal{B}})$ in terms of certain polytopes defined from \mathcal{A} and ω and their lattice points. This theorem generalizes Theorem 2.5 in [10] which gave a complete characterization of the associated primes of $\text{in}_{\omega}(I_{\mathcal{A}})$ in terms of certain *lattice-point-free* polytopes defined from \mathcal{A} and ω . Using Theorem 3.10, we describe the similarities and differences between the associated primes (standard pairs) of $\text{in}_{\omega}(I_{\mathcal{A}})$ and $\text{in}_{\omega}(J_{\mathcal{B}})$, and give an answer to Problem 1.4 (Theorem 3.17).

Proposition 3.1. *Let I and J be homogeneous ideals in $\mathbf{k}[\mathbf{x}]$ with $\sqrt{J} = I$. Then $\sqrt{\text{in}_{\omega}(J)} = \sqrt{\text{in}_{\omega}(I)}$ for all $\omega \in \mathbb{R}^n$.*

Proof. Since $\sqrt{J} = I$, $J \subseteq I$ which implies that $\text{in}_{\omega}(J) \subseteq \text{in}_{\omega}(I)$ and hence $\sqrt{\text{in}_{\omega}(J)} \subseteq \sqrt{\text{in}_{\omega}(I)}$ for all $\omega \in \mathbb{R}^n$. To prove the other inclusion we first observe that $\sqrt{\text{in}_{\omega}(I)}$ is

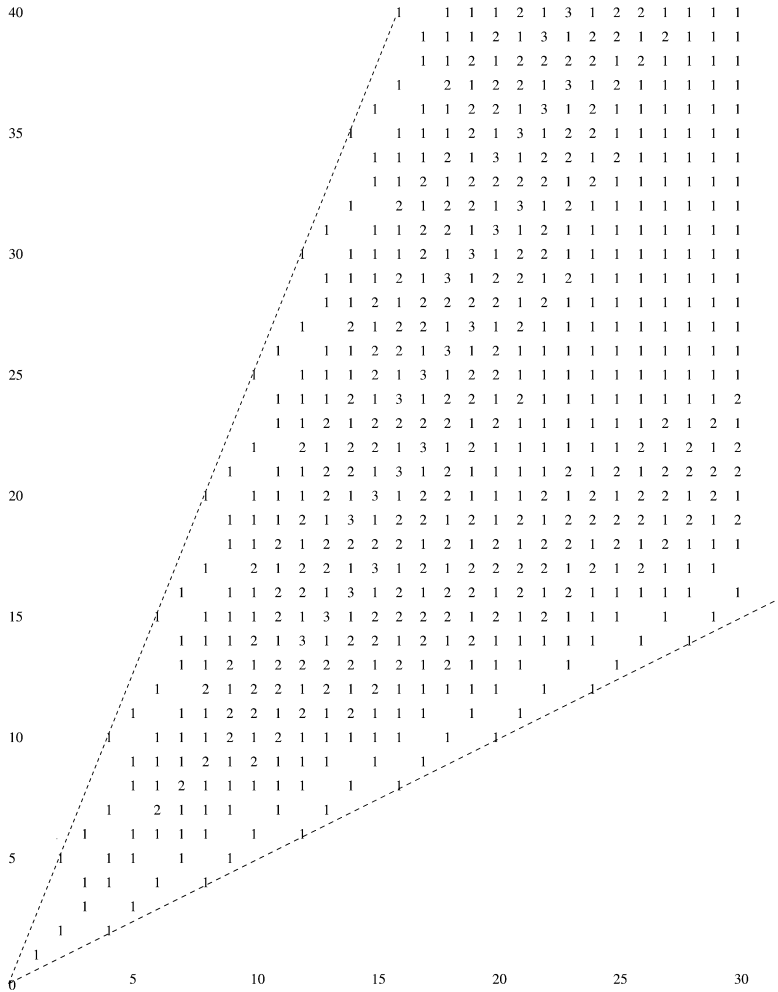


Fig. 1. The distribution of values of \mathbf{H}_{J_B} for the circuit ideal of the matrix A in Example 2.7.

ω -homogeneous since $\text{in}_\omega(I)$ is. Hence it suffices to show that any homogeneous element in $\sqrt{\text{in}_\omega(I)}$ is also in $\sqrt{\text{in}_\omega(J)}$. Let $f \in \sqrt{\text{in}_\omega(I)}$ be ω -homogeneous. Then there exists some m such that $f^m \in \text{in}_\omega(I)$. The polynomial f^m is also ω -homogeneous, so $f^m = \text{in}_\omega(F)$ for some $F \in I$. Since $\sqrt{J} = I$, $F^k \in J$ for some k , and $\text{in}_\omega(F^k) = \text{in}_\omega(F)^k = f^{mk}$. Hence, $f \in \sqrt{\text{in}_\omega(J)}$. \square

Definition 3.2. [16, Chapter 8]

- (1) The *regular triangulation* of \mathcal{A} with respect to ω is the simplicial complex Δ_ω on the vertex set $[n] = \{1, \dots, n\}$ such that $\{i_1, \dots, i_r\} \subseteq [n]$ is a face of Δ_ω if and only if there exists a vector $\mathbf{c} \in \mathbb{R}^d$ such that $\mathbf{a}_j \cdot \mathbf{c} = \omega_j$ if $j \in \{i_1, \dots, i_r\}$ and $\mathbf{a}_j \cdot \mathbf{c} < \omega_j$ if $j \notin \{i_1, \dots, i_r\}$.
- (2) The *Stanley–Reisner* ideal of a simplicial complex Δ on $[n]$ is the ideal in $\mathbf{k}[\mathbf{x}]$ generated by the monomials $\mathbf{x}_\sigma := \prod_{i \in \sigma} x_i$ for each minimal non-face σ of Δ .

Theorem 8.3 in [16] states that $\sqrt{\text{in}_\omega(I_{\mathcal{A}})}$ is the Stanley–Reisner ideal of the regular triangulation Δ_ω of \mathcal{A} . For a set $\sigma \subseteq [n]$ define $P_\sigma := \langle x_j : j \notin \sigma \rangle \subset \mathbf{k}[\mathbf{x}]$. Note that P_σ is a monomial prime ideal such that $\mathbf{k}[\mathbf{x}]/P_\sigma$ has Krull dimension $|\sigma|$.

Corollary 3.3. *For a basis ideal $J_{\mathcal{B}}$ with $\sqrt{J_{\mathcal{B}}} = I_{\mathcal{A}}$, the following hold.*

- (1) *All the associated primes of $\text{in}_\omega(J_{\mathcal{B}})$ are monomial ideals of the form P_σ where σ is a face of the simplicial complex Δ_ω .*
- (2) *The prime P_σ is a minimal prime of $\text{in}_\omega(J_{\mathcal{B}})$ if and only if σ is a maximal face of Δ_ω .*

Proof. If I is the Stanley–Reisner ideal of a simplicial complex Δ on $[n]$, then I has the irredundant prime decomposition $I = \bigcap_{\sigma \in \max(\Delta)} P_\sigma$ where $\max(\Delta)$ is the set of maximal faces of Δ [16, Chapter 8]. Thus the minimal primes of $\text{in}_\omega(J_{\mathcal{B}})$, which equal the minimal primes of $\sqrt{\text{in}_\omega(J_{\mathcal{B}})} = \sqrt{\text{in}_\omega(I_{\mathcal{A}})}$ (by Proposition 3.1), are the primes P_σ as σ varies in $\max(\Delta_\omega)$, proving (2). If P_τ is an embedded prime of $\text{in}_\omega(J_{\mathcal{B}})$, then $\tau \subset \sigma$ for some $\sigma \in \max(\Delta_\omega)$. This implies that τ is a lower-dimensional face of Δ_ω , proving (1). \square

If τ is a lower-dimensional face of Δ_ω , P_τ may or may not be an embedded prime of $\text{in}_\omega(J_{\mathcal{B}})$. Theorem 3.10 characterizes the lower-dimensional faces of Δ_ω that index embedded primes of $\text{in}_\omega(J_{\mathcal{B}})$.

Example 3.4. Let \mathcal{B} be the set of circuits of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}.$$

Using the program Gfan [12] we find that both $I_{\mathcal{A}}$ and $J_{\mathcal{B}}$ have eight distinct monomial initial ideals. Table 1 gives a representative weight vector ω for each pair of initial ideals and verifies Proposition 3.1.

Remark 3.5. If we drop the assumption that $\sqrt{J_{\mathcal{B}}} = I_{\mathcal{A}}$, then it need not be that $\sqrt{\text{in}_\omega(J_{\mathcal{B}})}$ is the Stanley–Reisner ideal of any regular triangulation of \mathcal{A} . For the matrix A in Example 3.4, the set $\mathcal{B} = \{(1, -2, 1, 0), (2, -3, 0, 1)\}$ spans the lattice $\mathcal{L}_{\mathcal{A}}$ and $J_{\mathcal{B}} = \langle ac - b^2, a^2d - b^3 \rangle$. The radical of $J_{\mathcal{B}}$ is $\langle bc - ad, b^2 - ac \rangle$, a proper subideal of $I_{\mathcal{A}}$. The grevlex initial ideal of $J_{\mathcal{B}}$ with $a > b > c > d$ is $\langle b^2, abc, a^2c^2 \rangle$ whose radical is $\langle b, ac \rangle$. This ideal is not listed in the last column of Table 1.

Table 1
Comparison of initial ideals of $I_{\mathcal{A}}$ and $J_{\mathcal{B}}$ from Example 3.4

ω	$\text{in}_\omega(I_{\mathcal{A}})$	$\text{in}_\omega(J_{\mathcal{B}})$	Radical of both initial ideals
(10, 0, 1, 3)	$\langle ac, ad, bd \rangle$	$\langle ac, a^2d, bd, ad^2 \rangle$	$\langle ac, ad, bd \rangle$
(10, 0, 3, 1)	$\langle ac, c^2, ad \rangle$	$\langle ac, c^2, a^2d, abd, ad^2 \rangle$	$\langle c, ad \rangle$
(3, 1, 10, 0)	$\langle ac, bc, c^2, a^2d \rangle$	$\langle ac, b^2c, c^2, a^2d, bcd \rangle$	"
(1, 3, 10, 0)	$\langle b^3, ac, bc, c^2 \rangle$	$\langle b^3, ac, b^2c, c^2, bcd \rangle$	$\langle b, c \rangle$
(1, 5, 3, 0)	$\langle b^2, bc, c^2 \rangle$	$\langle b^2, abc, c^2, bcd \rangle$	"
(0, 10, 3, 1)	$\langle b^2, bc, c^3, bd \rangle$	$\langle b^2, abc, bc^2, c^3, bd \rangle$	"
(1, 3, 0, 10)	$\langle b^2, ad, bd \rangle$	$\langle b^2, a^2d, bd, acd, ad^2 \rangle$	$\langle b, ad \rangle$
(3, 10, 0, 1)	$\langle b^2, bc, bd, ad^2 \rangle$	$\langle b^2, abc, bc^2, bd, ad^2 \rangle$	"

We now establish the necessary definitions and lemmas for Theorem 3.10. The associated primes of a monomial ideal M can be studied via a combinatorial construction introduced in [17] called the *standard pair decomposition* of M .

Definition 3.6. Let $M \subset \mathbf{k}[\mathbf{x}]$ be a monomial ideal, $\mathbf{x}^{\mathbf{u}}$ a standard monomial of M and $\sigma \subseteq [n]$. Then $(\mathbf{x}^{\mathbf{u}}, \sigma)$ is an *admissible pair* of M if:

- (1) $\text{supp}(\mathbf{u}) \cap \sigma = \emptyset$,
- (2) all monomials in $\mathbf{x}^{\mathbf{u}} \cdot \mathbf{k}[x_j : j \in \sigma]$ are standard monomials of M .

An admissible pair $(\mathbf{x}^{\mathbf{u}}, \sigma)$ of M is called a *standard pair* of M if there does not exist another admissible pair $(\mathbf{x}^{\mathbf{v}}, \tau)$ such that $\mathbf{v} \leq \mathbf{u}$ and $\text{supp}(\mathbf{u} - \mathbf{v}) \cup \sigma \subseteq \tau$.

The (unique) decomposition of the standard monomials of M given by its standard pairs is the *standard pair decomposition* of M . Let $\text{Ass}(I)$ denote the set of associated primes of an ideal I . Since M is a monomial ideal, all elements of $\text{Ass}(M)$ have the form P_σ for some $\sigma \subseteq [n]$. Standard pairs of M are related to $\text{Ass}(M)$ as follows.

Proposition 3.7. [17, Lemmas 3.3 and 3.5]

- (1) $P_\sigma \in \text{Ass}(M)$ if and only if M has a standard pair of the form (\cdot, σ) .
- (2) P_σ is a minimal prime of M if and only if $(1, \sigma)$ is a standard pair of M .

We now define the polytopes needed in Theorem 3.10. Fix a matrix $G \in \mathbb{Z}^{n \times (n-d)}$ whose columns form a basis for the lattice \mathcal{L}_A . Such a G is called a *Gale dual* of A . In particular, the columns of G span the kernel of A as an \mathbb{R} -vector space. For $\mathbf{u} \in \mathbb{N}^n$ let

$$Q_{\mathbf{u}} := \{\mathbf{z} \in \mathbb{R}^{n-d} : G\mathbf{z} \leq \mathbf{u}\}.$$

Recall that by assumption, $P_{\mathbf{b}} = \{\mathbf{x} \in \mathbb{R}_{\geq 0}^n : A\mathbf{x} = \mathbf{b}\}$ is a polytope for all $\mathbf{b} \in \mathbb{N}A$. The polyhedron $Q_{\mathbf{u}}$ is the image of $P_{A\mathbf{u}}$ under the isomorphism

$$\phi_{\mathbf{u}} : \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = A\mathbf{u}\} \rightarrow \mathbb{R}^{n-d} \quad \text{such that } \mathbf{x} \mapsto \mathbf{z} \quad \text{where } \mathbf{u} - G\mathbf{z} = \mathbf{x}.$$

For each $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} = A\mathbf{u}$, $\mathbf{u} - \mathbf{x} = G\mathbf{z}$ for some $\mathbf{z} \in \mathbb{R}^{n-d}$ since $\mathbf{u} - \mathbf{x} \in \ker(A) = \{G\mathbf{z} : \mathbf{z} \in \mathbb{R}^{n-d}\}$. Further, this \mathbf{z} is unique since the columns of G are linearly independent. The vector \mathbf{u} maps to $\mathbf{0}$ under $\phi_{\mathbf{u}}$ and hence $\mathbf{0} \in Q_{\mathbf{u}}$.

Next, define

$$Q_{\mathbf{u}, \omega} := Q_{\mathbf{u}} \cap \{\mathbf{z} \in \mathbb{R}^{n-d} : (-\omega G)\mathbf{z} \leq \mathbf{0}\},$$

the subpolytope of $Q_{\mathbf{u}}$ created by adding one new inequality depending on ω . For σ a face of Δ_{ω} , further define

$$Q_{\mathbf{u}, \omega}^{\bar{\sigma}} := \{\mathbf{z} \in \mathbb{R}^{n-d} : (G\mathbf{z} \leq \mathbf{u})^{\bar{\sigma}}, (-\omega G)\mathbf{z} \leq \mathbf{0}\},$$

where $(G\mathbf{z} \leq \mathbf{u})^{\bar{\sigma}}$ denotes the subsystem of inequalities indexed by $\bar{\sigma}$ in $G\mathbf{z} \leq \mathbf{u}$. Theorem 1 in [11] proves that $Q_{\mathbf{u}, \omega}^{\bar{\sigma}}$ is a polytope. It is a relaxation of $Q_{\mathbf{u}, \omega}$. Figure 2 shows pictures of the

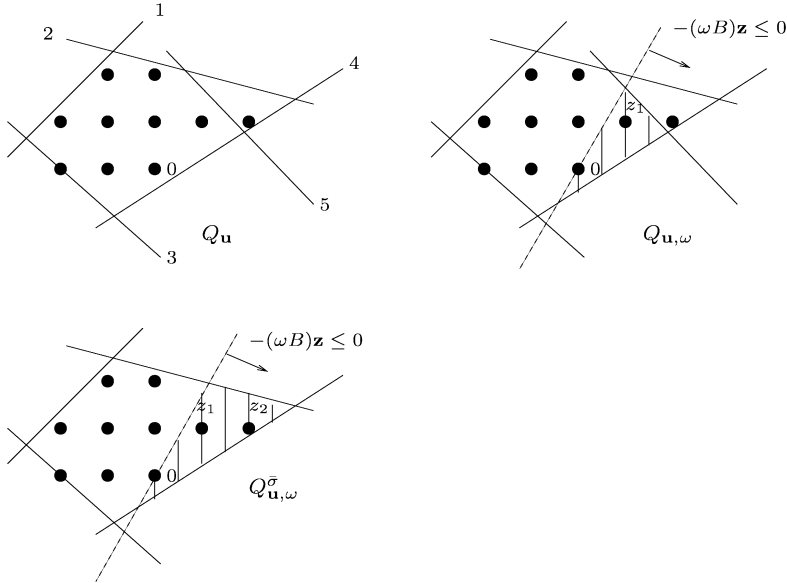


Fig. 2. The polytopes $Q_{\mathbf{u}}$, $Q_{\mathbf{u},\omega}$ and $Q_{\mathbf{u},\omega}^{\bar{\sigma}}$.

polytopes $Q_{\mathbf{u}}$, $Q_{\mathbf{u},\omega}$ and $Q_{\mathbf{u},\omega}^{\bar{\sigma}}$. The inequalities in $G\mathbf{z} \leq \mathbf{u}$ are numbered $1, \dots, 5$ and $Q_{\mathbf{u},\omega}^{\bar{\sigma}}$ is drawn for $\sigma = \{5\}$.

For a non-zero lattice point $\mathbf{z} \in Q_{\mathbf{u},\omega}^{\bar{\sigma}}$, set $\mathbf{m}_{\mathbf{z}} := (\mathbf{u} - G\mathbf{z})^-$. Let G_i denote the i th row of G .

Remark 3.8.

- (1) The i th component $(\mathbf{m}_{\mathbf{z}})_i > 0$ if and only if \mathbf{z} violates the i th inequality $G_i\mathbf{z} \leq u_i$ among the inequalities $G\mathbf{z} \leq \mathbf{u}$ defining $Q_{\mathbf{u},\omega}$.
- (2) Since every $\mathbf{z} \in Q_{\mathbf{u},\omega}^{\bar{\sigma}}$ satisfies $G_i\mathbf{z} \leq u_i$ for $i \notin \sigma$, the support of $\mathbf{m}_{\mathbf{z}}$ is contained in σ .
- (3) The vector $\mathbf{m}_{\mathbf{z}}$ is the component-wise smallest vector \mathbf{m} in \mathbb{N}^n with support in σ such that $\mathbf{z} \in Q_{\mathbf{u}+\mathbf{m},\omega}$.
- (4) By the definition of $\mathbf{m}_{\mathbf{z}}$, $\mathbf{u} + \mathbf{m}_{\mathbf{z}} - G\mathbf{z} \in \mathbb{N}^n$.

For instance, in Fig. 2, $\mathbf{z}_1 \in Q_{\mathbf{u},\omega}$ and hence $\mathbf{m}_{\mathbf{z}_1} = \mathbf{0}$, while \mathbf{z}_2 violates the inequality $G_5\mathbf{z} \leq u_5$ defining $Q_{\mathbf{u},\omega}$ and hence $\mathbf{m}_{\mathbf{z}_2}$ has a positive fifth component.

Theorem 3.10 will generalize the following theorem for toric ideals.

Theorem 3.9. [10, Theorem 2.5] Assume $\mathbf{u} \in \mathbb{N}^n$ and $\sigma \in \Delta_{\omega}$ such that $\text{supp}(\mathbf{u}) \cap \sigma = \emptyset$. Then $(\mathbf{x}^{\mathbf{u}}, \sigma)$ is a standard pair of $\text{in}_{\omega}(I_{\mathcal{A}})$ if and only if the following two conditions hold.

- (1) There are no non-zero lattice points in $Q_{\mathbf{u},\omega}^{\bar{\sigma}}$.
- (2) For every $i \in \bar{\sigma}$ there is a non-zero lattice point in $Q_{\mathbf{u},\omega}^{\bar{\sigma} \cup \{i\}}$.

Theorem 3.10 is analogous, but involves an algebraic component rather than being purely polyhedral. Recall that $\mathbf{x}_{\sigma} = \prod_{i \in \sigma} x_i$.

Theorem 3.10. Assume $\mathbf{u} \in \mathbb{N}^n$ and $\sigma \in \Delta_\omega$ such that $\text{supp}(\mathbf{u}) \cap \sigma = \emptyset$. Then $(\mathbf{x}^{\mathbf{u}}, \sigma)$ is a standard pair of $\text{in}_\omega(J_B)$ if and only if the following two conditions hold.

- (1) For each non-zero lattice point \mathbf{z} in $Q_{\mathbf{u}, \omega}^{\bar{\sigma}}$, $\mathbf{x}^{\mathbf{u}+\mathbf{m}_z} - \mathbf{x}^{\mathbf{u}+\mathbf{m}_z-G\mathbf{z}} \notin (J_B : \mathbf{x}_\sigma^\infty)$.
- (2) For each $i \in \bar{\sigma}$, there exists some non-zero lattice point $\mathbf{z} \in Q_{\mathbf{u}, \omega}^{\bar{\sigma} \cup \{i\}}$ such that $\mathbf{x}^{\mathbf{u}+\mathbf{m}_z} - \mathbf{x}^{\mathbf{u}+\mathbf{m}_z-G\mathbf{z}} \in (J_B : \mathbf{x}_{\sigma \cup \{i\}}^\infty)$.

We first use Theorem 3.10 to reprove Theorem 3.9.

Proof of Theorem 3.9. Since I_A is prime and monomial free, $(I_A : \mathbf{x}_\tau^\infty) = I_A$ for all $\tau \subseteq [n]$. Thus if \mathbf{z} is a non-zero lattice point in $Q_{\mathbf{u}, \omega}^{\bar{\sigma}}$, then $\mathbf{x}^{\mathbf{u}+\mathbf{m}_z} - \mathbf{x}^{\mathbf{u}+\mathbf{m}_z-G\mathbf{z}} \in I_A = (I_A : \mathbf{x}_\sigma^\infty)$. Hence, Theorem 3.10(1) holds if and only if there are no non-zero lattice points in $Q_{\mathbf{u}, \omega}^{\bar{\sigma}}$. Similarly, Theorem 3.10(2) holds in the toric situation if and only if for every $i \in \bar{\sigma}$ there is a non-zero lattice point in $Q_{\mathbf{u}, \omega}^{\bar{\sigma} \cup \{i\}}$. \square

Proof of Theorem 3.10. (\Rightarrow): Suppose $(\mathbf{x}^{\mathbf{u}}, \sigma)$ is a standard pair of $\text{in}_\omega(J_B)$. Then by Corollary 3.3, $\sigma \in \Delta_\omega$ and $\text{supp}(\mathbf{u}) \cap \sigma = \emptyset$. Suppose \mathbf{z} is a non-zero lattice point in $Q_{\mathbf{u}, \omega}^{\bar{\sigma}}$. Then $-(\omega G)\mathbf{z} \leq 0$, and because ω is generic, we may assume $-(\omega G)\mathbf{z} < 0$. For any $\mathbf{m} \in \mathbb{N}^n$ with support contained in σ , $\mathbf{x}^{\mathbf{u}+\mathbf{m}}$ is a standard monomial of $\text{in}_\omega(J_B)$ since $(\mathbf{x}^{\mathbf{u}}, \sigma)$ is a standard pair. If further, $\mathbf{m} \geq \mathbf{m}_z = (\mathbf{u} - G\mathbf{z})^-$, then $\mathbf{u} + \mathbf{m} - G\mathbf{z} \in \mathbb{N}^n$ and $A(\mathbf{u} + \mathbf{m} - G\mathbf{z}) = A(\mathbf{u} + \mathbf{m})$ since $AG = 0$. Also, $\omega \cdot (\mathbf{u} + \mathbf{m} - G\mathbf{z}) = \omega \cdot (\mathbf{u} + \mathbf{m}) - (\omega G)\mathbf{z} < \omega \cdot (\mathbf{u} + \mathbf{m})$ since $-(\omega G)\mathbf{z} < 0$. Therefore, by Lemma 2.5, $\mathbf{x}^{\mathbf{u}+\mathbf{m}} - \mathbf{x}^{\mathbf{u}+\mathbf{m}-G\mathbf{z}} \notin J_B$. In particular, $\mathbf{x}^{\mathbf{u}+\mathbf{m}_z} - \mathbf{x}^{\mathbf{u}+\mathbf{m}_z-G\mathbf{z}} \notin J_B$ and for all $\mathbf{m}' \in \mathbb{N}^n$ with support in σ , $\mathbf{x}^{\mathbf{m}'}(\mathbf{x}^{\mathbf{u}+\mathbf{m}_z} - \mathbf{x}^{\mathbf{u}+\mathbf{m}_z-G\mathbf{z}}) \notin J_B$. Rewriting, this gives $\mathbf{x}^{\mathbf{u}+\mathbf{m}_z} - \mathbf{x}^{\mathbf{u}+\mathbf{m}_z-G\mathbf{z}} \notin (J_B : \mathbf{x}_\sigma^\infty)$ and (1) holds.

Suppose $i \notin \sigma$. Then there exists some $\mathbf{m} \in \mathbb{N}^n$ with $\text{supp}(\mathbf{m}) \subseteq \sigma$ and $p > 0$ such that $\mathbf{x}^{\mathbf{u}+\mathbf{m}}x_i^p \in \text{in}_\omega(J_B)$. Let \mathbf{q} be the unique sink in the same component of $\mathcal{F}_\omega^B(A(\mathbf{u} + \mathbf{m} + p\mathbf{e}_i))$ as $\mathbf{u} + \mathbf{m} + p\mathbf{e}_i$. Note that $\mathbf{q} \neq \mathbf{u} + \mathbf{m} + p\mathbf{e}_i$ since $\mathbf{x}^{\mathbf{q}} \notin \text{in}_\omega(J_B)$. Let $\mathbf{z} \in \mathbb{Z}^{n-d}$ be such that $\mathbf{q} = \mathbf{u} + \mathbf{m} + p\mathbf{e}_i - G\mathbf{z}$. Then $\mathbf{u} + \mathbf{m} + p\mathbf{e}_i$ maps to $\mathbf{0}$ and \mathbf{q} maps to $\mathbf{z} \neq \mathbf{0}$ in $Q_{\mathbf{u}+\mathbf{m}+p\mathbf{e}_i}$ under the map $\phi_{\mathbf{u}+\mathbf{m}+p\mathbf{e}_i}$. Since $\omega \cdot \mathbf{q} = \omega \cdot (\mathbf{u} + \mathbf{m} + p\mathbf{e}_i - G\mathbf{z}) < \omega \cdot (\mathbf{u} + \mathbf{m} + p\mathbf{e}_i)$, we see that $-(\omega G)\mathbf{z} < 0$. Therefore, \mathbf{z} is a lattice point in $Q_{\mathbf{u}+\mathbf{m}+p\mathbf{e}_i, \omega}$ and hence in $Q_{\mathbf{u}, \omega}^{\bar{\sigma} \cup \{i\}}$ obtained by throwing away the inequalities of $G\mathbf{z} \leq \mathbf{u}$ indexed by $\sigma \cup \{i\}$ from $Q_{\mathbf{u}+\mathbf{m}+p\mathbf{e}_i, \omega}$. This is because $\text{supp}(\mathbf{m} + p\mathbf{e}_i) \subseteq \sigma \cup \{i\}$. By definition, $\mathbf{m}_z \leq \mathbf{m} + p\mathbf{e}_i$ since \mathbf{m}_z is the component-wise smallest vector \mathbf{m}' with support in $\sigma \cup \{i\}$ such that $\mathbf{z} \in Q_{\mathbf{u}+\mathbf{m}', \omega}^{\bar{\sigma} \cup \{i\}}$ and we know that $\mathbf{z} \in Q_{\mathbf{u}+\mathbf{m}+p\mathbf{e}_i}$. Since $\mathbf{q} = \mathbf{u} + \mathbf{m} + p\mathbf{e}_i - G\mathbf{z}$ lies in the same component of $\mathcal{F}_\omega^B(A(\mathbf{u} + \mathbf{m} + p\mathbf{e}_i))$ as $\mathbf{u} + \mathbf{m} + p\mathbf{e}_i$, by Lemma 2.1,

$$\mathbf{x}^{\mathbf{u}+\mathbf{m}+p\mathbf{e}_i} - \mathbf{x}^{\mathbf{u}+\mathbf{m}+p\mathbf{e}_i-G\mathbf{z}} = \mathbf{x}^{\mathbf{m}+p\mathbf{e}_i-\mathbf{m}_z}(\mathbf{x}^{\mathbf{u}+\mathbf{m}_z} - \mathbf{x}^{\mathbf{u}+\mathbf{m}_z-G\mathbf{z}}) \in J_B.$$

This implies that $\mathbf{x}^{\mathbf{u}+\mathbf{m}_z} - \mathbf{x}^{\mathbf{u}+\mathbf{m}_z-G\mathbf{z}} \in (J_B : \mathbf{x}_{\sigma \cup \{i\}}^\infty)$ and (2) holds.

(\Leftarrow): Suppose (1) and (2) hold for some $\sigma \in \Delta_\omega$ and some $\mathbf{u} \in \mathbb{N}^n$ with support in $\bar{\sigma}$. We first show that $\mathbf{x}^{\mathbf{u}+\mathbf{m}}$ is a standard monomial of $\text{in}_\omega(J_B)$ where $\mathbf{m} \in \mathbb{N}^n$ is an arbitrary vector with $\text{supp}(\mathbf{m}) \subseteq \sigma$. Suppose \mathbf{z} is a non-zero lattice point in $Q_{\mathbf{u}+\mathbf{m}, \omega}$. Then \mathbf{z} is also a non-zero

lattice point in the relaxation $Q_{\mathbf{u}+\mathbf{m},\omega}^{\bar{\sigma}} = Q_{\mathbf{u},\omega}^{\bar{\sigma}}$. Compute \mathbf{m}_z for this \mathbf{u} and \mathbf{z} . Since $\mathbf{z} \in Q_{\mathbf{u}+\mathbf{m},\omega}$, $\mathbf{m}_z \leq \mathbf{m}$. By (1), $(\mathbf{x}^{\mathbf{u}+\mathbf{m}_z} - \mathbf{x}^{\mathbf{u}+\mathbf{m}_z-G\mathbf{z}}) \notin (J_{\mathcal{B}} : \mathbf{x}_{\sigma}^{\infty})$ which implies that

$$\mathbf{x}^{\mathbf{m}-\mathbf{m}_z}(\mathbf{x}^{\mathbf{u}+\mathbf{m}_z} - \mathbf{x}^{\mathbf{u}+\mathbf{m}_z-G\mathbf{z}}) = \mathbf{x}^{\mathbf{u}+\mathbf{m}} - \mathbf{x}^{\mathbf{u}+\mathbf{m}-G\mathbf{z}} \notin J_{\mathcal{B}}.$$

Thus for each $\mathbf{z} \neq \mathbf{0}$ in $Q_{\mathbf{u}+\mathbf{m},\omega}$, the vector $\mathbf{u} + \mathbf{m} - G\mathbf{z}$ does not lie in the same component as $\mathbf{u} + \mathbf{m}$. This implies that $\omega \cdot \mathbf{v} > \omega \cdot (\mathbf{u} + \mathbf{m})$ for all \mathbf{v} in the same component as $\mathbf{u} + \mathbf{m}$. By Lemma 2.5, $\mathbf{x}^{\mathbf{u}+\mathbf{m}}$ is a standard monomial of $\text{in}_{\omega}(J_{\mathcal{B}})$. Since $\text{supp}(\mathbf{u}) \cap \sigma = \emptyset$ and \mathbf{m} is an arbitrary vector with support contained in σ , we conclude that $(\mathbf{x}^{\mathbf{u}}, \sigma)$ is an admissible pair of $\text{in}_{\omega}(J_{\mathcal{B}})$.

To show that $(\mathbf{x}^{\mathbf{u}}, \sigma)$ is a standard pair, we need to argue that the monomials of this pair are not properly contained in any other standard pair $(\mathbf{x}^{\mathbf{u}'}, \sigma')$ of $\text{in}_{\omega}(J_{\mathcal{B}})$. Suppose there is such a standard pair. We first argue that $\sigma = \sigma'$. By (2), if $i \notin \sigma$ then there exists some non-zero lattice point \mathbf{z} in $Q_{\mathbf{u},\omega}^{\bar{\sigma} \cup \{i\}}$ such that

$$\mathbf{x}^{\mathbf{u}+\mathbf{m}_z} - \mathbf{x}^{\mathbf{u}+\mathbf{m}_z-G\mathbf{z}} \in (J_{\mathcal{B}} : \mathbf{x}_{\sigma \cup \{i\}}^{\infty}).$$

This implies that there exists some $p \in \mathbb{N}$ and $\mathbf{m} \in \mathbb{N}^n$ with support in σ such that $x_i^p \mathbf{x}^{\mathbf{m}}(\mathbf{x}^{\mathbf{u}+\mathbf{m}_z} - \mathbf{x}^{\mathbf{u}+\mathbf{m}_z-G\mathbf{z}}) \in J_{\mathcal{B}}$. Since $(-\omega G)\mathbf{z} < 0$, $x_i^p \mathbf{x}^{\mathbf{m}}(\mathbf{x}^{\mathbf{u}+\mathbf{m}_z})$ is the leading term of $x_i^p \mathbf{x}^{\mathbf{m}}(\mathbf{x}^{\mathbf{u}+\mathbf{m}_z} - \mathbf{x}^{\mathbf{u}+\mathbf{m}_z-G\mathbf{z}}) \in J_{\mathcal{B}}$ and hence is in $\text{in}_{\omega}(J_{\mathcal{B}})$. This construction shows that not all monomials of the form $\mathbf{x}^{\mathbf{u}}\mathbf{x}^{\mathbf{q}}$ where the support of \mathbf{q} is contained in $\sigma \cup \{i\}$ are standard monomials of $\text{in}_{\omega}(J_{\mathcal{B}})$ and hence $(\mathbf{x}^{\mathbf{u}}, \sigma)$ is not contained in any admissible pair $(\mathbf{x}^{\mathbf{u}'}, \sigma')$ with $\sigma \subsetneq \sigma'$. To finish the argument, suppose $(\mathbf{x}^{\mathbf{u}}, \sigma)$ is contained in a standard pair of form $(\mathbf{x}^{\mathbf{u}'}, \sigma)$. Then $\mathbf{u} = \mathbf{m} + \mathbf{u}'$ for some \mathbf{m} whose support is contained in σ . However, because $(\mathbf{x}^{\mathbf{u}}, \sigma)$ is a standard pair, the support of \mathbf{u} must also be disjoint from σ . Thus $\mathbf{m} = \mathbf{0}$ and so $\mathbf{u} = \mathbf{u}'$. \square

We now apply Theorems 3.10 and 3.9 to study the difference between the two monomial ideals $\text{in}_{\omega}(I_{\mathcal{A}})$ and $\text{in}_{\omega}(J_{\mathcal{B}})$. This difference will be the key to our study of the associated primes of $J_{\mathcal{B}}$ itself in Section 4.

Definition 3.11. A $J_{\mathcal{B}}$ -specific standard pair (JSP) is a standard pair of $\text{in}_{\omega}(J_{\mathcal{B}})$ that is not also a standard pair of $\text{in}_{\omega}(I_{\mathcal{A}})$.

Corollary 3.12. Assume $\mathbf{u} \in \mathbb{N}^n$ and $\sigma \in \Delta_{\omega}$ such that $\text{supp}(\mathbf{u}) \cap \sigma = \emptyset$. Then $(\mathbf{x}^{\mathbf{u}}, \sigma)$ is a JSP if and only if the two conditions of Theorem 3.10 hold and there exists at least one non-zero lattice point $\mathbf{z} \in Q_{\mathbf{u},\omega}^{\bar{\sigma}}$.

Proof. If the two conditions of Theorem 3.10 hold then $(\mathbf{x}^{\mathbf{u}}, \sigma)$ is a standard pair of $\text{in}_{\omega}(J_{\mathcal{B}})$ and if there is a non-zero lattice point $\mathbf{z} \in Q_{\mathbf{u},\omega}^{\bar{\sigma}}$, then by Theorem 3.9, $(\mathbf{x}^{\mathbf{u}}, \sigma)$ is not a standard pair of $\text{in}_{\omega}(I_{\mathcal{A}})$. Thus it is a JSP. Conversely, if $(\mathbf{x}^{\mathbf{u}}, \sigma)$ is a JSP then the two conditions of Theorem 3.10 hold. Suppose there is no non-zero lattice point $\mathbf{z} \in Q_{\mathbf{u},\omega}^{\bar{\sigma}}$. Then condition (1) of Theorem 3.9 is true. But since condition (2) of Theorem 3.10 holds for this JSP, there is a non-zero lattice point in $Q_{\mathbf{u},\omega}^{\bar{\sigma} \cup \{i\}}$ for each $i \notin \sigma$, which is condition (2) of Theorem 3.9. This implies that $(\mathbf{x}^{\mathbf{u}}, \sigma)$ is a standard pair of $\text{in}_{\omega}(I_{\mathcal{A}})$, contradicting that it is a JSP. \square

Example 3.13. Consider the matrix A and weight vector ω given below:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 3 & 4 & 5 & 6 \\ 0 & 0 & 7 & 8 & 9 \end{pmatrix}, \quad \omega = (1000, 100, 10, 1, 0).$$

Let \mathcal{B} be the set of circuits of \mathcal{A} . The matrix

$$G = \begin{pmatrix} 2 & 4 \\ -1 & -2 \\ -5 & -9 \\ 1 & 0 \\ 3 & 7 \end{pmatrix}$$

is a Gale dual of A and $\omega G = (1851, 3710)$. We have

$$J_{\mathcal{B}} = \langle b^2c^9 - a^4e^7, bd^9 - a^2e^8, bc^8 - a^2d^7, ce - d^2 \rangle$$

and its initial ideal

$$\text{in}_{\omega}(J_{\mathcal{B}}) = \langle a^2d^7, ce, a^4d^6e^4, a^4d^4e^5, a^2d^6e^5, a^4d^2e^6, a^2d^4e^6, a^4e^7, a^2d^2e^7, a^2e^8 \rangle$$

has 58 standard pairs. These ideals and standard pairs were computed using Macaulay 2 [8]. Consider the standard pair $(d^4e^3, \{1, 2\})$ for which $\mathbf{u} = (0, 0, 0, 4, 3)$ and $\sigma = \{1, 2\}$. The monomial d^4e^3 is a standard monomial for the toric initial ideal $\text{in}_{\omega}(I_{\mathcal{A}})$ as well and $\mathcal{Q}_{\mathbf{u}, \omega} \cap \mathbb{Z}^2 = \{0\}$. However, the polytope

$$\mathcal{Q}_{\mathbf{u}, \omega}^{\tilde{\sigma}} = \left\{ \mathbf{z} \in \mathbb{Z}^2 : \begin{pmatrix} -5 & -9 \\ 1 & 0 \\ 3 & 7 \end{pmatrix} \mathbf{z} \leq \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix}, 1851z_1 + 3710z_2 \geq 0 \right\}$$

contains two more lattice points: $(1, 0)$ and $(3, -1)$. Thus $(\mathbf{x}^{\mathbf{u}}, \sigma)$ is not a standard pair of $\text{in}_{\omega}(I_{\mathcal{A}})$, so it is a JSP. Both points have $\mathbf{m}_{\mathbf{z}} = (2, 0, 0, 0, 0)$. For $(1, 0)$, $\mathbf{x}^{\mathbf{u}+\mathbf{m}_{\mathbf{z}}} - \mathbf{x}^{\mathbf{u}+\mathbf{m}_{\mathbf{z}}-G\mathbf{z}} = a^2d^4e^3 - bc^5d^3$ is not in $(J_{\mathcal{B}} : (ab)^{\infty})$ but lies in $(J_{\mathcal{B}} : (aby)^{\infty})$ for each $y \in \{c, d, e\}$. Similarly, for $(3, -1)$, $\mathbf{x}^{\mathbf{u}+\mathbf{m}_{\mathbf{z}}} - \mathbf{x}^{\mathbf{u}+\mathbf{m}_{\mathbf{z}}-G\mathbf{z}} = a^2d^4e^3 - bc^6de$ does not lie in $(J_{\mathcal{B}} : (ab)^{\infty})$ but lies in $(J_{\mathcal{B}} : (aby)^{\infty})$ for each $y \in \{c, d, e\}$.

We now use JSPs to give a precise description of the set $\mathcal{H} := \{\mathbf{b} \in \mathbb{N}\mathcal{A} : \mathbf{H}_{J_{\mathcal{B}}}(\mathbf{b}) > 1\}$. This description gives a new proof of the following theorem alluded to in Section 2 (cf. Fig. 1). The theorem also follows from [3, Corollary 5.3].

Theorem 3.14. For all $\mathbf{b} \in \mathbb{N}\mathcal{A}$ sufficiently far from the boundary of $\text{cone}(\mathcal{A})$, $\mathbf{H}_{J_{\mathcal{B}}}(\mathbf{b}) = 1$ and hence the graphs $\mathcal{F}^{\mathcal{B}}(\mathbf{b})$ and $\mathcal{F}_{\omega}^{\mathcal{B}}(\mathbf{b})$ are connected.

Recall that \mathbf{b} lies in \mathcal{H} if and only if for a generic ω , $\text{in}_{\omega}(J_{\mathcal{B}})$ has more than one standard monomial of degree \mathbf{b} . That is, $\mathcal{H} = \{A\mathbf{u} : \mathbf{x}^{\mathbf{u}} \in \text{in}_{\omega}(I_{\mathcal{A}}) \setminus \text{in}_{\omega}(J_{\mathcal{B}})\}$. Since all standard monomials of degree \mathbf{b} other than the toric standard monomial lie on JSPs of $\text{in}_{\omega}(J_{\mathcal{B}})$, it follows that \mathcal{H} is contained in the union of the images in $\mathbb{N}\mathcal{A}$ of the JSPs of $\text{in}_{\omega}(J_{\mathcal{B}})$ under the map $\pi : \mathbb{N}^n \rightarrow \mathbb{N}\mathcal{A}$, $\mathbf{u} \mapsto A\mathbf{u}$.

Lemma 3.15. *If (\mathbf{x}^u, σ) is a JSP of $\text{in}_\omega(J_B)$, then the set $\{\mathbf{a}_i: i \in \sigma\}$ is contained in a facet of $\text{cone}(\mathcal{A})$.*

Proof. Since (\mathbf{x}^u, σ) is a standard pair of $\text{in}_\omega(J_B)$, by Corollary 3.3, σ is a face of the triangulation Δ_ω . Suppose $\text{cone}(\mathcal{A}_\sigma)$ intersects the interior of $\text{cone}(\mathcal{A})$. Choose a monomial \mathbf{x}^α on the JSP (\mathbf{x}^u, σ) such that $\mathbf{x}^\alpha \in \text{in}_\omega(I_A)$. Let \mathbf{x}^β be the standard monomial of $\text{in}_\omega(I_A)$ of degree $A\alpha$. Then $\mathbf{x}^\alpha - \mathbf{x}^\beta \in I_A$ with leading term \mathbf{x}^α . Since $\mathbf{x}_\sigma^{\mathbf{m}} \mathbf{x}^\alpha \notin \text{in}_\omega(J_B)$ for any \mathbf{m} , the binomial $\mathbf{x}_\sigma^{\mathbf{m}}(\mathbf{x}^\alpha - \mathbf{x}^\beta) \notin J_B$ for any \mathbf{m} since its leading term $\mathbf{x}_\sigma^{\mathbf{m}} \mathbf{x}^\alpha$ would then be in $\text{in}_\omega(J_B)$. This implies that $(J_B : \mathbf{x}_\sigma^\infty) \neq I_A$. On the other hand, every embedded prime of J_B is of the form $P_\tau + I_A$ where τ indexes some proper face of $\text{cone}(\mathcal{A})$ (see Proposition 4.1). The monomial \mathbf{x}_σ lies in each of these embedded primes since σ is not contained in any proper face of $\text{cone}(\mathcal{A})$. This implies that for \mathbf{m} large enough, $\mathbf{x}_\sigma^{\mathbf{m}}$ lies in every primary component of J_B except I_A , which in turn implies that $(J_B : \mathbf{x}_\sigma^\infty) = I_A$, a contradiction. \square

Proof of Theorem 3.14. By Lemma 3.15, if (\mathbf{x}^u, σ) is a JSP of $\text{in}_\omega(J_B)$, then $A\mathbf{u} + \mathbb{N}\mathcal{A}_\sigma$, its image under π in $\mathbb{N}A$, is contained in some hyperplane parallel to a facet of $\text{cone}(\mathcal{A})$. Since there are only finitely many JSPs of $\text{in}_\omega(J_B)$, \mathcal{H} is contained in finitely many hyperplanes parallel to the facets of $\text{cone}(\mathcal{A})$. This implies that the maximum distance of a point in \mathcal{H} from the boundary of $\text{cone}(\mathcal{A})$ is bounded which proves the theorem. \square

We conclude this section with an answer to Problem 1.4.

Definition 3.16. A polytope $Q_{\mathbf{u}, \omega}^\sigma$ corresponding to a JSP (\mathbf{x}^u, σ) of $\text{in}_\omega(J_B)$ is called a *JSP polytope* of \mathcal{A} .

Note that JSP polytopes can be defined independently of standard pairs by the conditions of Corollary 3.12.

Theorem 3.17. *The following are equivalent.*

- (1) *The ideals I_A and J_B are not equal.*
- (2) *There is a generic $\omega \in \mathbb{R}^n$ for which \mathcal{A} has a JSP polytope.*
- (3) *For every generic $\omega \in \mathbb{R}^n$, \mathcal{A} has a JSP polytope.*

Proof. The ideal $I_A = J_B$ if and only if for any generic $\omega \in \mathbb{R}^n$, $\text{in}_\omega(I_A) = \text{in}_\omega(J_B)$ which is if and only if $\text{in}_\omega(J_B)$ has no JSPs. \square

4. Associated primes of the circuit ideal

In this section, we show how the associated primes of J_B relate to the JSP polytopes of its initial ideals discussed in Section 3. Again, we assume throughout this section that J_B is a basis ideal of \mathcal{A} and that $\sqrt{J_B} = I_A$. Recall that a face F of $\text{cone}(\mathcal{A})$ is recorded as the set $\sigma := \{j: \mathbf{a}_j \in F\} \subseteq [n]$.

Proposition 4.1. [5, Proposition 7.8] *All associated primes of J_B are of the form $P_\sigma + I_A$ for some face σ of $\text{cone}(\mathcal{A})$. The toric ideal $I_A = P_{[n]} + I_A$ is the unique minimal prime of J_B . However, not all proper faces of $\text{cone}(\mathcal{A})$ need index an associated prime of J_B .*

Definition 4.2. [1] Let I be any ideal in $k[\mathbf{x}]$ and let P be an ideal that contains I . Then P is an associated prime of I if P is prime and there exists some $f \in k[\mathbf{x}]$ such that $(I : f) = P$. We call f a *witness* for P .

Using Proposition 4.1, we can now state the main results of this section. We say that τ is the *type* of a standard pair of the form (\cdot, τ) .

Theorem 4.3. Let τ be any (possibly empty) proper face of $\text{cone}(\mathcal{A})$ and ω be a generic weight vector. If $P_\tau + I_{\mathcal{A}}$ is associated to $J_{\mathcal{B}}$ (and hence embedded), then there exists $\gamma \subseteq \tau$ and a $J_{\mathcal{B}}$ -specific standard pair of $\text{in}_\omega(J_{\mathcal{B}})$ of type γ such that

- (1) if σ is a face of $\text{cone}(\mathcal{A})$ properly contained in τ , then γ is not contained in σ , and
- (2) $|\gamma| = \dim \text{cone}(\mathcal{A}_\tau)$.

Furthermore, there is a witness for the prime $P_\tau + I_{\mathcal{A}}$ whose leading term with respect to ω lies on such a JSP.

We also prove a partial converse.

Theorem 4.4. For a generic ω , if $\text{in}_\omega(J_{\mathcal{B}})$ has a JSP of type γ , then $J_{\mathcal{B}}$ has an embedded prime $P_\sigma + I_{\mathcal{A}}$ for some face σ of $\text{cone}(\mathcal{A})$ such that $\sigma \supseteq \gamma$.

Before proving the theorems, we consider a few implications. We say that a face τ of $\text{cone}(\mathcal{A})$ is *simplicial* if $|\tau| = \dim \text{cone}(\mathcal{A}_\tau)$. If τ is a simplicial face of $\text{cone}(\mathcal{A})$, then no binomial in $I_{\mathcal{A}}$ is supported entirely on τ , so $P_\tau + I_{\mathcal{A}}$ is just the monomial prime $P_\tau = \langle x_i : i \notin \tau \rangle$. Then Theorem 4.3 specializes as follows.

Corollary 4.5. If $P_\tau + I_{\mathcal{A}}$ is an embedded prime of $J_{\mathcal{B}}$ and τ is a simplicial face of $\text{cone}(\mathcal{A})$, then for every generic ω , $\text{in}_\omega(J_{\mathcal{B}})$ has a JSP of type τ .

The situation is more complicated when non-simplicial faces of $\text{cone}(\mathcal{A})$ index embedded primes. In particular, Theorem 4.3 does not specify a particular $\gamma \subseteq [n]$ such that every monomial initial ideal of $J_{\mathcal{B}}$ must have a JSP of type γ .

Example 4.6. Let \mathcal{B} be the set of circuits of the matrix

$$A = \begin{pmatrix} 5 & 0 & 0 & 2 & 1 & 0 \\ 0 & 5 & 0 & 1 & 4 & 2 \\ 0 & 0 & 5 & 2 & 0 & 3 \end{pmatrix}.$$

The configuration \mathcal{A} labeled 1, ..., 6 in Fig. 3 spans the cone over a triangle in \mathbb{R}^3 , so by Proposition 4.1, there are seven possible embedded primes of $J_{\mathcal{B}}$ corresponding to the seven proper faces of $\text{cone}(\mathcal{A})$. All seven of these primes are indeed associated to $J_{\mathcal{B}}$. The two non-simplicial 2-dimensional faces, $\{1, 2, 5\}$ and $\{2, 3, 6\}$, index the primes $P_{\{1,2,5\}} + I_{\mathcal{A}} = \langle f, d, c, ab^4 - e^5 \rangle$ and $P_{\{2,3,6\}} + I_{\mathcal{A}} = \langle e, d, a, b^2c^3 - f^5 \rangle$. The third 2-face $\{1, 3\}$ is simplicial and indexes the prime $P_{\{1,3\}} = \langle b, d, e, f \rangle$. The remaining four primes $P_{\{1,2\}} = \langle c, d, e, f \rangle$, $P_{\{1,3\}} = \langle b, d, e, f \rangle$, $P_{\{2,3\}} = \langle a, d, e, f \rangle$, and $P_{\{\emptyset\}} = \langle a, b, c, d, e, f \rangle$ correspond to the three rays of $\text{cone}(\mathcal{A})$ and to the apex, all of which are trivially simplicial.

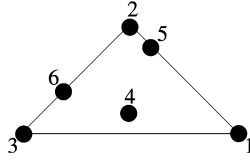


Fig. 3. The point configuration of Example 4.6.

By Corollary 4.5, each initial ideal $\text{in}_\omega(J_B)$ has JSPs of types $\{1, 3\}$, $\{3\}$, $\{2\}$, $\{1\}$, and \emptyset corresponding to the five simplicial faces of $\text{cone}(\mathcal{A})$. Since $P_{\{1,2,5\}} + I_{\mathcal{A}}$ is associated, Theorem 4.3 requires that $\text{in}_\omega(J_B)$ has a JSP of type $\{1, 2\}$, $\{1, 5\}$, or $\{2, 5\}$. Similarly, because of $P_{\{2,3,6\}} + I_{\mathcal{A}}$, there must be a JSP of type $\{2, 3\}$, $\{2, 6\}$, or $\{3, 6\}$. We list the types of JSPs that appear for two term orders.

- For lexicographic order with $f > e > \cdots > a$, $\text{in}_>(J_B)$ has the following types of JSPs: $\{1, 3\}$, $\{3\}$, $\{2\}$, $\{1\}$, \emptyset , $\{1, 2\}$, $\{2, 3\}$.
- For reverse lexicographic order with $a > b > \cdots > f$, $\text{in}_>(J_B)$ has the following types of JSPs: $\{1, 3\}$, $\{3\}$, $\{2\}$, $\{1\}$, \emptyset , $\{6\}$, $\{1, 5\}$, $\{3, 6\}$, $\{2, 6\}$, $\{2, 5\}$.

We now prove Theorem 4.3. The idea is to find a witness for the embedded prime $P_\tau + I_{\mathcal{A}}$, compute its normal form with respect to the reduced Gröbner basis $G_\omega(J_B)$, and show that the result is a witness whose ω -initial term lies on a JSP satisfying all of the desired properties.

Lemma 4.7. *If f is a witness for an embedded prime $P_\tau + I_{\mathcal{A}}$ of J_B , then the following hold.*

- (1) *The witness f is in the toric ideal $I_{\mathcal{A}}$.*
- (2) *For any $g \in J_B$, $f + g$ is also a witness for $P_\tau + I_{\mathcal{A}}$. In particular, the normal form of f with respect to $G_\omega(J_B)$ is a witness.*
- (3) *If $\mathbf{x}^{\mathbf{m}}$ is a monomial with $\text{supp}(\mathbf{m}) \subseteq \tau$, then $\mathbf{x}^{\mathbf{m}}f$ is also a witness.*

Proof. (1) Since τ is a proper face of $\text{cone}(\mathcal{A})$, there is some variable $x_i \in P_\tau + I_{\mathcal{A}}$, so $x_i f \in J_B \subset I_{\mathcal{A}}$. Since $I_{\mathcal{A}}$ is a prime ideal without monomials, $f \in I_{\mathcal{A}}$.

(2) Since $g \in J_B$, so is pg for any polynomial $p \in \mathbf{k}[\mathbf{x}]$, and thus $p(f + g) \in J_B \Leftrightarrow pf \in J_B$. Thus $(J_B : f + g) = (J_B : f) = P_\tau + I_{\mathcal{A}}$.

(3) If $h \in P_\tau + I_{\mathcal{A}}$, then $h(\mathbf{x}^{\mathbf{m}}f) = (\mathbf{x}^{\mathbf{m}}h)f$ is in J_B by the assumption that f is a witness. On the other hand, if $h \notin P_\tau + I_{\mathcal{A}}$, then neither is $\mathbf{x}^{\mathbf{m}}h$ because $\text{supp}(\mathbf{m}) \subseteq \tau$ and $P_\tau + I_{\mathcal{A}}$ is prime. Thus $\mathbf{x}^{\mathbf{m}}h \notin P_\tau + I_{\mathcal{A}} = (J_B : f)$, so $h \notin (J_B : \mathbf{x}^{\mathbf{m}}f)$. Thus $(J_B : \mathbf{x}^{\mathbf{m}}f) = (J_B : f) = P_\tau + I_{\mathcal{A}}$ as claimed. \square

Lemma 4.8. *If f is a witness for an embedded prime $P_\tau + I_{\mathcal{A}}$ of J_B and \bar{f} is the normal form of f with respect to $G_\omega(J_B)$, then $\text{in}_\omega(\bar{f})$ lies on a JSP (\cdot, γ) of $\text{in}_\omega(J_B)$ with $\gamma \subseteq \tau$.*

Proof. By Lemma 4.7, \bar{f} is also a witness for $P_\tau + I_{\mathcal{A}}$ and $\bar{f} \in I_{\mathcal{A}}$. This implies that $\text{in}_\omega(\bar{f}) \in \text{in}_\omega(I_{\mathcal{A}}) \setminus \text{in}_\omega(J_B)$, so $\text{in}_\omega(\bar{f})$ must lie on some JSP (\cdot, γ) of $\text{in}_\omega(J_B)$. Since $x_i \bar{f} \in J_B$ whenever $i \notin \tau$ because \bar{f} witnesses $P_\tau + I_{\mathcal{A}}$, it follows that $x_i \text{in}_\omega(\bar{f}) \in \text{in}_\omega(J_B)$ for $i \notin \tau$, so $\gamma \subseteq \tau$. \square

Proof of Theorem 4.3. Suppose $P_\tau + I_{\mathcal{A}}$ is an embedded prime of J_B and $e := \dim \text{cone}(\mathcal{A}_\tau)$. We first claim the following: there is a constant C such that for all sufficiently large N , $P_\tau + I_{\mathcal{A}}$

has at least N^e witnesses whose normal forms with respect to $G_\omega(J_B)$ have distinct leading terms, and each such leading term $\mathbf{x}^{\mathbf{p}}$ has the property that every component of \mathbf{p} is bounded above by CN .

Suppose the claim is true. By Lemma 4.8, each such monomial $\mathbf{x}^{\mathbf{p}}$ must lie on a JSP of type γ with $\gamma \subseteq \tau$. Each such standard pair contains at most $C^{|\gamma|}(N+1)^{|\gamma|}$ monomials $\mathbf{x}^{\mathbf{p}}$ such that p_i is bounded above by CN . Since there are only finitely many standard pairs for $\text{in}_\omega(J_B)$, all the standard pairs of type γ with $|\gamma| < e$ together cover only at most $\mathcal{O}(N^{e-1})$ of the monomials which is not enough to contain the N^e leading terms $\mathbf{x}^{\mathbf{p}}$. Thus some of these leading terms must lie on standard pairs (\cdot, γ) with $|\gamma| \geq e$. Since by Corollary 3.3, each γ is a face of the triangulation Δ_ω of \mathcal{A} , this is only possible if $|\gamma| = e$ and γ is not contained in any face σ of $\text{cone}(\mathcal{A})$ whose dimension is less than e . These are exactly the types of standard pairs specified by Theorem 4.3.

Now we prove the claim. Since $P_\tau + I_{\mathcal{A}}$ and J_B are both \mathcal{A} -homogeneous, there exists an \mathcal{A} -homogeneous witness f for $P_\tau + I_{\mathcal{A}}$. Set $\mathbf{x}^{\mathbf{u}} := \text{in}_\omega(f)$. Since $e = \dim \text{cone}(\mathcal{A}_\tau)$, we can find an e -subset α of τ such that the columns of A indexed by α are linearly independent. Thus if $\mathbf{m}_1 \neq \mathbf{m}_2$ are supported only on α , then $A\mathbf{m}_1 \neq A\mathbf{m}_2$.

Consider all polynomials of the form $\mathbf{x}^{\mathbf{m}}f$ where $0 \leq m_i < N$ for $i \in \alpha$ and $m_i = 0$ for $i \notin \alpha$. Such a polynomial is \mathcal{A} -homogeneous of \mathcal{A} -degree $A\mathbf{m} + A\mathbf{u}$ and so is its normal form with respect to $G_\omega(J_B)$ since J_B is an \mathcal{A} -homogeneous ideal. Thus the normal forms of these N^e polynomials are all \mathcal{A} -homogeneous of different degrees, so in particular they all have distinct leading terms. Furthermore, by parts (2) and (3) of Lemma 4.7, each such normal form is a witness for $P_\tau + I_{\mathcal{A}}$.

It remains to establish that if $\mathbf{x}^{\mathbf{p}}$ is the leading term of one of the normal forms, then each component of \mathbf{p} is bounded by a constant multiple of N . Let \mathbf{a} be a strictly positive vector in the rowspan of A . Such a vector exists since $\mathcal{L}_{\mathcal{A}} \cap \mathbb{N}^n = \{0\}$. By scaling, we can assume that the minimum component of \mathbf{a} is 1. Let R be its maximum component. Since $A\mathbf{p} = A(\mathbf{u} + \mathbf{m})$, it follows that $\mathbf{a} \cdot \mathbf{p} = \mathbf{a} \cdot (\mathbf{u} + \mathbf{m})$. Then

$$\begin{aligned} \|\mathbf{p}\|_1 &= \sum_{i=1}^n p_i \leq \sum_{i=1}^n a_i p_i = \sum_{i=1}^n a_i (u_i + m_i) \\ &\leq R \sum_{i=1}^n (u_i + m_i) = R \left(\sum_{i=1}^n u_i + \sum_{i=1}^n m_i \right) < R(\|\mathbf{u}\|_1 + nN). \end{aligned}$$

It follows that for any i , we have

$$p_i \leq \|\mathbf{p}\|_1 < RnN + R\|\mathbf{u}\|_1$$

which is a bound of the desired form. \square

We now prove Theorem 4.4. Recall the following algebraic fact.

Lemma 4.9. *If I is an ideal in $k[\mathbf{x}]$ and g is any polynomial, then the associated primes of $(I : g^\infty)$ are exactly the associated primes of I that do not contain g .*

Proposition 4.10. Recall that $\mathbf{x}_\tau = \prod_{i \in \tau} x_i$. The associated primes of $(J_{\mathcal{B}} : \mathbf{x}_\tau^\infty)$ are exactly the associated primes $P_\sigma + I_{\mathcal{A}}$ of $J_{\mathcal{B}}$ that satisfy $\sigma \supseteq \tau$.

Proof. We get $\mathbf{x}_\tau \in P_\sigma + I_{\mathcal{A}}$ if and only if some x_i with $i \in \tau$ lies in $P_\sigma + I_{\mathcal{A}}$, which occurs if and only if τ is not contained in σ . Now apply Lemma 4.9. \square

Proof of Theorem 4.4. Suppose (\mathbf{x}^u, γ) is a JSP of $\text{in}_\omega(J_{\mathcal{B}})$. Choose $f \in I_{\mathcal{A}}$ such that $\text{in}_\omega(f) = \mathbf{x}^u \mathbf{x}_\gamma^m$ for some $m \geq 0$. This is possible since every JSP of $\text{in}_\omega(J_{\mathcal{B}})$ contains non-standard monomials of $\text{in}_\omega(I_{\mathcal{A}})$. Since no monomial of the form $\text{in}_\omega(f) \cdot \mathbf{x}_\gamma^*$ lies in $\text{in}_\omega(J_{\mathcal{B}})$, no polynomial of the form $f \cdot \mathbf{x}_\gamma^*$ lies in $J_{\mathcal{B}}$. This implies that $(J_{\mathcal{B}} : \mathbf{x}_\gamma^\infty)$ does not contain f and is hence not equal to $I_{\mathcal{A}}$. However, since $(J_{\mathcal{B}} : \mathbf{x}_\gamma^\infty) \subsetneq I_{\mathcal{A}}$, $(J_{\mathcal{B}} : \mathbf{x}_\gamma^\infty)$ must have an embedded prime. This prime is also embedded in $J_{\mathcal{B}}$ by Proposition 4.10, and it has the form $P_\sigma + I_{\mathcal{A}}$ for some $\sigma \supseteq \gamma$. \square

Theorem 4.4 is only a partial converse to Theorem 4.3. It is not true for a given weight vector ω that the existence of a JSP (\mathbf{x}^u, γ) of $\text{in}_\omega(J_{\mathcal{B}})$ satisfying the conditions of Theorem 4.3 with respect to some proper face τ of $\text{cone}(\mathcal{A})$ implies that $P_\tau + I_{\mathcal{A}}$ is associated to $J_{\mathcal{B}}$.

Example 4.11. Let \mathcal{B} be the set of circuits of the matrix

$$A = \begin{pmatrix} 1 & 3 & 2 & 4 \\ 1 & 4 & 5 & 2 \end{pmatrix}.$$

The values of the \mathcal{A} -graded Hilbert function of this \mathcal{A} are shown in Fig. 1. The proper faces of $\text{cone}(\mathcal{A})$ are $\{3\}$, $\{4\}$, and \emptyset . Only the first two index associated primes of $J_{\mathcal{B}}$. However, if we take ω to represent the lexicographic term order with $a > b > c > d$, there are five JSPs of $\text{in}_\omega(J_{\mathcal{B}})$ of type \emptyset . On the other hand, if ω represents the \mathcal{A} -graded reverse lexicographic order with $a > b > c > d$, then there are no JSPs of type \emptyset .

Question 4.12. If τ is a face of $\text{cone}(\mathcal{A})$ such that for every generic ω there is a JSP of $\text{in}_\omega(J_{\mathcal{B}})$ satisfying the conditions of Theorem 4.3 with respect to τ , then is $P_\tau + I_{\mathcal{A}}$ necessarily associated to $J_{\mathcal{B}}$?

We conclude this section with an application of Theorem 4.3 to the specific case of circuit ideals of normal configurations. Let \mathcal{B} be the set of circuits of a configuration \mathcal{A} satisfying $\mathbb{Z}\mathcal{A} = \mathbb{Z}^d$ and whose vectors comprise the lattice points in a lattice polytope R . Further assume that the first row of A is $(1, \dots, 1)$, so R is at height 1. The polytope R defines a projective toric variety $X_{\mathcal{A}}$ and the faces $\{\tau\}$ of R (which are in bijection with the faces of $\text{cone}(\mathcal{A})$) index a canonical collection of affine charts $\{U_\tau\}$ covering $X_{\mathcal{A}}$ [7]. We investigate how smoothness of U_τ determines whether $P_\tau + I_{\mathcal{A}}$ is associated to $J_{\mathcal{C}_{\mathcal{A}}} = J_{\mathcal{B}}$.

Definition 4.13.

- (1) Let K be a convex rational polyhedral cone in \mathbb{R}^t that does not contain a line. We say that K is *smooth* if it is generated by primitive vectors that form part of a basis for \mathbb{Z}^t .
- (2) Let K_F denote the inner normal cone of a face F of a polytope Q . The face F is *smooth* if the restriction of K_F to the linear span of Q is smooth.

Remark 4.14.

- (1) If v is a smooth vertex of a polytope Q then there are exactly $\dim Q$ edges of Q incident to v . Further, the cone dual to K_v is also smooth [6, Theorem 2.10, Chapter V]. Note that this dual cone is the tangent cone of Q at v and contains Q .
- (2) A face F of a polytope Q is smooth if and only if the affine toric variety U_F is smooth [7].

Theorem 4.15. *Let A and R be as above. If \mathbf{a}_n is a smooth vertex of R , then $P_{\{n\}} (= P_{\{n\}} + I_A)$ is not an associated prime of $J_{\mathcal{C}_A}$.*

Proof. Suppose $P_{\{n\}}$ is associated. Since $\{n\}$ is a simplicial face of $\text{cone}(A)$, by Corollary 4.5, every monomial initial ideal $\text{in}_\omega(J_{\mathcal{C}_A})$ has a JSP of the form $(\mathbf{x}^{\mathbf{u}}, \{n\})$. In particular, let ω represent an elimination order with x_n most expensive. We may assume that each of $\mathbf{a}_1, \dots, \mathbf{a}_{d-1} \in A$ is the first lattice point from \mathbf{a}_n along one of the $d-1$ edges incident to \mathbf{a}_n . Let $\mathbf{y}_i := \mathbf{a}_i - \mathbf{a}_n$ for $i = 1, \dots, d-1$.

Since \mathbf{a}_n is smooth and R is contained in the tangent cone at \mathbf{a}_n , for each lattice point in R (i.e. column of A), there are unique $m_i \in \mathbb{N}$ such that $\mathbf{a}_j = \mathbf{a}_n + \sum_{i=1}^{d-1} m_i \mathbf{y}_i$. Rearranging terms, and setting $M = -1 + \sum_{i=1}^{d-1} m_i$, we get

$$\mathbf{a}_j + M\mathbf{a}_n = \sum_{i=1}^{d-1} m_i \mathbf{a}_i \quad (1)$$

with all coefficients non-negative. If $j = n$, (1) reduces to $0 = 0$, and if $1 \leq j \leq d-1$, it reduces to $\mathbf{a}_j = \mathbf{a}_j$. But in the non-trivial case where $d-1 < j < n$, (1) is a *circuit* because $\mathbf{a}_1, \dots, \mathbf{a}_{d-1}, \mathbf{a}_n$ form a maximal linearly independent set. Thus $x_j x_n^M - \prod_{i=1}^{d-1} x_i^{m_i} \in J_{\mathcal{C}_A}$. By choice of ω , its leading term is $x_j x_n^M$. Since $(\mathbf{x}^{\mathbf{u}}, \{n\})$ is a JSP, this term must not divide $x_n^N \mathbf{x}^{\mathbf{u}}$ for any N . Thus $j \notin \text{supp}(\mathbf{u})$.

For N sufficiently large, $x_n^N \mathbf{x}^{\mathbf{u}} \in \text{in}_\omega(I_A)$, so we can choose $\mathbf{x}^{\mathbf{v}} \notin \text{in}_\omega(I_A)$ such that $x_n^N \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in I_A$. Since I_A is prime, factor out any common monomial to get $\mathbf{x}^{\tilde{\mathbf{u}}} - \mathbf{x}^{\tilde{\mathbf{v}}} \in I_A$ where $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{v}}$ have disjoint supports. Since $\tilde{\mathbf{u}} - \tilde{\mathbf{v}} \in \mathcal{L}_A$, the convex hulls of $\{\mathbf{a}_i : i \in \text{supp}(\tilde{\mathbf{u}})\}$ and $\{\mathbf{a}_i : i \in \text{supp}(\tilde{\mathbf{v}})\}$ must intersect.

Since \mathbf{a}_n is smooth, we can assume by applying an invertible \mathbb{Z} -affine transformation that \mathbf{a}_n is the origin and \mathbf{a}_i is the i th standard basis vector in \mathbb{Z}^{d-1} for $1 \leq i \leq d-1$. That is, $\mathbf{a}_n, \mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{d-1}$ are the vertices of the standard simplex S in \mathbb{R}^{d-1} . Since $j \notin \text{supp}(\tilde{\mathbf{u}})$ for any $d-1 < j < n$, $U := \text{conv}(\mathbf{a}_i : i \in \text{supp}(\tilde{\mathbf{u}}))$ is a face of S . Since $\text{supp}(\tilde{\mathbf{v}}) \cap \text{supp}(\tilde{\mathbf{u}}) = \emptyset$ and S contains no lattice points except its vertices, $V := \text{conv}(\mathbf{a}_i : i \in \text{supp}(\tilde{\mathbf{v}}))$ consists only of vertices of S outside U along with lattice points in $R \setminus S$. Now S and $\overline{R \setminus S}$ are both convex, so U and V could intersect only on the boundary of S . But since the vertices in U and those in $V \cap S$ form disjoint faces of S , there is no intersection on this boundary, contradicting that $U \cap V \neq \emptyset$. \square

Example 4.16 (Non-smooth vertices of R may or may not index associated primes of $J_{\mathcal{C}_A}$). For the A below, the polytope R is a triangle in \mathbb{R}^3 .

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 3 & 4 & 5 & 5 \\ 0 & 1 & 1 & 1 & 2 \end{pmatrix}.$$

None of the three vertices $(0, 0)$, $(5, 1)$, and $(5, 2)$ of R are smooth. The vertices $(0, 0)$ and $(5, 2)$ both index associated primes, while the vertex $(5, 1)$ does not.

Example 4.17 (Smooth edges of R may index associated primes of J_{C_A}). Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \end{pmatrix}$$

where $\text{cone}(A)$ is the cone over a rectangular prism R . All edges of R are smooth, but we compute that four of them index associated primes of J_{C_A} .

5. Fans of toric and circuit ideals

The main goal of this section is to compare I_A and J_{C_A} via three polyhedral fans that can be associated to them. These results rely on Corollary 5.7 which states that for a generic ω , the top-dimensional components of $\text{in}_\omega(I_A)$ and $\text{in}_\omega(J_{C_A})$ are the same. We begin by providing a more general result.

Definition 5.1. [14, p. 112] Let $J \subseteq \mathbf{k}[\mathbf{x}]$ be an ideal and d be the Krull dimension of $\mathbf{k}[\mathbf{x}]/J$. We define $\text{top}(J)$ to be the intersection of all primary components of J of dimension d .

Note that $\text{top}(J)$ is well-defined since the d -dimensional primary components of J are minimal and are hence unique.

In projective space the degree of an irreducible variety is defined as the number of points in its intersection with a complementary dimensional subspace in general position. Recall the usual generalization to ideals.

Definition 5.2. [17, Section 1] Let $P \subseteq \mathbf{k}[\mathbf{x}]$ be a prime ideal. The *multiplicity* $\text{mult}(Q)$ of a P -primary ideal Q is the length of a maximal strictly increasing chain of P -primary ideals $Q \subset \cdots \subset P$. Let $I \subseteq \mathbf{k}[\mathbf{x}]$ be a homogeneous ideal in the total degree grading. The *degree* of I is defined as

$$\deg(I) = \sum_Q \text{mult}(Q) \deg(V(Q)),$$

where the sum is taken over the d -dimensional primary components Q in a minimal primary decomposition of I and $V(Q)$ denotes the variety defined by Q .

The degree is also characterized as the normalized leading coefficient of the Hilbert polynomial of $\mathbf{k}[\mathbf{x}]/I$ and thus does not change when going to initial ideals.

Lemma 5.3. Let $J \subseteq I \subseteq \mathbf{k}[\mathbf{x}]$ be ideals with the Krull dimension of $\mathbf{k}[\mathbf{x}]/J$ being d . Any d -dimensional associated prime P of I is an associated prime of J .

Proof. Clearly, $J \subseteq P$. We wish to show that P is a minimal prime of J . By Proposition 4.6 in [1] it suffices to prove that P is minimal with respect to inclusion among all prime ideals

containing J . Suppose P' is a prime ideal with $J \subseteq P' \subseteq P$ then, since J and P have the same dimension, $P = P'$ and P is indeed minimal. \square

Proposition 5.4. *Let $I, J \subseteq \mathbf{k}[\mathbf{x}]$ be homogeneous ideals in the total degree grading with the Krull dimensions of $\mathbf{k}[\mathbf{x}]/I$ and $\mathbf{k}[\mathbf{x}]/J$ both being d and with the degrees of I and J being equal. If $J \subseteq I$ then $\text{top}(I) = \text{top}(J)$.*

Proof. We consider minimal primary decompositions of I and J

$$\begin{aligned} I &= Q_1 \cap \cdots \cap Q_t \cap \cdots \cap Q_r, \\ J &= Q'_1 \cap \cdots \cap Q'_s, \end{aligned}$$

where Q_1, \dots, Q_t are the d -dimensional components in the decomposition of I . By Lemma 5.3 we may assume that $\sqrt{Q_i} = \sqrt{Q'_i} =: P_i$ for $i = 1, \dots, t$. Let $i \leq t$ and consider the primary components Q_i and Q'_i . As $J \subseteq I$ we have $Q'_i \subseteq Q_i$. By the definition of multiplicity, $\text{mult}(Q'_i) \geq \text{mult}(Q_i)$. The ideal J may have other d -dimensional components in its decomposition. Hence using Definition 5.2 we get

$$\deg(J) \geq \sum_{i=1}^t \text{mult}(Q'_i) \deg(V(Q_i)) \geq \sum_{i=1}^t \text{mult}(Q_i) \deg(V(Q_i)) = \deg(I).$$

Our assumption $\deg(J) = \deg(I)$ now implies that $\text{mult}(Q'_i) = \text{mult}(Q_i)$ for all $i \leq t$. Furthermore, we see that J cannot have more d -dimensional components. According to the definition of multiplicity the inclusion $Q'_i \subseteq Q_i$ cannot be strict. This proves that the d -dimensional components are the same in the two decompositions. As top is defined as their intersection, $\text{top}(I) = \text{top}(J)$. \square

Corollary 5.5. *If $J \subseteq I \subseteq \mathbf{k}[\mathbf{x}]$ are homogeneous ideals in the total degree grading with $\text{top}(I) = \text{top}(J)$ then for $\omega \in \mathbb{R}^n$*

$$\text{top}(\text{in}_\omega(I)) = \text{top}(\text{in}_\omega(J)).$$

Proof. Clearly, $\text{in}_\omega(J) \subseteq \text{in}_\omega(I)$. It follows from the definition of degree and top that I and J have the same degree and dimension. So do their initial ideals. We now apply Proposition 5.4 to $\text{in}_\omega(J)$ and $\text{in}_\omega(I)$. \square

Corollary 5.6. *Let $J \subseteq \mathbf{k}[\mathbf{x}]$ be a homogeneous ideal in the total degree grading. For $\omega \in \mathbb{R}^n$ we have*

$$\text{top}(\text{in}_\omega(\text{top}(J))) = \text{top}(\text{in}_\omega(J)).$$

Proof. Let $I = \text{top}(J)$ and apply Corollary 5.5. \square

Corollary 5.7. *If I_A is homogeneous in the total degree grading and J_B is a basis ideal with $\sqrt{J_B} = I_A$, then for $\omega \in \mathbb{R}^n$*

$$\text{top}(\text{in}_\omega(I_A)) = \text{top}(\text{in}_\omega(J_B)).$$

In particular,

$$\text{top}(\text{in}_\omega(I_A)) = \text{top}(\text{in}_\omega(J_{\mathcal{C}_A})).$$

Proof. By [5, Theorem 7.6], the unique minimal primary component of J_B is $(J_B : (x_1 \cdots x_n)^\infty)$ which equals I_A by Proposition 1.3. Thus $\text{top}(J_B) = I_A$ and the claim follows immediately from Corollary 5.6. \square

Corollary 5.8. *If $J \subseteq I \subseteq \mathbf{k}[\mathbf{x}]$ are homogeneous ideals in the total degree grading with $\text{top}(I) = \text{top}(J)$ then for generic $\omega \in \mathbb{R}^n$ the d -dimensional standard pairs of $\text{in}_\omega(I)$ and of $\text{in}_\omega(J)$ are the same.*

Proof. The claim follows from Corollary 5.5 if we can prove that any d -dimensional monomial ideal M has the same d -dimensional standard pairs as $\text{top}(M)$. Consider a minimal primary decomposition $M = Q_1 \cap \cdots \cap Q_t \cap \cdots \cap Q_r$ where Q_1, \dots, Q_t are the d -dimensional components. Now $\text{top}(M) = Q_1 \cap \cdots \cap Q_t$. Without loss of generality we may assume that each Q_i is a monomial primary ideal and hence of the form $Q_i = \langle x_j^{(\mathbf{v}_i)_j} \rangle_{j \notin \sigma_i} + \langle \mathbf{x}^{\mathbf{w}} \rangle_{\mathbf{w} \in S_i}$, for some $\sigma_i \subseteq [n]$, $\mathbf{v}_i \in \mathbb{N}^n$ and a collection S_i of vectors in \mathbb{N}^n with support of size at least two and contained in $\overline{\sigma_i}$. Here $(\mathbf{v}_i)_j$ denotes the j th entry of the vector \mathbf{v}_i . The exponent vectors of monomials not in Q_i are unbounded exactly on the entries indexed by σ_i .

Any d -dimensional standard pair of $\text{top}(M)$ is clearly admissible for M . Furthermore, since $\dim(M) = d$ it is also a standard pair of M . Conversely, if $(\mathbf{x}^{\mathbf{u}}, \sigma)$ is a d -dimensional standard pair of M then the monomials it represents are contained in $\overline{Q_1 \cap \cdots \cap Q_r} = \overline{Q_1} \cup \cdots \cup \overline{Q_r}$. As the exponent vector of such a monomial may be arbitrary large at the entries indexed by σ , for some i we must have $\sigma \subseteq \sigma_i$ with $\mathbf{x}^{\mathbf{u}} \notin Q_i$. Since $|\sigma_k| \leq d$ for all $k = 1, \dots, r$, we get $|\sigma_i| = d$ and $\dim(Q_i) = d$. Hence $(\mathbf{x}^{\mathbf{u}}, \sigma)$ is admissible for $\text{top}(M)$ and since $\dim(\text{top}(M)) = d$ it is also standard. \square

In particular, the above corollary can be applied to the circuit ideal and the toric ideal of a point configuration. A longer variant of our proof, not presented here, proves the following statement for generic ω .

Conjecture 5.9. *The equality in Corollary 5.7 holds for any $\omega \in \mathbb{R}^n$ even if the ideals are not homogeneous in the total degree grading.*

5.1. Polyhedral fans of $J_{\mathcal{C}_A}$

An ideal in $\mathbf{k}[\mathbf{x}]$ gives rise to several natural equivalence relations on \mathbb{R}^n some of which give rise to polyhedral fans. In this final part, we compare various equivalence relations and fans for toric and circuit ideals.

Definition 5.10. Let $I \subseteq \mathbf{k}[\mathbf{x}]$ be an ideal homogeneous with respect to grading by a positive vector $\mathbf{a} \in \mathbb{N}_{>0}^n$. We define three equivalence relations on \mathbb{R}^n :

- The *initial ideal* equivalence relation $\mathbf{u} \sim \mathbf{v} \Leftrightarrow \text{in}_{\mathbf{u}}(I) = \text{in}_{\mathbf{v}}(I)$.
- The *top* equivalence relation $\mathbf{u} \sim \mathbf{v} \Leftrightarrow \text{top}(\text{in}_{\mathbf{u}}(I)) = \text{top}(\text{in}_{\mathbf{v}}(I))$.
- The *radical* equivalence relation $\mathbf{u} \sim \mathbf{v} \Leftrightarrow \sqrt{\text{in}_{\mathbf{u}}(I)} = \sqrt{\text{in}_{\mathbf{v}}(I)}$.

In all three cases, the equivalence classes are invariant under translation by \mathbf{a} .

Definition 5.11. A collection C of polyhedra in \mathbb{R}^n is a *polyhedral complex* if:

- (1) all proper faces of a polyhedron $P \in C$ are in C , and
- (2) the intersection of any two polyhedra $A, B \in C$ is a face of A and B .

A polyhedral complex is a *fan* if it only consists of cones.

We say that an equivalence relation defines a fan F if the closures of its equivalence classes are exactly the cones in F .

Proposition 5.12. Let I be as in Definition 5.10. Then

- (1) the initial ideal equivalence relation defines the Gröbner fan of I ,
- (2) the radical equivalence relation does not define a fan in general.

A proof of the first claim is given in [16, Chapter 2]. See also [13]. The following example demonstrates the second claim.

Example 5.13. The radical equivalence classes of the homogeneous ideal $I = \langle c^4 - ba^3, ab^3 - ba^3 \rangle \subset \mathbf{k}[a, b, c]$ are not all convex. Four of the eight monomial initial ideals of I have radical $\langle ab, ac, bc \rangle$ and the others have radical $\langle ab, c \rangle$. The intersection of the Gröbner fan with the two-dimensional standard simplex is shown in Fig. 4 and the two radical equivalence classes appear in gray and white.

However, for toric ideals, all three equivalence relations of Definition 5.10 give rise to polyhedral fans.

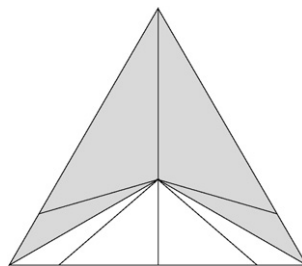


Fig. 4. The Gröbner fan from Example 5.13 and the two radical equivalence classes. The fan is drawn in the standard simplex with $(1, 0, 0)$ at the right bottom, $(0, 1, 0)$ at the left bottom and $(0, 0, 1)$ at the top.

Proposition 5.14.

- (1) *The radical equivalence relation of $I_{\mathcal{A}}$ defines the secondary fan of \mathcal{A} .*
- (2) *The top equivalence relation of $I_{\mathcal{A}}$ defines the hypergeometric fan of \mathcal{A} .*

Furthermore, the Gröbner fan of $I_{\mathcal{A}}$ is a refinement of the hypergeometric fan of \mathcal{A} , which is a refinement of the secondary fan of \mathcal{A} .

Proposition 5.14 may be taken as the definition of the hypergeometric and secondary fans of \mathcal{A} . The proposition is a collection of several known results [14, Proposition 3.3.1 and Corollary 3.3.2], [2], and [16, Chapter 8]. We now study the three equivalence classes for $J_{\mathcal{C}_{\mathcal{A}}}$.

Proposition 5.15. *The radical equivalence classes of $J_{\mathcal{C}_{\mathcal{A}}}$ form a polyhedral fan that coincides with the secondary fan of $I_{\mathcal{A}}$.*

Proof. This follows from Proposition 3.1 since $I_{\mathcal{A}} = \sqrt{J_{\mathcal{C}_{\mathcal{A}}}}$. \square

For the top equivalence relation of $J_{\mathcal{C}_{\mathcal{A}}}$ we need the following proposition which follows from Corollary 5.7.

Proposition 5.16. *The top equivalence relation for $I_{\mathcal{A}}$ and $J_{\mathcal{B}}$ are the same if $I_{\mathcal{A}}$ is homogeneous in the total degree grading and $I_{\mathcal{A}} = \sqrt{J_{\mathcal{B}}}$. It follows that the top equivalence relation of $J_{\mathcal{B}}$ defines the hypergeometric fan of $I_{\mathcal{A}}$. In particular, this holds if $\mathcal{B} = \mathcal{C}_{\mathcal{A}}$.*

We conjecture that the condition that $I_{\mathcal{A}}$ is homogeneous in the total degree grading can be left out. In contrast to Theorem 5.15 and Proposition 5.16, we have the following result for the initial ideal equivalence relation for $I_{\mathcal{A}}$ and $J_{\mathcal{C}_{\mathcal{A}}}$.

Proposition 5.17. *In general, neither is the Gröbner fan of $I_{\mathcal{A}}$ a refinement of the Gröbner fan of $J_{\mathcal{C}_{\mathcal{A}}}$, nor vice-versa.*

Proof. Let $A = (7 \ 9 \ 13 \ 15)$. It is easy to check that $\text{in}_{(0,16,27,1)}(J_{\mathcal{C}_{\mathcal{A}}}) = \text{in}_{(0,20,25,3)}(J_{\mathcal{C}_{\mathcal{A}}})$ while $\text{in}_{(0,16,27,1)}(I_{\mathcal{A}}) \neq \text{in}_{(0,20,25,3)}(I_{\mathcal{A}})$. Hence $(0, 16, 27, 1)$ and $(0, 20, 25, 3)$ lie in the same maximal cell of the Gröbner fan of $J_{\mathcal{C}_{\mathcal{A}}}$ but in different maximal cells of the Gröbner fan of $I_{\mathcal{A}}$. This proves that the Gröbner fan of $J_{\mathcal{C}_{\mathcal{A}}}$ does not refine the Gröbner fan of $I_{\mathcal{A}}$. On the other hand, $\text{in}_{(0,4,19,9)}(I_{\mathcal{A}}) = \text{in}_{(0,4,16,5)}(I_{\mathcal{A}})$ and $\text{in}_{(0,4,19,9)}(J_{\mathcal{C}_{\mathcal{A}}}) \neq \text{in}_{(0,4,16,5)}(J_{\mathcal{C}_{\mathcal{A}}})$. Hence the Gröbner fan of $I_{\mathcal{A}}$ does not refine the Gröbner fan of $J_{\mathcal{C}_{\mathcal{A}}}$. \square

The example in Proposition 5.17 would be best illustrated by a picture of the three-dimensional standard simplex in \mathbb{R}^4 intersected with the fans. Unfortunately we are limited to two dimensions in our drawing (Fig. 5). The hypergeometric fan of $I_{\mathcal{A}}$ is drawn at the end of the two Gröbner fans.

Corollary 5.18. *If $n - d = 2$ the Gröbner fan of $J_{\mathcal{C}_{\mathcal{A}}}$ refines that of $I_{\mathcal{A}}$.*

Proof. By Theorem 3.3.8 in [14], if $n - d = 2$ then the Gröbner fan of $I_{\mathcal{A}}$ equals the hypergeometric fan of \mathcal{A} . The corollary then follows from Proposition 5.16 and the fact that the Gröbner fan of $J_{\mathcal{C}_{\mathcal{A}}}$ refines the hypergeometric fan of \mathcal{A} . \square

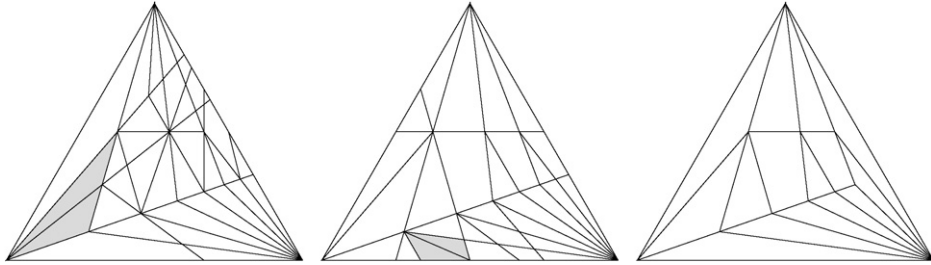


Fig. 5. The Gröbner fans in the proof of Proposition 5.17 intersected with the simplex with coordinates $(0, 1, 0, 0)$ (right), $(0, 0, 1, 0)$ (left) and $(0, 0, 0, 1)$ (top). The circuit fan is to the left, the toric fan in the middle and the hypergeometric fan at the right.

Acknowledgment

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