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# Pro-$p$ groups with waists ☆

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ABSTRACT

A waist $W$ of a pro-$p$ group $G$ is a subgroup which is comparable with any open normal subgroup of $G$. The position of $W$ with respect to the terms of a central series of $G$ is studied here. If $p$ is odd, with some natural hypotheses we show that $W$ is a term of both the lower and upper central series of $G$.


## 1. Introduction

In this paper we study the structure of pro-$p$ groups $G$ that have a nontrivial (open) normal subgroup $W$ which is comparable with any other (open) normal subgroup $N$: either $W \leqslant N$, or $N \leqslant W$. Equivalently, $W$ is the only (open) normal subgroup of $G$ of its own index.

A similar question was investigated in [1] for a module $M$ over an Artin ring: a submodule $N$ is called there a *waist* for $M$ if it is comparable with any other submodule of $M$. Thus, in the terminology of [1], our goal is to study pro-$p$ groups with waists.

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Examples of pro-$p$ groups with waists are easy to find:

- in a procyclic group all subgroups are waists;
- if a finite $p$-group $G$ has cyclic center then the minimal normal subgroup of $G$ is a waist.

There are indeed more interesting examples:

- in a pro-$p$ group $G$ of nilpotency class at least 3 and commutator subgroup of index $p^2$, the second and third terms of the lower central series are waists;
- the class of groups that have a nontrivial normal subgroup whose nontrivial cosets consist of conjugate elements was studied in [2,3] and other papers (see [4] for a rather complete list of references): such a subgroup, which is now called a Camina kernel, is easily seen to be a waist (in fact, some of our results below can be thought of as generalizations of facts on Camina groups);
- in pro-$p$ groups of maximal class all the terms of the lower central series are waists.

A pro-$p$ group in which all the terms of the lower central series are waists is said to have obliquity 0. The Nottingham group is an example of a pro-$p$ group of obliquity 0 which is not of maximal class. Such groups have received some interest in the literature (see [5]).

In Section 2 we give some general properties of waists and we show that in some sense a waist $W$ is almost a term of every central series of the group $G$. From our results it follows that $W$ is a term of every central series with elementary abelian sections.

In Section 3 we work under the hypothesis that $p$ is odd. We first show that $W$ is a term of the lower central series of $G$ unless $G$ is procyclic or $|W| = p$. Furthermore we show that, with the same exceptions, if $G$ is finite then $W$ is a term of the upper central series of $G$. Finally we prove that if $G_2$ is a waist then $G_3$ is also a waist.

Our notation is standard. For $i \geqslant 1$ we denote by $G_i$ the $i$-th term of the lower central series of $G$ and for $i \geqslant 0$ by $Z_i(G)$ the $i$-th term of the upper central series of $G$, where $Z_0(G) = 1$. By $\Omega_1(G)$ we denote the subgroup of $G$ generated by the elements of order $p$.

Throughout this paper when we write 'finitely generated' we mean 'topologically finitely generated'.

## 2. General results

**Definition 1.** A subgroup $W$ of a pro-$p$ group $G$ is said to be a waist if for every open normal subgroup $N$ of $G$ we have $N \leqslant W$ or $W \leqslant N$.

We collect here the first properties of waists:

**Proposition 2.** *Let $G$ be a pro-$p$ group with a waist $W$. Then*:

(i) *if $W \neq 1$ then $W$ is open in $G$*;
(ii) *if $N \trianglelefteqslant_c G$ then $WN/N$ is a waist in $G/N$*;
(iii) *if $W$ is a proper subgroup of $G$ then $G$ is finitely generated*;
(iv) *$W$ is characteristic in $G$ (in particular $W$ is normal)*;
(v) *for every closed normal subgroup $N$ of $G$ we have $W \leqslant N$ or $N \leqslant W$.*

**Proof.**

(i) If $W$ is not open then it does not contain any open normal subgroup of $G$. Therefore, $W$ is contained in every open normal subgroup of $G$: this implies $W = 1$.
(ii) This is trivial since the open normal subgroups of $G/N$ are of the form $MN/N$ with $M \trianglelefteqslant_O G$.
(iii) The open maximal subgroups of $G$ cannot be properly contained in $W$ and hence contain $W$. By statement (i), the index of $W$ in $G$ is finite, and then there is only a finite number of maximal

open subgroups in $G$: therefore the Frattini subgroup has finite index in $G$. By [6, Proposition 1.9] this means that $G$ is finitely generated.

(iv) We may assume that $W$ is a proper subgroup of $G$: by statement (iii), the group $G$ is finitely generated. By statement (i), the index $[G:W]$ is finite: let $N$ be an open normal subgroup of $G$ such that $[G:N]=[G:W]$. This forces $N=W$, i.e. $W$ is normal (and it is the only open normal subgroup of index $[G:W]$). If $\varphi$ is an automorphism of $G$, then $\varphi$ is continuous, by [6, Corollary 1.22], so that $W\varphi$ is an open normal subgroup of $G$ of index $[G:W]$. Therefore $W\varphi=W$.

(v) If $N \nleqslant W$ then $NM \nleqslant W$ for every open normal subgroup $M$ of $G$. As $NM \trianglelefteqslant_O G$, we then have $W \leqslant NM$. Since $N=\bar{N}=\bigcap_{M \trianglelefteqslant_O G} NM$, we find $W \leqslant N$. $\square$

**Corollary 3.** *If a pro-$p$ group $G$ has a base for the neighborhoods of the identity formed by waists then $G$ is either finite or just infinite.*

**Proof.** If $N \trianglelefteqslant_C G$ and $N \neq 1$ then there exists a proper waist $W$ of $G$ which does not contain $N$. By Proposition 2(v) the subgroup $W$ is contained in $N$ and Proposition 2(i) then implies that $N$ is open. $\square$

**Lemma 4.** *Let $W$ be an open normal subgroup of a pro-$p$ group $G$. Then the following are equivalent*:

(i) *$W$ is a waist in $G$*;
(ii) *$W/\overline{W^p[W,G]}$ is a waist in $G/\overline{W^p[W,G]}$*;
(iii) *$W/H$ is a waist in $G/H$ for every open normal subgroup $H$ of $G$ which is contained in $W$ with index $p$.*

**Proof.** Clearly (i) implies (ii) and (ii) implies (iii) by Proposition 2(ii).

Let us assume that (iii) holds and we prove that (i) holds. Let $N$ be an open normal subgroup of $G$ which does not contain $W$. We may then choose an open normal subgroup $H$ of $G$ which is contained in $W$ with index $p$ and contains $N \cap W$. By the modular law we have $H = H(N \cap W) = HN \cap W$. Therefore $HN/H$ does not contain $W/H$ and it is then contained in $W/H$. This implies that $N \leqslant W$. $\square$

**Lemma 5.** *Let $G$ be a pro-$p$ group with a waist $W$ and let $H$ and $L$ be closed normal subgroups of $G$ such that $H \gneqq W \gneqq L \geqslant [G,H]$. Then $H/L$ is procyclic.*

**Proof.** By factoring out $L$ we may assume that $L=1$ and that $H$ is central. Since $W$ is open by Proposition 2(i), the subgroup $H$ is open too. As $H$ is an open central subgroup of $G$, every open subgroup of $H$ is open and normal in $G$. In particular, if $N$ is an open subgroup of $H$ strictly contained in $W$, the quotient $H/N$ has exactly one subgroup of order $|W/N|$: this implies that $H/N$ is cyclic. This is true in general for every open subgroup $N$ of $H$ (simply intersect $N$ with an open normal subgroup of $G$ strictly contained in $W$): therefore $H$ is procyclic. $\square$

The open (normal) subgroups of a procyclic pro-$p$ group $G$ form a chain $G \geqslant G^p \geqslant G^{p^2} \geqslant \cdots$ so every open normal subgroup of $G$ is a waist. We investigate now waists in non-procyclic pro-$p$ groups.

**Proposition 6.** *If $G$ is a non-procyclic pro-$p$ group with a proper waist $W$, then the following hold*:

(i) $G_2 \geqslant W$;
(ii) *$G/W$ is not cyclic*;
(iii) *the terms $G_i$ of the lower central series of $G$ are open in $G$ (so they form a base for the neighborhoods of the identity)*;
(iv) *if $H \trianglelefteqslant_O G$ then $[G,H] \trianglelefteqslant_O G$.*

**Proof.** By Proposition 2(iii), the group $G$ is finitely generated. [6, Proposition 1.19] then implies that $G_2$ is closed in $G$, so, by Proposition 2(v), we have either $G_2 \geqslant W$ or $W \gneq G_2$. In the latter case, $G/G_2$ would be procyclic by Lemma 5: [6, Proposition 1.9] then implies that $G$ would be procyclic too, contradicting the hypothesis. This yields (i).

We already noted that $G/G_2$ is not cyclic: since $W \leqslant G_2$, the quotient $G/W$ is not cyclic. This proves (ii).

As $G_2$ has finite index in $G$, every $G_i$ has finite index in $G$ too: by [6, Theorem 1.17] $G_i$ is then open.

Finally, if $H \trianglelefteqslant_O G$ then $H$ contains $G_i$ for some $i$. Therefore $[G, H]$ contains $G_{i+1}$ and is then open. □

The finite $p$-groups which do not contain any elementary abelian normal subgroup of order $p^2$ are completely classified (see for instance [7, (4.3) and (4.5)]). In particular the non-cyclic ones are 2-groups of maximal class so we have:

**Lemma 7.** *Let $G$ be a non-cyclic finite $p$-group which does not contain an elementary abelian normal subgroup of order $p^2$. Then $H^p \leqslant [G, H]$ for every normal subgroup $H$ of $G$. In particular $G_i^p \leqslant G_{i+1}$ for every $i$.*

We now have the following:

**Proposition 8.** *If $G$ is a non-procyclic pro-$p$ group and $W$ is a proper waist of $G$ then $W^p \leqslant [G, W]$.*

**Proof.** By way of contradiction we assume that $W^p \nleqslant [G, W]$: as $[G, W]$ is open (Proposition 6(iv)), we may then choose an open normal subgroup $K$ of $G$ such that $W \geqslant K \geqslant [G, W]$ and $W/K$ is a cyclic group of order $p^2$. Since $W/K$ is the only normal subgroup of order $p^2$ of $G/K$, there is no elementary abelian normal subgroup of order $p^2$ in $G/K$. As $(W/K)^p \nleqslant [G/K, W/K] = 1$, Lemma 7 would imply that $G/K$ is a cyclic group. The quotient $G/W$ would be cyclic too, contradicting Proposition 6(ii). □

Using standard commutator calculus, Proposition 8 yields:

**Corollary 9.** *Let $G$ be a non-procyclic pro-$p$ group with a proper waist $W$. We set $W_1 := W$ and $W_i := [G, W_{i-1}]$ for $i > 1$. Then $W_i^p \leqslant W_{i+1}$, for every $i \geqslant 0$.*

We recall the following statement [8, Satz III.2.13]:

**Proposition 10.** *Let $G$ be a finite group. Then, for $i \geqslant 1$ we have that* $\exp(Z_{i+1}(G)/Z_i(G))$ *divides* $\exp(Z_i(G)/Z_{i-1}(G))$ *and* $\exp(G_{i+1}/G_{i+2})$ *divides* $\exp(G_i/G_{i+1})$.

Dual to Proposition 8 we have:

**Proposition 11.** *If $G$ is a non-procyclic pro-$p$ group with a proper waist $W$ and $|W| > p$ then $Z(G/W)^p = 1$.*

**Proof.** Let $K$ be an open normal subgroup of $G$ with index $p^2$ in $W$: by Proposition 6(ii), the quotient $G/K$ is not cyclic. We may then factor out $K$ and assume, without loss of generality, that $|W| = p^2$. By Lemma 7 we may assume that $G$ has an elementary abelian normal subgroup of order $p^2$: this is necessarily $W$. We claim that $W$ is a term of the upper central series of $G$: if not, $Z_{i+1}(G) \gneq W \gneq Z_i(G)$ for $i = 0$ or $i = 1$. By Lemma 5, the quotient $Z_{i+1}(G)/Z_i(G)$ would be a cyclic group of order at least $p^2$. Since $W$ is not cyclic, $Z_i(G) \neq 1$, that is $i \neq 0$. Therefore $Z_i(G)$ is the center of $G$ and has order $p$. By Proposition 10 we have that $Z_{i+1}(G)/Z_i(G)$ has then exponent $p$, a contradiction. Since $W$ is a term of the upper central series it contains $Z(G)$: this has exponent $p$, so that, using again Proposition 10, every section of the upper central series has exponent $p$: in particular $Z(G/W)$ has exponent $p$. □

We can now strengthen Lemma 5.

**Proposition 12.** *Let $G$ be a non-procyclic pro-$p$ group with a proper waist $W$ and let $H$ and $L$ be subgroups of $G$ such that $H \gneqq W \gneqq L \geqslant [G,H]$. Then*:

(i) *$H$ and $L$ are both open and normal in $G$*;
(ii) *$H/L$ is cyclic*;
(iii) $[W:L]=p$;
(iv) $L=[G,H]$;
(v) *$H$ is not a waist in $G$*;
(vi) $W^p[W,G]=L^p[L,G]$.

*If, moreover, $|W|>p$ then*:

(vii) $[H:W]=p$;
(viii) $H/L=Z(G/L)$;
(ix) *$L$ is not a waist in $G$.*

**Proof.** Both $H$ and $L$ are normal since $[G,L]\leqslant[G,H]\leqslant L\leqslant H$. The subgroup $H$ is open, since it contains $W$. By Proposition 6(iv), $[G,H]$ is then open, so $L$ is open too. This proves (i).

We now apply Lemma 5 to the chain $H\gneqq W\gneqq[G,H]$ and we see that $H/[G,H]$ is cyclic. Since $L\geqslant[G,H]$ we have (ii).

In particular $W/[G,H]$ is cyclic. Since $[G,H]\geqslant[G,W]$, Proposition 8 then implies that $W/[G,H]$ has order $p$. This forces $L=[G,H]$, as $W\gneqq L\geqslant[G,H]$ and we have (iii) and (iv).

Now note that $H\neq G$ for, otherwise, $W$ would strictly contain $[G,H]=G_2$, contradicting Proposition 6(i). Since $H/[G,H]$ is cyclic of order at least $p^2$, it follows that $H^p\nleqslant[G,H]$, so Proposition 8 implies that $H$ is not a waist in $G$ as claimed in (v).

By Proposition 8, $W^p[W,G]=[W,G]$. As $H/L$ is cyclic, it follows that $W\leqslant H^pL$, so:

$$[W,G]\leqslant\bigl[H^pL,G\bigr]=\bigl[H^p,G\bigr][L,G]\leqslant[H,G]^p[H,G,G][L,G]=L^p[L,G],$$

proving (vi).

We now suppose that $|W|>p$. There is no loss of generality if we assume that $|W|=p^2$. Let $Z(G/L)=K/L$. Clearly $K\geqslant H$. We now apply Lemma 5 to the chain $K\gneqq W\gneqq L$ and we see that $K/L$ is cyclic. In particular $K/W$ is cyclic. Since $Z(G/W)\geqslant K/W$, Proposition 11 then implies that $K/W$ has order $p$. This forces $K=H$, as $K\geqslant H\gneqq W$. This gives (vii) and (viii).

To prove (ix) note that since $W^p[W,G]=L^p[L,G]=1$, $W$ is central and has exponent $p$. Every subgroup of $W$ of order $p$ is then normal in $G$, so there exists more than one normal subgroup of order $p$: in particular $L$ is not a waist in $G$. $\square$

**Corollary 13.** *Let $G$ be a non-procyclic pro-$p$ group with two proper waists $W\geqslant X$. We put $W_1:=W$ and $W_i:=[G,W_{i-1}]$ for $i>1$. Then $X=W_i$ for some $i$. If, moreover, $|X|>p$ then $W/X=Z_{i-1}(G/X)$.*

**Proof.** Let $i$ be the maximum integer such that $W_i\geqslant X$. By Corollary 9 we know that $W_i^p\leqslant W_{i+1}$. If $W_i\neq X$ then, by Proposition 12, the quotient $W_i/W_{i+1}$ would be a cyclic group of order at least $p^2$, a contradiction.

Assume now that $|X|>p$. Since $X=W_i$ and $X\neq W_{i-1}$, it follows that $W/X\leqslant Z_{i-1}(G/X)$ and $W/X\nleqslant Z_{i-2}(G/X)$. Since $W/X$ is a waist in $G/X$ we have then $Z_{i-1}(G/X)\geqslant W/X\gneqq Z_{i-2}(G/X)$. If $W/X\neq Z_{i-1}(G/X)$ then, by Proposition 12, $Z_{i-1}(G/X)/Z_{i-2}(G/X)$ would be a cyclic group of order at least $p^2$: however, by Proposition 11 we know that $Z(G/X)^p=1$, so that $Z_{i-1}(G/X)^p\leqslant Z_{i-2}(G/X)$ by Proposition 10, a contradiction. $\square$

**Corollary 14.** *If $G$ is a pro-$p$ group with a non-cyclic waist $W$ then $W\geqslant Z(G)$.*

**Proof.** As $Z(G) \trianglelefteq_C G$, by Proposition 2(v) either $W \geqslant Z(G)$ or $Z(G) \gneq W$. In the latter case Proposition 12 (with $H = Z(G)$ and $L = 1$) implies that $Z(G)$ would be cyclic and $W$ would be cyclic too. □

## 3. Main results

The results of the previous section hold for all primes $p$: in particular Proposition 12 bounds a central section containing a waist. If $p$ is odd we can say much more.

**Theorem 15.** *Let $G$ be a non-procyclic pro-$p$ group with a proper waist $W$, and let $p > 2$. If $|W| \neq p$ then $W$ is a term of the lower central series of $G$.*

**Proof.** We suppose, by way of contradiction, that $W$ is not a term of the lower central series of $G$. By Proposition 6(iii) the terms of the lower central series are open normal subgroups of $G$ so there exists an integer $c$ such that $G_{c-1} \gneq W \gneq G_c$. By Proposition 12 we then have $[W : G_c] = p$. Factoring out an open normal subgroup of $G$ of index $p$ in $G_c$ we may assume, without loss of generality, that $W$ has order $p^2$ and $G_c$ has order $p$ (hence $c$ is the nilpotency class of the finite $p$-group $G$). Proposition 12 gives $W^p[W, G] = G_c^p G_{c+1} = 1$ so $W$ is central and elementary abelian: Corollary 14 then implies that $W = Z(G)$.

For $0 \leqslant i \leqslant c-1$, let $\varphi_i$ be the map from $G_{c-i}$ into $G$ that sends $x$ into $x^{p^i}$. Clearly $\varphi_0$ is a homomorphism whose image is contained in $Z(G)$. For $1 \leqslant i \leqslant c-1$, we set $H_i := \prod_{j=0}^{i} G_{c-j+1}^{p^j}$: since $p > 2$, a standard application of P. Hall's collection formula (see [6, Appendix A]) yields, for $x \in G_{c-i}$ and $g \in G$, that $(xg)^{p^i} \equiv x^{p^i} g^{p^i} \bmod H_i$ and $[x, g]^{p^{i-1}} \equiv [x^{p^{i-1}}, g] \bmod H_i$. As $Z(G)^p = 1$, by Proposition 10, for $1 \leqslant h \leqslant c$, we have $G_{c-h+1}^{p^{h-1}} \leqslant Z_h(G)^{p^{h-1}} \leqslant Z(G)$ and $G_{c-h+1}^{p^h} \leqslant Z_h(G)^{p^h} = 1$. In particular $H_i = 1$, so $\varphi_i$ is a homomorphism from $G_{c-i}$ into $Z(G)$ also for $1 \leqslant i \leqslant c-1$. Moreover we have

$$[x, g]^{p^{i-1}} = \left[x^{p^{i-1}}, g\right] \quad \text{for } 1 \leqslant i \leqslant c-1 \text{ and every } x \in G_{c-i},\ g \in G. \tag{1}$$

We now claim that, for every $0 \leqslant i \leqslant c-1$, we have $\ker \varphi_i = G_{c-i} \cap Z_i(G)$ and $[G_{c-i} : G_{c-i} \cap Z_i(G)] = p$. For $i = c-1$, this will provide our final contradiction, since $G/Z_{c-1}(G)$ cannot be cyclic.

We proceed by induction on $i$, the case $i = 0$ being trivial. By inductive hypothesis the claim holds for $i-1$, so $\operatorname{Im} \varphi_{i-1}$ is a subgroup of $W$ of order $[G_{c-i+1} : \ker \varphi_{i-1}] = p$. Therefore, if we consider the homomorphism $\tilde{\varphi}_i$ from $G_{c-i}$ into $W/\operatorname{Im} \varphi_{i-1}$ which is the composition of $\varphi_i$ with the canonical projection of $W$ onto $W/\operatorname{Im} \varphi_{i-1}$ we find that $[G_{c-i} : \ker \tilde{\varphi}_i] \leqslant |W/\operatorname{Im} \varphi_{i-1}| = p$. On the other hand, since $Z_i(G)^{p^i} = 1$, it follows that $G_{c-i} \cap Z_i(G) \leqslant \ker \varphi_i \leqslant \ker \tilde{\varphi}_i$. As $[G_{c-i} : G_{c-i} \cap Z_i(G)] \geqslant p$ (for, otherwise, $G_c = 1$), to get our claim it is enough to show that $\ker \tilde{\varphi}_i \leqslant G_{c-i} \cap Z_i(G)$. So let $x \in \ker \tilde{\varphi}_i$, that is $x^{p^i} \in \operatorname{Im} \varphi_{i-1}$: as $|\operatorname{Im} \varphi_{i-1}| = p$, the subgroup $H := \langle x^{p^{i-1}} \rangle \operatorname{Im} \varphi_{i-1}$ has order at most $p^2$. By (1), $[x^{p^{i-1}}, G] \leqslant G_{c-i+1}^{p^{i-1}} = \operatorname{Im} \varphi_{i-1}$ so $H$ is normal in $G$: as $|H| \leqslant p^2$, it follows that $H$ is contained in $W = Z(G)$ and therefore $x^{p^{i-1}} \in Z(G)$. By (1) again, we have $[x, g]^{p^{i-1}} = [x^{p^{i-1}}, g] = 1$ for every $g \in G$: as $[x, g] \in G_{c-i+1}$ this means that $[x, g] \in \ker \varphi_{i-1}$, that is, by inductive hypothesis, that $[x, g] \in Z_{i-1}(G)$. Therefore $x \in Z_i(G)$, as claimed. □

It is quite natural to ask whether a similar result holds for the upper central series. Let us consider an infinite non-procyclic pro-$p$ group $G$ with a proper waist $W$: by Proposition 6(iii) the terms of the lower central series of $G$ have finite index, so that $G$ is not nilpotent. Therefore a proper waist of a non-procyclic pro-$p$ group $G$ can be a term of the upper central series only if $G$ is finite. We need the following lemma:

**Lemma 16.** *Let $p > 2$ and let $G$ be a finite $p$-group with a proper waist $W$ of order $p^3$ and exponent $p^2$. Then $G$ has maximal class.*

**Proof.** A normal subgroup of $G$ of exponent $p$ cannot contain $W$ and it is then strictly contained in $W$: therefore its order is at most $p^2$. By [9, Theorem 1.1] $G$ is either of maximal class or regular.

Let us exclude the latter case. If $G$ is regular then $\Omega_1(G)$ has exponent $p$ and it is then contained in $W$, so its order is at most $p^2$ (actually it is exactly $p^2$). As $G$ is regular we have $[G : G^p] = |\Omega_1(G)| \leqslant p^2$. By [8, Satz III.11.4], the group $G$ is then metacyclic. Let $N$ be a normal cyclic subgroup of $G$. Since $N$ cannot contain the non-cyclic subgroup $W$, it is (strictly) contained in $W$. The quotient group $G/N$ cannot then be cyclic, since $G/W$ is not cyclic by Proposition 6(ii). Therefore $G$ cannot be metacyclic, a contradiction. $\square$

**Theorem 17.** *Let $G$ be a non-cyclic finite $p$-group with a proper waist $W$, and let $p > 2$. If $|W| \neq p$ then $W$ is a term of the upper central series of $G$.*

**Proof.** We suppose, by way of contradiction, that $W$ is not a term of the upper central series of $G$. Therefore there exists an integer $i$ such that $Z_{i+1}(G) \gneq W \gneq Z_i(G)$. By Proposition 12, $[W : Z_i(G)] = p$, so $Z_i(G) \neq 1$, and moreover $Z_{i+1}(G)/Z_i(G)$ is a cyclic group of order $p^2$. In particular. By factoring out $Z_{i-1}(G)$ we may assume that $Z_2(G) \gneq W \gneq Z(G)$. Since $Z_2(G)/Z(G)$ has exponent $p^2$, by Proposition 10 the exponent of $Z(G)$ is at least $p^2$. Let $H$ be a subgroup of $Z(G)$ such that $Z(G)/H$ is cyclic of order $p^2$. We note that $W/H$ is not cyclic, for otherwise $W^pH$ would strictly contain $Z(G)^pH$, while Proposition 12 ensures that $W^p[W, G] = Z(G)^p(Z(G), G) = Z(G)^p$. The quotient group $G/H$ has then a waist $W/H$ of order $p^3$ which is neither cyclic nor of exponent $p$, for it contains the cyclic group $Z(G)/H$ of order $p^2$. By Lemma 16 the quotient group $G/H$ has maximal class. This is a contradiction since the center of $G/H$ contains $Z(G)/H$ and has then order at least $p^2$. $\square$

At this point the reader may think that, for $p$ odd, a waist is a term of every central series so the situation described in Proposition 12 never appears. This is not true: as an example consider the extension $G$ of the abelian group $A$ of type $(p^2, p^2)$ with the automorphism represented by the matrix $\left(\begin{smallmatrix} 1 & p\nu \\ p & 1 \end{smallmatrix}\right)$ where $\nu$ is not a square modulo $p$. It is easy to show, as in [10], that $G_2$ is a waist of order $p^2$. Furthermore if $a$ is an element of $A - G_2$ and $H := \langle a \rangle G_2$, then $H \gneq G_2 \gneq [G, H]$.

**Theorem 18.** *Let $G$ be a pro-$p$ group with $p > 2$. If $G_2$ is a waist then $G_3$ is a waist too.*

**Proof.** By Lemma 4 we may assume, without loss of generality, that $G_3$ has order $p$ (in particular $G$ is finite). If $Z(G) = G_3$ then there exists exactly one normal subgroup of order $p$, that is $G_3$, and we are done. By way of contradiction we then assume that $Z(G) \neq G_3$: by [11, Theorem 1.1] the quotient $G/G_3$ is not a Camina group. Moreover, since $G_2$ is not contained in $Z(G)$, it strictly contains $Z(G)$: therefore $|G_2| > p^2$ and $[G_2 : G_3] > p$. Hence [10, Proposition 2.1] implies that $G_2 = G_3G^p$ and that there exists a positive integer $n$ such that $[G : G_2] = p^{2n+1}$ and $[G_2 : G_3] \leqslant p^{n+1}$. By Proposition 11 we know that $G^p \leqslant G_2$ and by [8, Satz III.2.13.b] we have $G_2^p \leqslant G_3$. Therefore the $p$-th power map induces a homomorphism from $G/G_2$ into $G_2/G_3$. Since $[G : G_2] > [G_2 : G_3]$ the kernel of this homomorphism is not trivial, that is, there exists an element $x$ in $G - G_2$ such that $x^p \in G_3$. Let us consider the group $H := G_2\langle x \rangle$. It is a normal subgroup of class at most 2 such that $H^p = G_2^p\langle x^p \rangle \leqslant G_3$. Since the class of $H$ is less than $p$, the group $H$ is regular and $|\Omega_1(H)| = [H : H^p] \geqslant [H : G_3] = |G_2|$. Therefore the normal subgroup $\Omega_1(H)$ contains $G_2$. As $H$ is regular, $\Omega_1(H)$ has exponent $p$ and $G_2$ has exponent $p$ too. Since $G^p \leqslant G_2$ this implies that $G$ has exponent at most $p^2$. Let us consider a generic element $y$ of $G$ and let $K := G_2\langle y \rangle$. Once again $K$ is a normal subgroup of class at most 2: as a consequence $K^p = G_2^p\langle y^p \rangle = \langle y^p \rangle$ is a normal subgroup of order at most $p$ and then it is central. By the arbitrariness of $y$ it follows that $G^p \leqslant Z(G)$. Since $G_2 = G_3G^p$, it would follow that $G_2$ is central and $G_3 = 1$, a contradiction. $\square$

**Remark 19.** In this section the assumption $p \neq 2$ was used quite extensively. Pro-2 groups seem to require an entirely different approach.

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