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# How to compute the Stanley depth of a monomial ideal ☆

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**Abstract**

Let $J \subset I$ be monomial ideals. We show that the Stanley depth of $I/J$ can be computed in a finite number of steps. We also introduce the fdepth of a monomial ideal which is defined in terms of prime filtrations and show that it can also be computed in a finite number of steps. In both cases it is shown that these invariants can be determined by considering partitions of suitable finite posets into intervals.


*Keywords:* Stanley depth; Stanley decomposition; Partitions; Prime filtrations

## Introduction

Let $K$ be a field, $S = K[x_1, \dots, x_n]$ be the polynomial ring in $n$ variables, and $M$ be a finitely generated $\mathbb{Z}^n$-graded $S$-module. Let $u \in M$ be a homogeneous element in $M$ and $Z$ a subset of $\{x_1, \dots, x_n\}$. We denote by $uK[Z]$ the $K$-subspace of $M$ generated by all elements $uv$ where $v$ is a monomial in $K[Z]$. The $\mathbb{Z}^n$-graded $K$-subspace $uK[Z] \subset M$ is called a *Stanley space of dimension* $|Z|$, if $uK[Z]$ is a free $K[Z]$-module.

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A *Stanley decomposition* of $M$ is a presentation of the $\mathbb{Z}^n$-graded $K$-vector space $M$ as a finite direct sum of Stanley spaces

$$\mathscr{D}\colon\ M = \bigoplus_{i=1}^{m} u_i K[Z_i]$$

in the category of $\mathbb{Z}^n$-graded $K$-vector spaces. In other words, each of the summands is a $\mathbb{Z}^n$-graded $K$-subspace of $M$ and the decomposition is compatible with the $\mathbb{Z}^n$-grading, i.e. for each $a \in \mathbb{Z}^n$ we have $M_a = \bigoplus_{i=1}^{m}(u_i K[Z_i])_a$. The number $\operatorname{sdepth}\mathscr{D} = \min\{|Z_i|\colon\ i = 1, \dots, m\}$ is called the *Stanley depth of* $\mathscr{D}$. The *Stanley depth* of $M$ is defined to be

$$\operatorname{sdepth} M = \max\{\operatorname{sdepth}\mathscr{D}\colon\ \mathscr{D} \text{ is a Stanley decomposition of } M\}.$$

It is conjectured by Stanley [20] that $\operatorname{depth} M \leqslant \operatorname{sdepth} M$ for all $\mathbb{Z}^n$-graded $S$-modules $M$, see also [21,22]. The conjecture is widely open (see however [1,10,11,18]). A priori it is not clear how one can compute $\operatorname{sdepth} M$. We will discuss this question in a special case.

Let $J \subset I \subset S$ be two monomial ideals. Then $I/J$ is a $\mathbb{Z}^n$-graded $S$-module. One of the aims of this paper is to show that $\operatorname{sdepth} I/J$ can be computed in a finite number of steps. Let $x^{a_1}, \dots, x^{a_m}$ be a monomial set of generators of $I$, and $x^{b_1}, \dots, x^{b_r}$ a monomial set of generators of $J$. Here we denote as usual by $x^a$ the monomial $x_1^{a(1)} \cdots x_n^{a(n)}$. Now we fix an integer vector $g \in \mathbb{Z}^n$ with the property $a_i \leqslant g$ and $b_j \leqslant g$ for all $i$ and $j$, where $\leqslant$ denotes the partial order in $\mathbb{Z}^n$ which is given by componentwise comparison. Given these data, we define the *characteristic poset* $P_{I/J}^g$ of $I/J$ with respect to $g$ as the subposet

$$P_{I/J}^g = \bigl\{a \in \mathbb{Z}^n\colon\ x^a \in I \setminus J,\ a \leqslant g\bigr\}$$

of $\mathbb{Z}^n$.

As one of the main results of this paper we show in Theorem 2.1 that each partition of $P_{I/J}^g$ into intervals induces a Stanley decomposition of $I/J$, and show that for any Stanley decomposition of $I/J$ there exists one induced by a partition of $P_{I/J}^g$ whose Stanley depth is greater than or equal to the given one. These two facts together imply that the Stanley depth can be computed by considering the finitely many different partitions of $P_{I/J}^g$.

Being able to compute the Stanley depth in a finite number of steps does however not mean that we have an efficient algorithm to compute the Stanley depth. The known algorithms (see [1,14,16]) to compute at least one Stanley decomposition, among them the Janet algorithm, practically never provides a Stanley decomposition whose Stanley depth coincides with the Stanley depth of the module. For example, if we take the graded maximal ideal $\mathfrak{m} = (x_1, \dots, x_n)$. Then the Janet algorithm gives a decomposition of Stanley depth 1. On the other hand, by using our methods we can show that $\operatorname{sdepth}\mathfrak{m} = \lceil n/2 \rceil$ for $n \leqslant 9$. Probably this is true for all $n$, but we do not know the general result. To prove this one would have to find appropriate partitions of $P_{\mathfrak{m}}$. To find the general strategy to get such partitions in this particular case is an interesting combinatorial problem which we could not yet solve.

There is a natural lower bound for both, $\operatorname{depth} M$ and $\operatorname{sdepth} M$. In order to describe this bound, let

$$\mathscr{F}\colon\ 0 = M_0 \subset M_1 \subset \cdots \subset M_m = M$$

be a chain of $\mathbb{Z}^n$-graded submodules of $M$. Then $\mathcal{F}$ is called a *prime filtration* of $M$ if $M_i/M_{i-1} \cong (S/P_i)(-a_i)$ where $a_i \in \mathbb{Z}^n$ and where each $P_i$ is a monomial prime ideal. We call the set of prime ideals $\{P_1, \dots, P_m\}$ the *support* of $\mathcal{F}$ and denote it by $\operatorname{supp}\mathcal{F}$. Furthermore we set $\operatorname{fdepth}\mathcal{F} = \min\{\dim S/P\colon\ P \in \operatorname{supp}\mathcal{F}\}$ and

$$\operatorname{fdepth} M = \max\{\operatorname{fdepth}\mathcal{F}\colon\ \mathcal{F} \text{ is a prime filtration of } M\}.$$

It is then very easy to see that $\operatorname{fdepth} M \leqslant \operatorname{depth} M, \operatorname{sdepth} M$. Again it is not at all obvious how to actually compute the fdepth of a module. Similarly as for the sdepth we show however that the fdepth of $I/J$ can be computed in a finite number of steps. This result is a consequence of Theorem 2.4. Indeed, this theorem implies that $\operatorname{fdepth} I/J$ can be computed by considering among the partitions of $P^g_{I/J}$ into intervals precisely those partitions which satisfy the condition that their partial unions in a suitable order are order filters in $P^g_{I/J}$, see Corollary 2.5 for details.

In the last section of this paper we present a few applications of the general theory developed in Section 2 and give some classes of examples. In particular we prove in Proposition 3.2 that any ideal monomial complete intersection satisfies Stanley's conjecture, and in Proposition 3.7 that any ideal of Borel type satisfies Stanley's conjecture. In the case of a complete intersection we actually show that the fdepth coincides with the depth. The proof of Proposition 3.7 is based on two results shown before in this section. The first result (Proposition 3.4) says that the sdepth of a monomial ideal is bounded below by $n - m + 1$ where $n$ is the number of variables of the ambient polynomial ring and where $m$ is the number of generators of the ideal. The second result needed in the proof of Proposition 3.7 says that the sdepth of the extension of a monomial ideal in a polynomial extension goes up by the number of variables which are adjoined in this extension, see Proposition 3.6. We also compute the Stanley depth of any complete intersection generated by three elements. It turns out that its Stanley depth is always equal to $n - 1$. In a final observation we show that the conjecture of Soleyman Jahan [19] concerning a lower bound for the regularity of a $\mathbb{Z}^n$-graded module implies the following conjecture: there exists a partition $P^g_{I/J} = \bigcup_{I=1}^r [c_i, d_i]$ of $P^g_{I/J}$ with the property that $|c_i| \leqslant \operatorname{reg} I/J$ for all $i$. Here $|c|$ denotes the sum of the components of the vector $c$.

## 1. Stanley decompositions and prime filtrations

In this section we shall discuss the relationship between Stanley decompositions and prime filtrations. We will also recall some basic upper and lower bounds for the Stanley depth.

Let $K$ be a field. Throughout the paper $S$ will denote the polynomial ring $K[x_1, \dots, x_n]$ in $n$ variables over $K$. Fig. 1 displays a Stanley decomposition of $S/I$ and of $I$ for the monomial ideal $I = (x_1x_2^3, x_1^3x_2)$. The gray area represents the $K$-vector space spanned by the monomials in $I$. The hatched area, the fat lines and the isolated fat dots represent Stanley spaces of dimension 2, 1, and 0, respectively. According to Fig. 1 we have the following Stanley decompositions

$$I = x_1x_2^3K[x_1, x_2] \oplus x_1^3x_2^2K[x_1] \oplus x_1^3x_2K[x_1],$$

and

$$S/I = K[x_2] \oplus x_1K[x_1] \oplus x_1x_2K \oplus x_1x_2^2K \oplus x_1^2x_2K \oplus x_1^2x_2^2K.$$

Here we identify $S/I$ with the $K$-subspace of $S$ generated by all monomials $u \in S \setminus I$.

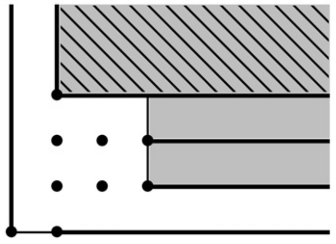

Fig. 1.

We first note

**Lemma 1.1.** *Any finitely generated $\mathbb{Z}^n$-graded $S$-module $M$ admits a Stanley decomposition.*

The proof is based on the fact that any prime filtration of $M$ yields a Stanley decomposition. We call a chain of $\mathbb{Z}^n$-graded submodules

$$\mathcal{F}\colon\ 0 = M_0 \subset M_1 \subset \cdots \subset M_m = M$$

a *prime filtration* of $M$ if $M_i/M_{i-1} \cong (S/P_i)(-a_i)$ where $a_i \in \mathbb{Z}^n$ and where each $P_i$ is a monomial prime ideal. We call the set of prime ideals $\{P_1, \dots, P_m\}$ the *support* of $\mathcal{F}$ and denote it by $\operatorname{supp}(\mathcal{F})$.

It is well known that at least one such prime filtration always exists. Indeed, let $P \in \operatorname{Ass}(M)$. Then $P$ is a monomial prime ideal and there exists a homogeneous element $u \in M$, say of degree $a \in \mathbb{Z}^n$, such that $uS \cong (S/P)(-a)$, cf. [6, Lemma 1.5.6]. We set $M_1 = uS$, and apply the same reasoning to $M/M_1$. Noetherian induction completes the proof.

Each prime filtration $\mathcal{F}$ of $M$ gives rise to a Stanley decomposition $\mathcal{D}(\mathcal{F})$ as follows: Since $M_i/M_{i-1} \cong S/P_i(-a_i)$, there exists a homogeneous element $u_i \in M_i$ of degree $a_i$, whose residue class modulo $M_{i-1}$ generates $M_i/M_{i-1}$ and such that $u_iK[Z_i] \cong M_i/M_{i-1}$, where $Z_i = \{x_j\colon\ x_j \notin P_i\}$ and where $u_iK[Z_i]$ is a free $K[Z_i]$-module. The filtration $\mathcal{F}$ provides a decomposition $M = \bigoplus_{i=1}^m M_i/M_{i-1}$ as direct sum of $K$-vector spaces. Since each of the factors $M_i/M_{i-1}$ is a Stanley space $u_iK[Z_i]$ we obtain the decomposition $\mathcal{D}(\mathcal{F}) = \bigoplus_{i=1}^m u_iK[Z_i]$, as desired. We say that $\mathcal{D}(\mathcal{F})$ is the Stanley decomposition induced by the prime filtration $\mathcal{F}$.

Not all Stanley decompositions of $M$ are induced by prime filtrations. Indeed, a prime filtration $\mathcal{F}$ of $M$ is essentially the same thing as a sequence of homogeneous generators $u_1, \dots, u_m$ of $M$ with the property that the colon ideal $(u_1, \dots, u_{i-1})\colon u_i$ is generated by a subset of the variables, in other words, is a monomial prime ideal, say $P_i$. We call such a sequence in $M$ a *sequence with linear quotients*. We say that $M$ has *linear quotients* if there exists a minimal set of homogeneous generators of $M$ which is a sequence with linear quotients. From our discussions so far it follows that a Stanley decomposition $\mathcal{D}\colon\ M = \bigoplus_{i=1}^m u_iK[Z_i]$ is induced by a prime filtration of $M$ if and only if, after a suitable renumbering of the direct summands, we have $(u_1, \dots, u_{i-1})\colon u_i = (x_j\colon\ x_j \notin Z_i)$.

Consider for example the Stanley decomposition of the ideal

$$(x_1, x_2, x_3) = x_1x_2x_3K[x_1, x_2, x_3] \oplus x_1K[x_1, x_2] \oplus x_2K[x_2, x_3] \oplus x_3K[x_1, x_3]. \qquad (1)$$

This Stanley decomposition is not induced by a prime filtration of $(x_1, x_2, x_3)$. In fact, no order of the elements $x_1x_2x_3, x_1, x_2, x_3$ is a sequence with linear quotients.

For later applications we will give the following simple characterization of Stanley decompositions induced by a prime filtration which was first given by Soleyman Jahan [18, Proposition 2.7] in the case that $M = S/I$.

**Proposition 1.2.** *Let $M$ be a finitely generated $\mathbb{Z}^n$-graded $S$-module and $\mathscr{D}$: $M = \bigoplus_{i=1}^m u_i K[Z_i]$ a Stanley decomposition of $M$. Then the following conditions are equivalent*:

(a) *$\mathscr{D}$ is induced by a prime filtration.*
(b) *After a suitable relabeling of the summands in $\mathscr{D}$ we have $M_j = \bigoplus_{i=1}^j u_i K[Z_i]$ is a $\mathbb{Z}^n$-graded submodule of $M$ for $j = 1, \dots, m$.*

**Proof.** (a) $\Rightarrow$ (b) follows immediately from the construction of a Stanley decomposition which is induced by a prime filtration.

(b) $\Rightarrow$ (a): We claim that $\mathscr{F}$: $0 \subset M_1 \subset M_2 \subset \cdots \subset M_m = M$ is a prime filtration of $M$. First notice that for each $j$, the module $M_j/M_{j-1}$ is a cyclic module generated by the residue class $\bar{u}_j = u_j + M_{j-1}$. Indeed, each element $u \in M_j$ can be written as $u = \sum_{k=1}^j u_k f_k$ with $f_k \in K[Z_k]$ for $k = 1, \dots, j$. Therefore $\bar{u} = \bar{u}_j f_j$.

Next we claim that the annihilator of $\bar{u}_j$ is equal to the monomial prime ideal $P$ generated by the variables $x_k \notin Z_j$. In fact, if $x_k \notin Z_j$, then $\deg x_k u_j \neq \deg u_j v$ for all monomials $v \in K[Z_j]$. Therefore, since $M_j = \bigoplus_{i=1}^j u_i K[Z_i]$ is a decomposition of $\mathbb{Z}^n$-graded $K$-vector spaces, it follows that $x_k u_j \in M_{j-1}$. This implies that $x_k \bar{u}_j = 0$ and shows that $P$ is contained in the annihilator of $\bar{u}_j$. On the other hand, if $v$ is a monomial in $S \setminus P$, then $v \in K[Z_j]$ and so $u_j v$ is a nonzero element in $u_j K[Z_j]$. This implies that $v$ does not belong to the annihilator of $\bar{u}_j$ and shows that $P$ is precisely the annihilator of $\bar{u}_j$. From all this we conclude that $\mathscr{D}$ is induced by $\mathscr{F}$. $\square$

**Proposition 1.3.** *Let $M$ be a finitely generated $\mathbb{Z}^n$-graded $S$-module, and let $\mathscr{F}$ be a prime filtration of $M$. Then*

$$\min\{\dim S/P\colon\ P \in \mathscr{F}\} \leqslant \operatorname{depth} M, \operatorname{sdepth} M \leqslant \min\bigl\{\dim S/P\colon\ P \in \operatorname{Ass}(M)\bigr\}.$$

**Proof.** The bounds for the depth are well known. For the convenience of the reader we give the references. One has $\operatorname{depth} M \leqslant \dim S/P$ for all $P \in \operatorname{Ass} M$, see [6, Proposition 1.2.13]. This gives the upper bound for the depth.

Let $\mathscr{F}$: $0 = M_0 \subset M_1 \subset \cdots \subset M_m = M$ be the given prime filtration of $M$. The exact sequence $0 \to M_1 \to M \to M/M_1 \to 0$ yields the inequality $\operatorname{depth} M \geqslant \min\{\operatorname{depth} M_1, \operatorname{depth} M/M_1\}$, see [6, Proposition 1.2.9]. Therefore the lower bound for the depth follows by induction on the length of the filtration.

The lower bound for $\operatorname{sdepth} M$ is due to the fact that any filtration induces a Stanley decomposition. The upper bound for $\operatorname{sdepth} M$ has been shown by Apel [2] in case that $M = S/I$ where $I$ is a monomial ideal. By the same reasoning one can show the upper bound for general $M$, see [19]. $\square$

It is clear that whenever $\operatorname{depth} M$ attains the lower bound given in Proposition 1.3, then Stanley's conjecture holds for $M$. This situation happens of course if the upper and lower bound given in Proposition 1.3 coincide. This is the case if $M$ admits a prime filtration $\mathcal{F}$ with $\operatorname{supp}(\mathcal{F}) = \operatorname{Ass}(M)$ in which case $M$ is said to be *almost clean*. According to Dress [7] the module $M$ is called *clean*, if there exists a prime filtration with $\operatorname{supp}(\mathcal{F}) = \operatorname{Min}(M)$. The combinatorial significance of this notion is that the Stanley–Reisner ring $K[\Delta]$ of a simplicial complex is clean if and only if $\Delta$ is shellable, see [7, Theorem]. This result has been extended in [11] to $K$-algebras $S/I$ where $I$ is a monomial ideal, not necessarily squarefree. This is achieved by introducing pretty clean modules. A $\mathbb{Z}^n$-graded $S$-module $M$ is called *pretty clean* if $M$ admits a filtration $\mathcal{F}$: $0 = M_0 \subset M_1 \subset \cdots \subset M_m = M$ with $M_i/M_{i-1} \cong S/P_i$ and such that for all $i < j$ with $P_i \subset P_j$ it follows that $P_i = P_j$. It is easy to see that a pretty clean module is almost clean (see [11, Corollary 3.4]), so that pretty clean modules satisfy Stanley's conjecture. In case $M = S/I$ where $I$ is a monomial ideal, the property of being pretty clean is equivalent to say that the associated multicomplex is shellable, see [11, Theorem 10.5]. Thus we have the following implications:

$$\text{shellable} \quad \Leftrightarrow \quad \text{clean} \quad \Rightarrow \quad \text{pretty clean} \quad \Rightarrow \quad \text{almost clean},$$

and each of these conditions implies that depth = sdepth. On the other hand, the inequalities in Proposition 1.3 may all be strict. For example, let $M = \mathfrak{m} = (x_1, x_2, x_3)$ be the maximal ideal of $S = K[x_1, x_2, x_3]$ and $\mathcal{F}$ the prime filtration of $\mathfrak{m}$ corresponding to the sequence $x_1, x_2, x_3^2, x_3$ with linear quotients 0: $x_1 = 0$, $(x_1)$: $x_2 = (x_1)$, $(x_1, x_2)$: $x_3^2 = (x_1, x_2)$ and $(x_1, x_2, x_3^2)$: $x_3 = (x_1, x_2, x_3)$. Then

$$\min\{\dim S/P\colon\ P \in \mathcal{F}\} = 0 < \operatorname{depth} \mathfrak{m} = 1 < \operatorname{sdepth} \mathfrak{m} = 2 < \min\bigl\{\dim S/P\colon\ P \in \operatorname{Ass}(\mathfrak{m})\bigr\} = 3.$$

The only question is why the Stanley depth of $\mathfrak{m}$ is equal to 2. To see this, we first observe that for a monomial ideal $I \subset K[x_1, \ldots, x_n]$ we have $\operatorname{sdepth} I = n$, if and only if $I$ is a principal ideal. Indeed, if $I = (u)$, then $I = uK[x_1, \ldots, x_n]$ is a Stanley decomposition. On the other hand, if $I$ is not principal at least two Stanley spaces are needed to cover $I$. Obviously any two Stanley spaces of dimension $n$ intersect, so that one of the summands in the Stanley decomposition must have dimension smaller than $n$.

Thus we have $\operatorname{sdepth} \mathfrak{m} \leqslant 2$. Since (1) is a Stanley decomposition of $\mathfrak{m}$ of Stanley depth 2, we see that $\operatorname{sdepth} \mathfrak{m} = 2$.

In our example, the prime filtration $\mathcal{F}$ was not very well chosen. If we replace $\mathcal{F}$ by the prime filtration $\mathcal{F}'$ which is induced by the sequence $x_1, x_2, x_3$, then $\min\{\dim S/P\colon\ P \in \mathcal{F}'\} = \operatorname{depth} \mathfrak{m} = 1$. Thus $\operatorname{fdepth} \mathfrak{m} = \operatorname{depth} \mathfrak{m}$ in this case.

It is clear Stanley's conjecture holds for $M$ if $\operatorname{fdepth} M = \operatorname{depth} M$. In general however, we may have $\operatorname{fdepth} M < \operatorname{depth} M$ as the following result shows.

**Proposition 1.4.** *Let $K$ be a field, $\Delta$ be a simplicial complex and $K[\Delta]$ its Stanley–Reisner ring. Suppose that $K[\Delta]$ is Cohen–Macaulay. Then* $\operatorname{fdepth} K[\Delta] = \operatorname{depth} K[\Delta]$ *if and only if $\Delta$ is shellable.*

**Proof.** We have $\operatorname{fdepth} K[\Delta] = \operatorname{depth} K[\Delta]$ if and only if there exists a prime filtration $\mathcal{F}$ of $K[\Delta]$ with $\dim S/P \geqslant \operatorname{depth} K[\Delta] = \dim K[\Delta]$ for all $P \in \operatorname{supp} \mathcal{F}$. This is the case if and only if $\operatorname{supp} \mathcal{F}$ is equal to the set of minimal prime ideals of $I_\Delta$. By the theorem of Dress [7] this condition is satisfied if and only if $\Delta$ is shellable. $\square$

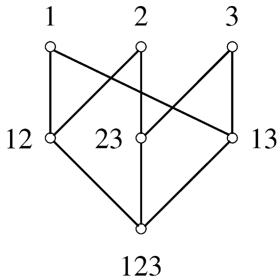


Fig. 2.

Assume $K[\Delta]$ is not necessarily Cohen–Macaulay. In view of Proposition 1.4 one may ask whether $\Delta$ is shellable in the non-pure sense [5], provided $\operatorname{fdepth} K[\Delta] = \operatorname{depth} K[\Delta]$. Unfortunately this is not always the case as the following simple example shows: let $\Delta$ be the simplicial complex on the vertex set [1–4] with facets $\{1,2\}$ and $\{3,4\}$. Then $I_\Delta = (x_1x_3, x_1x_4, x_2x_3, x_2x_4)$ and $\operatorname{depth} K[\Delta] = 1$. Denote by $y_i$ the residue class of $x_i$ modulo $I_\Delta$. The sequence $y_1, y_3, y_4, 1$ with linear quotients $0 : y_1 = (y_3, y_4)$, $(y_1) : y_3 = (y_1, y_2, y_4)$, $(y_1, y_2) : y_4 = (y_1, y_3)$ and $(y_1, y_2, y_4) : 1 = (y_1, y_3, y_4)$ shows that $\operatorname{fdepth} K[\Delta] \geqslant 1$. Since, on the other hand, one always has $\operatorname{fdepth} K[\Delta] \leqslant \operatorname{depth} K[\Delta]$, we see that $\operatorname{fdepth} K[\Delta] = \operatorname{depth} K[\Delta] = 1$. However, $\Delta$ is not shellable.

## 2. Stanley decompositions and partitions

Let $I \subset S$ be a monomial ideal. In this section we show that the Stanley depth of $I$ and of $S/I$ can be determined in a finite number of steps. In order to treat both cases simultaneously we will show this more generally for $\mathbb{Z}^n$-graded modules of the form $I/J$ where $J \subset I$ are monomial ideals in $S$.

We define a natural partial order on $\mathbb{N}^n$ as follows: $a \leqslant b$ if and only if $a(i) \leqslant b(i)$ for $i = 1, \dots, n$. Note that $x^a \mid x^b$ if and only if $a \leqslant b$. Here, for any $c \in \mathbb{N}^n$ we denote as usual by $x^c$ the monomial $x_1^{c(1)} x_2^{c(2)} \cdots x_n^{c(n)}$. Observe that $\mathbb{N}^n$ with the partial order introduced is a distributive lattice with meet $a \wedge b$ and join $a \vee b$ defined as follows: $(a \wedge b)(i) = \min\{a(i), b(i)\}$ and $(a \vee b)(i) = \max\{a(i), b(i)\}$. We also denote by $\varepsilon_j$ the $j$th canonical unit vector in $\mathbb{Z}^n$.

Suppose $I$ is generated by the monomials $x^{a_1}, \dots, x^{a_r}$ and $J$ by the monomials $x^{b_1}, \dots, x^{b_s}$. We choose $g \in \mathbb{N}^n$ such that $a_i \leqslant g$ and $b_j \leqslant g$ for all $i$ and $j$, and let $P_{I/J}^g$ be the set of all $c \in \mathbb{N}^n$ with $c \leqslant g$ and such that $a_i \leqslant c$ for some $i$ and $c \ngeqslant b_j$ for all $j$. The set $P_{I/J}^g$ viewed as a subposet of $\mathbb{N}^n$ is a finite poset. We call it the *characteristic poset* of $I/J$ with respect to $g$. There is a natural choice for $g$, namely the join of all the $a_i$ and $b_j$. For this $g$, the poset $P_{I/J}^g$ has the least number of elements, and we denote it simply by $P_{I/J}$. Note that if $\Delta$ is a simplicial complex on the vertex set $[n]$, then $P_{S/I_\Delta}$ is just the face poset of $\Delta$.

Fig. 2 shows the characteristic poset for the maximal ideal $\mathfrak{m} = (x_1, x_2, x_3) \subset K[x_1, x_2, x_3]$. (The following figures represent the Hasse diagram of the dual poset of the characteristic poset. Recall that the dual poset $P^*$ of a poset $P$ has the same underlying set as $P$, but $x \leqslant y$ in $P^*$ if and only if $y \leqslant x$ in $P$.) The elements of this poset correspond to the squarefree monomials $x_1$, $x_2$, $x_3$, $x_1x_2$, $x_2x_3$, $x_1x_3$ and $x_1x_2x_3$. Thus the corresponding labels in Fig. 2 should be $(1,0,0), (0,1,0), \dots, (1,1,1)$. In the squarefree case, like in this example, it is however more convenient and shorter to replace the $(0,1)$-vectors (which label the vertices in the characteristic

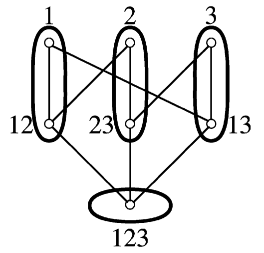


Fig. 3.

poset) by their support. In other words, each (0, 1)-vector with support $\{i_1 < i_2 < \cdots < i_k\}$ is replaced by $i_1 i_2 \cdots i_k$, as done in Fig. 2.

Given any poset $P$ and $a, b \in P$, we set $[a, b] = \{c \in P\colon\ a \leqslant c \leqslant b\}$ and call $[a, b]$ an *interval*. Of course, $[a, b] \neq \emptyset$ if and only if $a \leqslant b$. Suppose $P$ is a finite poset. A *partition* of $P$ is a disjoint union

$$\mathscr{P}\colon\ P = \bigcup_{i=1}^{r} [a_i, b_i]$$

of intervals.

Fig. 3 displays a partition of the poset given in Fig. 2. The framed regions in Fig. 3 indicate that $P_{\mathfrak{m}} = [1, 12] \cup [2, 23] \cup [3, 13] \cup [123, 123]$.

We will show that each partition of $P^g_{I/J}$ gives rise to a Stanley decomposition of $I/J$.

In order to describe the Stanley decomposition of $I/J$ coming from a partition of $P^g_{I/J}$ we shall need the following notation: for each $b \in P^g_{I/J}$, we set $Z_b = \{x_j\colon\ b(j) = g(j)\}$. We also introduce the function

$$\rho : P^g_{I/J} \to \mathbb{Z}_{\geqslant 0}, \quad c \mapsto \rho(c),$$

where $\rho(c) = |\{j\colon\ c(j) = g(j)\}|$ $(= |Z_c|)$. We then have

**Theorem 2.1.**

(a) *Let* $\mathscr{P}\colon\ P^g_{I/J} = \bigcup_{i=1}^{r} [c_i, d_i]$ *be a partition of* $P^g_{I/J}$. *Then*

$$\mathscr{D}(\mathscr{P})\colon\ I/J = \bigoplus_{i=1}^{r} \left( \bigoplus_{c} x^c K[Z_{d_i}] \right) \tag{2}$$

*is a Stanley decomposition of* $I/J$, *where the inner direct sum is taken over all* $c \in [c_i, d_i]$ *for which* $c(j) = c_i(j)$ *for all* $j$ *with* $x_j \in Z_{d_i}$. *Moreover,* $\operatorname{sdepth} \mathscr{D}(\mathscr{P}) = \min\{\rho(d_i)\colon\ i = 1, \ldots, r\}$.

(b) *Let* $\mathscr{D}$ *be a Stanley decomposition of* $I/J$. *Then there exists a partition* $\mathscr{P}$ *of* $P^g_{I/J}$ *such that*

$$\operatorname{sdepth} \mathscr{D}(\mathscr{P}) \geqslant \operatorname{sdepth} \mathscr{D}.$$

*In particular,* sdepth $I/J$ *can be computed as the maximum of the numbers* sdepth $\mathscr{D}(\mathscr{P})$*, where $\mathscr{P}$ runs over the (finitely many) partitions of $P^g_{I/J}$.*

For the proof of this theorem we use functors introduced by Miller [15, Definition 2.7]. Let $\mathscr{M}$ be the category of finitely generated $\mathbb{Z}^n$-graded $S$-modules, and let $g \in \mathbb{N}^n$. The functors $\mathscr{A}_g : \mathscr{M} \to \mathscr{M}$ and $\mathscr{B}_g : \mathscr{M} \to \mathscr{M}$ are defined as follows:

(1) $\mathscr{A}_g(M)_a = M_{a \wedge g}$ for all $a \in \mathbb{Z}^n$ with $S$-action

$$\mathscr{A}_g(M)_a \xrightarrow{x_i} \mathscr{A}_g(M)_{a+\varepsilon_i} = \begin{cases} \mathrm{id}, & \text{if } a(i) \geqslant g(i), \\ x_i : M_{a\wedge g} \to M_{(a+\varepsilon_i)\wedge b}, & \text{if } a(i) < g(i). \end{cases}$$

(2) $\mathscr{B}_g(M) = \bigoplus_{a \in [0,g]} M_a$, viewed as the subquotient of $M$ bounded in the interval $[0, g]$.

We list a few properties of these functors which are relevant to our proofs. Let $J \subset I$ be monomial ideals in $S$ and let $g$ be chosen as described in the introduction of this section. Then

(i) $\mathscr{B}_g(I/J) = \bigoplus_{c \in P^g_{I/J}} K x^c$ and $\mathscr{A}_g(\mathscr{B}_g(I/J)) = I/J$.
(ii) Let $[a, b] \subset P^g_{I/J}$, and let $K[a, b]$ be the subquotient of $S$ bounded in the interval $[a, b]$. Then $\mathscr{A}_g(K[a, b]) = K[a, b'] \otimes_K K[Z_b]$, where $b'(j) = a(j)$ if $b(j) = g(j)$, and $b'(j) = b(j)$ otherwise, and where $K[a, b']$ is viewed as a subquotient of $K[Y_b]$ where $Y_b$ is the set of variables not belonging to $Z_b$, so that $K[a, b'] \otimes_K K[Z_b]$ has a natural $S = K[Y_b] \otimes_K K[Z_b]$ module structure.
(iii) The functors $\mathscr{A}_g$ and $\mathscr{B}_g$ are exact [15, Lemma 2.9].

The properties (i) and (ii) of the functors $\mathscr{A}_g$ and $\mathscr{B}_g$ listed above follow immediately from the definitions.

**Proof of Theorem 2.1.** (a) The partition $\mathscr{P}$ induces a decomposition $\mathscr{B}_g(I/J) = \bigoplus_{i=1}^r K[c_i, d_i]$ of $\mathscr{B}_g(I/J)$ as a direct sum of $K$-vector spaces. Since $\mathscr{A}_g$ is a $K$-linear functor, (i) and (ii) yield the decomposition $I/J = \bigoplus_{i=1}^r K[c_i, d_i'] \otimes_K K[Z_{d_i}]$. Since

$$K\big[c_i, d_i'\big] \otimes_K K[Z_{d_i}] = \bigoplus_{c \in [c_i, d_i']} x^c K[Z_{d_i}],$$

as $K$-vector space, we obtain the desired decomposition (2). The statement about the Stanley depth of $\mathscr{D}(\mathscr{P})$ follows immediately from the definitions.

(b) Let $M = x^a K[Z]$ be a Stanley space. Then $M$ is a $K[Z]$-module and may be viewed as an $S$-module via the natural $K$-algebra homomorphism $S \to K[Z]$, and

$$\mathscr{B}_g(M) = \begin{cases} 0, & \text{if } a \nleqslant g, \\ K[a, b], & \text{if } a \leqslant g, \end{cases}$$

where $b(j) = a(j)$ for $x_j \notin Z$, and $b(j) = g(j)$ otherwise. In particular, $\rho(b) \geqslant |Z|$.

Therefore, if $\mathscr{D}$: $I/J = \bigoplus_{i=1}^r x^{a_i} K[Z_i]$ is a Stanley decomposition of $I/J$, then

$$\mathscr{B}_g(I/J) = \bigoplus_{i=1}^{r} \mathscr{B}_g\bigl(x^{a_i} K[Z_i]\bigr) = \bigoplus_{\substack{i \\ a_i \leqslant g}} K[a_i, b_i]$$

for suitable $b_i$ with $\rho(b_i) \geqslant |Z_i|$ for all $i$ such that $a_i \leqslant g$. It follows that $\mathscr{P}$: $P_{I/J}^g = \bigcup_{i, a_i \leqslant g} [a_i, b_i]$ is a partition of $P_{I/J}^g$, and (a) implies that sdepth $\mathscr{D}(\mathscr{P}) \geqslant$ sdepth $\mathscr{D}$. □

As an immediate consequence of Theorem 2.1 we have

**Corollary 2.2.** *Let $J \subset I$ be monomial ideals. Then*

$$\operatorname{sdepth} I/J = \max\bigl\{\operatorname{sdepth} \mathscr{D}(\mathscr{P}):\ \mathscr{P} \text{ is a partition of } P_{I/J}^g\bigr\}.$$

*In particular, there exists a partition $\mathscr{P}$: $P_{I/J}^g = \bigcup_{i=1}^r [c_i, d_i]$ of $P_{I/J}^g$ such that*

$$\operatorname{sdepth} I/J = \min\bigl\{\rho(d_i):\ i = 1, \ldots, r\bigr\}.$$

We consider three examples to illustrate Theorem 2.1. As a first example, consider the partition of the poset $P_{\mathfrak{m}}$ given in Fig. 3. According to Theorem 2.1 the Stanley decomposition corresponding to this partition is exactly the one given in (1).

The second, slightly more involved example, is displayed in Fig. 4. In the first picture the hatched region corresponds to the $K$-vector space spanned by all monomials in $I \setminus J$ where $I = (x_1^2x_2^4, x_1^3x_2^3, x_1^5x_2)$ and $J = (x_1^4x_2^5, x_1^6x_2^2)$. The second picture shows a partition of $P_{I/J}^g$ where $g = (7, 6)$. The partition is the following:

$$P_{I/J}^g = \bigl[(2,4),(3,6)\bigr] \cup \bigl[(4,3),(5,4)\bigr] \cup \bigl[(5,1),(7,1)\bigr] \cup \bigl[(3,3),(3,3)\bigr] \cup \bigl[(5,2),(5,2)\bigr].$$

To this partition corresponds by Theorem 2.1 the following Stanley decomposition

$$\begin{aligned} I/J = {} & \bigl(x_1^2x_2^4K[x_2] \oplus x_1^3x_2^4K[x_2]\bigr) \oplus \bigl(x_1^4x_2^3K \oplus x_1^5x_2^3K \oplus x_1^4x_2^4K \oplus x_1^5x_2^4K\bigr) \\ & \oplus x_1^5x_2K[x_1] \oplus x_1^3x_2^3K \oplus x_1^5x_2^2K \end{aligned}$$

which is shown in the third picture of Fig. 4.

The last example demonstrates part (b) of Theorem 2.1. Let $I = (x_1^2, x_2^2) \subset K[x_1, x_2]$, then $P_I = \{(2,0), (0,2), (2,1), (1,2), (2,2)\}$. Consider the following Stanley decomposition

$$\mathscr{D}:\ I = x_1^2K[x_1] \oplus x_1^2x_2K[x_1, x_2] \oplus x_1x_2^2K[x_2] \oplus x_2^2K \oplus x_2^3K[x_2]$$

with sdepth$(\mathscr{D}) = 0$. We apply the construction given in the proof of Theorem 2.1. the functor $\mathscr{B}_g$ applied to the summands of $\mathscr{D}$ yields the assignments: $x_1^2K[x_1] \mapsto K[(2,0),(2,0)]$, $x_1^2x_2K[x_1,x_2] \mapsto K[(2,1),(2,2)]$, $x_1x_2^2K[x_2] \mapsto K[(1,2),(1,2)]$, $x_2^2K \mapsto K[(0,2),(0,2)]$ and $x_2^3K[x_2] \mapsto 0$. Thus we obtain the partition

$$\mathscr{P}:\ P_{S/I}^g = \bigl[(2,0),(2,0)\bigr] \cup \bigl[(2,1),(2,2)\bigr] \cup \bigl[(1,2),(1,2)\bigr] \cup \bigl[(0,2),(0,2)\bigr]$$

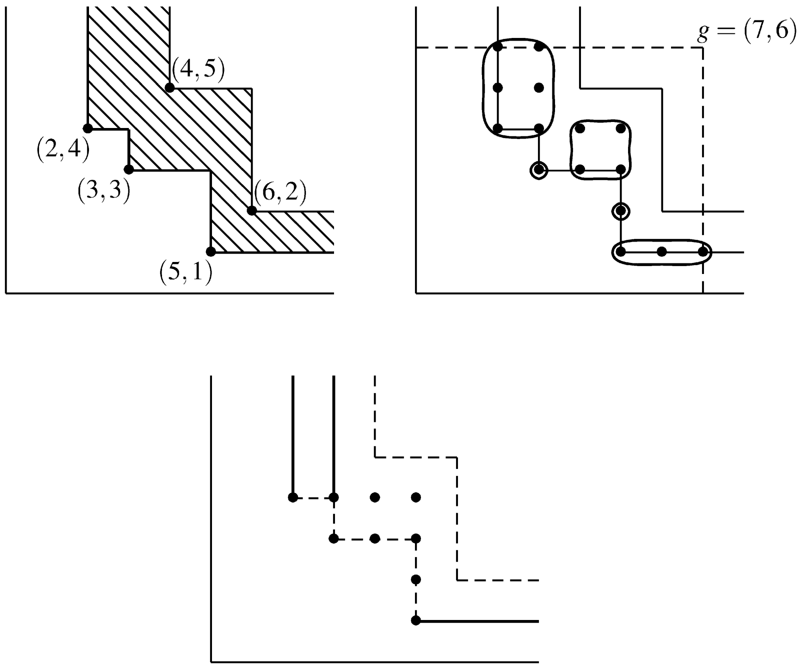


Fig. 4.

which, according to Theorem 2.1(a), gives the following Stanley decomposition

$$\mathscr{D}(\mathscr{P})\colon\ I = x_1^2K[x_1] \oplus x_1^2x_2K[x_1,x_2] \oplus x_1x_2^2K[x_2] \oplus x_2^2K[x_2]$$

with $\operatorname{sdepth}(\mathscr{P}) = 1$. In general the theorem asserts that $\operatorname{sdepth}\mathscr{D}(\mathscr{P}) \geqslant \operatorname{sdepth}\mathscr{P}$. The example shows that it may indeed be bigger.

If we want to use Corollary 2.2 in concrete cases to compute the Stanley depth, it is advisable to choose $g$ such that the poset $P_{I/J}^g$ is as small as possible. If $G(I) = \{x^{a_1}, \dots, x^{a_r}\}$ and $G(J) = \{x^{b_1}, \dots, x^{b_s}\}$, then with $g = a_1 \vee \dots \vee a_r \vee b_1 \vee \dots \vee b_s$ the poset $P_{I/J}^g$ has the least number of elements.

The following examples demonstrate the power of Corollary 2.2 and also show that in general it is very hard to compute the Stanley depth of a monomial ideal, even though it can be done in a finite number of steps.

**Example 2.3.** Let $\mathfrak{m}$ be the graded maximal ideal of $S = K[x_1, \dots, x_n]$. Then $\operatorname{sdepth}\mathfrak{m} = \lceil n/2 \rceil$ for $n \leqslant 9$, where $\lceil n/2 \rceil$ denotes the smallest integer $\geqslant n/2$. We expect this to be true for all integers $n$, but do not have a general proof yet. Here we give a proof for $n = 4$ and 5 to demonstrate the kind of arguments we use. We use the same notation as used in Fig. 2 where a set $\{i_1 < i_2 < \dots < i_k\}$ is written as $i_1i_2\cdots i_k$.

(a) Let $n = 4$. Then $P_{\mathfrak{m}}$ is the following collection of subsets of the set 1234.

$$\begin{array}{cccccc} & & 1 \quad 2 \quad 3 \quad 4 & & & \\ 12 & 13 & 14 & 23 & 24 & 34 \\ & 123 & 124 & 134 & 234 & \\ & & 1234 & & & \end{array}$$

Let $A = [1, 12] \cup [2, 23] \cup [3, 34] \cup [4, 14]$. Then $A \cup \bigcup_{a \in P_{\mathfrak{m}} \setminus A} [a, a]$ is a partition of $P_{\mathfrak{m}}$ and by Corollary 2.2 we obtain that $\operatorname{sdepth} \mathfrak{m} \geqslant 2$. On the other hand, since $\mathfrak{m}$ is not principal we have $\operatorname{sdepth} \mathfrak{m} \leqslant 3$. Assume that $\operatorname{sdepth} \mathfrak{m} = 3$. By Corollary 2.2 there exists a partition of $P_{\mathfrak{m}}$ into disjoint intervals such that the end point of each interval is at least a 3-set of the poset shown above. If one of these intervals is $[i, 1234]$, say $[1, 1234]$, then one of the intervals $[2, 234]$, $[3, 234]$, $[4, 234]$ would have to cover the rest, a contradiction. Otherwise we have four disjoint intervals of type $[i, ijk]$, where $1 \leqslant i \leqslant 4$ and $ijk$ runs over the set $\{123, 124, 134, 234\}$. Therefore the number of 2-sets in $P_{\mathfrak{m}}$ is at least $4 \times 2 = 8$, a contradiction. Hence, our assumption is false and consequently $\operatorname{sdepth} \mathfrak{m} = 2 = \lceil 4/2 \rceil$.

(b) Let $n = 5$. Obviously $A = [1, 123] \cup [2, 234] \cup [3, 345] \cup [4, 145] \cup [5, 125]$ is a disjoint union of intervals which contains all 1- and 2-sets of $P_{\mathfrak{m}}$. Then $A \cup \bigcup_{a \in P_{\mathfrak{m}} \setminus A} [a, a]$ is a partition of $P_{\mathfrak{m}}$ and applying Corollary 2.2 we obtain that $\operatorname{sdepth} \mathfrak{m} \geqslant 3$. With the same arguments given in (a) one can show that $\operatorname{sdepth} \mathfrak{m} \neq 4$. Hence $\operatorname{sdepth} \mathfrak{m} = 3 = \lceil 5/2 \rceil$.

The next result clarifies for which partitions $\mathscr{P}$ of $P^g_{I/J}$ the Stanley decomposition $\mathscr{D}(\mathscr{P})$ of $I/J$ is induced by a prime filtration and shows that $\operatorname{fdepth} I/J$ can be computed in a finite number of steps. Recall that a subset $S$ of a poset $P$ is called an *order filter* if for all $x \in S$ and all $y \geqslant x$ one has $y \in S$ as well.

**Theorem 2.4.**

(a) *Let* $\mathscr{P}$: $P^g_{I/J} = \bigcup_{i=1}^r [c_i, d_i]$ *be a partition of* $P^g_{I/J}$ *with the property that for all* $j$ *the union* $\bigcup_{i=1}^j [c_i, d_i]$ *is an order filter in* $P^g_{I/J}$. *Then* $\mathscr{D}(\mathscr{P})$ *is induced by a prime filtration.*

(b) *Let* $\mathscr{D}$ *be a Stanley decomposition of* $I/J$ *induced by a prime filtration of* $I/J$. *Then there exists a partition* $\mathscr{P}$ *of* $P^g_{I/J}$ *with the property that* $\mathscr{D}(\mathscr{P})$ *is induced by a prime filtration and such that* $\operatorname{sdepth} \mathscr{D}(\mathscr{P}) \geqslant \operatorname{sdepth} \mathscr{D}$.

**Proof.** (a) Set $M_j = \bigoplus_{i=1}^j K[c_i, d_i]$. The assumption on the partition implies that $M_j$ is a $\mathbb{Z}^n$-graded submodule of $\mathscr{B}_g(I/J)$ for $j = 1, \dots, r$. Thus we obtain a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_r = \mathscr{B}_g(I/J)$ with factors $M_j/M_{j-1} \cong K[c_j, d_j]$. We set $N_j = \mathscr{A}_g(M_j)$. Since $\mathscr{A}_g$ is an exact functor, we obtain, by using properties (i) and (ii) of the functors $\mathscr{A}_g$ and $\mathscr{B}_g$, a filtration $0 = N_0 \subset N_1 \subset \cdots \subset N_r = I/J$ of $I/J$ with factors $K[a_j, b'_j] \otimes_K [Z_b]$. Since each of these factors has a natural prime filtration, we see that $\mathscr{D}(\mathscr{P})$ is induced by a prime filtration of $I/J$.

(b) By Proposition 1.2 the Stanley decomposition $\mathscr{D}$ is of the form $I/J = \bigoplus_{i=1}^r x^{a_i} K[Z_i]$ such that $N_j = \bigoplus_{i=1}^j x^{a_i} K[Z_i]$ is a $\mathbb{Z}^n$-graded submodule of $I/J$ for $j = 1, \dots, r$. We construct a partition $\mathscr{P}$ of $P^g_{I/J}$ from the given Stanley decomposition $\mathscr{D}$ as it is done in the proof of Theorem 2.1(b). Then $\operatorname{sdepth} \mathscr{D}(\mathscr{P}) \geqslant \operatorname{sdepth} \mathscr{D}$, and it remains to be shown that $\mathscr{P}$ satisfies the properties formulated in part (a) of this theorem.

With the notation introduced in the proof of Theorem 2.1(b) we have $\mathscr{B}_g(N_j) = \bigcup_{i\leqslant j,\, a_i\leqslant g}^{j} K[a_i, b_i']$. We note that $\mathscr{B}_g(N_j)$ is a $\mathbb{Z}^n$-graded submodule of $\mathscr{B}_g(I/J)$ since $N_j$ is a $\mathbb{Z}^n$-graded submodule of $I/J$. This is equivalent to saying that $\bigcup_{i\leqslant j,\, a_i\leqslant g}^{j}[a_i, b_i']$ is an order filter in $P_{I/J}^g$. It follows that $\mathscr{P}$: $P_{I/J}^g = \bigcup_{a_i\leqslant g}[a_i, b_i']$ satisfies the desired properties. $\square$

**Corollary 2.5.** *Let $J \subset I$ be monomial ideals. Then* fdepth $I/J$ *is the maximum of the numbers* sdepth $\mathscr{D}(\mathscr{P})$*, where the maximum is taken over all partitions $\mathscr{P}$: $P_{I/J}^g = \bigcup_{i=1}^r [c_i, d_i]$ of $P_{I/J}^g$ with the property that for all $j$, the union $\bigcup_{i=1}^j [c_i, d_i]$ is an order filter in $P_{I/J}^g$.*

Theorem 2.4 can be used to compute the Krull dimension of $I/J$.

**Corollary 2.6.** $\dim I/J = \max\{\rho(c)\colon\ c \in P_{I/J}^g\}$.

**Proof.** Let $\mathscr{F}$ be any prime filtration of $I/J$. Then $\dim I/J = \max\{\dim S/P\colon\ P \in \operatorname{supp}(\mathscr{F})\}$. Now consider the canonical partition $\mathscr{P}$: $P_{I/J}^g = \bigcup_{c\in P_{I/J}^g}[c, c]$. We choose a total order $\succ$ of the intervals with the property that $[c, c] \succ [d, d]$ implies that $|d| \leqslant |c|$. Then the union of any initial sequence of these intervals is an order filter in $P_{I/J}^g$. Therefore it follows from Theorem 2.4 that $\mathscr{D}(\mathscr{P})$: $I/J = \bigoplus_{c\in P_{I/J}^g} x^c K[Z_c]$ is induced by a prime filtration $\mathscr{F}$ of $I/J$.

It follows that

$$\begin{aligned}\dim I/J &= \max\bigl\{\dim S/P\colon\ P \in \operatorname{supp}(\mathscr{F})\bigr\}\\ &= \max\bigl\{|Z_c|\colon\ c \in P_{I/J}^g\bigr\} = \max\bigl\{\rho(c)\colon\ c \in P_{I/J}^g\bigr\}. \qquad \square\end{aligned}$$

## 3. Applications and examples

As shown in the previous section, the sdepth as well as the fdepth of $I/J$ for monomial ideals $J \subset I$ can be computed by considering the partitions of the (finite) characteristic poset $P_{I/J}^g$. This does not mean that these invariants can be computed in practice, because the number of possible partitions can easily become very huge. In this section we will show that the techniques of the previous section nevertheless allow us to give bounds and in some cases even to compute these invariants.

The following proposition reassembles some observations we implicitly made in the previous sections.

**Proposition 3.1.** *Let $J \subset I$ be monomial ideals. Then*

(a) fdepth $S/I$ = depth $S/I$, *if $S/I$ is pretty clean*;
(b) fdepth $I$ = depth $I$, *if $I$ has linear quotients*;
(c) fdepth $I/J \geqslant \min\{\rho(c)\colon\ c \in P_{I/J}\}$. *In particular, if $I$ is a squarefree monomial ideal, then* fdepth $I \geqslant \min\{\deg u\colon\ u \in G(I)\}$.

**Proof.** (a) Let $\mathscr{F}$ be a pretty clean filtration of $S/I$. As we mentioned already in Section 1, we have $\operatorname{Ass}(S/I) = \operatorname{supp}\mathscr{F}$. Thus it follows from Proposition 1.3 that fdepth $S/I$ = depth $S/I$.

(b) By assumption, $G(I) = \{u_1, \ldots, u_r\}$ and $P_i = (u_1, \ldots, u_{i-1})\colon u_i$ is generated by a subset of $\{x_1, \ldots, x_n\}$ for each $i$. Let $m_i$ be the number of generators of $P_i$. It is shown in [13] that $\operatorname{proj}\dim I = \max\{m_1, \dots, m_r\}$, so that $\operatorname{depth} I = n - \max\{m_1, \dots, m_r\} = \min\{n - m_1, \dots, n - m_r\}$. On the other hand, $\mathscr{F}$: $(0) \subset (u_1) \subset (u_1, u_2) \subset \cdots \subset I$ is a prime filtration of $I$ with $\operatorname{supp} \mathscr{F} = \{P_1, \dots, P_r\}$. Hence $\operatorname{fdepth} I \geqslant \min\{\dim S/P_1, \dots, \dim S/P_r\} = \operatorname{depth} I$. Since we always have $\operatorname{fdepth} I \leqslant \operatorname{depth} I$, the assertion follows.

(c) We already observed in the proof of Corollary 2.6 that $\mathscr{P}$: $I/J = \bigcup_{c \in P_{I/J}} [c, c]$ induces a prime filtration $\mathscr{F}$. It follows from the definitions that

$$\min\{\dim S/P\colon P \in \operatorname{supp} \mathscr{F}\} = \min\bigl\{\rho(c)\colon c \in P_{I/J}\bigr\}.$$

This yields the desired inequality. In the squarefree case, $\rho(c) = |c| = \deg x^c$. This implies the second part of statement (c). □

We would like to mention that Soleyman Jahan [19] proved with the same arguments that $\operatorname{sdepth} I \geqslant \operatorname{depth} I$ if $I$ has linear quotients.

As an example, consider the ideal $I_{n,d}$ generated by all squarefree monomials of degree $d$ in $n$ variables. $I_{n,d}$ is the Stanley–Reisner ideal of the $(d-1)$-skeleton of the $n$-simplex. Since all the skeletons of the $n$-simplex are shellable, it follows from Proposition 3.1(a) and the discussions in Section 1 that $\operatorname{fdepth} S/I_{n,d} = \operatorname{sdepth} S/I_{n,d} = \operatorname{depth} S/I_{n,d} = d - 1$.

It is known [9] that $I_{n,d}$ has linear quotients since $I_{n,d}$ is a polymatroidal ideal. Therefore Proposition 3.1(b) implies that $\operatorname{fdepth} I_{n,d} = \operatorname{depth} I_{n,d} = d$. This fact one could also deduce from Proposition 3.1(c), since all generators of $I_{n,d}$ are of degree $d$.

To compute $\operatorname{sdepth} I_{n,d}$ is much harder. Even for the graded maximal ideal $\mathfrak{m} = I_{n,1}$, we cannot compute the Stanley depth in general, see Example 2.3.

Monomial complete intersections are generic monomial ideals. Apel ([1, Theorem 2] and [2, Theorem 3]) showed that for a generic monomial ideal $I$, Stanley's conjecture holds for $I$ and $S/I$. Here we give a short proof of this result in case that $I$ is a complete intersection.

**Proposition 3.2.** *Let $I \subset S$ be a monomial complete intersection. Then* $\operatorname{fdepth} S/I = \operatorname{depth} S/I$ *and* $\operatorname{fdepth} I = \operatorname{depth} I$. *In particular, Stanley's conjecture holds for $S/I$ and $I$.*

**Proof.** The equality $\operatorname{fdepth} S/I = \operatorname{depth} S/I$ follows from the fact that $S/I$ is pretty clean, as shown in [10].

Let $G(I) = \{u_1, \dots, u_r\}$. In order to compute the fdepth of $I$ we consider the filtration

$$(0) \subset (u_1) \subset (u_1, u_2) \subset \cdots \subset (u_1, \dots, u_r) = I.$$

We have

$$(u_1, \dots, u_i)/(u_1, \dots, u_{i-1}) \cong S/(u_1, \dots, u_{i-1})\colon u_i = S/(u_1, \dots, u_{i-1}),$$

for all $i$, since $u_1, \dots, u_r$ is a regular sequence. It follows that

$$\begin{aligned}\operatorname{fdepth} I &\geqslant \min\bigl\{\operatorname{fdepth} S/(u_1, \dots, u_i)\colon i = 1, \dots, r-1\bigr\}\\ &= \min\bigl\{\operatorname{depth} S/(u_1, \dots, u_i)\colon i = 1, \dots, r-1\bigr\}\\ &= \operatorname{depth} S/(u_1, \dots, u_{r-1}) = n - r + 1 = \operatorname{depth} I.\end{aligned}$$

Therefore $\operatorname{fdepth} I = \operatorname{depth} I$. □

After these examples one might have the impression that one always has $\operatorname{fdepth} I = \operatorname{depth} I$. This is however not the case as the following example shows: let $\Delta$ be the simplicial complex on the vertex set $\{1,\dots,6\}$, associated to a triangulation of the real projective plane $\mathbb{P}^2$, whose facets are

$$\mathscr{F}(\Delta) = \{125, 126, 134, 136, 145, 234, 235, 246, 356, 456\}.$$

Then the Stanley–Reisner ideal of $\Delta$ is

$$I_\Delta = (x_1x_2x_3, x_1x_2x_4, x_1x_3x_5, x_1x_4x_6, x_1x_5x_6, x_2x_3x_6, x_2x_4x_5, x_2x_5x_6, x_3x_4x_5, x_3x_4x_6).$$

It is known that $\operatorname{depth} I_\Delta = 4$ if $\operatorname{char} K \neq 2$ and $\operatorname{depth} I_\Delta = 3$ if $\operatorname{char} K = 2$. Since the inequality $\operatorname{fdepth} I_\Delta \leqslant \operatorname{sdepth} I_\Delta$ holds independent of the characteristic of the base field, we obtain that $\operatorname{fdepth} I_\Delta \leqslant 3$. On the other hand it follows from Proposition 3.1(c) that $\operatorname{fdepth} I_\Delta \geqslant 3$. Therefore $\operatorname{fdepth} I_\Delta = 3$ and $\operatorname{fdepth} I_\Delta < \operatorname{depth} I_\Delta$ for any field $K$ with $\operatorname{char} K \neq 2$.

We now give a lower bound for the sdepth of a monomial ideal by using a strategy which is modeled after the Janet algorithm (see [14] and [16]) and which allows to use induction on the number of variables. Let $I \subset S$ be a monomial ideal with $G(I) = \{x^{a_1},\dots,x^{a_m}\}$. We set $a = a_1 \vee a_2 \vee \cdots \vee a_m$. Then we can write $P_I$ as a disjoint union $P_I = \bigcup_{j=p}^q A_j$, where $p = \min\{a_1(n),\dots,a_m(n)\}$, $q = a(n)$ and $A_j = \{c \in P_I\colon c(n) = j\}$. For all $j$ with $p \leqslant j \leqslant q$ we let $I_j$ be the monomial ideal of $K[x_1,\dots,x_{n-1}]$ such that $I \cap x_n^j K[x_1,\dots,x_{n-1}] = x_n^j I_j$. Then for all $j$ with $p \leqslant j \leqslant q$, we have $A_j = \{(c, j)\colon c \in P_{I_j}^g\}$ with $g = (a(1),\dots,a(n-1))$.

**Proposition 3.3.** *With the notation introduced we have*

$$\operatorname{sdepth} I \geqslant \min\{\operatorname{sdepth} I_p,\dots,\operatorname{sdepth} I_{q-1}, \operatorname{sdepth} I_q + 1\}.$$

**Proof.** By Corollary 2.2 there exists for each $j \in \{p,\dots,q\}$ a partition $P_{I_j}^g = \bigcup_{k=1}^{r_j}[c_{jk}, d_{jk}]$ of $P_{I_j}^g$ with $\operatorname{sdepth} I_j = \min\{\rho(d_{jk})\colon k = 1,\dots,r_j\}$. Since $P_I$ is the disjoint union of the $A_j$ it follows that $P_I = \bigcup_{j=p}^q \bigcup_{k=1}^{r_j}[(c_{jk}, j), (d_{jk}, j)]$ is a partition of $P_I$. We have

$$\rho(d_{jk}, j) = \begin{cases} \rho(d_{jk}), & \text{if } j < q, \\ \rho(d_{jk}) + 1, & \text{if } j = q. \end{cases}$$

Hence the conclusion follows from Theorem 2.1. □

Now we are ready to prove

**Proposition 3.4.** *Let $I \subset S$ be a monomial ideal generated by $m$ elements. Then*

$$\operatorname{sdepth} I \geqslant \max\{1, n - m + 1\}.$$

**Proof.** We may assume that $m$ is the number of minimal monomial generators of $I$. Then we proceed by induction on $n$. If $n = 1$, then $I = (u)$ is a principal ideal with Stanley decomposition $I = uK[x_1]$. Therefore, $\operatorname{sdepth} I = 1$. For the induction step we shall use Proposition 3.3. Indeed, we already have that $I_j$ is a monomial ideal of $K[x_1,\dots,x_{n-1}]$ for all $j$, with $p \leqslant j \leqslant q$. In addition, one can easily see that $|G(I_j)| < m$ for all $j$ such that $j < q$, and $|G(I_q)| \leqslant m$. Hence, by induction hypothesis we have $\operatorname{sdepth} I_j \geqslant \max\{1, n - |G(I_j)|\} \geqslant \max\{1, n - m + 1\}$ for all $j$ with $j < q$, and similarly the induction hypothesis implies that $\operatorname{sdepth} I_q \geqslant \max\{1, n - m\}$, so that $\operatorname{sdepth} I_q + 1 \geqslant \max\{2, n - m + 1\} \geqslant \max\{1, n - m + 1\}$. Applying now Proposition 3.3 we obtain the desired inequality. □

**Corollary 3.5.** *Let $I \subset S$ be a monomial ideal minimally generated by 2 elements. Then* $\operatorname{fdepth} I = \operatorname{depth} I = \operatorname{sdepth} I = n - 1$.

**Proof.** Let $G(I) = \{u_1, u_2\}$. Then $I = v(v_1, v_2)$ where $v = \gcd(u_1, u_2)$ and $v_1, v_2$ is a regular sequence. It follows that, up to shift, the $\mathbb{Z}^n$-graded modules $I$ and $(v_1, v_2)$ are isomorphic. Thus the equality $\operatorname{fdepth} I = \operatorname{depth} I$ follows from Proposition 3.2. The last equality is a consequence of Proposition 3.4. □

Next we will show that ideals of Borel type satisfy Stanley's conjecture. For the proof we shall need

**Lemma 3.6.** *Let $J \subset I$ be monomial ideals of S, and let $T = S[x_{n+1}]$ be the polynomial ring over S in the variable $x_{n+1}$. Then*

$$\operatorname{depth} IT/JT = \operatorname{depth} I/J + 1, \qquad \operatorname{fdepth} IT/JT = \operatorname{fdepth} I/J + 1,$$
$$\operatorname{sdepth} IT/JT = \operatorname{sdepth} I/J + 1.$$

**Proof.** The statement about the depth is obvious since $x_{n+1}$ is regular on $IT/JT$. In order to prove the other two equations we consider the characteristic poset $P_{I/J}$ of $I/J$ as well as the characteristic poset $P_{IT/JT}$ of $IT/JT$. The map $P_{I/J} \to P_{IT/JT}$, $c \mapsto c^* = (c(1), \dots, c(n), 0)$ is an isomorphism of posets with the additional property that $\rho(c) = \rho(c^*) - 1$. In particular, if $\mathscr{P}$: $P_{I/J} = \bigcup_{i=1}^r [c_i, d_i]$ is a partition of $P_{I/J}$, then $\mathscr{P}^*$: $P_{IT/JT} = \bigcup_{i=1}^r [c_i^*, d_i^*]$ is a partition of $P_{IT/JT}$, and the assignment $\mathscr{P} \mapsto \mathscr{P}^*$ establishes a bijection between partitions of $P_{I/J}$ and $P_{IT/JT}$. Since $\rho(d_i) = \rho(d_i^*) - 1$ we see that $\operatorname{sdepth} \mathscr{D}(\mathscr{P}) = \operatorname{sdepth} \mathscr{D}(\mathscr{P}^*) - 1$ for all partitions $\mathscr{P}$ of $P_{I/J}$. Therefore the desired equations follow from Corollaries 2.2 and 2.5. □

We would like to remark that Rauf [17] proved a similar result for $S/I$.

A monomial ideal is called of *Borel type* if it satisfies one of the following equivalent conditions:

(i) For each monomial $u \in I$ and all integers $i, j, s$ with $1 \leqslant j < i \leqslant n$ and $s > 0$ such that $x_i^s \mid u$ there exists an integer $t \geqslant 0$ such that ${x_j}^t (u/x_i^s) \in I$.
(ii) If $P \in \operatorname{Ass}(S/I)$, then $P = (x_1, \dots, x_j)$ for some $j$.

This class of ideals includes all Borel-fixed ideals (see [8]) as well as the squarefree strongly stable ideals [3]. Some authors call these ideals also ideals of nested type [4]. In [1] Apel proved that Borel-fixed ideals satisfy Stanley's conjecture. However we could not follow all the steps of his proof. The next result generalizes his statement. For the proof we shall need the following notation: for a monomial $u$ we set $m(u) = \max\{i\colon x_i \text{ divides } u\}$, and for a monomial ideal $I \neq 0$ we set $m(I) = \max\{m(u)\colon u \in G(I)\}$.

**Proposition 3.7.** *Let $I \subset S$ be an ideal of Borel type. Then* $\operatorname{sdepth} S/I \geqslant \operatorname{depth} S/I$ *and* $\operatorname{sdepth} I \geqslant \operatorname{depth} I$. *In particular, Stanley's conjecture holds for $I$ and $S/I$.*

**Proof.** It is shown in [11, Proposition 5.2] that $S/I$ is pretty clean. This implies $\operatorname{sdepth} S/I \geqslant \operatorname{depth} S/I$. In order to prove the second inequality, we use the fact that $S/I$ is sequentially Cohen–Macaulay as was shown in [12, Corollary 2.5]. Indeed there exists a chain of ideals $I = I_0 \subset I_1 \subset \cdots \subset I_r = S$ with the properties that $I_j/I_{j-1}$ is Cohen–Macaulay and $\dim(I_j/I_{j-1}) < \dim(I_{j+1}/I_j)$ for all $j$. This chain of ideals is constructed recursively as follows: let $I_0 = I$ and $n_0 = m(I_0)$. Suppose that $I_l$ is already defined. If $I_l = S$, then the chain ends. Otherwise, let $n_l = m(I_l)$ and set $I_{l+1} = I_l : x_{n_l}^{\infty}$. We notice that $n \geqslant n_0 > n_1 > \cdots > n_r = 0$. It is shown in [12, Corollary 2.6] that $\operatorname{Ext}_S^i(S/I, S) \neq 0$ if and only if $i \in \{n_0, n_1, \ldots, n_{r-1}\}$.

Observing that $\operatorname{depth} S/I = \min\{i\colon \operatorname{Ext}_S^{n-i}(S/I, S) \neq 0\}$, it follows that $\operatorname{depth} S/I = n - n_0$. Therefore $\operatorname{depth} I = n - n_0 + 1$. Since $G(I) \subset K[x_1, \ldots, x_{n_0}]$, we obtain by applying Lemma 3.6 and Proposition 3.4 that $\operatorname{sdepth} I \geqslant n - n_0 + 1$. Hence we have $\operatorname{sdepth} I \geqslant \operatorname{depth} I$, as desired. $\square$

Next we compute the sdepth of an ideal in a special case.

**Proposition 3.8.** *Let $I \subset S$ be a monomial complete intersection ideal minimally generated by 3 elements. Then* $\operatorname{sdepth} I = n - 1$.

**Proof.** Since $I$ is not principal we have $\operatorname{sdepth} I \leqslant n - 1$. In order to prove the statement it is enough, via Corollary 2.2, to find a partition $\mathscr{P}$ of $P_I$ such that $\operatorname{sdepth} \mathscr{D}(\mathscr{P}) = n - 1$. Let $G(I) = \{x^b, x^c, x^d\}$. Since $I$ is a monomial complete intersection we may assume, after a suitable renumbering of the variables, that $b = (a_1, \ldots, a_i, 0, \ldots, 0)$, $c = (0, \ldots, 0, a_{i+1}, \ldots, a_{i+j}, 0, \ldots, 0)$ and $d = (0, \ldots, 0, a_{i+j+1}, \ldots, a_n)$ with $1 \leqslant i, j, n - i - j$. We may also assume that $a_k \geqslant 1$ for all $k = 1, \ldots, n$. Indeed, if one of the $a_k$ is zero, then we may use Lemma 3.6 and the proof follows immediately by induction on $n$.

Let $a = b \vee c \vee d = (a_1, \ldots, a_n)$. We claim that $\mathscr{P}$: $P_I = B \cup C \cup D \cup [a, a]$ is a partition of $P_I$, where

$$B = \bigcup_{k=1}^{j} \left[ b + \sum_{l=1}^{k-1} a_{i+l}\varepsilon_{i+l}, a - \varepsilon_{i+k} \right],$$

$$C = \bigcup_{k=1}^{n-i-j} \left[ c + \sum_{l=1}^{k-1} a_{i+j+l}\varepsilon_{i+j+l}, a - \varepsilon_{i+j+k} \right]$$

and

$$D = \bigcup_{k=1}^{i} \left[ d + \sum_{l=1}^{k-1} a_l\varepsilon_l, a - \varepsilon_k \right].$$

It follows then, using Corollary 2.2, that $\operatorname{sdepth} \mathscr{D}(\mathscr{P}) = n - 1$, as desired.

In order to prove our claim we first show that the intervals in $\mathscr{P}$ cover $P_I$. In fact, let $e \in P_I$. If $e = a$, then $e \in [a, a]$. Otherwise $e \neq a$ and we may assume that $e \geqslant b$. Then $e = (a_1, \ldots, a_i, x_{i+1}, \ldots, x_n)$ with $x_k \leqslant a_k$ for all $k$. Since $e \leqslant a$ and $e \neq a$ there exists a $k_0 \in$

$\{i+1,\dots,n\}$ such that $x_{k_0} < a_{k_0}$, and $k_0$ is minimal with this property. If $k_0 \in \{i+1,\dots,i+j\}$ then $e \in [b+\sum_{l=1}^{k_0-1} a_{i+l}\varepsilon_{i+l}, a-\varepsilon_{i+k_0}] \subset B$. Otherwise $e \in C$ by similar arguments.

It remains to be shown that the intervals in $\mathscr{P}$ are pairwise disjoint. For this we show: (i) the intervals in each of $B$, $C$ and $D$ are pairwise disjoint, and (ii) $B \cap C = B \cap D = C \cap D = \emptyset$.

For the proof of (i) consider for example the set $B$ (the arguments for $C$ and $D$ are the same). If $j=1$ then we are done. Otherwise choose two arbitrary intervals in $B$, say $[b+\sum_{l=1}^{k-1} a_{i+l}\varepsilon_{i+l}, a-\varepsilon_{i+k}]$ and $[b+\sum_{l=1}^{p-1} a_{i+l}\varepsilon_{i+l}, a-\varepsilon_{i+p}]$ with $1 \leqslant k < p \leqslant j$. Since the $(i+k)$th component of any vector of the first interval is $< a_{i+k}$ and the $(i+k)$th component of any vector in the second interval is $a_{i+k}$, it follows that $[b+\sum_{l=1}^{k-1} a_{i+l}\varepsilon_{i+l}, a-\varepsilon_{i+k}] \cap [b+\sum_{l=1}^{p-1} a_{i+l}\varepsilon_{i+l}, a-\varepsilon_{i+p}] = \emptyset$.

It remains to prove (ii). Let $e=(e_1,\dots,e_n) \in B \cap C$. Since $e \in C$, we have $e_k = a_k$ for all $k$ with $k \in \{i+1,\dots,i+j\}$. On the other hand $e \in B$ implies that there exists $k \in \{i+1,\dots,i+j\}$ such that $e_k < a_k$, a contradiction. Hence $B \cap C = \emptyset$. A similar argument can be used to show $B \cap D = C \cap D = \emptyset$. □

We close our paper by stating a conjecture on partitions which follows from a conjecture of Soleyman Jahan [19].

We denote by $\operatorname{reg} M$ the regularity of the graded $S$-module $M$.

**Conjecture 3.9.** *Let $J \subset I$ be monomial ideals. Then there exists a partition $\mathscr{P}$: $P_{I/J} = \bigcup_{i=1}^{r}[c_i, d_i]$ of the characteristic poset $P_{I/J}$ such that $|c_i| \leqslant \operatorname{reg}(I/J)$ for all $i$.*

The original conjecture of Soleyman Jahan says that for $I/J$ there exists a Stanley decomposition $\mathscr{D}$: $I/J = \bigoplus_{i=1}^{r} x^{c_i}K[Z_i]$ such that $|c_i| = \deg x^{c_i} \leqslant \operatorname{reg}(I/J)$ for all $i$. Let $\mathscr{P}$ be the partition of $P_{I/J}$ constructed in Theorem 2.1 with the property that $\operatorname{sdepth}\mathscr{D}(\mathscr{P}) \geqslant \operatorname{sdepth}\mathscr{D}$. It follows from the construction of $\mathscr{P}$ that for each interval $[c,d]$ of this partition we have $c \in \{c_1,\dots,c_r\}$. This shows that Soleyman Jahan's conjecture implies Conjecture 3.9.

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## References

[1] J. Apel, On a conjecture of R.P. Stanley; Part I – Monomial ideals, J. Algebraic Combin. 17 (2003) 39–56.
[2] J. Apel, On a conjecture of R.P. Stanley; Part II – Quotients modulo monomial ideals, J. Algebraic Combin. 17 (2003) 57–74.
[3] A. Aramova, J. Herzog, T. Hibi, Squarefree lexsegment ideals, Math. Z. 228 (1998) 353–378.
[4] I. Bermejo, P. Gimenez, Saturation and Castelnuovo–Mumford regularity, J. Algebra 303 (2006) 592–617.
[5] A. Björner, M. Wachs, Shellable nonpure complexes and posets. I, Trans. Amer. Math. Soc. 349 (1997) 3945–3975.
[6] W. Bruns, J. Herzog, Cohen Macaulay Rings, revised ed., Cambridge Univ. Press, 1996.
[7] A. Dress, A new algebraic criterion for shellability, Beitrage Algebra Geom. 34 (1) (1993) 45–55.
[8] D. Eisenbud, Commutative Algebra; with a View Towards Algebraic Geometry, Grad. Texts in Math., Springer, 1995.
[9] A. Conca, J. Herzog, Castelnuovo–Mumford regularity of products of ideals, Collect. Math. 54 (2003) 137–152.

[10] J. Herzog, A. Soleyman Jahan, S. Yassemi, Stanley decompositions and partitionable simplicial complexes, J. Algebraic Combin., in press.
[11] J. Herzog, D. Popescu, Finite filtrations of modules and shellable multicomplexes, Manuscripta Math. 121 (2006) 385–410.
[12] J. Herzog, D. Popescu, M. Vladoiu, On the Ext-modules of ideals of Borel type, in: Contemp. Math., vol. 331, 2003, pp. 171–186.
[13] J. Herzog, Y. Takayama, Resolutions by mapping cones, in: The Roos Festschrift volume Nr. 2(2), Homology, Homotopy Appl. 4 (2002) 277–294.
[14] M. Janet, Les modules des formes algébriques et la théorie générale des systèmes différentiels, Ann. Sci. École Norm. Sup. 41 (3) (1924) 27–65.
[15] E. Miller, The Alexander duality functors and local duality with monomial support, J. Algebra 231 (2000) 180–234.
[16] W. Plesken, D. Robertz, Janet's approach to presentations and resolutions for polynomials and linear pdes, Arch. Math. 84 (2005) 22–37.
[17] A. Rauf, Stanley decompositions, pretty clean filtrations and reductions modulo regular elements, Bull. Math. Soc. Sci. Math. Roumaine 50 (98) (2007) 347–354.
[18] A. Soleyman Jahan, Prime filtrations of monomial ideals and polarizations, J. Algebra 312 (2007) 1011–1032.
[19] A. Soleyman Jahan, Stanley decompositions of squarefree modules and Alexander duality, arXiv: math.AC/0709.4145.
[20] R.P. Stanley, Linear Diophantine equations and local cohomology, Invent. Math. 68 (1982) 175–193.
[21] R.P. Stanley, Combinatorics and Commutative Algebra, Birkhäuser, 1983.
[22] R.P. Stanley, Positivity Problems and Conjectures in Algebraic Combinatorics, in: V. Arnold, M. Atiyah, P. Lax, B. Mazur (Eds.), Mathematics: Frontiers and Perspectives, American Mathematical Society, Providence, RI, 2000, pp. 295–319.