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Stanley depth of complete intersection monomial ideals and upper-discrete partitions

Yi Huang Shen

Department of Mathematics, Purdue University, West Lafayette, IN 47907, USA

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ABSTRACT

Let I be an m-generated complete intersection monomial ideal in $S = K[x_1, \ldots, x_n]$. We show that the Stanley depth of I is $n - \lfloor \frac{m}{2} \rfloor$. We also study the upper-discrete structure for monomial ideals and prove that if I is a squarefree monomial ideal minimally generated by 3 elements, then the Stanley depth of I is $n - 1$.

1. Introduction

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the set of non-negative integers. Let K be a field and $S = K[x_1, \ldots, x_n]$ be a polynomial ring over K. Suppose M is a finitely generated $\mathbb{Z}^n$-graded S-module. If $u \in M$ is a homogeneous element and Z is a subset of $\{x_1, \ldots, x_n\}$, then the K-subspace $uK[Z]$ of M is called a *Stanley space*. A *Stanley decomposition of* M is a partition $\mathcal{D} : M = \bigoplus_{i=1}^m u_i K[Z_i]$ in the category of $\mathbb{Z}^n$-graded K-vector spaces. The *Stanley depth of* $\mathcal{D}$ is $\operatorname{sdepth}(\mathcal{D}) = \min\{|Z_i|\colon\ 1 \leqslant i \leqslant m\}$ and the *Stanley depth of* M is

$$\operatorname{sdepth}(M) = \max\bigl\{\operatorname{sdepth}(\mathcal{D})\colon\ \mathcal{D} \text{ is a Stanley decomposition of } M\bigr\}.$$

The interest in finding Stanley decompositions and Stanley depths can be traced back to the pioneering paper of Stanley [6]. There it was conjectured that $\operatorname{depth}(M) \leqslant \operatorname{sdepth}(M)$. In [4] it was shown that if M allows a prime filtration $\mathcal{F}$ with $\operatorname{supp}(\mathcal{F}) = \min(M)$, then this conjecture holds. And if $I \subset S$ is a Gorenstein monomial ideal with $\dim(S) \leqslant 5$, then [3] showed that this conjecture is also true for $M = S/I$. However, in spite of the many supporting facts, the conjecture still remains open.

E-mail address: yshen@math.purdue.edu.

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One of the main obstacles for verifying the Stanley's conjecture lies in the difficulty of computing Stanley depths. Even with the method of Herzog, Vladoiu and Zheng which we will discuss immediately, it is still practically very difficult to find the Stanley depth for modules from general monomial ideals. The strongest result known to us that is pertinent to our work concerns the homogeneous maximal ideal $(x_1, \dots, x_n) \subset S$, which will be mentioned in Theorem 2.1 below.

In this paper, we will focus on the case where $M = I$ is a monomial ideal in S. Let $G(I) = \{v_1, \dots, v_m\}$ be the set of minimal monomial generators of I, and for $c = (c(1), \dots, c(n)) \in \mathbb{N}^n$, denote $x^c = \prod_i x_i^{c(i)}$. For a fixed $g \in \mathbb{N}^n$ such that $\operatorname{lcm}(v_1, \dots, v_m)$ divides x^g, Herzog, Vladoiu and Zheng introduced in [5] the associated poset $P_I^g = \{c \in \mathbb{N}^n\colon\ c \leqslant g \text{ and } v_i | x^c \text{ for some } i\}$ for I. Here $\leqslant$ is the natural partial order in $\mathbb{Z}^n$ by componentwise comparison. For $a, b \in P_I^g$, define the interval $[a, b]$ to be $\{c \in P_I^g\colon\ a \leqslant c \leqslant b\}$. Corresponding to each (disjoint) partition $\mathcal{P} : P_I^g = \bigcup_{i=1}^r [c_i, d_i]$, there is a Stanley decomposition $\mathcal{D}(\mathcal{P})$ of I. They showed in [5, Corollary 2.5] that there is a partition $\mathcal{P}$ such that $\operatorname{sdepth}(I) = \operatorname{sdepth}(\mathcal{D}(\mathcal{P}))$.

Recently, Cimpoeaş studied Stanley decomposition of complete intersection ideals. He proved in [2, Theorem 2.1] that the Stanley depth of a complete intersection monomial ideal is equal to the Stanley depth of its radical. Therefore, the focus of research is directed to squarefree monomial ideals. Recall that a Stanley space $uK[Z]$ is called squarefree, if u is squarefree and $\operatorname{supp}(u) \subset Z$. If I is a squarefree monomial ideal, we can take $g = (1, \dots, 1)$ and write P_I^g simply as P_I. Recall that a vector $d \in \mathbb{Z}^n$ is *squarefree* if $d(i) = 0$ or 1, for all $1 \leqslant i \leqslant n$. If $d \in \mathbb{N}^n$ is squarefree, write $Z_d = \{x_j\colon\ d(j) = 1\}$. Then for any partition $\mathcal{P} : P_I = \bigcup_i [c_i, d_i]$, $\mathcal{D}(\mathcal{P}) : I = \bigoplus_i x^{c_i} K[Z_{d_i}]$ is the associated Stanley decomposition of I introduced in [5]. Meanwhile $\operatorname{sdepth}(\mathcal{D}(\mathcal{P})) = \min\{|d_i|\colon\ 1 \leqslant i \leqslant r\}$. Here $|d_i|$ is the sum of components in d_i. The Stanley decomposition $\mathcal{D}(\mathcal{P})$ is clearly squarefree. This observation shows in particular that

$$\operatorname{sdepth}(I) = \max\bigl\{\operatorname{sdepth}(\mathcal{D})\colon\ \mathcal{D} \text{ is a squarefree Stanley decomposition of } I\bigr\}.$$

This paper proceeds as follows. We compute in Theorem 2.4 the Stanley depth of complete intersection monomial ideals. It turns out that the Stanley depth depends only on the dimension of the polynomial ring and the minimal number of generators. The third section studies the upper-discrete partition of squarefree monomial ideals. And in the last section, we prove that the Stanley depth of a squarefree monomial ideal minimally generated by 3 elements is $n - 1$. For 4-generated squarefree monomial ideals, the lower bound of Stanley depth is $n - 2$.

2. Stanley depth of complete intersection monomial ideals

The Stanley depth of the monomial maximal ideal is known.

Theorem 2.1. *(See [1, Theorem 2.2].) Let* $\mathfrak{m} = (x_1, \dots, x_n)$ *be the maximal ideal in* $S = K[x_1, \dots, x_n]$, *then* $\operatorname{sdepth}(\mathfrak{m}) = \lceil \frac{n}{2} \rceil$.

Herzog, Vladoiu and Zheng computed the Stanley depth of 3-generated complete intersection monomial ideals.

Proposition 2.2. *(See [5, Proposition 3.8].) Let* $I \subset S$ *be a complete intersection monomial ideal minimally generated by* 3 *elements. Then* $\operatorname{sdepth}(I) = n - 1$.

We want to generalize the above two results and answer Conjecture 2.5 in [2]. For simplicity of notation, we identify any squarefree vector $c \in \mathbb{Z}^n$ with $\{i \mid c(i) = 1\}$.

Lemma 2.3. *Let* $v_1, \dots, v_m$ *be squarefree monomials in* $K[x_1, \dots, x_{n-1}]$. *If* $I = (v_1, \dots, v_{m-1}, v_m x_n)$ *and* $I' = (v_1, \dots, v_{m-1}, v_m x_n x_{n+1})$ *are ideals in* $S = K[x_1, \dots, x_n]$ *and* $S' = S[x_{n+1}]$ *respectively, then* $\operatorname{sdepth}(I') = \operatorname{sdepth}(I) + 1$.

Proof. By assumption, there is a partition $\mathcal{P} : P_I = \bigcup_i [c_i, d_i]$ for I such that $\operatorname{sdepth}(\mathcal{D}(\mathcal{P})) = \operatorname{sdepth}(I)$. By [2, Corollary 2.3], $\operatorname{sdepth}(I') \leqslant \operatorname{sdepth}(I) + 1$. Now it suffices to construct a partition $\mathcal{P}'$ for $P_{I'}$ with $\operatorname{sdepth}(\mathcal{D}(\mathcal{P}')) = \operatorname{sdepth}(I) + 1$.

For each interval $B = [c, d]$ in $\mathcal{P}$, we define the corresponding interval B':

(1) If $n \in c$, which by our identification means $c(n) = 1$, let $B^1 = [c \cup \{n+1\}, d \cup \{n+1\}]$.
(2) If $n \notin c$, let $B^2 = [c, d \cup \{n+1\}]$. Furthermore, if $n \notin d$, let $B^3 = [c \cup \{n\}, d \cup \{n\}]$.

Let B' be the union of those B^k's defined. Hence $B' = B^1$, $B' = B^2$ or $B' = B^2 \cup B^3$. B' is a subset of $P_{I'}$. We claim that $\mathcal{P}' : P_{I'} = \bigcup_{i=1}^r B'_i$ is a partition for $P_{I'}$ with $\operatorname{sdepth}(\mathcal{D}(\mathcal{P}')) = \operatorname{sdepth}(I) + 1$.

First, we prove that the intervals B'_i cover $\mathcal{P}'$. Let u be a proper subset of $\{1, \dots, n+1\}$ in $P_{I'}$. Depending on whether $n+1 \in u$, we have two cases.

(1) If $n+1 \in u$, let $u' = u \setminus \{n+1\}$. We have $u' \in P_I$, hence there is an interval $B = [c, d]$ in $\mathcal{P}$ such that $u' \in B$. If $n \in c$, then $u \in B^1$. Otherwise, $n \notin c$, and $u \in B^2$.
(2) If $n+1 \notin u$, then x^u is divisible by some $v_i \neq v_m$. Consequently, $u \in P_I$ and there is an interval $B = [c, d]$ in $\mathcal{P}$ with $u \in B$.
   (a) If $n \notin c$, then $u \in B^2$.
   (b) If $n \in c$, then $n \in u$ as well. Let $u' = u \setminus \{n\}$ and again we have $u' \in P_I$. There is an interval $\tilde{B} = [\tilde{c}, \tilde{d}]$ in $\mathcal{P}$ with $u' \in \tilde{B}$. Since $n \notin u'$, $n \notin \tilde{c}$. Now depending on whether $n \in \tilde{d}$ or $n \notin \tilde{d}$, $u \in \tilde{B}^2$ or $u \in \tilde{B}^3$.

Now we show that the intervals in $\mathcal{P}'$ are pairwise disjoint. Suppose $B_1 = [c_1, d_1]$ and $B_2 = [c_2, d_2]$ are intervals in $\mathcal{P}$. We prove by contradiction that B_1^i and B_2^j are disjoint for $1 \leqslant i \neq j \leqslant 3$.

Suppose that $u \in B_1^1 \cap B_2^2$, then $n+1 \in u$. Let $u' = u \setminus \{n+1\}$. Then $u' \in B_1 \cap B_2$, hence $B_1 = B_2$. But $n \in c_1$, $n \notin c_2$ and $c_1 = c_2$. This is a contradiction.

Suppose that $u \in B_1^1 \cap B_2^3$, then $n+1 \in u$. But $n+1 \notin d$, and $x^{d \cup \{n\}}$ is divisible by x^u. As a result, $n+1 \notin u$. This is a contradiction.

Suppose that $u \in B_1^2 \cap B_2^3$, then $n \in u$ and $n+1 \notin u$. Let $u' = u \setminus \{n\}$, then $u' \in B_2$. Since $n \notin c_1$ and x^u is divisible by x^{c_1}, $x^{u'}$ is also divisible by x^{c_1}. Meanwhile, since $n+1 \notin u$, $n+1 \notin u'$. Since $x^{d_1 \cup \{n+1\}}$ is divisible by x^u, x^{d_1} is divisible by $x^{u'}$. Thus $u' \in B_1$ as well. Hence $B_1 = B_2$. Now since $u \in B_2^3$, $n \in u$ and $n \notin d_2$. Since $d_1 = d_2$ and $u \in B_1^2$, $n \notin u$. This is a contradiction.

Now let $i = j$. If $u \in B_1^1 \cap B_2^1$ or $B_1^2 \cap B_2^2$, let $u' = u \setminus \{n+1\}$, then $u \in B_1 \cap B_2$ and $B_1 = B_2$. Likewise, if $u \in B_1^3 \cap B_2^3$, let $u' = u \setminus \{n\}$. Notice that $n+1 \notin u'$ and $u' \in B_1 \cap B_2$. Hence $B_1 = B_2$. $\square$

Theorem 2.4. *Let $I \subset S = K[x_1, \dots, x_n]$ be a complete intersection monomial ideal minimally generated by m elements. Then* $\operatorname{sdepth}(I) = n - \lfloor \frac{m}{2} \rfloor$.

Proof. Following [2, Theorem 2.1] and [5, Lemma 3.6], one can assume that I is squarefree and every ring variable shows up in exactly one monomial generator of I. We fix m and prove the theorem by induction on $n \geqslant m$.

The base case is when $n = m$ and hence $I = (x_1, \dots, x_m)$ is the maximal ideal. The validity now follows from Theorem 2.1. Notice that $\lceil \frac{m}{2} \rceil = m - \lfloor \frac{m}{2} \rfloor$.

Now let $n \geqslant m$ and assume that the theorem holds for n. We want to prove that it also holds for $n+1$. Without loss of generality, we consider a squarefree complete intersection monomial ideal I' in $S' = S[x_{n+1}]$, minimally generated by monomials $v_1, \dots, v_{m-1}, v_m x_{n+1}$ and assume that x_n divides v_m. Then the ideal $I = (v_1, \dots, v_m)$ in S is also a squarefree complete intersection monomial ideal. Therefore, by the induction hypothesis, $\operatorname{sdepth}(I) = n - \lfloor \frac{m}{2} \rfloor$. Now by Lemma 2.3, $\operatorname{sdepth}(I') = \operatorname{sdepth}(I) + 1$ and this completes the proof. $\square$

3. Upper-discrete partitions

In this section, we introduce the upper-discrete partitions. It will be the main tool in the next section to study the 3-generated squarefree monomial ideals.

Definition 3.1. Let P be the associated poset of monomials in S. P is called *upper-discrete of degree k*, if there is a partition $\mathcal{P} : P = \bigcup_i [c_i, d_i]$, such that $|d_i| \geqslant k$ for all i, and $c_i = d_i$ when $|d_i| > k$. And this partition is called an *upper-discrete partition of degree k*.

Example 3.2. We use the notations in figure 2 of [5] and consider the ideal $I = (x_1x_2, x_2x_3, x_1x_3) \subset K[x_1, x_2, x_3]$. It is readily seen that $P_I = [12, 12] \cup [23, 23] \cup [13, 13] \cup [123, 123]$ gives an upper-discrete partition of degree 2. However, a shorter one $P_I = [12, 123] \cup [23, 23] \cup [13, 13]$ does not.

Proposition 3.3. *If I is a squarefree monomial ideal in S, then the poset P_I is upper-discrete of degree k for $k \leqslant \operatorname{sdepth}(I)$.*

Proof. Let $\mathcal{P} : P_I = \bigcup_i [c_i, d_i]$ be a Stanley decomposition with $\operatorname{sdepth}(\mathcal{D}(\mathcal{P})) = \operatorname{sdepth}(I)$. Hence $|d_i| \geqslant \operatorname{sdepth}(I) \geqslant k$. Now it suffices to show that each interval $[c_i, d_i]$ allows an upper-discrete partition of degree k. This is equivalent to say that the interval $[\emptyset, d_i \setminus c_i]$ admits an upper-discrete partition of degree $k - |c_i|$, where $[\emptyset, d_i \setminus c_i]$ is an interval in the poset P_S for the unit ideal S. Since $[\emptyset, d_i \setminus c_i]$ is isomorphic to the poset $P_{S'}$ where $S' = K[x_1, \ldots, x_{|d_i| - |c_i|}]$, it is enough to show that the poset P_S is upper-discrete of degree k for $0 \leqslant k \leqslant n$.

We prove by induction on n. The base cases when $n = 0$ or $n = 1$ are trivial. Now let $n \geqslant 2$ and suppose the claim holds for $n - 1$. The cases when $k = 0$ or $k = n$ are clear. Hence we may assume that $1 \leqslant k \leqslant n - 1$ and let $S' = K[x_1, \ldots, x_{n-1}]$. Then $P_{S'}$ have two upper-discrete partitions $\mathcal{P}^1 : P_{S'} = \bigcup_i [c_i^1, d_i^1]$ and $\mathcal{P}^2 : P_{S'} = \bigcup_i [c_i^2, d_i^2]$ of degrees k and $k - 1$, respectively. Clearly

$$\mathcal{P} : P_S = \left(\bigcup_i [c_i^1, d_i^1]\right) \cup \left(\bigcup_i [c_i^2 \cup \{n\}, d_i^2 \cup \{n\}]\right)$$

is an upper-discrete partition of degree k. And this completes the proof. $\square$

Remark 3.4. Let $I \subset S$ be a squarefree complete intersection monomial ideal with minimal monomial generating set $G(I) = \{v_1, \ldots, v_m\}$. We further assume that x_n divides v_m. Let $I' = (v_1, \ldots, v_m x_{n+1}) \subset S' = S[x_{n+1}]$. If P_I has an upper-discrete partition $\mathcal{P}$ of degree k, then the proof of Lemma 2.3 can be modified as follows to give an upper-discrete partition of $P_{I'}$ of degree $k + 1$.

Let $B = [c, d]$ be an interval in $\mathcal{P}$. We construct the interval B' in the following way:

(1) If $n \in c$, let $B^1 = [c \cup \{n+1\}, d \cup \{n+1\}]$.
(2) If $n \notin c$,
   (a) if $|c| \leqslant k$, let $B^2 = [c, d \cup \{n+1\}]$. Furthermore, if $n \notin d$, let $B^3 = [c \cup \{n\}, d \cup \{n\}]$;
   (b) if $|c| > k$, hence $c = d$, then let
      - $B^4 = B$,
      - $B^5 = [c \cup \{n\}, c \cup \{n\}]$,
      - $B^6 = [c \cup \{n+1\}, c \cup \{n+1\}]$.

Let B' be the union of those B^k defined. Hence either $B' = B^1$, $B' = B^2$, $B' = B^2 \cup B^3$, or $B' = B^4 \cup B^5 \cup B^6$. The rest of the proof is essentially the same.

4. Squarefree monomial ideals

If I is not a complete intersection, the formula in Theorem 2.4 will fail in general. For instance, let $I = (x_1x_2x_3, x_1x_2x_4, x_1x_3x_4, x_2x_3x_4)$ in $S = K[x_1, \ldots, x_4]$. Then $\operatorname{sdepth}(I) = 3$ instead of $4 - \lfloor \frac{4}{2} \rfloor = 2$. However, when $m = 3$, the situation is different.

Theorem 4.1. *Let I be a 3-generated squarefree monomial ideal in $S = K[x_1, \ldots, x_n]$. Then* $\operatorname{sdepth}(I) \geqslant n - 1$. *In particular, if I is not principal,* $\operatorname{sdepth}(I) = n - 1$.

Proof. Let I be generated by monomials v_1, v_2 and v_3. For any ring variable x_j, we say x_j is of type i, if there are exactly i of the three generators involve the variable x_j.

If x_n is of type 0, then for the ideal $I' = (v_1, v_2, v_3)$ in $K[x_1, \dots, x_{n-1}]$, we have $\operatorname{sdepth}(I') = \operatorname{sdepth}(I) - 1$ by [5, Lemma 3.6].

In a like manner, if x_n is of type 3, then for the ideal $I' = (v_1/x_n, v_2/x_n, v_3/x_n)$ in S, it is readily seen that I' is naturally isomorphic to I in the category of $\mathbb{Z}^n$-graded K-vector spaces up to degree shifting. Thus, $\operatorname{sdepth}(I') = \operatorname{sdepth}(I)$. But then, x_n is of type 0 for I'.

Hence it suffices to prove the result for the case when all ring variables are of type either 1 or 2. We call variable x_j to be of type 1-(i), if x_j is of type 1 and v_i involves x_j. By Lemma 2.3, we may assume that for every i, $1 \leqslant i \leqslant 3$, there is at most one ring variable to be of type 1-(i).

After these reductions, it is easily seen that the proof is done once we can show the following:

(I) Fix $n \geqslant 0$ and let I be any ideal in $S = K[x_1, \dots, x_n]$ generated by squarefree monomials v_1, v_2 and v_3, such that all ring variables are of type 2 for I. We prove that $\operatorname{sdepth}(I) \geqslant n - 1$.

(II) For any fixed I in (I), we also consider ideals $I_1 = (v_1 x_{n+1}, v_2, v_3)$ in $S_1 = S[x_{n+1}]$, $I_2 = (v_1 x_{n+1}, v_2 x_{n+2}, v_3)$ in $S_2 = S_1[x_{n+2}]$, and $I_3 = (v_1 x_{n+1}, v_2 x_{n+2}, v_3 x_{n+3})$ in $S_3 = S_2[x_{n+3}]$. We prove that $\operatorname{sdepth}(I_i) \geqslant n - 1 + i$ for $1 \leqslant i \leqslant 3$.

The proof is then carried out in 4 steps.

Step 0. To begin with, we investigate the ideal I in case (I) and assume that all ring variables are of type 2. We prove by induction on $n \in \mathbb{N}$ that $\operatorname{sdepth}(I) \geqslant n - 1$.

The base cases when $n \leqslant 1$ are easy to verify. Now we assume that the formula holds for a fixed $n \geqslant 1$ and consider the ideal $I' = (v_1 x_{n+1}, v_2 x_{n+1}, v_3)$ in $S' = S[x_{n+1}]$. Here v_1, v_2 and v_3 are squarefree monomials in $S = K[x_1, \dots, x_n]$, and all ring variables of S are of type 2 for $I = (v_1, v_2, v_3)$ in S. We want to show that $\operatorname{sdepth}(I') \geqslant n$.

By induction hypothesis, $\operatorname{sdepth}(I) \geqslant n - 1$. Thus we can find $\mathcal{P}: P_I = \bigcup_i [c_i, d_i]$, an upper-discrete partition of degree $n - 1$. For each interval $B = [c, d]$ in $\mathcal{P}$, define B' as follows.

(1) Suppose $|d| = n - 1$. If v_3 divides x^c, let $B^1 = [c, d \cup \{n+1\}]$. Otherwise, $v_3 \nmid x^c$, and let $B^2 = [c \cup \{n+1\}, d \cup \{n+1\}]$.

(2) If $|d| = n$, then $c = d = \{1, \dots, n\}$. Let $B^3 = [c, c]$ and $B^4 = [c \cup \{n+1\}, c \cup \{n+1\}]$.

Let B' be B^1, B^2 or $B^3 \cup B^4$ correspondingly. B' is a subset of $P_{I'}$. We claim that $\mathcal{P}': P_{I'} = \bigcup B'_i$ is an upper-discrete partition of degree n.

We first show that the intervals B'_i cover $P_{I'}$. Let $u \in P_{I'}$, if $u = \{1, \dots, n+1\}$, then $u \in B^4$. Otherwise, we may assume that $|u| \leqslant n$.

(1) If $n + 1 \notin u$, then v_3 divides u. For this reason, we have $u \in P_I$, and $u \in B = [c, d]$ in $\mathcal{P}$. If $u = \{1, \dots, n\}$, then $u \in B^3$. Otherwise, $|u| \leqslant n - 1$. We claim that $v_3 \mid x^c$, hence $u \in B^1$. If $v_3 \nmid x^c$, we have v_1 or v_2 dividing x^c. As a result, x^d is divisible by v_1 or v_2. But x^d is also divisible by v_3, thus $\operatorname{supp}(v_1 v_3) = \operatorname{supp}(v_2 v_3) = \{1, \dots, n\} \subset d$. At the same time, $|d| = n - 1$ and this is a contradiction.

(2) If $n + 1 \in u$, let $u' = u \setminus \{n+1\}$ and we have $u' \in P_I$. Hence there is an interval $B = [c, d]$ in $\mathcal{P}$ with $u' \in B$. Then depending on whether $v_3 \mid x^c$ or $v_3 \nmid x^c$, $u \in B^1$ or $u \in B^2$.

Now we show the intervals in $\mathcal{P}'$ are pairwise disjoint. It is straightforward to check that $\{1, \dots, n\}$ and $\{1, \dots, n+1\}$ are only in intervals B^3 and B^4, respectively.

Now suppose $u \in B_1^1 \cap B_2^2 \neq \emptyset$ for $B_1 = [c_1, d_1]$ and $B_2 = [c_2, d_2]$ in $\mathcal{P}$. According to the construction, $v_3 \mid x^{c_1}$ and $c_1 \leqslant u \leqslant d_2 \cup \{n+1\}$. Hence $v_3 \mid x^{d_2 \cup \{n+1\}}$. At the same time, $v_3 \nmid x^{c_2}$, hence $v_1 \mid x^{c_2}$ or $v_2 \mid x^{c_2}$. Thus x^{d_2} is divisible by either v_1 or v_2, and $d_2 = \{1, \dots, n\}$. This is against the assumption that $|d_2| = n - 1$.

Likewise, if $u \in B_1^1 \cap B_2^1$ or $B_1^2 \cap B_2^2$, then $u \setminus \{n+1\} \in B_1 \cap B_2$. Hence $B_1 = B_2$. This completes the proof for the claim.

Step 1. Let I be the ideal in case (I). Then in Step 0, we showed that $\operatorname{sdepth}(I) \geqslant n-1$. As a result, we have an upper-discrete partition $\mathcal{P}: P_I = \bigcup_i [c_i, d_i]$ of degree $n-1$. Now we construct an upper-discrete partition of degree n for P_{I_1}, where I_1 is constructed in case (II).

For each $B = [c, d]$ in $\mathcal{P}$, we define B' as follows.

(1) Suppose $|d| = n-1$. If v_1 divides x^c, then let $B^1 = [c \cup \{n+1\}, d \cup \{n+1\}]$. Otherwise, $v_1 \nmid x^c$, and let $B^2 = [c, d \cup \{n+1\}]$.
(2) Suppose $|d| = n$, then $c = d = \{1, \ldots, n\}$. Let $B^3 = [c, c]$ and $B^4 = [c \cup \{n+1\}, c \cup \{n+1\}]$.

Define $B' = B^1$, $B' = B^2$ or $B' = B^3 \cup B^4$ correspondingly. B' is a subset of P_{I_1}. We claim that $\mathcal{P}_1 : P_{I_1} = \bigcup_i B'_i$ is a partition that satisfies the requirement.

We first show that intervals B'_i cover P_{I_1}. Let $u \in P_{I_1}$. If $|u| = n+1$, then $u = \{1, \ldots, n+1\}$, and $u \in B^4$. Otherwise, we may assume that $|u| \leqslant n$.

(1) If $v_1 x_{n+1}$ divides x^u, then $n+1 \in u$. Let $u' = u \setminus \{n+1\}$, then v_1 divides $x^{u'}$ and $u' \in P_I$. Thus there is an interval $B = [c, d]$ in $\mathcal{P}$ such that $u' \in B$. Since $|u'| \leqslant n-1$, $|d| = n-1$. We claim that $v_1 \mid x^c$, hence $u \in B^1$. Otherwise, v_2 or v_3 divides x^c. Say it is v_2, then v_2 also divides x^d. On the other hand, v_1 divides $x^{u'}$, hence x^d is also divisible by v_1. Thus $\operatorname{supp}(v_1 v_2) = \{1, \ldots, n\} \subset d$ and $|d| \geqslant n$. However $|d| = n-1$ and this is a contradiction.
(2) If v_1 does not divide x^u, then neither does $v_1 x_{n+1}$. Therefore, v_2 or v_3 divides x^u. Let $u' = u \setminus \{n+1\}$. Then v_2 or v_3 divides $x^{u'}$, and we have $u' \in P_I$. Let $u' \in B = [c, d]$, an interval in $\mathcal{P}$. Since $v_1 \nmid x^{u'}$, we have $v_1 \nmid x^c$ and $u \in B^2$.
(3) If v_1 divides x^u, but $v_1 x_{n+1}$ does not, then $n+1 \notin u$. Since $u \in P_{I_1}$, x^u is divisible by v_2 or v_3. Since $\operatorname{supp}(v_1 v_2) = \operatorname{supp}(v_1 v_3) = \{1, \ldots, n\}$, this would force $u = \{1, \ldots, n\}$, and $u \in B^3$.

Now we show that $\mathcal{P}_1 : P_{I_1} = \bigcup_i B'_i$ is a disjoint union. Since $\mathcal{P}$ is an upper-discrete partition, if $u_1 = \{1, \ldots, n+1\}$, B^4 is the only interval containing u_1.

Consider $u_2 = \{1, \ldots, n\}$ and suppose that $u_2 \in B^1$ for some $B = [c, d]$ in $\mathcal{P}$. Then $n+1 \in u_2$ and this is impossible. On the other hand, suppose $u_2 \in B^2$ for some $B = [c, d]$ in $\mathcal{P}$. Then $c \leqslant u_2 \leqslant d \cup \{n+1\}$. Since $n+1 \notin u_2$, we have $c \leqslant u_2 \leqslant d$ and $u_2 \in B$. Hence $|d| \geqslant |u_2| = n$. On the other hand, by our assumption on B^2, $|d| = n-1$ and this is a contradiction.

Let $B_1 = [c_1, d_1]$ and $B_2 = [c_2, d_2]$ be intervals in $\mathcal{P}$. If $u \in B_1^1 \cap B_2^2 \neq \emptyset$, then $u \setminus \{n+1\} \in B_1 \cap B_2$. Hence $B_1 = B_2$. Meanwhile, $v_1 \mid x^{c_1}$, while $v_1 \nmid x^{c_2}$. This is a contradiction.

Similarly, if $u \in B_1^1 \cap B_2^1$ or $B_1^2 \cap B_2^2$, then $u \setminus \{n+1\} \in B_1 \cap B_2$. Thus $B_1 = B_2$.

Step 2. Using partition $\mathcal{P}_1$ in Step 1, we construct an upper-discrete partition $\mathcal{P}_2$ for P_{I_2} with degree $n+1$. For each $B = [c, d]$ in $\mathcal{P}_1$, we define B' as follows.

(1) Suppose $|d| = n$. If v_2 divides x^c, then let $B^1 = [c \cup \{n+2\}, d \cup \{n+2\}]$. Otherwise, $v_2 \nmid x^c$, and let $B^2 = [c, d \cup \{n+2\}]$.
(2) Suppose $|d| = n+1$, then $c = d = \{1, \ldots, n+1\}$. Let $B^3 = [\{1, \ldots, n\}, c]$ and $B^4 = [c \cup \{n+2\}, c \cup \{n+2\}]$.

Define $B' = B^1$, $B' = B^2$ or $B' = B^3 \cup B^4$ correspondingly. B' is a subset of P_{I_2}. We claim $\mathcal{P}_2 : P_{I_2} = \bigcup_i B'_i$ is a partition that satisfies the requirement.

We first show that intervals B'_i cover P_{I_2}. Let $u \in P_{I_2}$. If $|u| = n+2$, then $u = \{1, \ldots, n+2\}$, and $u \in B^4$. Otherwise, we may assume that $|u| \leqslant n+1$.

(1) If $v_2 x_{n+2}$ divides x^u, then $n+2 \in u$. Let $u' = u \setminus \{n+2\}$, then v_2 divides $x^{u'}$ and $u' \in P_{I_1}$. Thus there is an interval $B = [c, d]$ in $\mathcal{P}_1$ that $u' \in B$. Since $|u'| \leqslant n$, $|d| = n$. We claim that $v_2 \mid x^c$, hence $u \in B^1$. Otherwise, $v_1 x_{n+1}$ or v_3 divides x^c.
   (a) If $v_1 x_{n+1}$ divides x^c, then x^d is divisible by both $v_1 x_{n+1}$ and v_2. Thus $\operatorname{supp}(v_1 x_{n+1} v_2) = \{1, \ldots, n+1\} \subset d$ and $|d| \geqslant n+1$. Nevertheless, $|d| = n$ and this is a contradiction.

 (b) If v_3 divides x^c, then x^d is divisible by both v_2 and v_3. Thus $\operatorname{supp}(v_2v_3) = \{1, \dots, n\} \subset d$. Since $|d| = n$, $d = \{1, \dots, n\}$. Thus by the construction of $\mathcal{P}_1$, $c = d$. We still have $v_2 \mid x^c$.

(2) If v_2 does not divide x^u, then neither does v_2x_{n+2}. Hence v_1x_{n+1} or v_3 divides x^u. Let $u' = u \setminus \{n+2\}$. Then v_1x_{n+1} or v_3 divides $x^{u'}$, and we have $u' \in P_{I_1}$. Let $u' \in B = [c, d]$, an interval in $\mathcal{P}_1$. Since $v_2 \nmid x^{u'}$, we have $v_2 \nmid x^c$ and $u \in B^2$.

(3) If v_2 divides x^u, but v_2x_{n+2} does not, then $n+2 \notin u$. Thus x^u is divisible by v_1x_{n+1} or v_3.
 (a) If x^u is divisible by v_1x_{n+1}, since $\operatorname{supp}(v_1x_{n+1}v_2) = \{1, \dots, n+1\}$, this would force $u = \{1, \dots, n+1\}$, and $u \in B^3$.
 (b) Otherwise, x^u is divisible by v_3. At this moment, $\operatorname{supp}(v_2v_3) = \{1, \dots, n\} \subset u$. Since $|u| \leqslant n+1$ and $n+1, n+2 \notin u$, this forces $u = \{1, \dots, n\}$, and $u \in B^3$.

Now we need to show that $\mathcal{P}_2 : P_{I_2} = \bigcup_i B'_i$ is a disjoint union. The proof is similar to that in Step 1. However, one still need to consider $u_3 = \{1, \dots, n\}$.

If $u_3 \in B^1$ for some $B = [c, d] \in \mathcal{P}_1$, then $n+2 \in u_3$. This is impossible. If $u_3 \in B^2$, then $c \leqslant u_3 \leqslant d \cup \{n+2\}$. Since $n+2 \notin u_3$, this implies that $c \leqslant u_3 \leqslant d$, i.e., $u_3 \in B$. By our construction of $\mathcal{P}_1$, $c = d$. Thus v_2 divides x^c, and instead of B^2, we should construct B^1. This is a contradiction.

Step 3. Using partition $\mathcal{P}_2$ in Step 2, we construct an upper-discrete partition $\mathcal{P}_3$ for P_{I_3} with degree $n+2$. For each $B = [c, d]$ in $\mathcal{P}_2$, we define B' as follows.

(1) Suppose $\{1, \dots, n\} \not\subset d$, then $|d| = n+1$. If v_3 divides x^c, then let $B^1 = [c \cup \{n+3\}, d \cup \{n+3\}]$. Otherwise, $v_3 \nmid x^c$, and let $B^2 = [c, d \cup \{n+3\}]$.

(2) Suppose $\{1, \dots, n\} \subset d$. Then according to the construction of $\mathcal{P}_2$, B is one of the following intervals:
 - $[\{1, \dots, n\}, \{1, \dots, n+1\}]$,
 - $[\{1, \dots, n, n+2\}, \{1, \dots, n, n+2\}]$,
 - $[\{1, \dots, n+2\}, \{1, \dots, n+2\}]$.

 In particular, $\{1, \dots, n\} \subset c$. Now define

$$B^3 = \bigl[\{1, \dots, n+1\}, \{1, \dots, n+2\}\bigr],$$
$$B^4 = \bigl[\{1, \dots, n, n+2\}, \{1, \dots, n, n+2, n+3\}\bigr],$$
$$B^5 = \bigl[\{1, \dots, n, n+3\}, \{1, \dots, n+1, n+3\}\bigr],$$

 and

$$B^6 = \bigl[\{1, \dots, n+3\}, \{1, \dots, n+3\}\bigr].$$

Define $B' = B^1$, $B' = B^2$ or $B' = B^3 \cup B^4 \cup B^5 \cup B^6$ correspondingly. B' is a subset of P_{I_3}. We claim $\mathcal{P}_3 : P_{I_3} = \bigcup_i B'_i$ is a partition that satisfies the requirement.

We first show that intervals B'_i cover P_{I_3}. Let $u \in P_{I_3}$. If $\{1, \dots, n\} \subset u$, then $|u| \geqslant n+1$ and u is in exactly one of the B^i for $3 \leqslant i \leqslant 6$. Otherwise, we have $|u| \leqslant n+2$ and x^u is divisible by exactly one of the monomial generators v_ix_{n+i} for I_3.

(1) If v_3x_{n+3} divides x^u, then $n+3 \in u$. Let $u' = u \setminus \{n+3\}$, then v_3 divides $x^{u'}$ and $u' \in P_{I_2}$. Thus there is an interval $B = [c, d]$ in $\mathcal{P}_2$ that $u' \in B$. If v_1x_{n+1} or v_2x_{n+2} divides x^c, then x^u is also divisible by it, which is impossible. Hence $v_3 \mid x^c$. Since $\{1, \dots, n\} \not\subset u$, by our construction of $\mathcal{P}_2$, $\{1, \dots, n\} \not\subset d$ and $u \in B^1$.

(2) If v_3 does not divide x^u, then neither does v_3x_{n+3}. Hence v_1x_{n+1} or v_2x_{n+2} divides x^u. Let $u' = u \setminus \{n+3\}$. Then v_1x_{n+1} or v_2x_{n+2} divides $x^{u'}$, and we have $u' \in P_{I_2}$. Let $u' \in B = [c, d]$, an interval in $\mathcal{P}_2$. Since $v_3 \nmid x^{u'}$, we have $v_3 \nmid x^c$ and $u \in B^2$.

(3) If v_3 divides x^u, but v_3x_{n+3} does not, then $n+3 \notin u$. Thus x^u is divisible by v_1x_{n+1} or v_2x_{n+2}. Hence $\operatorname{supp}(v_1v_3) = \operatorname{supp}(v_2v_3) = \{1,\dots,n\} \subset u$, which is a contradiction.

At this stage, we have to show that $\mathcal{P}_2 : P_{I_2} = \bigcup_i B'_i$ is a disjoint union. Let $u \in P_{I_3}$. If $\{1,\dots,n\} \subset u$, then $|u| \geqslant n+1$ and $u \in B^3 \cup B^4 \cup B^5 \cup B^6$. Suppose $u \in \tilde{B}^1$ or $\tilde{B}^2$ for some interval $\tilde{B} = [c,d]$ in $\mathcal{P}_2$. Then $\{1,\dots,n\} \subset d \cup \{n+3\}$. This implies that $\{1,\dots,n\} \subset d$, which is against the construction of $\tilde{B}^1$ or $\tilde{B}^2$. The rest of the proof is similar to that in Step 1. $\square$

As shown by the example at the beginning of this section, the Stanley depth of 4-generated squarefree monomial ideal is not necessarily $n-2$. Nevertheless, $n-2$ is the sharp lower bound.

Proposition 4.2. *Let $I \subset S = K[x_1,\dots,x_n]$ be a squarefree monomial ideal generated by 4 elements. Then* $\operatorname{sdepth}(I) \geqslant n-2$.

Proof. We apply the technique in [5, Proposition 3.4] and use their notations. Hence we prove by induction on n, with $n=1$ being trivial. Now consider $n \geqslant 2$ and assume that the claim holds for $n-1$. Suppose the minimal monomial generating set is $G(I) = \{x^{a_1},\dots,x^{a_4}\}$. Without loss of generality, we may assume that $a_1 \vee \cdots \vee a_4 = (1,\dots,1)$. Then there is a disjoint union $P_I = A_0 \cup A_1$, where $A_i = \{c \in P_I\colon\ c(n) = i\}$ for $0 \leqslant i \leqslant 1$.

It is observed in [5] that $A_i = \{(c,i)\colon\ c \in P^g_{I_i}\}$ with $g = (1,\dots,1) \in \mathbb{N}^{n-1}$, and I_i is the monomial ideal in $K[x_1,\dots,x_{n-1}]$ such that

$$I \cap x_n^i K[x_1,\dots,x_{n-1}] = x_n^i I_i.$$

I_0 and I_1 are still squarefree. Furthermore, $|G(I_0)| \leqslant 3$ and $|G(I_1)| \leqslant 4$. Now by Theorem 4.1, $\operatorname{sdepth}(I_0) \geqslant (n-1)-1$, and by induction hypothesis $\operatorname{sdepth}(I_1) \geqslant (n-1)-2$. Therefore, by [5, Proposition 3.3], $\operatorname{sdepth}(I) \geqslant \min\{\operatorname{sdepth}(I_0), \operatorname{sdepth}(I_1)+1\} \geqslant n-2$. $\square$

To conclude, we ask the following question for squarefree monomial ideals.

Question 4.3. Let I be an m-generated squarefree monomial ideal in $S = K[x_1,\dots,x_n]$. Is it true that $\operatorname{sdepth}(I) \geqslant n - \lfloor \frac{m}{2} \rfloor$?

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