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The dynamics of Leavitt path algebras

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ABSTRACT

Recently it was shown that the notion of flow equivalence of shifts of finite type in symbolic dynamics is related to the Morita theory and the Grothendieck group in the theory of Leavitt path algebras (Abrams et al., 2011, [4]). In this paper we show that the notion of the conjugacy of shifts of finite type is closely related to the *graded* Morita theory and consequently the *graded* Grothendieck group. This fits into the general framework we have in these two theories: Conjugacy yields the flow equivalence, and the graded Morita equivalence can be lifted to the Morita equivalence. Starting from a finite directed graph, the observation that the graded Grothendieck group of the Leavitt path algebra associated to E coincides with the Krieger dimension group of the shift of finite type associated to E provides a link between the theory of Leavitt path algebras and symbolic dynamics. It has been conjectured that the ordered graded Grothendieck group as $\mathbb{Z}[x, x^{-1}]$-module (we call this the graded dimension group) classifies the unital Leavitt path algebras completely (Hazrat, 2013, [20]). Via the above correspondence, utilising the results from symbolic dynamics, we prove that for two purely infinite simple unital Leavitt path algebras, if their graded dimension groups are isomorphic, then the algebras are isomorphic.

1. Introduction

One of the central objects in the theory of symbolic dynamics is a shift of finite type (i.e., a topological Markov chain). Every finite directed graph E with no sinks and sources gives rise to a shift of

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finite type X_E by considering the set of bi-infinite paths and the natural shift of the paths to the left. This is called an edge shift. Conversely any shift of finite type is conjugate to an edge shift.

There are two notions of equivalences in the classifications of shifts of finite type: the conjugacy and the weaker notion of flow equivalence. Two shifts of finite type X_E and X_F are conjugate if E can be obtained from F by a series of in/out-splitting and their inverses (see [26, Theorem 7.1.2, Corollary 7.1.5]). Furthermore, X_E and X_F are flow equivalent if E can be obtained from F by a series of in/out-splitting, expansion and their inverses (see [26, p. 456] and [28]).

On the other hand, to a directed graph one can associate an analytical object, called a graph C^*-algebra [29] and an algebraic object called a Leavitt path algebra [1,6].

The relations between symbolic dynamics and (graph) C^*-algebras were explored in the work of Cuntz and Krieger [12] and later Bates and Pask [10], among others. The algebraic counter part of graph C^*-algebras, i.e., Leavitt path algebras, are related to symbolic dynamics by the recent work of Abrams, Louly, Pardo and Smith [4]. In both settings, it was observed that the notion of flow equivalence is related to the theory of Morita equivalence and the Grothendieck group K_0. It was shown that for two essential finite graphs E and F, if X_E is flow equivalent to X_F, then $C^*(E)$ is Morita equivalent to $C^*(F)$ and $\mathcal{L}(E)$ is Morita equivalent to $\mathcal{L}(F)$ (see [10] and [4], respectively).

Inspired by this connection between symbolic dynamics and graph algebras, in this note we show that, whereas, the flow equivalence is related to the Morita theory and the Grothendieck group K_0, the notion of the conjugacy in symbolic dynamics is closely related to the *graded* Morita theory and consequently the *graded* Grothendieck group K_0^{gr} in the setting of Leavitt path algebras. This fits into the general framework we have in these two theories: Conjugacy yields the flow equivalence and the graded Morita equivalence can be lifted to the Morita equivalence (see Section 2).

Thanks to the deep work of Williams [37], there is a matrix criterion when two edge shifts are conjugate; X_E and X_F are conjugate if and only if the adjacency matrices of E and F are strongly shift equivalent (see Section 6). Williams further introduced a weaker notion of the shift equivalence. Inspired by the success of K-theory in classification of AF C^*-algebras [14], Krieger introduced a variation of K_0 which he showed is a complete invariant for the shift equivalence [24]. It can be observed that Krieger's dimension group (and Wagoner's dimension module [35,36]) coincides with the graded Grothendieck group of associated Leavitt path algebras. This provides a bridge from the theory of Leavitt path algebras to symbolic dynamics.

Throughout this paper, we only will consider finite graphs with no sinks (and no sources in several occasions). The reason is, considering the edge shift X_E of a graph E, no arrow that begins at a source and no arrow that ends at a sink appear in any bi-infinite path (see the paragraph before the definition of essential graphs [26, Definition 2.2.9]). Thus only graphs with no sinks and sources appear in symbolic dynamics. Furthermore, one of the most interesting classes of Leavitt path algebras, i.e., purely infinite simple unital algebras are within this class of graphs.

The results of the paper are summarised as follows. Let E and F be two finite directed graphs with no sinks and sources and A_E and A_F be their adjacency matrices, respectively.

Corollary 12: The matrices A_E and A_F are shift equivalent if and only if there is an order preserving $\mathbb{Z}[x,x^{-1}]$-module isomorphism $K_0^{\mathrm{gr}}(\mathcal{L}(E)) \cong K_0^{\mathrm{gr}}(\mathcal{L}(F))$.

Proposition 15: If A_E and A_F are strongly shift equivalent then $\mathcal{L}(E)$ and $\mathcal{L}(F)$ are graded Morita equivalent. Conversely, if $\mathcal{L}(E)$ and $\mathcal{L}(F)$ are graded Morita equivalent, then A_E and A_F are shift equivalent.

Example 18: The strongly shift equivalence does not imply graded isomorphisms between Leavitt path algebras.

Theorem 21: Suppose $\mathcal{L}(E)$ and $\mathcal{L}(F)$ are purely infinite simple unital algebras. Then $\mathcal{L}(E) \cong \mathcal{L}(F)$ if there is an order preserving $\mathbb{Z}[x,x^{-1}]$-module isomorphism

$$\big(K_0^{\mathrm{gr}}\big(\mathcal{L}(E)\big), \big[\mathcal{L}(E)\big]\big) \cong \big(K_0^{\mathrm{gr}}\big(\mathcal{L}(F)\big), \big[\mathcal{L}(F)\big]\big).$$

For a graph E, let $\mathcal{P}(E)$ be the associated path algebra and $\mathrm{Gr}\text{-}\mathcal{P}(E)$ be the category of $\mathbb{Z}$-graded right $\mathcal{P}(E)$-modules and $\mathrm{Fdim}\text{-}\mathcal{P}(E)$ be its full (Serre) subcategory of modules that are the sum of their finite-dimensional submodules. Paul Smith [31–33] recently studied the quotient category

$$\mathrm{QGr}\text{-}\mathcal{P}(E) := \mathrm{Gr}\text{-}\mathcal{P}(E)/\,\mathrm{Fdim}\text{-}\mathcal{P}(E).$$

Theorem 23: There is an ordered abelian group isomorphism $K_0^{\mathrm{gr}}(\mathcal{L}(E)) \cong K_0^{\mathrm{gr}}(\mathcal{L}(F))$ if and only if $\mathrm{QGr}\text{-}\mathcal{P}(E) \approx \mathrm{QGr}\text{-}\mathcal{P}(F)$.

The main aim of this paper is to provide evidence that the graded Grothendieck group is a capable invariant which could eventually provide a bridge between the theories of graph C^*-algebras and Leavitt path algebras via symbolic dynamics. This group comes equipped with a pre-ordered abelian group structure, plus an action of $\mathbb{Z}$ on it (which corresponds to the shifting in a shift space), which makes it a $\mathbb{Z}[x, x^{-1}]$-module, and a distinguished element, namely the identity (see Section 4). In Section 8 we show that there is a product formula for this K-group. Section 6 relates it to conjugacy in symbolic dynamics. And in Section 7 we show that only the pre-order part of this group is sufficient to classify the quotient category of path algebras.

The paper is organised as follows. In Section 2 we recall the notion of graded Morita theory. For an arbitrary group Γ, one can equip the Leavitt path algebra $\mathcal{L}(E)$ associated to the graph E with a Γ-graded structure. This is recalled in Section 3. The graded Grothendieck group as an invariant for classification of Leavitt path algebras was first considered in [20]. In Section 4 we recall this group. In fact the graded Grothendieck group is not only an ordered abelian group, but has $\mathbb{Z}[x, x^{-1}]$-modules structure. The action of x on the group captures the shifting in the corresponding shift of finite type. In Section 5 we observe that for a finite directed graph, the graded Grothendieck group of the Leavitt path algebra associated to E coincides with the Krieger dimension group of the shift of finite type associated to E. This provides a link between the theory of Leavitt path algebras and symbolic dynamics. This has also been recently observed by Ara and Pardo, using a different approach in [5]. In fact Ara and Pardo settle the graded conjecture [20,21] positively for the class of graphs with no sinks and sources.

Section 6 is the main part of the paper, where the relations between strongly shift equivalence and graded Morita equivalence are studied. Section 7 shows that the graded Grothendieck group is a complete invariant for the quotient category of path algebras. In fact, here, we only need the structure of ordered abelian group (not the module structure) of the graded Grothendieck group to classify the quotient categories.

In Section 8 we study the behaviour of the graded Grothendieck group on the product of the graphs. We will establish a formula to express K_0^{gr} of the product of the graphs as the tensor product of K_0^{gr} of the graphs. We close the paper with Appendix A.

2. Graded Morita theory

For a graded ring A, the graded Grothendieck group $K_0^{\mathrm{gr}}(A)$ is constructed from the category of graded finitely generated projective A-modules. Thus it is natural to consider categories of graded modules which are equivalent, which in turn induces isomorphic graded Grothendieck groups. We will see that the graded equivalence of categories are closely related to the notion of shift equivalence in symbolic dynamics in Section 6.

In this section we gather results on graded Morita theory that we need in the paper. For the theory of graded rings, we refer the reader to [27] and for (nongraded) Morita theory to [25].

For an abelian group Γ and a Γ-graded ring A, by $\mathrm{Gr}\text{-}A$, we denote the category consisting of graded right A-modules as objects and graded homomorphisms as the morphisms. For $\alpha \in \Gamma$, the *α-suspension functor* or *shift functor*

$$\mathcal{T}_\alpha : \mathrm{Gr}\text{-}A \longrightarrow \mathrm{Gr}\text{-}A,$$
$$M \longmapsto M(\alpha)$$

is an isomorphism with the property $\mathcal{T}_\alpha \mathcal{T}_\beta = \mathcal{T}_{\alpha+\beta}$, $\alpha, \beta \in \Gamma$. Throughout the note all functors are additive functors.

Definition 1. Let A and B be Γ-graded rings.

(1) A functor $\phi : \text{Gr-}A \to \text{Gr-}B$ is called a *graded functor* if $\phi \mathcal{T}_\alpha = \mathcal{T}_\alpha \phi$.
(2) A graded functor $\phi : \text{Gr-}A \to \text{Gr-}B$ is called a *graded equivalence* if there is a graded functor $\psi : \text{Gr-}B \to \text{Gr-}A$ such that $\psi\phi \cong 1_{\text{Gr-}A}$ and $\phi\psi \cong 1_{\text{Gr-}B}$.
(3) If there is a graded equivalence between $\text{Gr-}A$ and $\text{Gr-}B$, we say A and B are *graded equivalent* or *graded Morita equivalent* and we write $\text{Gr-}A \approx_{\text{gr}} \text{Gr-}B$, or $\text{Gr}^\Gamma\text{-}A \approx_{\text{gr}} \text{Gr}^\Gamma\text{-}B$ to emphasis the categories are Γ-graded.
(4) A functor $\phi : \text{Mod-}A \to \text{Mod-}B$ is called a *graded functor* if there is a graded functor $\phi' : \text{Gr-}A \to \text{Gr-}B$ such that the following diagram, where the vertical functors are forgetful functors, commutes:

$$\begin{tikzcd} \text{Gr-}A \arrow[r, "\phi'"] \arrow[d, "U"'] & \text{Gr-}B \arrow[d, "U"] \\ \text{Mod-}A \arrow[r, "\phi"] & \text{Mod-}B. \end{tikzcd} \tag{1}$$

The functor ϕ' is called an *associated graded functor* of ϕ.
(5) A functor $\phi : \text{Mod-}A \to \text{Mod-}B$ is called a *graded equivalence* if it is graded and an equivalence.

Note that throughout the paper, if two graded rings A and B are graded isomorphic, we write $A \cong_{\text{gr}} B$, whereas if they are graded Morita equivalent, we write $\text{Gr-}A \approx_{\text{gr}} \text{Gr-}B$.

For a ring A, and a full idempotent element $e \in A$ (i.e., $e^2 = e$ and $AeA = A$), it is well known that the ring A is Morita equivalent to eAe. In Example 2 we establish a similar statement in the graded setting which will be used in Proposition 13 and Theorem 30.

Example 2. Let A be a graded ring and e be a full homogeneous idempotent of A, i.e., $e^2 = e$ and $AeA = A$. Clearly e has degree zero. Consider $P = eA$. One can readily see that P is a right graded progenerator. Then $P^* = \operatorname{Hom}_A(eA, A) \cong_{\text{gr}} Ae$ as graded left A-module and $B = \operatorname{End}_A(eA, eA) \cong_{\text{gr}} eAe$ as graded rings. The A–A-bimodule graded homomorphism $\phi : Ae \otimes_{eAe} eA \to A$ and the eAe–eAe-bimodule graded homomorphism $\psi : eA \otimes_A Ae \to eAe$ are isomorphism. Consequently one can check that the functors $- \otimes_A Ae : \text{Gr-}A \to \text{Gr-}eAe$ and $- \otimes_{eAe} eA : \text{Gr-}eAe \to \text{Gr-}A$ are inverse of each other. Thus we get a (graded) equivalence between $\text{Gr-}A$ and $\text{Gr-}eAe$ which lifts to a (graded) equivalence between $\text{Mod-}A$ and $\text{Mod-}eAe$, as it is shown in the diagram below:

$$\begin{tikzcd} \text{Gr-}A \arrow[r, "-\otimes_A Ae"] \arrow[d, "U"'] & \text{Gr-}eAe \arrow[d, "U"] \\ \text{Mod-}A \arrow[r, "-\otimes_A Ae"] & \text{Mod-}eAe. \end{tikzcd}$$

Example 2 shows that a graded equivalence between the categories $\text{Gr-}A$ and $\text{Gr-}eAe$ can be lifted to an equivalence between the categories $\text{Mod-}A$ and $\text{Mod-}eAe$. This lifting of the graded equivalence is a general phenomenon as proved by Gordon and Green in the case of $\mathbb{Z}$-graded rings [18, Proposition 5.3, Theorem 5.4]. Since we need this result in several occasions, we record it here.

Theorem 3. *Let A and B be two Γ-graded rings. The following are equivalent*:

(1) Mod-A *is graded equivalent to* Mod-B;
(2) Gr-A *is graded equivalent to* Gr-B;
(3) $B \cong_{\mathrm{gr}} \mathrm{End}_A(P)$ *for a graded A-progenerator P*;
(4) $B \cong_{\mathrm{gr}} e\mathbb{M}_n(A)(\bar{\delta})e$ *for a full homogeneous idempotent $e \in \mathbb{M}_n(A)(\bar{\delta})$, where $\bar{\delta} = (\delta_1, \dots, \delta_n)$, $\delta_i \in \Gamma$.*

3. Grading on Leavitt path algebras

For an arbitrary group Γ, one can equip $\mathcal{L}(E)$ with a Γ-graded structure. This will be needed in the note (see the proof of Theorem 30). We first recall the definition of a Leavitt path algebra associated to a directed graph (see [1,6]) and then discuss the grading on these algebras.

A *directed graph* $E = (E^0, E^1, r, s)$ consists of two countable sets E^0, E^1 and maps $r, s : E^1 \to E^0$. The elements of E^0 are called *vertices* and the elements of E^1 *edges*. If $s^{-1}(v)$ is a finite set for every $v \in E^0$, then the graph is called *row-finite*. In this setting, if the number of vertices, i.e., $|E^0|$, is finite, then the number of edges, i.e., $|E^1|$, is finite as well and we call E a *finite* graph. In this paper we only consider finite graphs.

For a graph $E = (E^0, E^1, r, s)$, a vertex v for which $s^{-1}(v)$ is empty is called a *sink*, while a vertex w for which $r^{-1}(w)$ is empty is called a *source*. An edge with the same source and range is called a *loop*. A path μ in a graph E is a sequence of edges $\mu = \mu_1 \dots \mu_k$, such that $r(\mu_i) = s(\mu_{i+1})$, $1 \leqslant i \leqslant k-1$. In this case, $s(\mu) := s(\mu_1)$ is the *source* of μ, $r(\mu) := r(\mu_k)$ is the *range* of μ, and k is the *length* of μ which is denoted by $|\mu|$. We consider a vertex $v \in E^0$ as a *trivial* path of length zero with $s(v) = r(v) = v$. By E^n, $n \in \mathbb{N}$, we denote the set of paths of length n. If μ is a nontrivial path in E, and if $v = s(\mu) = r(\mu)$, then μ is called a *closed path based at* v. If $\mu = \mu_1 \dots \mu_k$ is a closed path based at $v = s(\mu)$ and $s(\mu_i) \neq s(\mu_j)$ for every $i \neq j$, then μ is called a *cycle*.

Definition 4 *(Leavitt path algebras).* For a row-finite graph E and a ring R with identity, the *Leavitt path algebra of E*, denoted by $\mathcal{L}_R(E)$, is the algebra generated by the sets $\{v \mid v \in E^0\}$, $\{\alpha \mid \alpha \in E^1\}$ and $\{\alpha^* \mid \alpha \in E^1\}$ with the coefficients in R, subject to the relations

(1) $v_i v_j = \delta_{ij} v_i$ for every $v_i, v_j \in E^0$;
(2) $s(\alpha)\alpha = \alpha r(\alpha) = \alpha$ and $r(\alpha)\alpha^* = \alpha^* s(\alpha) = \alpha^*$ for all $\alpha \in E^1$;
(3) $\alpha^* \alpha' = \delta_{\alpha\alpha'} r(\alpha)$, for all $\alpha, \alpha' \in E^1$;
(4) $\sum_{\{\alpha \in E^1,\, s(\alpha) = v\}} \alpha\alpha^* = v$ for every $v \in E^0$ for which $s^{-1}(v)$ is non-empty.

Here the ring R commutes with the generators $\{v, \alpha, \alpha^* \mid v \in E^0,\ \alpha \in E^1\}$. Throughout this note the coefficient ring is a fixed field K and we simply write $\mathcal{L}(E)$ instead of $\mathcal{L}_K(E)$. The elements α^* for $\alpha \in E^1$ are called *ghost edges*. One can show that $\mathcal{L}(E)$ is a ring with identity if and only if the graph E is finite (otherwise, $\mathcal{L}(E)$ is a ring with local identities).

Recall that a ring A is called a *Γ-graded ring*, or simply a *graded ring*, if $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$, where Γ is an (abelian) group, each A_γ is an additive subgroup of A and $A_\gamma A_\delta \subseteq A_{\gamma+\delta}$ for all $\gamma, \delta \in \Gamma$.

The set $A^h = \bigcup_{\gamma \in \Gamma} A_\gamma$ is called the set of *homogeneous elements* of A. The non-zero elements of A_γ are called *homogeneous of degree* γ and we write $\deg(a) = \gamma$ if $a \in A_\gamma \backslash \{0\}$. We call the set

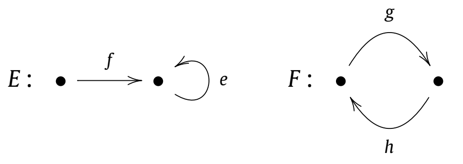
$$\Gamma_A = \{\gamma \in \Gamma \mid A_\gamma \neq 0\}$$

the *support* of A. We say A has a *trivial grading*, or A is *concentrated in degree zero* if the support of A is the trivial group, i.e., $A_0 = A$ and $A_\gamma = 0$ for $\gamma \in \Gamma \backslash \{0\}$. For a Γ-graded ring A (with identity element 1), one can prove that 1 is a homogeneous element of degree 0, A_0 is a subring of A and for an invertible element $a \in A_\gamma$, its inverse a^{-1} is homogeneous of degree $-\gamma$, i.e., $a^{-1} \in A_{-\gamma}$.

A Γ-graded ring $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$ is called a *strongly graded ring* if $A_\gamma A_\delta = A_{\gamma+\delta}$ for all $\gamma, \delta \in \Gamma$. One can show that A is strongly graded if and only if $1 \in A_\gamma A_{-\gamma}$ for any $\gamma \in \Gamma$. For the theory of graded rings, we refer the reader to [22,27].

Let Γ be an arbitrary group with the identity element e. Let $w : E^1 \to \Gamma$ be a *weight* map and further define $w(\alpha^*) = w(\alpha)^{-1}$, for $\alpha \in E^1$ and $w(v) = e$ for $v \in E^0$. The free K-algebra generated by the vertices, edges and ghost edges is a Γ-graded K-algebra. Furthermore, the Leavitt path algebra is the quotient of this algebra by relations in Definition 4 which are all homogeneous. Thus $\mathcal{L}_K(E)$ is a Γ-graded K-algebra.

Example 5 *(Different gradings on a Leavitt path algebras).* Consider the graphs



Assigning 0 to vertices and 1 to edges in the graphs in the usual manner, by [19, Theorem 4.2] we obtain $\mathcal{L}(E) \cong_{\mathrm{gr}} \mathbb{M}_2(K[x, x^{-1}])(0, 1)$ whereas $\mathcal{L}(F) \cong_{\mathrm{gr}} \mathbb{M}_2(K[x^2, x^{-2}])(0, 1)$ and one can easily observe that $\mathcal{L}_K(E) \not\cong_{\mathrm{gr}} \mathcal{L}_K(F)$.

However assigning 1 for the degree of f and 2 for the degree of e in E and 1 for the degrees of g and h in F, the proof of [19, Theorem 4.2] shows that $\mathcal{L}_K(E) \cong \mathbb{M}_2(K[x^2, x^{-2}])(0, 1)$ and $\mathcal{L}_K(F) \cong \mathbb{M}_2(K[x^2, x^{-2}])(0, 1)$. So with these gradings, $\mathcal{L}_K(E) \cong_{\mathrm{gr}} \mathcal{L}_K(F)$.

The natural and standard grading given to a Leavitt path algebra is a $\mathbb{Z}$-grading by setting $\deg(v) = 0$, for $v \in E^0$, $\deg(\alpha) = 1$ and $\deg(\alpha^*) = -1$ for $\alpha \in E^1$. If $\mu = \mu_1 \dots \mu_k$, where $\mu_i \in E^1$, is an element of $\mathcal{L}(E)$, then we denote by μ^* the element $\mu_k^* \dots \mu_1^* \in \mathcal{L}(E)$. Further we define $v^* = v$ for any $v \in E^0$. Since $\alpha^* \alpha' = \delta_{\alpha\alpha'} r(\alpha)$, for all $\alpha, \alpha' \in E^1$, any word in the generators $\{v, \alpha, \alpha^* \mid v \in E^0,\ \alpha \in E^1\}$ in $\mathcal{L}(E)$ can be written as $\mu\gamma^*$ where μ and γ are paths in E (vertices are considered paths of length zero). The elements of the form $\mu\gamma^*$ are called *monomials*.

Taking the grading into account, one can write $\mathcal{L}(E) = \bigoplus_{k \in \mathbb{Z}} \mathcal{L}(E)_k$ where

$$\mathcal{L}(E)_k = \left\{ \sum_i r_i \alpha_i \beta_i^* \;\middle|\; \alpha_i, \beta_i \text{ are paths},\ r_i \in K,\ \text{and } |\alpha_i| - |\beta_i| = k \text{ for all } i \right\}.$$

The following theorem was proved in [19] which determined finite graphs whose associated Leavitt path algebras are strongly $\mathbb{Z}$-graded (see also [21] for another proof by realising Leavitt path algebras as corner skew Laurent polynomial rings).

Theorem 6. *Let E be a finite graph. Then $L(E)$ is strongly graded if and only if E does not have sinks.*

This theorem along with Dade's theorem (see Section 4.2) will be used throughout this paper to pass from the graded K-theory to the nongraded K-theory of ring of homogeneous elements of degree zero.

4. Graded Grothendieck groups and graded dimension groups

For an abelian monoid V, we denote by V^+ the group completion of V. This gives a left adjoint functor to the forgetful functor from the category of abelian groups to abelian monoids. When the monoid V has a Γ-module structure, where Γ is a group, then V^+ inherits a natural Γ-module structure, or equivalently, $\mathbb{Z}[\Gamma]$-module structure.

The graded Grothendieck group of a graded ring is constructed as the completion of the abelian monoid of isomorphic classes of graded finitely generated projective modules (i.e., graded finitely generated modules which are also projective). Namely, for a Γ-graded ring A and a graded finitely generated projective (right) A-module P, let $[P]$ denote the class of graded finitely generated projective modules graded isomorphic to P. Then the monoid

$$\mathcal{V}^{\mathrm{gr}}(A)=\big\{[P]\;\big|\;P\text{ is graded finitely generated projective }A\text{-module}\big\} \tag{2}$$

has a Γ-module structure defined as follows: for $\gamma\in\Gamma$ and $[P]\in\mathcal{V}^{\mathrm{gr}}(A)$, $\gamma.[P]=[P(\gamma)]$. The group $\mathcal{V}^{\mathrm{gr}}(A)^{+}$ is called the *graded Grothendieck group* and is denoted by $K_0^{\mathrm{gr}}(A)$, which as the above discussion shows is a $\mathbb{Z}[\Gamma]$-module. This extra $\mathbb{Z}[\Gamma]$-module carries a substantial information about the graded ring A.

The main aim of this note is to concentrate on the graded Grothendieck group of Leavitt path algebras as a capable invariant for these algebras. This line of study started in [20].

4.1. Let V be an abelian monoid, Γ a group and V be a (left) Γ-module. Let $\geqslant$ be a reflexive and transitive relation on V which respects the monoid and the module structures, i.e., for $\gamma\in\Gamma$ and $x,y,z\in V$, if $x\geqslant y$, then $x+z\geqslant y+z$ and $\gamma x\geqslant\gamma y$. We call V a *Γ-pre-ordered module*. We call V a *pre-ordered module* when Γ is clear from the context. The *cone* of V is defined as $\{x\in V\mid x\geqslant 0\}$ and denoted by V_+. The set V_+ is a Γ-submonoid of V, i.e., a submonoid which is closed under the action of Γ. In fact, V is a Γ-pre-ordered module if and only if there exists a Γ-submonoid of V. (Since V is a Γ-module, it can be considered as a $\mathbb{Z}[\Gamma]$-module.) An element $u\in V_+$ is called an *order-unit* if for any $x\in V$, there are $\alpha_1,\dots,\alpha_n\in\Gamma$, $n\in\mathbb{N}$, such that

$$\sum_{i=1}^{n}\alpha_i u\geqslant x. \tag{3}$$

As usual, in this setting, we only consider homomorphisms which preserve the pre-ordering, i.e., a Γ-homomorphism $f:V\to W$, such that $f(V_+)\subseteq W_+$.

For a Γ-graded ring A, $K_0^{\mathrm{gr}}(A)$ is a pre-ordered abelian group with the set of isomorphism classes of graded finitely generated projective right A-modules as the cone of ordering, denoted by $K_0^{\mathrm{gr}}(A)_+$ (i.e., the image of $\mathcal{V}^{\mathrm{gr}}(A)$ under the natural homomorphism $\mathcal{V}^{\mathrm{gr}}(A)\to K_0^{\mathrm{gr}}(A)$). Furthermore, the following shows that $[A]$ is an order-unit for the pre-ordered group $K_0^{\mathrm{gr}}(A)$. If $x\in K_0^{\mathrm{gr}}(A)$, then there are graded finitely generated projective modules P and P' such that $x=[P]-[P']$. But there is a graded module Q such that $P\oplus Q\cong A^n(\overline{\alpha})$, where $\overline{\alpha}=(\alpha_1,\dots,\alpha_n)$, $\alpha_i\in\Gamma$. Now

$$\big[A^n(\overline{\alpha})\big]-x=[P]+[Q]-[P]+\big[P'\big]=[Q]+\big[P'\big]=\big[Q\oplus P'\big]\in K_0^{\mathrm{gr}}(A)^{+}.$$

This shows that $\sum_{i=1}^{n}\alpha_i[A]=[A^n(\overline{\alpha})]\geqslant x$ which satisfies (3).

We call the triple $(K_0^{\mathrm{gr}}(A),K_0^{\mathrm{gr}}(A)_+,[A])$ the *graded dimension group* (see Section 5 for some background on dimension groups).

In [20] it was conjectured that the graded dimension group is a complete invariant for unital Leavitt path algebras. Namely, for finite graphs E and F, $\mathcal{L}(E)\cong_{\mathrm{gr}}\mathcal{L}(F)$ if and only if there is an order preserving $\mathbb{Z}[x,x^{-1}]$-module isomorphism

$$\phi:K_0^{\mathrm{gr}}\big(\mathcal{L}(E)\big)\to K_0^{\mathrm{gr}}\big(\mathcal{L}(F)\big) \tag{4}$$

such that $\phi([\mathcal{L}(E)])=\mathcal{L}(F)$. We denote the existence of such an order isomorphism by

$$\big(K_0^{\mathrm{gr}}\big(\mathcal{L}(E)\big),K_0^{\mathrm{gr}}\big(\mathcal{L}(E)\big)_+,\big[\mathcal{L}(E)\big]\big)\cong\big(K_0^{\mathrm{gr}}\big(\mathcal{L}(F)\big),K_0^{\mathrm{gr}}\big(\mathcal{L}(F)\big)_+,\big[\mathcal{L}(F)\big]\big).$$

It was shown in [20] that the conjecture is valid for the so-called polycephaly graphs, which include acyclic, comets and multi-headed graphs.

4.2. Let A be a strongly Γ-graded ring. By Dade's theorem (see [13, Theorem 2.8] and [27, Theorem 3.1.1]), the functor $(-)_0 : \text{Gr-}A \to \text{Mod-}A_0$, $M \mapsto M_0$, is an additive functor with an inverse $- \otimes_{A_0} A : \text{Mod-}A_0 \to \text{Gr-}A$ so that it induces an equivalence of categories. This implies that $K_0^{\mathrm{gr}}(A) \cong K_0(A_0)$, where A_0 is the ring of homogeneous elements of degree zero. Furthermore, under this isomorphism, $K_0^{\mathrm{gr}}(A)_+$ maps onto $K_0(A_0)_+$ and $[A]$ to $[A_0]$.

In the case of Leavitt path algebras, combining Dade's theorem with Theorem 6, we get the following corollary.

Corollary 7. *Let E be a finite graph with no sinks. Then*

(1) $\text{Gr-}\mathcal{L}(E)$ *is equivalent to* $\text{Mod-}\mathcal{L}(E)_0$ *via the functor* $M \mapsto M_0$;
(2) *the equivalence in* (1) *induces an isomorphism of pre-ordered groups*

$$\big(K_0^{\mathrm{gr}}\big(\mathcal{L}(E)\big), K_0^{\mathrm{gr}}\big(\mathcal{L}(E)\big)_+, \big[\mathcal{L}(E)\big]\big) \cong \big(K_0\big(\mathcal{L}(E)_0\big), K_0\big(\mathcal{L}(E)_0\big)_+, \big[\mathcal{L}(E)_0\big]\big).$$

Since the ring $\mathcal{L}(E)_0$ of homogeneous part of degree zero is an ultramatricial K-algebra (see Section 4.3), the above corollary coupled with Goodearl–Handelman version [16] of Elliott's theorem [14] below gives us a tool for the classification of Leavitt path algebras.

Theorem 8. *Let A and B be ultramatricial K-algebras. Then*

(1) *A is Morita equivalent to B if and only if $K_0(A) \cong K_0(B)$ are pre-ordered abelian groups*;
(2) *A is isomorphic to B as a K-algebra if and only if*

$$\big(K_0(A), K_0(A)_+, [A]\big) \cong \big(K_0(B), K_0(B)_+, [B]\big).$$

4.3. For the Leavitt path algebra $\mathcal{L}(E)$, the structure of the ring of homogeneous elements of degree zero, $\mathcal{L}(E)_0$, is known and can be represented by a stationary Bratteli diagram. Ordering the vertices of the finite graph E, say, $\{u_1, u_2, \dots, u_n\}$, then there are $A_E(i,j)$-lines connecting u_i from the one row of the Bratteli diagram to the u_j of the next row. Here A_E is the adjacency matrix of E. Since we need to calculate $K_0(\mathcal{L}(E)_0)$, we recall the description of $\mathcal{L}(E)_0$ in the setting of finite graphs with no sinks (see the proof of Theorem 5.3 in [6] which is inspired by [12, Proposition 2.3] in the setting of graph C^*-algebras). Let $L_{0,n}$ be the linear span of all elements of the form pq^* with $r(p) = r(q)$ and $|p| = |q| \leqslant n$. Then

$$\mathcal{L}(E)_0 = \bigcup_{n=0}^{\infty} L_{0,n}, \tag{5}$$

where the transition inclusion $L_{0,n} \to L_{0,n+1}$ is to identify pq^* with $r(p) = v$ by

$$\sum_{\{\alpha \mid s(\alpha)=v\}} p\alpha(q\alpha)^*.$$

Note that since E does not have sinks, for any $v \in E_0$ the set $\{\alpha \mid s(\alpha) = v\}$ is not empty.

For a fixed $v \in E^0$, let $L_{0,n}^v$ be the linear span of all elements of the form pq^* with $|p| = |q| = n$ and $r(p) = r(q) = v$. Arrange the paths of length n with the range v in a fixed order $p_1^v, p_2^v, \dots, p_{k_n^v}^v$, and observe that the correspondence of $p_i^v p_j^{v*}$ to the matrix unit e_{ij} gives rise to a ring isomorphism $L_{0,n}^v \cong \mathbb{M}_{k_n^v}(K)$. Furthermore, $L_{0,n}^v$, $v \in E^0$, form a direct sum. This implies that

$$L_{0,n} \cong \bigoplus_{v \in E^0} \mathbb{M}_{k_n^v}(K),$$

where k_n^v, $v \in E^0$, is the number of paths of length n with the range v. The inclusion map $L_{0,n} \to L_{0,n+1}$ is

$$A_E^t : \bigoplus_{v \in E^0} \mathbb{M}_{k_n^v}(K) \longrightarrow \bigoplus_{v \in E^0} \mathbb{M}_{k_{n+1}^v}(K). \tag{6}$$

This means $(A_1, \dots, A_l) \in \bigoplus_{v \in E^0} \mathbb{M}_{k_n^v}(K)$ is sent to

$$\left(\sum_{j=1}^{l} n_{j1} A_j, \dots, \sum_{j=1}^{l} n_{jl} A_j \right) \in \bigoplus_{v \in E^0} \mathbb{M}_{k_{n+1}^v}(K),$$

where n_{ji} is the number of edges connecting v_j to v_i and

$$\sum_{j=1}^{l} k_j A_j = \begin{pmatrix} A_1 & & & \\ & A_1 & & \\ & & \ddots & \\ & & & A_l \end{pmatrix}$$

in which each matrix is repeated k_j times down the leading diagonal and if $k_j = 0$, then A_j is omitted.

Writing $\mathcal{L}(E)_0 = \varinjlim_n L_{0,n}$, since the Grothendieck group K_0 respects the direct limit, we have $K_0(\mathcal{L}(E)_0) \cong \varinjlim_n K_0(L_{0,n})$. Since K_0 of (Artinian) simple algebras are $\mathbb{Z}$, the ring homomorphism $L_{0,n} \to L_{0,n+1}$ induces the group homomorphism

$$\mathbb{Z}^{E^0} \xrightarrow{A_E^t} \mathbb{Z}^{E^0},$$

where $A_E^t : \mathbb{Z}^{E^0} \to \mathbb{Z}^{E^0}$ is multiplication from left which is induced by the homomorphism (6).

For a finite graph E with no sinks, with n vertices and the adjacency matrix A, by Theorem 6, $K_0^{\mathrm{gr}}(\mathcal{L}(E)) \cong K_0(\mathcal{L}(E)_0)$. Thus $K_0^{\mathrm{gr}}(\mathcal{L}(E))$ is the direct limit of the ordered direct system

$$\mathbb{Z}^n \xrightarrow{A^t} \mathbb{Z}^n \xrightarrow{A^t} \mathbb{Z}^n \xrightarrow{A^t} \cdots, \tag{7}$$

where the ordering in $\mathbb{Z}^n$ is defined point-wise (i.e., the positive cone is $\mathbb{N}^n$).

4.4. Since for a finite graph with no sinks, the graded Grothendieck group of its associated Leavitt path algebra is the direct limit of the form (7), here we recall two different presentations for the direct limit of abelian groups. This will be used in Example 10 and Lemma 11.

In Section 4.3 it was shown that for a finite graph with no sinks its graded Grothendieck group is a direct limit of a direct system of ordered free abelian groups with a matrix A (the transpose of the adjacency matrix of the graph) acting as an order preserving group homomorphism (from the left) as follows

$$\mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^n \xrightarrow{A} \cdots, \tag{8}$$

where the ordering in $\mathbb{Z}^n$ is defined point-wise. The direct limit of this system, $\varinjlim_A \mathbb{Z}^n$, is an ordered group and can be described as follows. Consider the pair (a, k), where $a \in \mathbb{Z}^n$ and $k \in \mathbb{N}$, and define the equivalence relation $(a, k) \sim (b, k')$ if $A^{k''-k} a = A^{k''-k'} b$ for some $k'' \in \mathbb{N}$. Let $[a, k]$ denote the equivalence class of (a,k). Clearly $[A^n a, n+k]=[a,k]$. Then it is not difficult to show that the direct limit of the system (8) is the abelian group consisting of equivalence classes $[a,k]$, $a\in\mathbb{Z}^n$, $k\in\mathbb{N}$, with addition defined by

$$[a,k]+\big[b,k'\big]=\big[A^{k'}a+A^k b, k+k'\big].$$

The positive cone of this ordered group is the set of elements $[a,k]$, where $a\in{\mathbb{Z}^+}^n$, $k\in\mathbb{N}$. Furthermore, there is automorphism $\delta_A:\varinjlim_A \mathbb{Z}^n\to\varinjlim_A \mathbb{Z}^n$ defined by $\delta_A([a,k])=[Aa,k]$.

There is another presentation for $\varinjlim_A \mathbb{Z}^n$ which is sometimes easier to work with. Consider the set

$$\Delta_A=\big\{v\in A^n\mathbb{Q}^n \bigm| A^k v\in\mathbb{Z}^n,\ \text{for some } k\in\mathbb{N}\big\}. \tag{9}$$

The set Δ_A forms an ordered abelian group with the usual addition of vectors and the positive cone

$$\Delta_A^+=\big\{v\in A^n\mathbb{Q}^n \bigm| A^k v\in{\mathbb{Z}^+}^n,\ \text{for some } k\in\mathbb{N}\big\}. \tag{10}$$

Furthermore, there is automorphism $\delta_A:\Delta_A\to\Delta_A$ defined by $\delta_A(v)=Av$. The map

$$\begin{aligned}\phi:\Delta_A &\longrightarrow \varinjlim_A \mathbb{Z}^n\\ v&\longmapsto \big[A^k v,k\big],\end{aligned} \tag{11}$$

where $k\in\mathbb{N}$ such that $A^k v\in\mathbb{Z}^n$, is an isomorphism which respects the action of A and the ordering, i.e., $\phi(\Delta_A^+)=(\varinjlim_A \mathbb{Z}^n)^+$ and $\phi(\delta_A(v))=\delta_A\phi(v)$.

In the case of finite graphs with no sinks, there is a good description of the action of the group on the graded Grothendieck group which we recall here.

Lemma 9. *Let E be a finite graph with no sinks and let $\mathcal{A}=\mathcal{L}(E)$.*

(1) *Any graded finitely generated projective right $\mathcal{A}$-module is generated by $u\mathcal{A}(i)$ up to isomorphism, where $u\in E^0$ and $i\in\mathbb{Z}$, i.e.,*

$$\mathcal{V}^{\mathrm{gr}}(\mathcal{A})=\big\langle\big[u\mathcal{A}(i)\big]\bigm| u\in E^0,\ i\in\mathbb{Z}\big\rangle.$$

(2) *For $i\geqslant 0$,*

$$x\big[u\mathcal{A}(i)\big]=\big[u\mathcal{A}(i+1)\big]=\sum_{\{\alpha\in E^1|s(\alpha)=u\}}\big[r(\alpha)\mathcal{A}(i)\big]. \tag{12}$$

Proof. (1) We first observe that for $\alpha\in E^1$ we have

$$\begin{aligned}\alpha\mathcal{A}&\cong_{\mathrm{gr}} r(\alpha)\mathcal{A},\\ \alpha^*\mathcal{A}&\cong_{\mathrm{gr}} r(\alpha)\mathcal{A}(-1).\end{aligned} \tag{13}$$

For the first isomorphism, consider the right $\mathcal{A}$-module homomorphism

$$\theta : \alpha\mathcal{A} \longrightarrow r(\alpha)\mathcal{A}$$
$$\alpha a \longmapsto \alpha^*\alpha a.$$

Clearly θ is an isomorphism. Furthermore, for $n \in \mathbb{Z}$, and $\alpha a \in (\alpha\mathcal{A})_n = \alpha\mathcal{A}_n$, a is homogeneous of degree n. So $\theta(\alpha a) = \alpha^*\alpha a = r(\alpha)a \in r(\alpha)\mathcal{A}_n = (r(\alpha)\mathcal{A})_n$. This shows that θ is a graded $\mathcal{A}$-module isomorphism.

For the second isomorphism in (13), consider the identity map

$$\alpha^*\mathcal{A} \longrightarrow r(\alpha)\mathcal{A}(-1)$$
$$\alpha^* a \longmapsto r(\alpha)\alpha^* a.$$

One can see that this map respects the grading and this gives the graded isomorphism between the two modules. Putting the two isomorphisms of (13) together, one can easily see that, for two paths α and β with $r(\alpha) = r(\beta)$ one has

$$\alpha\beta^*\mathcal{A} \cong_{\mathrm{gr}} r(\beta)\mathcal{A}\bigl(-|\beta|\bigr). \tag{14}$$

Now let P be a graded finitely generated projective $\mathcal{A}$-module. Since $\mathcal{A}$ is strongly graded (Theorem 6), P_0 is a finitely generated projective $\mathcal{A}_0$-module such that $P_0 \otimes_{\mathcal{A}_0} \mathcal{A} \cong_{\mathrm{gr}} P$ (see Section 4.2). On the other hand since $\mathcal{A}_0$ is an ultramatricial algebra (see (5)), P_0 extends from a finitely generated projective module of a ring in the direct limit of $\mathcal{A}_0$, i.e., there is a finitely generated projective $L_{0,n}$ module P' such that $P' \otimes_{L_{0,n}} \mathcal{A}_0 = P_0$ (see [16, Lemma 15.10]). On the other hand since $L_{0,n}$ is a semisimple algebra, P' is generated by the projective modules $pq^*L_{0,n}$ for paths p and q of length n with $r(p) = r(q)$. Then by (14)

$$\bigl(pq^*L_{0,n} \otimes_{L_{0,n}} \mathcal{A}_0\bigr) \otimes_{\mathcal{A}_0} \mathcal{A} \cong_{\mathrm{gr}} pq^*\mathcal{A} \cong_{\mathrm{gr}} r(q)\mathcal{A}(-n).$$

This shows that, up to isomorphism, P is generated by $u\mathcal{A}(i)$, where $u \in E^0$ and $i \in \mathbb{Z}$.

(2) First notice that for $i \geqslant 0$,

$$\mathcal{A}_{i+1} = \sum_{\alpha \in E^1} \alpha\mathcal{A}_i.$$

It follows

$$u\mathcal{A}_{i+1} = \bigoplus_{\{\alpha \in E^1 \mid s(\alpha) = u\}} \alpha\mathcal{A}_i$$

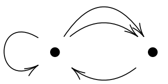
as $\mathcal{A}_0$-modules. Using the fact that $\mathcal{A}_n \otimes_{\mathcal{A}_0} \mathcal{A} \cong \mathcal{A}(n)$, $n \in \mathbb{Z}$, and the fact that $\alpha\mathcal{A}_i \cong r(\alpha)\mathcal{A}_i$ as $\mathcal{A}_0$-module, we get

$$u\mathcal{A}(i+1) \cong \bigoplus_{\{\alpha \in E^1 \mid s(\alpha) = u\}} r(\alpha)\mathcal{A}(i)$$

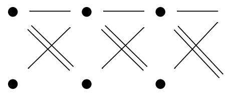
as graded $\mathcal{A}$-modules. This gives (12). $\square$

Recall that $K_0^{\mathrm{gr}}(\mathcal{A})$ is the group completion of $\mathcal{V}^{\mathrm{gr}}(\mathcal{A})$. The action of $\mathbb{N}[x, x^{-1}]$ on $\mathcal{V}^{\mathrm{gr}}(\mathcal{A})$ and thus the action of $\mathbb{Z}[x, x^{-1}]$ on $K_0^{\mathrm{gr}}(\mathcal{A})$ is defined on generators by $x^j[u\mathcal{A}(i)] = [u\mathcal{A}(i+j)]$, where $i, j \in \mathbb{Z}$.

Example 10. For the graph

$E:$ 

with the adjacency $A_E = \left(\begin{smallmatrix} 1 & 2 \\ 1 & 0 \end{smallmatrix}\right)$, the Bratteli diagram associated to $\mathcal{L}(E)_0$ is



and $\mathcal{L}(E)_0$ is the direct limit of the system

$$K \oplus K \xrightarrow{A_E^t} \mathbb{M}_2(K) \oplus \mathbb{M}_2(K) \xrightarrow{A_E^t} \mathbb{M}_4(K) \oplus \mathbb{M}_4(K) \xrightarrow{A_E^t} \cdots$$
$$(a,b) \mapsto \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \oplus \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

So $K_0^{\mathrm{gr}}(\mathcal{L}(E))$ is the direct limit of the direct system

$$\mathbb{Z}^2 \xrightarrow{A_E^t} \mathbb{Z}^2 \xrightarrow{A_E^t} \mathbb{Z}^2 \xrightarrow{A_E^t} \cdots.$$

Since $\det(A_E^t) = -2$, one can easily calculate that

$$K_0^{\mathrm{gr}}\big(\mathcal{L}(E)\big) \cong \mathbb{Z}[1/2] \oplus \mathbb{Z}[1/2].$$

Furthermore $[\mathcal{L}(E)] \in K_0^{\mathrm{gr}}(\mathcal{L}(E))$ is represented by $(1,1) \in \mathbb{Z}[1/2] \oplus \mathbb{Z}[1/2]$. Adopting (9) for the description of $K_0^{\mathrm{gr}}(\mathcal{L}(E))$, since the action of x on $K_0^{\mathrm{gr}}(\mathcal{L}(E))$ represented by action of A_E^t from the left, we have $x(a,b) = (a+b, 2a)$. Furthermore, considering (10) for the positive cone, ${A_E^t}^k(a,b)$ is eventually positive, if $v(a,b) > 0$, where $v = (2,1)$ is the Perron eigenvector of A_E (see [26, Lemma 7.3.8]). It follows that

$$K_0^{\mathrm{gr}}\big(\mathcal{L}(E)\big)^+ = \Delta^+_{A_E^t} = \big\{(a,b) \in \mathbb{Z}[1/2] \oplus \mathbb{Z}[1/2] \bigm| 2a+b > 0\big\} \cup \big\{(0,0)\big\}.$$

5. Krieger's dimension groups and Wagoner's dimension modules

The Grothendieck group K_0 is a pre-ordered abelian group with the set of isomorphism classes of finitely generated projective modules as its positive cone. For the category of ultramatricial algebras, K_0 along with its positive cone and the position of the identity is a complete invariant (see [14] and [17, §15]). Motivated by this, Krieger in [24] defined an invariant for the irreducible shifts of finite type. In general, a nonnegative integral $n \times n$ matrix A gives rise to a stationary system. This in turn gives a direct system of ordered free abelian groups with A acting as an order preserving group homomorphism as follows

$$\mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^n \xrightarrow{A} \cdots,$$

where the ordering in $\mathbb{Z}^n$ is defined point-wise. The direct limit of this system, $\Delta_A := \varinjlim_A \mathbb{Z}^n$ (i.e., the K_0 of the stationary system), along with its positive cone, Δ^+, and the automorphism which is induced by multiplication by A on the direct limit, $\delta_A : \Delta_A \to \Delta_A$, is the invariant considered by Krieger, now known as Krieger's dimension group. Following [26], we denote this triple by $(\Delta_A, \Delta_A^+, \delta_A)$. It can be shown that two matrices A and B are shift equivalent (see Section 6) if and only if their associated Krieger's dimension groups are isomorphic ([24, Theorem 4.2] and [26, Theorem 7.5.8], see also [26, §7.5] for a detailed algebraic treatment). Wagoner noted that the induced structure on Δ_A by the automorphism δ_A makes Δ_A a $\mathbb{Z}[x,x^{-1}]$-module (where the action of δ_A is multiplication by x^{-1}, see Lemma 11) which was systematically used in [35,36] (see also [11, §3]).

The graded Grothendieck group of a $\mathbb{Z}$-graded ring has a natural $\mathbb{Z}[x,x^{-1}]$-module structure (see Section 4) and the following observation (Lemma 11) shows that the graded Grothendieck group of the Leavitt path algebra associated to a matrix A coincides with the Krieger dimension group of the shift of finite type associated to A^t, i.e., the graded dimension group of a Leavitt path algebra coincides with Krieger's dimension group,

$$\big(K_0^{\mathrm{gr}}\big(\mathcal{L}(E)\big), K_0^{\mathrm{gr}}\big(\mathcal{L}(E)\big)^+\big) \cong \big(\Delta_{A^t}, \Delta_{A^t}^+\big).$$

This will provide a link between the theory of Leavitt path algebras and symbolic dynamics. This lemma was also proved differently by Ara and Pardo in [5].

Lemma 11. *Let E be a finite graph with no sinks with the adjacency matrix A. Then there is an isomorphism $\phi : K_0^{\mathrm{gr}}(\mathcal{L}(E)) \to \Delta_{A^t}$ such that $\phi(x\alpha) = \delta_{A^t}\phi(\alpha)$, $\alpha \in \mathcal{L}(E)$, $x \in \mathbb{Z}[x,x^{-1}]$ and $\phi(K_0^{\mathrm{gr}}(\mathcal{L}(E))^+) = \Delta_{A^t}^+$.*

Proof. Since by Theorem 6, $\mathcal{L}(E)$ is strongly graded, there is an ordered isomorphism $K_0^{\mathrm{gr}}(\mathcal{L}(E)) \to K_0(\mathcal{L}(E)_0)$. Thus by Section 4.3, (see (7)) the ordered group $K_0^{\mathrm{gr}}(\mathcal{L}(E))$ coincides with the ordered group Δ_{A^t}. We only need to check that their module structures are compatible. It is enough to show that the action of x on K_0^{gr} coincides with the action of A^t on $K_0(\mathcal{L}(E)_0)$, i.e., $\phi(x\alpha) = \delta_{A^t}\phi(\alpha)$.

Set $\mathcal{A} = \mathcal{L}(E)$. Since graded finitely generated projective modules are generated by $u\mathcal{A}(i)$, where $u \in E^0$ and $i \in \mathbb{Z}$, it suffices to show that $\phi(x[u\mathcal{A}]) = \delta_{A^t}\phi([u\mathcal{A}])$. Since the image of $u\mathcal{A}$ in $K_0(\mathcal{A}_0)$ is $[u\mathcal{A}_0]$, and $\mathcal{A}_0 = \bigcup_{n=0}^{\infty} L_{0,n}$, (see (5)) using the presentation of K_0 given in Section 4.4, we have

$$\phi\big([u\mathcal{A}]\big) = [u\mathcal{A}_0] = [uL_{0,0}, 1] = [u, 1].$$

Thus

$$\delta_{A^t}\phi\big([u\mathcal{A}]\big) = \delta_{A^t}\big([u,1]\big) = \big[A^t u, 1\big] = \sum_{\{\alpha \in E^1 \mid s(\alpha) = u\}} \big[r(\alpha), 1\big].$$

On the other hand,

$$\begin{aligned}\phi\big(x[u\mathcal{A}]\big) = \phi\big(\big[u\mathcal{A}(1)\big]\big) &= \phi\bigg(\sum_{\{\alpha \in E^1 \mid s(\alpha) = u\}} \big[r(\alpha)\mathcal{A}\big]\bigg) = \sum_{\{\alpha \in E^1 \mid s(\alpha) = u\}} \big[r(\alpha)\mathcal{A}_0\big] \\ &= \sum_{\{\alpha \in E^1 \mid s(\alpha) = u\}} \big[r(\alpha)L_{0,0}, 1\big] = \sum_{\{\alpha \in E^1 \mid s(\alpha) = u\}} \big[r(\alpha), 1\big]. \end{aligned} \tag{15}$$

Thus $\phi(x[u\mathcal{A}]) = \delta_{A^t}\phi([u\mathcal{A}])$. This finishes the proof. □

It is easy to see that two matrices A and B are shift equivalent if and only if A^t and B^t are shift equivalent. Combining this with Lemma 11 and the fact that Krieger's dimension group is a complete invariant for shift equivalent we have achieved our first main result.

Corollary 12. *Let E and F be finite graphs with no sinks and A_E and A_F be their adjacency matrices, respectively. Then A_E is shift equivalent to A_F if and only if there is an order preserving $\mathbb{Z}[x, x^{-1}]$-module isomorphism $K_0^{\mathrm{gr}}(\mathcal{L}(E)) \cong K_0^{\mathrm{gr}}(\mathcal{L}(F))$.*

6. Conjugacy, shift equivalence and graded Morita equivalence

6.1. The notion of the shift equivalence for matrices was introduced by Williams [37] (see also [26, §7]) in an attempt to provide a computable machinery for determining the conjugacy between two shifts of finite type. Recall that two square nonnegative integer matrices A and B are called *elementary shift equivalent*, and denoted by $A \sim_{ES} B$, if there are nonnegative matrices R and S such that $A = RS$ and $B = SR$. The equivalence relation $\sim_S$ on square nonnegative integer matrices generated by elementary shift equivalence is called *strong shift equivalence*. The weaker notion of shift equivalence is defined as follows. The nonnegative integer matrices A and B are called *shift equivalent* if there are nonnegative matrices R and S such that $A^l = RS$ and $B^l = SR$, for some $l \in \mathbb{N}$, and $AR = RB$ and $SA = BS$.

6.2. We recall the type of the graphs which are of interest in symbolic dynamics. It turns out the Leavitt path algebras associated to this class of graphs are very interesting algebras (i.e., purely infinite simple unital algebras [2]).

Let E be a finite directed graph. Then E is *irreducible* if given any two vertices v and w in E, there is a path from v to w. E is called *essential* if there are neither sources nor sinks in E, and E is *trivial* if E consists of a single cycle with no other vertices or edges (see [26, Definition 2.2.13] and [15]). A set of graphs which is simultaneously irreducible, essential, and nontrivial is of great interest in the theory of shifts of finite type. Indeed, an edge that begins at a source or ends at a sink does not appear in any bi-infinite path so the only part of an arbitrary finite graph E which appears in symbolic dynamic is the graph with no sources and sinks which is obtained by repeatedly removing all the sources and sinks from E (see [26, the remark after Example 2.2.8 and Proposition 2.2.10]). The following result connects this class of graphs to a very interesting class of Leavitt path algebras. Let E be a finite graph. Then E is irreducible, nontrivial, and essential if and only if E contains no sources, and $\mathcal{L}(E)$ is purely infinite simple (see [4, Lemma 1.17]).

Starting from a graph E, and a partition $\mathcal{P}$ of the edges, one can obtain new graphs, called *out-splitting*, denoted by $E_s(\mathcal{P})$, and *in-splitting*, denoted by $E_r(\mathcal{P})$. The converse of these processes are called *out-amalgamation* and *in-amalgamation* (see [26, §2.4] and [4, Definitions 1.9 and 1.12]). Furthermore, when the graph has a source, say v, there is a source elimination graph $E_{\setminus v}$, which is obtained by removing v and all the edges emitting from v from E (see [4, Definition 1.2]).

The following observation will be used in Proposition 15. Let E be a finite graph, let $v \in E^0$, and let $\mathcal{P}$ be a partition of the edges of E. Then E is essential (resp. nontrivial, resp. irreducible) if and only if $E_s(\mathcal{P})$, $E_r(\mathcal{P})$, and $E_{\setminus v}$ are each essential (resp. nontrivial, resp. irreducible) (see [4, Lemma 1.16]).

In [4, Proposition 1.4] it was shown that for a finite graph E such that $\mathcal{L}(E)$ is simple, removing a source vertex would not change the category of the corresponding Leavitt path algebra up to the Morita equivalence. We need a similar result in the graded setting without an extra assumption of simplicity (see [4, Proposition 3.1] for the general case in the nongraded setting). Thanks to Theorem 3, once this is proved it gives the nongraded statement naturally.

Proposition 13. *Let E be a finite graph with no sinks and at least two vertices. Let $v \in E^0$ be a source. Then $\mathcal{L}(E_{\setminus v})$ is graded Morita equivalent to $\mathcal{L}(E)$. Consequently $\mathcal{L}(E_{\setminus v})$ is Morita equivalent to $\mathcal{L}(E)$.*

Proof. Since $E_{\setminus v}$ is a complete subgraph of E, there is a (non-unital) graded algebra homomorphism $\phi : \mathcal{L}(E_{\setminus v}) \to \mathcal{L}(E)$, such that $\phi(u) = u$, $\phi(e) = e$ and $\phi(e^*) = e^*$, where $u \in E^0 \backslash \{v\}$ and $e \in E^1$. The graded uniqueness theorem [34, Theorem 4.8] implies ϕ is injective. Thus $\mathcal{L}(E_{\setminus v}) \cong_{\mathrm{gr}} \phi(\mathcal{L}(E_{\setminus v}))$. It is not difficult to see that $\phi(\mathcal{L}(E_{\setminus v})) = p\mathcal{L}(E)p$, where $p = \sum_{u \in E^0_{\setminus v}} u$. This immediately implies that $\mathcal{L}(E_{\setminus v})$ is graded Morita equivalent to $p\mathcal{L}(E)p$. On the other hand, the (graded) ideal generated by $p = \sum_{u \in E^0_{\setminus v}} u$ coincides with the (graded) ideal generated by $\{u \mid u \in E^0_{\setminus v}\}$. But the smallest hereditary and saturated subset of E^0 containing $E^0_{\setminus v}$ is E^0. Thus by [6, Theorem 5.3] the ideal generated by $\{u \mid u \in E^0_{\setminus v}\}$ is $\mathcal{L}(E)$. This shows p is a full homogeneous idempotent in $\mathcal{L}(E)$. Thus $p\mathcal{L}(E)p$ is graded Morita equivalent to $\mathcal{L}(E)$ (see Example 2). Putting these together we get $\mathcal{L}(E_{\setminus v})$ is graded Morita equivalent to $\mathcal{L}(E)$. The last part of the proposition follows from the Green–Gordon theorem (see Section 2). $\square$

Remark 14. The Morita equivalence of Proposition 13 induces an isomorphism between $K_0^{\mathrm{gr}}(\mathcal{L}(E_{\setminus v}))$ and $K_0^{\mathrm{gr}}(\mathcal{L}(E))$. However this isomorphism does not take $[\mathcal{L}(E_{\setminus v})]$ to $[\mathcal{L}(E)]$.

We are in a position to relate the notion of (strongly) shift equivalent in matrices with the graded Morita theory of Leavitt path algebras associated to those matrices, so that we achieve our second main result.

Proposition 15.

(1) *Let E be an essential graph and F be a graph obtained from an in-splitting or out-splitting of the graph E. Then $\mathcal{L}(E)$ is graded Morita equivalent to $\mathcal{L}(F)$.*
(2) *For essential graphs E and F, if the adjacency matrices A_E and A_F are strongly shift equivalent then $\mathcal{L}(E)$ is graded Morita equivalent to $\mathcal{L}(F)$.*
(3) *For graphs E and F with no sinks, if $\mathcal{L}(E)$ is graded Morita equivalent to $\mathcal{L}(F)$, then the adjacency matrices A_E and A_F are shift equivalent.*

Proof. (1) First suppose E is an essential graph and $E_r(\mathcal{P})$ the in-split graph from E using a partition $\mathcal{P}$. For each $v \in E^0$, define $Q_v = v_1$, which exists by the assumption that E has no sources. For $e \in \mathcal{E}_i^v$, define $T_e = \sum_{f \in s^{-1}(v)} e_1 f_i f_1^*$ and $T_e^* = \sum_{f \in s^{-1}(v)} f_1 f_i^* e_1^*$. In [4, Proposition 1.11], it was proved that $\{Q_v, T_e, T_e^* \mid v \in E^0,\ e \in E^1\}$ is an E-family which in turn induces a K-algebra homomorphism

$$\begin{aligned} \pi : \mathcal{L}(E) &\longrightarrow \mathcal{L}\big(E_r(\mathcal{P})\big), \\ v &\longmapsto Q_v = v_1, \\ e &\longmapsto T_e = \sum_{f \in s^{-1}(v)} e_1 f_i f_1^*, \\ e^* &\longmapsto T_e^* = \sum_{f \in s^{-1}(v)} f_1 f_i^* e_1^*. \end{aligned} \tag{16}$$

Furthermore, it was shown that $\pi(\mathcal{L}(E)) = p\mathcal{L}(E_r(\mathcal{P}))p$ where $p = \pi(1_{\mathcal{L}(E)}) = \sum_{v \in E^0} v_1$.

Since $\pi(v) = v_1 \neq 0$ (see [16, Lemma 1.5]), the graded uniqueness theorem [34, Theorem 4.8] implies π is injective. Furthermore (16) shows that π is a graded map. Thus

$$\mathcal{L}(E) \cong_{\mathrm{gr}} p\mathcal{L}\big(E_r(\mathcal{P})\big)p.$$

We will show that p is a full idempotent in $\mathcal{L}(E_r(\mathcal{P}))$. The (graded) ideal generated by $p = \sum_{v \in E^0} v_1$ coincides with the (graded) ideal generated by $\{v_1 \mid v \in E^0\}$. But the smallest hereditary and saturated subset of $E_r(\mathcal{P})^0$ containing $\{v_1 \mid v \in E^0\}$ is $E_r(\mathcal{P})^0$. Thus by [6, Theorem 5.3] the ideal generated by $\{v_1 \mid v \in E^0\}$ is $\mathcal{L}(E_r(\mathcal{P}))$. This shows p is a full homogeneous idempotent in $\mathcal{L}(E)$. Now in Theorem 3(4) by setting $e = p$, $n = 1$, $B = \mathcal{L}(E)$ and $A = \mathcal{L}(E_r(\mathcal{P}))$, we get that $\mathcal{L}(E)$ is graded Morita equivalent to $\mathcal{L}(E_r(\mathcal{P}))$.

On the other hand, if $E_s(\mathcal{P})$ is the out-split graph from E using a partition $\mathcal{P}$, then by [3, Theorem 2.8], there is a graded K-algebra isomorphism $\pi : \mathcal{L}(E) \to \mathcal{L}(E_s(\mathcal{P}))$. Again, Theorem 3(4) implies that $\mathcal{L}(E)$ is graded Morita equivalent to $\mathcal{L}(E_s(\mathcal{P}))$.

(2) If A_E is strongly shift equivalent to A_F, a combination of the Williams theorem [26, Theorem 7.2.7] and the Decomposition theorem [26, Theorem 7.1.2, Corollary 7.1.5] implies that the graph F can be obtained from E by a sequence of out-splittings, in-splittings and the inverses of these, namely, out-amalgamations and in-amalgamation. All the graphs which appear in this sequence are essential (see Section 6.2). Now a repeated application of part (1) gives that $\mathcal{L}(E)$ is graded Morita equivalent to $\mathcal{L}(F)$.

(3) Since $\text{Gr-}\mathcal{L}(E) \approx_{\text{gr}} \text{Gr-}\mathcal{L}(F)$, there is an order preserving $\mathbb{Z}[x, x^{-1}]$-module isomorphism $K_0^{\text{gr}}(\mathcal{L}(E)) \cong_{\text{gr}} K_0^{\text{gr}}(\mathcal{L}(F))$ (see Section 2). Thus by Corollary 12, A_E and A_F are shift equivalent. □

Remark 16. Proposition 15(3) shows that if $\mathcal{L}(E) \approx_{\text{gr}} \mathcal{L}(F)$ then the adjacency matrices of E and F are shift equivalent. One thinks that the converse of this statement is also valid. In fact, if A_E is shift equivalent to A_F then by Corollary 12, there is an order preserving $\mathbb{Z}[x, x^{-1}]$-module isomorphism $K_0^{\text{gr}}(\mathcal{L}(E)) \cong K_0^{\text{gr}}(\mathcal{L}(F))$. Since $\mathcal{L}(E)$ and $\mathcal{L}(F)$ are strongly graded, this implies $K_0(\mathcal{L}(E)_0) \cong K_0(\mathcal{L}(F)_0)$ as partially ordered abelian groups. Since $\mathcal{L}(E)_0$ and $\mathcal{L}(F)_0$ are ultramatricial algebras, by [17, Corollary 15.27] $\mathcal{L}(E)_0$ is Morita equivalent to $\mathcal{L}(F)_0$. Now the following diagram shows that $\text{Gr-}\mathcal{L}(E)$ is equivalent to $\text{Gr-}\mathcal{L}(F)$:

$$\begin{tikzcd}
\text{Mod-}\mathcal{L}(E)_0 \arrow[r] \arrow[d, "-\otimes\mathcal{L}(E)"'] & \text{Mod-}\mathcal{L}(F)_0 \arrow[d, "-\otimes\mathcal{L}(F)"] \\
\text{Gr-}\mathcal{L}(E) \arrow[r] & \text{Gr-}\mathcal{L}(F).
\end{tikzcd}$$

However it is not clear whether this equivalence is graded as in Definition 1(1).

Recall that for a graph, the associated Leavitt path algebra is purely infinite simple unital, if and only if the graph is finite, any vertex is connected to a cycle and any cycle has an exit (see [2] and [4, p. 205]). Note that for a finite graph, the condition of not having a sink is equivalent to any vertex be connected to a cycle. Thus by Theorem 6, purely infinite simple unital Leavitt path algebras are strongly graded.

Example 17 *(Purely infinite simple and its transpose are not graded Morita equivalent).* We have seen most of the results already proved in the literature on Morita equivalence, such as in-splitting, out-splitting, and removing of the sources can be extended to a stronger graded Morita equivalence. However this is not always the case. In [4, Proposition 3.10], it was shown that for a finite graph E without sources such that $\mathcal{L}(E)$ is purely infinite simple, $\mathcal{L}(E)$ and $\mathcal{L}(E^{\text{op}})$ are Morita equivalent. (Here E^{op} is the *opposite* or *transpose* of the graph E, i.e., E^{op} is obtained from E by reversing the arrows, so $A_E^t = A_{E^{\text{op}}}$. In [4], E^{op} is denoted by E^t.) However there are examples of the graph E such that $\mathcal{L}(E)$ and $\mathcal{L}(E^{\text{op}})$ are not graded Morita equivalent. Consider the graph E with the adjacency matrix

$$A_E = \begin{pmatrix} 19 & 5 \\ 4 & 1 \end{pmatrix}.$$

The Leavitt path algebra $\mathcal{L}(E)$ is a purely infinite simple unital algebra with no sources. If $\mathcal{L}(E)$ is graded Morita equivalent to $\mathcal{L}(E^{\text{op}})$, by Proposition 15(3), A_E and $A_{E^{\text{op}}} = A_E^t$ are shift equivalent. However, it is known that A_E and A_E^t are not shift equivalent (see [26, Example 7.4.19]).

Example 18 *(Strongly shift equivalence does not imply isomorphism).* By Proposition 15 if two essential graphs are strongly shift equivalent then their associated Leavitt path algebras are graded Morita equivalent. The following example shows that the strongly shift equivalence does not however imply that the Leavitt path algebras are isomorphic.

Consider the graphs

$$E: \begin{tikzcd} \bullet \arrow[loop left] \arrow[r, bend left] \arrow[r, bend left=50] & \bullet \arrow[l, bend left] \end{tikzcd} \qquad E^{\mathrm{op}}: \begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend left=50] & \bullet \arrow[l, bend left] \arrow[loop right] \end{tikzcd} \tag{17}$$

The adjacency matrices of E and E^{op} are strongly shift equivalent as the following computation shows. Thus by Proposition 15, $\mathcal{L}(E) \approx_{\mathrm{gr}} \mathcal{L}(E^{\mathrm{op}})$. However we will show $\mathcal{L}(E) \not\cong_{\mathrm{gr}} \mathcal{L}(E^{\mathrm{op}})$.

First, $A_E = \left(\begin{smallmatrix} 1 & 2 \\ 1 & 0 \end{smallmatrix}\right)$ and $A_{E^{\mathrm{op}}} = \left(\begin{smallmatrix} 1 & 1 \\ 2 & 0 \end{smallmatrix}\right)$. Notice that $A_E = A^t_{E^{\mathrm{op}}}$. Let $R_1 = \left(\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$ and $S_1 = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Then $A_E = R_1 S_1$ and set $E_1 := S_1 R_1 = \left(\begin{smallmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{smallmatrix}\right)$. Let $R_2 = \left(\begin{smallmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$ and $S_2 = \left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{smallmatrix}\right)$. Then $E_1 = R_2 S_2$ and set $E_2 := S_2 R_2 = \left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{smallmatrix}\right)$. Finally, let $R_3 = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ and $S_3 = \left(\begin{smallmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{smallmatrix}\right)$. Then $E_2 = R_3 S_3$ and $A_{E^{\mathrm{op}}} = S_3 R_3$. This shows

$$A_E \sim_{ES} E_1 \sim_{ES} E_2 \sim_{ES} A_{E^{\mathrm{op}}}.$$

Thus $A_E \sim_S A_{E^{\mathrm{op}}}$.

Now suppose $\mathcal{L}(E) \cong_{\mathrm{gr}} \mathcal{L}(E^{\mathrm{op}})$ which induces an order preserving $\mathbb{Z}[x, x^{-1}]$-module isomorphism $\phi : K_0^{\mathrm{gr}}(\mathcal{L}(E)) \to K_0^{\mathrm{gr}}(\mathcal{L}(E^{\mathrm{op}}))$, such that $\phi([\mathcal{L}(E)]) = [\mathcal{L}(E^{\mathrm{op}})]$. Since as abelian groups

$$K_0^{\mathrm{gr}}\big(\mathcal{L}(E)\big) \cong K_0^{\mathrm{gr}}\big(\mathcal{L}\big(E^{\mathrm{op}}\big)\big) \cong \mathbb{Z}[1/2] \oplus \mathbb{Z}[1/2],$$

ϕ is an invertible 2×2 matrix of the form $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, where $a, b, c, d \in \mathbb{Q}$. Furthermore, $[\mathcal{L}(E)] = (1, 1) \in \mathbb{Z}[1/2] \oplus \mathbb{Z}[1/2]$ and similarly, $[\mathcal{L}(E^{\mathrm{op}})] = (1, 1) \in \mathbb{Z}[1/2] \oplus \mathbb{Z}[1/2]$. Since $\phi([\mathcal{L}(E)]) = [\mathcal{L}(E^{\mathrm{op}})]$, using the matrix representation we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{18}$$

But then (see also Example 10)

$$\phi\big(A_{E^t}(1, 1)\big) = A_{E^{\mathrm{op}t}} \phi(1, 1) = A_{E^{\mathrm{op}t}}(1, 1).$$

Replacing the corresponding matrices into this equation we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{19}$$

Comparing (18) and (19) leads to a contradiction.

This example shows that although there is an order isomorphism $K_0^{\mathrm{gr}}(\mathcal{L}(E)) \cong K_0^{\mathrm{gr}}(\mathcal{L}(E^{\mathrm{op}}))$ as $\mathbb{Z}[x, x^{-1}]$-modules, but $\mathcal{L}(E) \not\cong_{\mathrm{gr}} \mathcal{L}(E^{\mathrm{op}})$. This implies that in the classification conjecture (see Section 4.1), the assumption of pointed isomorphisms can't be relaxed.

Theorem 21 below uses two main results of [4] which we recall here for the convenience of the reader.

Theorem 19. *(See [4].) Let E and F be finite graphs. If*

(1) *$\mathcal{L}(E)$ and $\mathcal{L}(F)$ are purely infinite simple rings, and*
(2) *$\det(I - A_E^t) = \det(I - A_F^t)$, and*
(3) *$K_0(\mathcal{L}(E)) \cong K_0(\mathcal{L}(F))$,*

then $\mathcal{L}(E)$ is Morita equivalent to $\mathcal{L}(F)$.

The next result depends on the work of Huang [23] on shifts of finite type.

Theorem 20. *(See [4].) Let E and F be finite graphs. If*

(1) *$\mathcal{L}(E)$ and $\mathcal{L}(F)$ are purely infinite simple rings, and*
(2) *$\mathcal{L}(E)$ is Morita equivalent to $\mathcal{L}(F)$, and*
(3) *$K_0(L(E)) \cong K_0(L(F))$ via an isomorphism which sends $[\mathcal{L}(E)]$ to $[\mathcal{L}(F)]$,*

then $\mathcal{L}(E)$ is isomorphic to $\mathcal{L}(F)$.

We are in a position to settle the graded conjecture (see Section 4.1 and [20, Conjecture 1]) for the case of purely infinite simple Leavitt path algebras up to grading. Namely, we are able to show that if the graded dimension groups are isomorphic, then the Leavitt path algebras are isomorphic. The theorem guarantees an isomorphism between the algebras, but not a graded isomorphism.

Theorem 21. *Let E and F be graphs such that $\mathcal{L}(E)$ and $\mathcal{L}(F)$ are purely infinite simple unital algebras. Then $\mathcal{L}(E) \cong \mathcal{L}(F)$ if there is an order preserving $\mathbb{Z}[x, x^{-1}]$-module isomorphism*

$$\big(K_0^{\mathrm{gr}}\big(\mathcal{L}(E)\big), K_0^{\mathrm{gr}}\big(\mathcal{L}(E)\big)_+, \big[\mathcal{L}(E)\big]\big) \cong \big(K_0^{\mathrm{gr}}\big(\mathcal{L}(F)\big), K_0^{\mathrm{gr}}\big(\mathcal{L}(F)\big)_+, \big[\mathcal{L}(F)\big]\big). \tag{20}$$

Proof. Eq. (20) gives an isomorphism $\phi : K_0^{\mathrm{gr}}(\mathcal{L}(E)) \to K_0^{\mathrm{gr}}(\mathcal{L}(F))$, where $[\mathcal{L}(E)]$ is sent to $[\mathcal{L}(F)]$. The main result of [21] shows that the diagram below is commutative, where U is the forgetful functor and T_1 is the suspension functor and the isomorphism ϕ_1 is induced from the commutativity of the left diagram (note that on the level of category of graded projective modules, $(T_1 - \mathrm{id})(P) = P(1) - P$, which induces an endomorphism on the graded Grothendieck group):

$$\begin{tikzcd}
K_0^{\mathrm{gr}}(\mathcal{L}(E)) \arrow[r, "T_1-\mathrm{id}"] \arrow[d, "\phi"] & K_0^{\mathrm{gr}}(\mathcal{L}(E)) \arrow[r, "U"] \arrow[d, "\phi"] & K_0(\mathcal{L}(E)) \arrow[r] \arrow[d, "\phi_1"] & 0 \\
K_0^{\mathrm{gr}}(\mathcal{L}(F)) \arrow[r, "T_1-\mathrm{id}"] & K_0^{\mathrm{gr}}(\mathcal{L}(F)) \arrow[r, "U"] & K_0(\mathcal{L}(F)) \arrow[r] & 0.
\end{tikzcd}$$

Thus we have an induced isomorphism

$$\begin{aligned} \phi_1 : K_0\big(\mathcal{L}(E)\big) &\longrightarrow K_0\big(\mathcal{L}(F)\big) \\ \big[\mathcal{L}(E)\big] &\longmapsto \big[\mathcal{L}(F)\big]. \end{aligned} \tag{21}$$

Next we show that $\mathcal{L}(E)$ is Morita equivalent to $\mathcal{L}(F)$. Let E' and F' be graphs with no sources by repeatedly removing the sources from E and F, respectively. By repeated application of Proposition 13 we have $\mathrm{Gr}\text{-}\mathcal{L}(E) \approx_{\mathrm{gr}} \mathrm{Gr}\text{-}\mathcal{L}(E')$ and $\mathrm{Gr}\text{-}\mathcal{L}(F) \approx_{\mathrm{gr}} \mathrm{Gr}\text{-}\mathcal{L}(F')$, which in turn shows that there are order preserving $\mathbb{Z}[x, x^{-1}]$-modules isomorphisms $K_0^{\mathrm{gr}}(\mathcal{L}(E)) \cong K_0^{\mathrm{gr}}(\mathcal{L}(E'))$ and $K_0^{\mathrm{gr}}(\mathcal{L}(F)) \cong K_0^{\mathrm{gr}}(\mathcal{L}(F'))$. Combining these with Eq. (20), we get an order preserving $\mathbb{Z}[x, x^{-1}]$-modules isomorphism $K_0^{\mathrm{gr}}(\mathcal{L}(E')) \cong K_0^{\mathrm{gr}}(\mathcal{L}(F'))$. Now Corollary 12 implies that $A_{E'}$ is shift equivalent to $A_{F'}$. By [26, Theorem 7.4.17], $\mathrm{BF}(A_{E'}) \cong \mathrm{BF}(A_{F'})$, where BF stands for the Bowen–Franks groups. On the other hand, by [26, Exercise 7.4.4, for $p(t) = 1 - t$], $\det(1 - A_{E'}) = \det(1 - A_{F'})$. Since the matrices $A_{E'}$ and $A_{F'}$ are irreducible, the main theorem of Franks [15] gives that $A_{E'}$ is flow equivalent to $A_{F'}$. Thus $A_{F'}$ can be obtained from $A_{E'}$ by a finite sequence of in/out-splitting and expansion of graphs (see [28]). Each of these transformations preserves the Morita equivalence (see the proof of Theorem 1.25 in [4]). So $\mathcal{L}(E')$ is Morita equivalent to $\mathcal{L}(F')$. Again using Proposition 13, we get that $\mathcal{L}(E)$ is Morita equivalent to $\mathcal{L}(F)$. Now by Theorem 20, this Morita equivalence together with K-group isomorphism (21), gives $\mathcal{L}(E) \cong \mathcal{L}(F)$. □

Very recently Ara and Pardo [5] settled the graded conjecture for the class of graphs with no sinks and sources using the theory of fractional skew monoid rings. Recall that a graded Morita equivalence between two graded rings induces an order preserving module isomorphism between their graded Grothendieck groups. So one hoped Proposition 13 combined with the Ara–Pardo theorem would give the graded conjecture for graphs with no sinks. However, since in Proposition 13 the position of identity is not preserved (see Remark 14) one cannot use this argument directly.

7. Noncommutative algebraic geometry

Let K be a field. If R is a commutative K-algebra which is generated by a finite number of elements of degree 1, then by the celebrated work of Serre [30], the category of quasi-coherent sheaves on the scheme $\operatorname{Proj}(R)$ is equivalent to $\operatorname{QGr}\text{-}R := \operatorname{Gr}\text{-}R/\operatorname{Fdim}\text{-}R$, where $\operatorname{Gr}\text{-}R$ is the category of $\mathbb{Z}$-graded modules over R and $\operatorname{Fdim}\text{-}R$ is the Serre subcategory of (direct limit of) finite-dimensional submodules. In particular when $R = K[x_0, x_1, \dots, x_n]$, then $\operatorname{QCoh}\mathbb{P}^n$ is equivalent to $\operatorname{QGr}\text{-}K[x_0, x_1, \dots, x_n]$.

Inspired by this, noncommutative algebraic geometry associates to a $\mathbb{Z}$-graded K-algebra A, with support $\mathbb{N}$, a "noncommutative scheme" $\operatorname{Proj}_{nc}(A)$ that is defined implicitly by declaring that the category of "quasi-coherent sheaves" on $\operatorname{Proj}_{nc}(A)$ is $\operatorname{QGr}\text{-}A := \operatorname{Gr}\text{-}A/\operatorname{Fdim}\text{-}A$. When A is coherent with $\operatorname{gr}\text{-}A$ its category of finitely presented graded modules then $\operatorname{qgr}\text{-}A := \operatorname{gr}\text{-}A/\operatorname{fdim}\text{-}A$ is viewed as the category of "coherent sheaves" on $\operatorname{Proj}_{nc}(A)$ (see [30,7,31] for more precise statements).

For a finite graph E, Paul Smith [32] gave a description of the category $\operatorname{QGr}\text{-}\mathcal{P}(E)$, where $\mathcal{P}(E)$ is the path algebra associated to E, in terms of easier to study categories of graded modules over Leavitt path algebras and ultramatricial algebras. Note that free algebras (on n generators) are examples of path algebras (of the graph with one vertex and n loops). Further, in [33] he showed that for two finite graphs E and F with no sinks or sources, if their adjacency matrices are shift equivalent, then the "noncommutative schemes" represented by their path algebras are the same, i.e., $\operatorname{QGr}\text{-}\mathcal{P}(E) \cong \operatorname{QGr}\text{-}\mathcal{P}(F)$.

Recall that, by assigning 1 to edges and 0 to vertices, the path algebra $\mathcal{P}(E)$ is a $\mathbb{Z}$-graded algebra with support $\mathbb{N}$. The category of $\mathbb{Z}$-graded right $\mathcal{P}(E)$-modules with degree-preserving homomorphisms is denoted by $\operatorname{Gr}\text{-}\mathcal{P}(E)$ and we write $\operatorname{Fdim}\text{-}\mathcal{P}(E)$ for its full subcategory of modules that are the sum of their finite-dimensional submodules. Since $\operatorname{Fdim}\text{-}\mathcal{P}(E)$ is a localising subcategory of $\operatorname{Gr}\text{-}\mathcal{P}(E)$ we can form the quotient category

$$\operatorname{QGr}\text{-}\mathcal{P}(E) := \operatorname{Gr}\text{-}\mathcal{P}(E)/\operatorname{Fdim}\text{-}\mathcal{P}(E).$$

Theorem 22. *(See [32, Theorem 1.3].) Let E be a finite graph and let E' be the graph without sources and sinks that is obtained by repeatedly removing all sources and sinks from E. Then*

$$\operatorname{QGr}\text{-}\mathcal{P}(E) \approx \operatorname{Gr}\text{-}\mathcal{L}\big(E'\big) \approx \operatorname{Mod}\text{-}\mathcal{L}\big(E'\big)_0.$$

The following theorem shows that the graded Grothendieck group can be considered as a complete invariant for the quotient category of path algebras.

Theorem 23. *Let E and F be graphs with no sinks. Then $K_0^{\mathrm{gr}}(\mathcal{L}(E)) \cong K_0^{\mathrm{gr}}(\mathcal{L}(F))$ as ordered abelian groups if and only if $\operatorname{QGr}\text{-}\mathcal{P}(E) \approx \operatorname{QGr}\text{-}\mathcal{P}(F)$.*

Proof. Using Dade's theorem (see Section 4.2) and the fact that $\mathcal{L}(E)$ and $\mathcal{L}(F)$ are strongly graded (Theorem 6), as ordered abelian groups, $K_0^{\mathrm{gr}}(\mathcal{L}(E)) \cong K_0^{\mathrm{gr}}(\mathcal{L}(F))$ if and only if $K_0(\mathcal{L}(E)_0) \cong K_0(\mathcal{L}(F)_0)$. On the other hand, since $\mathcal{L}(E)_0$ and $\mathcal{L}(F)_0$ are ultramatricial algebras (see Section 4.3), by [17, Corollary 15.27] $K_0(\mathcal{L}(E)_0) \cong K_0(\mathcal{L}(F)_0)$ as ordered abelian groups if and only if $\mathcal{L}(E)_0$ is Morita equivalent to $\mathcal{L}(F)_0$. Furthermore, a repeated application of Proposition 13 shows that repeatedly removing all the sources from E and F (call the new graphs obtained in this way with no sources E' and F', respectively) do not alter the corresponding categories modulo equivalence, i.e., $\mathcal{L}(E) \approx_{\mathrm{gr}} \mathcal{L}(E')$ and $\mathcal{L}(F) \approx_{\mathrm{gr}} \mathcal{L}(F')$, which implies $\mathcal{L}(E)_0 \approx \mathcal{L}(E')_0$ and $\mathcal{L}(F)_0 \approx \mathcal{L}(F')_0$.

Now combining this with Theorem 22 we have $K_0^{\mathrm{gr}}(\mathcal{L}(E)) \cong K_0^{\mathrm{gr}}(\mathcal{L}(F))$ if and only if $\mathrm{QGr}\text{-}\mathcal{P}(E) \approx \mathrm{QGr}\text{-}\mathcal{P}(F)$. $\square$

Theorem 23 gives the following corollary which is the main result of [33].

Corollary 24. *Let E and F be graphs with no sinks. If A_E is shift equivalent to A_F then* $\mathrm{QGr}\text{-}\mathcal{P}(E) \approx \mathrm{QGr}\text{-}\mathcal{P}(F)$.

Proof. Since A_E is shift equivalent to A_F, the Krieger dimension groups associated with E and F are isomorphic. But by Lemma 11 the Krieger dimension group coincides with the graded Grothendieck group, and so the corollary follows from Theorem 23. $\square$

The converse of Corollary 24 is not valid as the following example shows. Let E be a graph with one vertex and two loops and F be a graph with one vertex and four loops. Then $K_0^{\mathrm{gr}}(E) = \mathbb{Z}[1/2]$ and $K_0^{\mathrm{gr}}(F) = \mathbb{Z}[1/2]$. So the identity map gives an order preserving group isomorphism between the K_0^{gr}-groups. (Note that this isomorphism is not $\mathbb{Z}[x, x^{-1}]$-module isomorphism.) Then Theorem 23 shows that $\mathrm{QGr}\text{-}\mathcal{P}(E) \approx \mathrm{QGr}\text{-}\mathcal{P}(F)$. However one can readily see that A_E is not shift equivalent to A_F.

8. Product of graphs and graded Grothendieck groups

In [36, p. 149], Wagoner considered the product of two shift spaces and showed that the dimension module of the product is isomorphic to the tensor product of the dimension module of the shift spaces. In this section we carry over this to the case of Leavitt path algebras and graded Grothendieck groups.

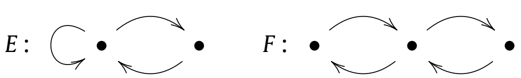
Let A and B be the adjacency matrices of the graphs E and F, respectively, where $|E^0| = m$ and $|F^0| = n$. Note that A and B can be considered as endomorphisms in $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}^m)$ and $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}^n)$, respectively. We define the *product of the graphs* E and F, denoted by $E \otimes F$, to be the graph associated to the matrix $A \otimes B \in \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}^{m+n})$. Concretely, if $A = (a_{ij})_{1 \leqslant i,j \leqslant m}$ and $B = (b_{lk})_{1 \leqslant l,k \leqslant n}$, then

$$A \otimes B = \big(a_{ij}(b_{lk})_{1 \leqslant l,k \leqslant n}\big)_{1 \leqslant i,j \leqslant m}, \tag{22}$$

i.e., the ijth entry of $A \otimes B$ is the matrix block $(a_{ij}b_{lk})_{1 \leqslant l,k \leqslant n}$. Note that this representation is independent of considering A and B as matrices acting from left or right.

If the matrices A and B have no zero rows, i.e., their associated Leavitt path algebras are strongly graded (Theorem 6), then so is the Leavitt path algebra associated to $A \otimes B$.

Example 25. Consider the graphs



with their adjacency matrices

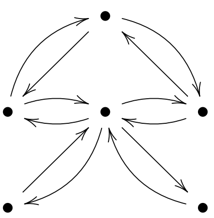
$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then

$$A \otimes B = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and the graph associated to this matrix is

$E \otimes F :$



In the following theorem we consider the tensor product of two pre-ordered $\mathbb{Z}[x, x^{-1}]$-modules G_1 and G_2, over $\mathbb{Z}$. We define the action of x on $G_1 \otimes_{\mathbb{Z}} G_2$ diagonally, i.e., $x(g_1 \otimes g_2) = xg_1 \otimes xg_2$ and extend it to the whole $\mathbb{Z}[x, x^{-1}]$ naturally. This makes $G_1 \otimes_{\mathbb{Z}} G_2$ a $\mathbb{Z}[x, x^{-1}]$-module. Further, the monoid $G_1^+ \otimes G_2^+$ (i.e., the set of direct sums of images of G_1^+ and G_2^+ in $G_1 \otimes_{\mathbb{Z}} G_2$), makes $G_1 \otimes_{\mathbb{Z}} G_2$ a pre-order group respecting the module structure.

Theorem 26. *Let $\mathcal{L}(E)$ and $\mathcal{L}(F)$ be Leavitt path algebras associated to finite graphs with no sinks E and F. Then there is an order preserving $\mathbb{Z}[x, x^{-1}]$-module isomorphism*

$$K_0^{\mathrm{gr}}\big(\mathcal{L}(E \otimes F)\big) \cong K_0^{\mathrm{gr}}\big(\mathcal{L}(E)\big) \otimes K_0^{\mathrm{gr}}\big(\mathcal{L}(F)\big),$$

which sends $[\mathcal{L}(E \otimes F)]$ to $[\mathcal{L}(E)] \otimes [\mathcal{L}(F)]$.

Proof. Let A_E and A_F be the adjacency matrices of the graphs E and F, respectively, where $|E^0| = m$ and $|F^0| = n$. Set $A = A_E^t$ and $B = A_F^t$. Observe that $A \otimes B = (A_E \otimes A_F)^t$. Now by (7) the vertical maps of the following diagram are isomorphisms:

$$\begin{tikzcd} K_0^{\mathrm{gr}}(\mathcal{L}(E)) \otimes K_0^{\mathrm{gr}}(\mathcal{L}(F)) \arrow[r, dotted] \arrow[d] & K_0^{\mathrm{gr}}(\mathcal{L}(E \otimes F)) \arrow[d] \\ \varinjlim_A \mathbb{Z}^m \otimes \varinjlim_B \mathbb{Z}^n \arrow[r, "\phi"] & \varinjlim_{A \otimes B} \mathbb{Z}^{m+n}. \end{tikzcd}$$

Using the description of Section 4.4 for the direct limits, define ϕ on generators as follows:

$$\phi\big([a, k] \otimes [b, l]\big) = \big[A^l a \otimes B^k b, k + l\big].$$

One checks easily that this map is well-defined and is a homomorphism of groups. Further,

$$\begin{aligned} x\phi\big([a, k] \otimes [b, l]\big) &= x\big[A^l a \otimes B^k b, k + l\big] \\ &= \big[(A \otimes B)\big(A^l a \otimes B^k b\big), k + l\big] = \big[A^{l+1} a \otimes B^{k+1} b, k + l\big] \\ &= \phi\big([Aa, k] \otimes [Bb, l]\big) = \phi\big(x\big([a, k] \otimes [b, l]\big)\big) \end{aligned}$$

shows that ϕ is a $\mathbb{Z}[x, x^{-1}]$-module homomorphism. Define

$$\begin{aligned}\psi : \varinjlim_{A\otimes B} \mathbb{Z}^{m+n} &\longrightarrow \varinjlim_{A} \mathbb{Z}^{m} \otimes \varinjlim_{B} \mathbb{Z}^{n}\\ [c,k] &\longmapsto \sum_i [a_i,k]\otimes[b_i,k],\end{aligned}$$

where $f(c) = \sum_i a_i \otimes b_i$ under a natural isomorphism $f : \mathbb{Z}^{m+n} \to \mathbb{Z}^m \otimes \mathbb{Z}^n$. One can check that ψ is indeed well-defined and $\phi\psi$ and $\psi\phi$ are the identity maps of the corresponding groups.

Finally, $[\mathcal{L}(E)]\otimes[\mathcal{L}(F)]$ is represented by $[\bar{1},0]\otimes[\bar{1},0]$ in $\varinjlim_A \mathbb{Z}^m \otimes \varinjlim_B \mathbb{Z}^n$ and $\phi([\bar{1},0]\otimes[\bar{1},0]) = [\bar{1},0]$ which represents $[\mathcal{L}(E\otimes F)]$ in $K_0^{\mathrm{gr}}(\mathcal{L}(E\otimes F))$. $\square$

Example 27. For the graph E in Example 25, one can calculate its dimension group as follows:

$$\begin{aligned}K_0^{\mathrm{gr}}\big(\mathcal{L}(E)\big) &\cong \mathbb{Z}\oplus\mathbb{Z};\\ K_0^{\mathrm{gr}}\big(\mathcal{L}(E)\big)^{+} &\cong \left\{(a,b) \;\middle|\; \frac{1+\sqrt{5}}{2}a + b \geqslant 0\right\};\\ \big[\mathcal{L}(E)\big] &= (1,1);\\ x(a,b) &= (a+b,a).\end{aligned}$$

Furthermore for the graph F, we have

$$\begin{aligned}K_0^{\mathrm{gr}}\big(\mathcal{L}(F)\big) &\cong \mathbb{Z}[1/2]\oplus\mathbb{Z}[1/2];\\ K_0^{\mathrm{gr}}\big(\mathcal{L}(F)\big)^{+} &\cong \mathbb{N}[1/2]\oplus\mathbb{N}[1/2];\\ \big[\mathcal{L}(F)\big] &= (2,1);\\ x(a,b) &= (2b,a).\end{aligned}$$

This information along with Theorem 26, will easily determine the graded dimension group associated to the graph $E\otimes F$ in Example 25.

Appendix A

In Proposition 15 we observed that the Leavitt path algebras associated to out-splitting and in-splitting graphs are graded Morita equivalent to the Leavitt path algebra of the original graph; these have been shown separately. Using Ashton and Bates notion of elementary shift equivalence defined for two graphs, one can uniformly show the above results. Namely, if two graphs E_1 and E_2 are elementary shift equivalent via a graph E_3, then $\mathcal{L}(E_1)$ and $\mathcal{L}(E_2)$ are graded Morita equivalent. It is not difficult to see that the out-splitting and in-splitting are elementary shift equivalent to the original graph [10]. However, the price to be paid for this unified approach is, due to the construction of E_3, we need to change the grading of Leavitt path algebras and the eventual graded Morita equivalent is $(1/2)\mathbb{Z}$-graded.

The following definition given in [8,9] for directed graphs with no sinks, provides graph theoretical conditions, when the adjacency matrices of two graphs are elementary shift equivalent (see also [26, p. 227]). Throughout, we work with finite graphs although the results can be extended to arbitrary graphs as in [9,10].

Definition 28. Let $E_i = (E_i^0, E_i^1, r_i, s_i)$, for $i = 1, 2$, be graphs. Suppose there is a graph $E_3 = (E_3^0, E_3^1, r_3, s_3)$ such that

(1) $E_3^0 = E_1^0 \cup E_2^0$ and $E_1^0 \cap E_2^0 = \emptyset$,
(2) $E_3^1 = E_{12}^1 \cup E_{21}^1$, where $E_{ij}^1 := \{e \in E_3^1 \mid s_3(e) \in E_i^0,\ r_3(e) \in E_j^0\}$,
(3) for $i = \{1, 2\}$, there are range and source-preserving bijections $\theta_i : E_i^1 \to E_3^2(E_i^0, E_i^0)$, where for $i \in \{1, 2\}$, $E_3^2(E_i^0, E_i^0) := \{\alpha \in E_3^2 \mid s_3(\alpha) \in E_i^0,\ r_3(\alpha) \in E_i^0\}$.

Then we say that E_1 and E_2 are *elementary shift equivalent* ($E_1 \sim_{ES} E_2$) *via* E_3.

Remark 29. In [8,9], for Definition 28, the term *elementary strong shift equivalent* is used.

One can prove that $E_1 \sim_{ES} E_2$ via a graph E_3 if and only if $A_{E_1} \sim_{ES} A_{E_2}$ (see [8, Proposition 3.10]).

The equivalence relation $\sim_S$ on directed graphs generated by elementary strong shift equivalence is called *strong shift equivalence*. In [9, Theorem 5.2] it was shown that if two row-finite graphs are strongly shift equivalent then their associated graph C^*-algebras are strongly Morita equivalent. We establish a similar statement in the setting of graded Leavitt path algebras.

In the following by $(1/2)\mathbb{Z}$ we denote the cyclic subgroup of rational numbers $\mathbb{Q}$ generated by $1/2$.

Theorem 30. *Let E_1 and E_2 be graphs with no sinks which are elementary strong shift equivalent via the graph E_3. Then $\mathcal{L}(E_1)$ and $\mathcal{L}(E_2)$ are $(1/2)\mathbb{Z}$-graded Morita equivalent.*

Proof. First observe that if $E_1 \sim_{ES} E_2$ via E_3 and E_1 and E_2 have no sinks, then E_3 does not have sinks either.

We first give a $(1/2)\mathbb{Z}$-graded structure to the Leavitt path algebras $\mathcal{L}(E_i)$, $i = 1, 2, 3$. We then show that $\mathcal{L}(E_1)$ and $\mathcal{L}(E_2)$ are $(1/2)\mathbb{Z}$-graded Morita equivalent to $\mathcal{L}(E_3)$. This implies that $\mathcal{L}(E_1)$ is $(1/2)\mathbb{Z}$-graded Morita equivalent to $\mathcal{L}(E_2)$.

Let $\Gamma = (1/2)\mathbb{Z}$ and set $\deg(v) = 0$, for $v \in E_i^0$, $\deg(\alpha) = 1$ and $\deg(\alpha^*) = -1$ for $\alpha \in E_i^1$, where $i = 1, 2$. By Section 3, we then obtain a natural $(1/2)\mathbb{Z}$-grading on $\mathcal{L}(E_1)$ and $\mathcal{L}(E_2)$ with support $\mathbb{Z}$.

Furthermore, set $\deg(v) = 0$, for $v \in E_3^0$, $\deg(\alpha) = 1/2$ and $\deg(\alpha^*) = -1/2$ for $\alpha \in E_3^1$. Thus we get a natural $(1/2)\mathbb{Z}$-grading on $\mathcal{L}(E_3)$, with support $(1/2)\mathbb{Z}$.

Next we construct a $(1/2)\mathbb{Z}$-graded ring homomorphism $\phi : \mathcal{L}(E_1) \to \mathcal{L}(E_3)$ such that $\phi(\mathcal{L}(E_1)) = p\mathcal{L}(E_3)p$, where $p = \sum_{v \in E_1^0} v \in \mathcal{L}(E_3)$. Define $\phi(v) = v$ for $v \in E_1^0$ and $\phi(l) = ef$, where $l \in E_1^1$ (and is of degree 1) and ef is a path of length 2 in E_3 (and is of degree 1) assigned uniquely to l via θ_1 in Definition 28(3). We check that the set $\{\phi(v), \phi(l) \mid v \in E_1^0,\ l \in E_1^1\}$ is an E-family in $\mathcal{L}(E_3)$.

Let $v \in E_1^0$ and $\phi(v) = v \in E_3^0$. Then

$$\begin{aligned} v &= \sum_{\{e \in E_3^1 \mid s_3(e) = v\}} ee^* \\ &= \sum_{\{ef \in E_3^2(E_1^0, E_1^0) \mid s_3(ef) = v\}} ef(ef)^* \\ &= \sum_{\{l \in E_1^1 \mid s_1(l) = v\}} \phi(l)\phi(l)^* \end{aligned}$$

where the last equality uses condition (3) in Definition 28. On the other hand for $l \in E_1^1$,

$$\phi(l)^*\phi(l) = (ef)^*(ef) = f^*e^*ef = f^*f = r(f) = \phi\bigl(r(l)\bigr),$$

since the map θ_1 in Definition 28 is range-preserving bijection. The rest of the relations of an E-family is easily checked and thus there is a map $\phi: \mathcal{L}(E_1) \to \mathcal{L}(E_3)$.

Clearly $\phi(\mathcal{L}(E_1)) \subseteq p\mathcal{L}(E_3)p$. Since $p\mathcal{L}(E_3)p$ is generated by elements of the form $p\alpha\beta^*p$, where $\alpha, \beta \in E_3^*$, we check that $p\alpha\beta^*p \in \phi(\mathcal{L}(E_1))$. Clearly either $p\alpha\beta^*p = \alpha\beta^*$ if $s_3(\alpha), s_3(\beta) \in E_1^0$ or $p\alpha\beta^*p = 0$ otherwise. Furthermore, for $\alpha\beta^* \neq 0$, we should have $r_3(\alpha) = r_3(\beta)$. Thus for an element $p\alpha\beta^*p \neq 0$, we have either $s_3(\alpha), s_3(\beta), r_3(\alpha), r_3(\beta) \in E_1^0$ or $s_3(\alpha), s_3(\beta) \in E_1^0$ and $r_3(\alpha), r_3(\beta) \in E_2^0$. In the first case, by the construction of E_3, the lengths of α and β are even and since θ_1 is bijective, α and β are in the images of the map ϕ. In the second case, since E_3 has no sinks, we have

$$\alpha\beta^* = \sum_{\{e \in E_3^1 \mid s(e) = r(\alpha)\}} \alpha e e^* \beta^* = \sum_{\{e \in E_3^1 \mid s(e) = r(\alpha)\}} \alpha e (\beta e)^*.$$

Now, again by the construction of E_3, for each e, $s_3(\alpha e), s_3(\beta e), r_3(\alpha e), r_3(\beta e) \in E_1^0$ and by the first case, they are in the image of the map ϕ. So $\phi(\mathcal{L}(E_1)) = p\mathcal{L}(E_3)p$.

Since $\phi(v) = v \neq 0$ (see [16, Lemma 1.5]) by the graded uniqueness theorem [34, Theorem 4.8] (which is valid for $(1/2)\mathbb{Z}$-graded Leavitt path algebras as well) ϕ is injective. Thus $\mathcal{L}(E_1) \cong_{\mathrm{gr}} p\mathcal{L}(E_3)p$. This immediately implies $\mathcal{L}(E_1)$ is $(1/2)\mathbb{Z}$-graded Morita equivalent to $p\mathcal{L}(E_3)p$. On the other hand, the (graded) ideal generated by $p = \sum_{v \in E_1^0} v$ in $\mathcal{L}(E_3)$ coincides with the (graded) ideal generated by $\{v \mid v \in E_1^0\}$. But the smallest hereditary and saturated subset of E_3^0 containing E_1^0 is E_3^0. Thus by [6, Theorem 5.3] the ideal generated by E_1^0 is $\mathcal{L}(E_3)$. This shows p is a full homogeneous idempotent in $\mathcal{L}(E_3)$. Thus $p\mathcal{L}(E_3)p$ is graded Morita equivalence to $\mathcal{L}(E_3)$ (see Example 2). Putting these together we get $\mathcal{L}(E_1) \approx_{\mathrm{gr}} \mathcal{L}(E_3)$. In a similar manner $\mathcal{L}(E_2) \approx_{\mathrm{gr}} \mathcal{L}(E_3)$. Thus $\mathcal{L}(E_1) \approx_{\mathrm{gr}} \mathcal{L}(E_2)$ as $(1/2)\mathbb{Z}$-graded rings. This finishes the proof. $\square$

Remark 31. Since throughout this paper we are interested in purely infinite simple Leavitt path algebras, which are (graded) simple, we could have specialised Theorem 30 to graphs associated to these algebras, and instead of using [16, Lemma 1.5] and [6, Theorem 5.3] in the proof, we could only use the simplicity of the algebras.

In [4], it was shown that if E is a finite graph with no sources and sinks such that $\mathcal{L}(E)$ is simple, then the Leavitt path algebras obtained from the in-splittings of E [4, Proposition 1.11] or out-splittings of E [4, Proposition 1.14] are Morita equivalent to $\mathcal{L}(E)$. For a graph E with no sinks, it was shown that an in-splitting of a graph E [10, Proposition 6.2] or an out-splitting of E [10, Proposition 6.3] is elementary shift equivalent to the graph E. Now using Theorems 30 and 3, we can obtain these results in a unified manner. Finally, one would like to know whether in Theorem 30, $\mathcal{L}(E_1)$ and $\mathcal{L}(E_2)$ are actually $\mathbb{Z}$-graded Morita equivalent.

References

[1] G. Abrams, G. Aranda Pino, The Leavitt path algebra of a graph, J. Algebra 293 (2) (2005) 319–334.
[2] G. Abrams, G. Aranda Pino, Purely infinite simple Leavitt path algebras, J. Pure Appl. Algebra 207 (3) (2006) 553–563.
[3] G. Abrams, P.N. Ánh, A. Louly, E. Pardo, The classification question for Leavitt path algebras, J. Algebra 320 (5) (2008) 1983–2026.
[4] G. Abrams, A. Louly, E. Pardo, C. Smith, Flow invariants in the classification of Leavitt path algebras, J. Algebra 333 (2011) 202–231.
[5] P. Ara, E. Pardo, K-Theoretic characterization of graded isomorphisms between Leavitt path algebras, arXiv:1210.3127.
[6] P. Ara, M.A. Moreno, E. Pardo, Nonstable K-theory for graph algebras, Algebr. Represent. Theory 10 (2) (2007) 157–178.
[7] M. Artin, J.J. Zhang, Noncommutative projective schemes, Adv. Math. 109 (2) (1994) 228–287.
[8] B. Ashton, Morita equivalence of graph C^*-algebras, Honours Thesis, Newcastle University, Australia, 1996.
[9] T. Bates, Applications of the gauge-invariant uniqueness theorem for the Cuntz–Krieger algebras of directed graphs, Bull. Aust. Math. Soc. 65 (2002) 57–67.
[10] T. Bates, D. Pask, Flow equivalence of graph algebras, Ergodic Theory Dynam. Systems 24 (2) (2004) 367–382.
[11] M. Boyle, Symbolic dynamics and matrices, in: R. Brualdi, S. Friedland, V. Klee (Eds.), Combinatorial and Graph-Theoretic Problems in Linear Algebra, in: IMA Vol. Math. Appl., vol. 50, Springer, 1993, pp. 1–38.
[12] J. Cuntz, W. Krieger, A class of C^*-algebras and topological Markov chains, Invent. Math. 56 (1980) 251–268.
[13] E. Dade, Group-graded rings and modules, Math. Z. 174 (3) (1980) 241–262.

[14] G.A. Elliott, On the classification of inductive limits of sequences of semisimple finite-dimensional algebras, J. Algebra 38 (1) (1976) 29–44.
[15] J. Franks, Flow equivalence of subshifts of finite type, Ergodic Theory Dynam. Systems 4 (1984) 53–66.
[16] K.R. Goodearl, Leavitt path algebras and direct limits, in: Contemp. Math., vol. 480, 2009, pp. 165–187.
[17] K.R. Goodearl, Von Neumann Regular Rings, second ed., Krieger Publishing Co., Malabar, FL, 1991.
[18] R. Gordon, E.L. Green, Graded Artin algebras, J. Algebra 76 (1982) 111–137.
[19] R. Hazrat, The graded structure of Leavitt path algebras, Israel J. Math., http://dx.doi.org/10.1007/s11856-012-0138-5, in press.
[20] R. Hazrat, The graded Grothendieck group and classification of Leavitt path algebras, Math. Ann. 355 (1) (2013) 273–325.
[21] R. Hazrat, A note on the isomorphism conjectures for Leavitt path algebras, J. Algebra 375 (2013) 33–40.
[22] R. Hazrat, Graded rings and graded Grothendieck groups, rhazrat.wordpress.com/preprints.
[23] D. Huang, Automorphisms of Bowen–Franks groups of shifts of finite type, Ergodic Theory Dynam. Systems 21 (2001) 1113–1137.
[24] W. Krieger, On dimension functions and topological Markov chains, Invent. Math. 56 (1980) 239–250.
[25] T.Y. Lam, Lectures on Modules and Rings, Springer-Verlag, 1999.
[26] D. Lind, B. Marcus, An Introduction to Symbolic Dynamics and Coding, Cambridge University Press, 1995.
[27] C. Năstăsescu, F. Van Oystaeyen, Methods of Graded Rings, Lecture Notes in Math., vol. 1836, Springer-Verlag, Berlin, 2004.
[28] W. Parry, D. Sullivan, A topological invariant of flows on 1-dimensional spaces, Topology 14 (1975) 297–299.
[29] I. Raeburn, Graph Algebras, CBMS Reg. Conf. Ser. Math., vol. 103, 2005.
[30] J.-P. Serre, Faisceaux alǵebrique cohérents, Ann. of Math. 61 (1955) 197–278.
[31] S. Paul Smith, The non-commutative scheme having a free algebra as a homogeneous coordinate ring, arXiv:1104.3822.
[32] S. Paul Smith, Category equivalences involving graded modules over path algebras of quivers, Adv. Math. 230 (2012) 1780–1810.
[33] S. Paul Smith, Shift equivalence and a category equivalence involving graded modules over path algebras of quivers, arXiv:1108.4994.
[34] M. Tomforde, Uniqueness theorems and ideal structure for Leavitt path algebras, J. Algebra 318 (1) (2007) 270–299.
[35] J.B. Wagoner, Markov partitions and K_2, Publ. Math. Inst. Hautes Etudes Sci. 65 (1987) 91–129.
[36] J.B. Wagoner, Topological Markov chains, C^*-algebras and K_2, Adv. Math. 71 (2) (1988) 133–185.
[37] R.F. Williams, Classification of subshifts of finite type, Ann. of Math. 98 (2) (1973) 120–153.