

## Addendum

# Class number of an abelian group

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### Abstract

The groups in this paper are abelian. In this Addendum to [T.G. Faticoni, The class number of an abelian group, J. Algebra 314 (2007) 978–1008] we show that a problem in the direct sum decompositions of torsion-free finite rank groups implies several important problems of number theory.

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### 1. Unique factorization

Given a ring  $R$ , and let  $u(R)$  be the units of  $R$ . Let  $\mathbf{k}$  be an algebraic number field, let  $\overline{E}$  be the ring of algebraic integers in  $\mathbf{k}$ , let  $f = [\mathbf{k} : \mathbb{Q}]$ , and let  $\Omega(\overline{E}) = \{\text{rtffr groups } G \mid \text{End}(G) \cong \overline{E} \text{ as rings}\}$ . Let  $h(\overline{E})$  denote the number of isomorphism classes of invertible fractional  $\overline{E}$ -ideals, and let  $h(\mathbf{k}) = h(\overline{E})$ . Let  $\overline{E}$  be the ring of algebraic integers in  $\mathbf{k}$  and for each prime  $p$  let  $E(p) = \mathbb{Z} + p\overline{E}$ , and let  $L(p) = \text{card}(u(\overline{E})/u(E(p)))$ .

Let  $h(G)$  denote the number of isomorphism classes of groups  $H$  that are locally isomorphic to  $G$ . We say that  $G$  has *power cancellation* if  $G^n \cong H^n$  implies  $G \cong H$  for groups  $H$  and integers  $n > 0$ . We say that  $G$  has  *$\Sigma$ -unique decomposition* if for each integer  $n > 0$ ,  $G^n$  has a unique direct sum decomposition.

**Theorem 1.1.** *The following are equivalent for an algebraic number field  $\mathbf{k}$ .*

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- (1) Each  $G \in \Omega(\bar{E})$  possesses the power cancellation property.
- (2) Each  $G \in \Omega(\bar{E})$  possesses a  $\Sigma$ -unique decomposition.
- (3)  $\mathbf{k}$  has the unique factorization property.
- (4) The sequence  $\{L(p)h(E(p)) \mid \text{rational primes } p\}$  is asymptotically equal to  $\{p^{f-1} \mid \text{rational primes } p\}$ .

**Proof.** By [2, Theorems 5.6, 6.1], parts (1) and (2) are equivalent to

- (5) Each  $G \in \Omega(\bar{E})$  satisfies  $h(G) = 1$ .

By [2, Corollary 3.2],  $h(G) = h(\bar{E}) = h(\mathbf{k})$ , and it is known that  $h(\mathbf{k}) = 1$  iff  $\mathbf{k}$  has unique factorization. Thus parts (5) and (3) are equivalent.

By [2, Theorem 9.4],  $h(\mathbf{k}) = 1$  iff the sequence  $\{h(E(p))L(p) \mid \text{rational primes } p\}$  is asymptotically equal to  $\{p^{f-1} \mid \text{rational primes } p\}$ . This completes the proof.  $\square$

**Corollary 1.2.** *The following are equivalent for a quadratic number field  $\mathbf{k}$ .*

- (1) Each  $G \in \Omega(\bar{E})$  possesses the power cancellation property.
- (2) Each  $G \in \Omega(\bar{E})$  possesses a  $\Sigma$ -unique decomposition.
- (3)  $\mathbf{k}$  has the unique factorization property.
- (4) The sequence  $\{L(p)h(E(p)) \mid \text{rational primes } p\}$  is asymptotically equal to the sequence of rational primes.

**Remark 1.3.**  $\Omega(\bar{E}) \neq \emptyset$  by Corner's Theorem [3] and Butler's Theorem [1]. Carl F. Gauss asked for a characterization of those algebraic number fields with unique factorization [4]. We recall that unique factorization in  $\mathbf{k}$  can be used to prove Fermat's Last Theorem for regular primes [4]. The sequence of rational primes is at the center of many important problems in number theory. This explains why many of the direct sum decomposition problems in torsion-free finite rank groups are hard. These problems imply several hard problems in number theory.

## References

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