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Cellularity of wreath product algebras and A -Brauer algebras[☆]

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ABSTRACT

A cellular algebra is called *cyclic cellular* if all cell modules are cyclic. Most important examples of cellular algebras appearing in representation theory are in fact cyclic cellular. We prove that if A is a cyclic cellular algebra, then the wreath product algebras $A \wr \mathfrak{S}_n$ are also cyclic cellular. We also introduce A -Brauer algebras, for algebras A with an involution and trace. This class of algebras includes, in particular, G -Brauer algebras for non-abelian groups G . We prove that if A is cyclic cellular then the A -Brauer algebras $D_n(A)$ are also cyclic cellular.

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1. Introduction

The concept of cellularity of algebras was introduced by Graham and Lehrer [13]. Cellularity is useful for analyzing the representation theory of important classes of algebras such as Hecke algebras, q -Schur algebras, Brauer and BMW algebras, etc.

In this paper, we introduce a variant of the notion of cellularity: A cellular algebra A is called *cyclic cellular* if all of its cell modules are cyclic A -modules. Although cyclic cellularity is nominally stronger than cellularity, most important classes of cellular algebras appearing in representation theory are in

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fact cyclic cellular. For example, the algebras mentioned above – Hecke algebras of type A , q -Schur algebras, Brauer algebras and BMW algebras – are cyclic cellular.

It appears that the requirement of cyclic cell modules eliminates some potential pathology allowed by the general notion of cellularity; for example, an abelian cellular algebra is cyclic cellular if and only if all the cell modules have rank 1.

Our main theorem regarding cyclic cellular algebras is the following: if A is a cyclic cellular algebra, then the wreath product of A with the symmetric group $\mathfrak{S}_n$ is also cyclic cellular.

In the second part of the paper, we introduce a new class of algebras, the A -Brauer algebras, which are a sort of wreath product of an algebra A with the Brauer algebra. The definition of the A -Brauer algebras requires that A be an algebra with involution $*$ and with a $*$ -invariant trace. The class of A -Brauer algebras includes in particular G -Brauer algebras for non-abelian groups G , generalizing the construction for abelian groups G by Parvathi and Savithri [23].

Our second main theorem is that if A is a cyclic cellular algebra with an involution-invariant trace, then the A -Brauer algebras $D_n(A)$ for $n \geq 1$ are also cyclic cellular.

2. Cellular algebras

2.1. Definition of cellularity

We recall the definition of *cellularity* from Graham and Lehrer [13]. The version given here is slightly weaker than the original definition in [13].

Definition 2.1. (See [13].) Let R be an integral domain and A a unital R -algebra. A *cell datum* for A consists of an R -linear algebra involution $a \mapsto a^*$; a finite partially ordered set $(\Gamma, \geq)$; for each $\gamma \in \Gamma$ a finite index set $\mathcal{T}(\gamma)$; and a subset

$$\mathcal{C} = \{c_{s,t}^\gamma : \gamma \in \Gamma \text{ and } s, t \in \mathcal{T}(\gamma)\} \subseteq A$$

with the following properties:

- (1) $\mathcal{C}$ is an R -basis of A .
- (2) For each $\gamma \in \Gamma$, let $\bar{A}^\gamma$ be the span of the $c_{s,t}^\mu$ with $\mu > \gamma$. Given $\gamma \in \Gamma$, $s \in \mathcal{T}(\gamma)$, and $a \in A$, there exist coefficients $r_v^s(a) \in R$ such that for all $t \in \mathcal{T}(\gamma)$:

$$ac_{s,t}^\gamma \equiv \sum_v r_v^s(a) c_{v,t}^\gamma \pmod{\bar{A}^\gamma}.$$

- (3) $(c_{s,t}^\gamma)^* \equiv c_{t,s}^\gamma \pmod{\bar{A}^\gamma}$ for all $\gamma \in \Gamma$ and $s, t \in \mathcal{T}(\gamma)$.

For brevity, we will write that $(\mathcal{C}, \Gamma)$ is a cellular basis of A . When we need to be more explicit, we write that $(A, *, \Gamma, \geq, \mathcal{T}, \mathcal{C})$ is a cell datum.

Note that it follows from Definition 2.1(2) and (3) that

$$c_{s,t}^\gamma a \equiv \sum_w r_w^t(a^*) c_{s,w}^\gamma \pmod{\bar{A}^\gamma}. \quad (2.1)$$

The original definition of Graham and Lehrer requires that $(c_{s,t}^\gamma)^* = c_{t,s}^\gamma$ for all $\gamma \in \Gamma$ and $s, t \in \mathcal{T}(\gamma)$. All the statements and arguments of [13] remain valid with the weaker Definition 2.1. The two definitions are equivalent in case 2 is invertible in the ground ring R . The advantages of the weaker definition are discussed in [11,12].

In a few places in this paper where we have occasion to compare the weaker definition of cellularity adopted here with the original definition of Graham and Lehrer in [13], we will refer to the original condition of Graham and Lehrer as *strict cellularity*.

Basis free characterizations of cellularity have been given in [17,16,11].

2.2. Structures related to cellularity

We recall some basic structures related to cellularity, see [13]. Let A be a cellular algebra with cell datum $(A, *, \Gamma, \geq, \mathcal{T}, \mathcal{C})$. Given $\gamma \in \Gamma$, let A^γ denote the span of the $c_{s,t}^\mu$ with $\mu \geq \gamma$. It follows that both A^γ and $\bar{A}^\gamma$ (defined above) are $*$ -invariant two-sided ideals of A .

For $\gamma \in \Gamma$, the *left cell module* Δ^γ is defined as follows: as an R -module, Δ^γ is free with basis indexed by $\mathcal{T}(\gamma)$, say $\{c_s^\gamma : s \in \mathcal{T}(\gamma)\}$; for each $a \in A$, the action of a on Δ^γ is defined by $ac_s^\gamma = \sum_v r_v^s(a)c_v^\gamma$ where the elements $r_v^s(a) \in R$ are the coefficients in Definition 2.1(2). Note that for any $t \in \mathcal{T}(\gamma)$, the assignment $c_s^\gamma \mapsto c_{s,t}^\gamma + \bar{A}^\gamma$ defines an injective A -module homomorphism from Δ^γ to $A^\gamma/\bar{A}^\gamma$.

If A and B are R -algebras with involutions denoted by $*$, then we have a functor $M \mapsto M^*$ from A - B bimodules to B - A bimodules, as follows. As an R -module, M^* is just a copy of M with elements marked by $*$. The B - A bimodule structure of M^* is determined by $bx^*a = (a^*xb^*)^*$. We have a natural isomorphism $M^{**} \cong M$, via $x^{**} \mapsto x$. In particular, taking B to be R with the trivial involution, we get a functor from left A -modules to right A -modules. Similarly, we get a functor from right A -modules to left A -modules. (If $\Delta \subset A$ is a left or right ideal, we have two meanings for Δ^* , namely application of the functor $*$, or application of the involution in A , but these agree.) If ${}_A M$ is a left A -module and N_A is a right A -module, then

$$(M \otimes_R N)^* \cong N^* \otimes_R M^*,$$

as A - A bimodules, with the isomorphism determined by $(m \otimes n)^* \mapsto n^* \otimes m^*$. In particular if ${}_A M$ is a left A -module and we identify M^{**} with M and $(M \otimes M^*)^*$ with $M^{**} \otimes M^* = M \otimes M^*$, then we have $(x \otimes y^*)^* = y \otimes x^*$.

Now we apply these observations with A a cellular algebra and Δ^γ a cell module. The assignment

$$\alpha_\gamma : c_{s,t}^\gamma + \bar{A}^\gamma \mapsto c_s^\gamma \otimes (c_t^\gamma)^*$$

determines an A - A bimodule isomorphism from $A^\gamma/\bar{A}^\gamma$ to $\Delta^\gamma \otimes_R (\Delta^\gamma)^*$. Moreover, we have $* \circ \alpha_\gamma = \alpha_\gamma \circ *$, which reflects the cellular algebra axiom $(c_{s,t}^\gamma)^* \equiv c_{t,s}^\gamma \pmod{\bar{A}^\gamma}$. The importance of the maps α_γ for the structure of cellular algebras was stressed by König and Xi [17,16].

There exists a symmetric R -valued bilinear form $\langle, \rangle$ on Δ^γ such that for all $s, t, u, v \in \mathcal{T}(\gamma)$,

$$c_{s,t}^\gamma c_{u,v}^\gamma \equiv \langle c_t^\gamma, c_u^\gamma \rangle c_{s,v}^\gamma \pmod{\bar{A}^\gamma}.$$

This bilinear form plays an essential role in the theory of cellular algebras, see [13].

2.3. Equivalent cellular bases

A cellular algebra A always admits many different cellular bases. In fact, any choice of an R -basis in each cell module can be globalized to a cellular basis of A , see Lemma 2.3 below.

Let A be a cellular algebra with cell datum $(A, *, \Gamma, \geq, \mathcal{T}, \mathcal{C})$. We say that a cellular basis

$$\mathcal{B} = \{b_{s,t}^\gamma : \gamma \in \Gamma \text{ and } s, t \in \mathcal{T}(\gamma)\}$$

is *equivalent* to the original cellular basis $\mathcal{C}$ if it determines the same ideals A^γ and the same cell modules as does $\mathcal{C}$. More precisely, the requirement is that

(1) for all $\gamma \in \Gamma$,

$$A^\gamma = \text{span}\{b_{s,t}^{\gamma'}: \gamma' \geq \gamma \text{ and } s, t \in \mathcal{T}(\gamma')\}, \quad \text{and}$$

(2) for all $\gamma \in \Gamma$ and $t \in \mathcal{T}(\gamma)$,

$$\text{span}\{b_{s,t}^\gamma + \bar{A}^\gamma: s \in \mathcal{T}(\gamma)\} \cong \Delta^\gamma,$$

as A -modules.

Lemma 2.2. Let A be a cellular algebra with cell datum $(A, *, \Gamma, \geq, \mathcal{T}, C)$. Let $\gamma \in \Gamma$ and let $b \in \Delta^\gamma$ be non-zero. Then $x \mapsto x \otimes b^*$ is an A -module isomorphism of Δ^γ onto $\Delta^\gamma \otimes_R \Delta^\gamma \otimes_R (\Delta^\gamma)^*$.

Proof. Since $(\Delta^\gamma)^*$ is a free R -module, it is torsion free; hence

$$\Delta^\gamma \cong \Delta^\gamma \otimes_R R \cong \Delta^\gamma \otimes_R Rb^* = \Delta^\gamma \otimes b^*$$

as A -modules. Explicitly, the isomorphism is $x \mapsto x \otimes b^*$. $\square$

Lemma 2.3. Let A be a cellular algebra with cell datum $(A, *, \Gamma, \geq, \mathcal{T}, C)$. For each $\gamma \in \Gamma$, fix an A - A bi-module isomorphism $\beta_\gamma: A^\gamma / \bar{A}^\gamma \rightarrow \Delta^\gamma \otimes_R (\Delta^\gamma)^*$ satisfying $* \circ \beta_\gamma = \beta_\gamma \circ *$, and let $\{b_t: t \in \mathcal{T}(\gamma)\}$ be an R -basis of Δ^γ . Finally, for each $\gamma \in \Gamma$ and each pair $s, t \in \mathcal{T}(\gamma)$, let $b_{s,t}^\gamma$ be an arbitrary lifting of $\beta_\gamma^{-1}(b_s \otimes b_t^*)$ in A^γ . Then

$$\mathcal{B} = \{b_{s,t}^\gamma: \gamma \in \Gamma \text{ and } s, t \in \mathcal{T}(\gamma)\}$$

is a cellular basis of A equivalent to the original cellular basis C .

Proof. It follows from [12, Lemma 2.3], that $\mathcal{B}$ is a cellular basis. It remains to show that $\mathcal{B}$ is equivalent to the original cellular basis C . For each fixed $\gamma \in \Gamma$, $\{b_s \otimes b_t^*: s, t \in \mathcal{T}(\gamma)\}$ is an R -basis of $\Delta^\gamma \otimes_R (\Delta^\gamma)^*$. It follows that

$$\text{span}\{b_{s,t}^\gamma: s, t \in \mathcal{T}(\gamma)\} + \bar{A}^\gamma = A^\gamma.$$

By induction on the partial order of Γ , we obtain that

$$A^\gamma = \text{span}\{b_{s,t}^{\gamma'}: \gamma' \geq \gamma \text{ and } s, t \in \mathcal{T}(\gamma')\}.$$

By definition of $b_{s,t}^\gamma$, we have that $\beta_\gamma(b_{s,t}^\gamma + \bar{A}^\gamma) = b_s \otimes b_t^*$. Thus for fixed $t \in \mathcal{T}(\gamma)$, the restriction of β_γ is an isomorphism from $\text{span}\{b_{s,t}^\gamma + \bar{A}^\gamma: s \in \mathcal{T}(\gamma)\}$ onto $\Delta^\gamma \otimes b_t^*$. By Lemma 2.2, $\Delta^\gamma \otimes b_t^* \cong \Delta^\gamma$ as A -modules. Thus we have

$$\text{span}\{b_{s,t}^\gamma + \bar{A}^\gamma: s \in \mathcal{T}(\gamma)\} \cong \Delta^\gamma. \quad \square$$

2.4. Cyclic cellular algebras

Definition 2.4. A cellular algebra is said to be *cyclic cellular* if every cell module of A is cyclic.

We will also say that a cellular basis is cyclic cellular if the cell modules defined via this basis are cyclic.

Lemma 2.5. Let A be a cellular algebra over an integral domain R with cell datum $(A, *, \Gamma, \geq, \mathcal{T}, C)$. The following are equivalent:

- (1) A is cyclic cellular.
- (2) For each $\gamma \in \Gamma$, there exists an element $y_\gamma \in A^\gamma$ with the properties:
 - (a) $y_\gamma \equiv y_\gamma^* \pmod{\bar{A}^\gamma}$.
 - (b) $A^\gamma = Ay_\gamma A + \bar{A}^\gamma$.
 - (c) $(Ay_\gamma + \bar{A}^\gamma)/\bar{A}^\gamma \cong \Delta^\gamma$, as A -modules.

Proof. Suppose that A is cyclic cellular. For each $\gamma \in \Gamma$, choose a generator δ^γ of the cell module Δ^γ . Let y_γ be any lifting in A^γ of $\alpha_\gamma^{-1}(\delta^\gamma \otimes (\delta^\gamma)^*)$. Then property (2a) holds because $(\delta^\gamma \otimes (\delta^\gamma)^*)^* = \delta^\gamma \otimes (\delta^\gamma)^*$. Property (2b) holds because $A(\delta^\gamma \otimes (\delta^\gamma)^*)A = \Delta^\gamma \otimes_R (\Delta^\gamma)^*$.

The restriction of α_γ yields an A -module isomorphism $ay_\gamma + \bar{A}^\gamma \mapsto a\delta^\gamma \otimes (\delta^\gamma)^*$. By Lemma 2.2, $x \otimes (\delta^\gamma)^* \mapsto x$ is an A -module isomorphism from $\Delta^\gamma \otimes (\delta^\gamma)^*$ onto Δ^γ . By composing these isomorphisms, we obtain the isomorphism $ay_\gamma + \bar{A}^\gamma \mapsto a\delta^\gamma$ from $(Ay_\gamma + \bar{A}^\gamma)/\bar{A}^\gamma$ to Δ^γ . Thus property (2c) holds.

Conversely, if (2) holds, then in particular (2c) implies that each cell module is cyclic. $\square$

For the remainder of this section, let A be a cyclic cellular algebra with cell datum $(A, *, \Gamma, \geq, \mathcal{T}, C)$, and write $C = \{c_{s,t}^\gamma: \gamma \in \Gamma \text{ and } s, t \in \mathcal{T}(\gamma)\}$.

Notation 2.6. For each $\gamma \in \Gamma$, let δ^γ be a generator of the cell module Δ^γ , and let y_γ be a lifting in A^γ of $\alpha_\gamma^{-1}(\delta^\gamma \otimes (\delta^\gamma)^*)$. Let $\{c_t^\gamma: t \in \mathcal{T}(\gamma)\}$ be the standard basis of the cell module Δ^γ derived from the cellular basis C of A . Since Δ^γ is cyclic, there exist elements $v_t \in A$ such that $c_t^\gamma = v_t \delta^\gamma$. We denote

$$V^\gamma = \{v_t: t \in \mathcal{T}(\gamma)\}.$$

Lemma 2.7.

- (1) $Ay_\gamma + \bar{A}^\gamma = \text{span}\{vy_\gamma: v \in V^\gamma\} + \bar{A}^\gamma$.
- (2) $y_\gamma A + \bar{A}^\gamma = \text{span}\{y_\gamma v^*: v \in V^\gamma\} + \bar{A}^\gamma$.

Proof. Considering the isomorphism $ay_\gamma + \bar{A}^\gamma \mapsto a\delta^\gamma$ from $(Ay_\gamma + \bar{A}^\gamma)/\bar{A}^\gamma$ to Δ^γ , we see that $\{vy_\gamma + \bar{A}^\gamma: v \in V^\gamma\}$ is a basis of $(Ay_\gamma + \bar{A}^\gamma)/\bar{A}^\gamma$. Lifting to A^γ , we have that $\text{span}\{vy_\gamma: v \in V^\gamma\} + \bar{A}^\gamma = Ay_\gamma + \bar{A}^\gamma$. Applying the involution $*$ and using that $y_\gamma^* \equiv y_\gamma \pmod{\bar{A}^\gamma}$, we obtain statement (2). $\square$

Lemma 2.8.

- (1) For $\gamma \in \Gamma$ and $s, t \in \mathcal{T}(\gamma)$, we have

$$v_s y_\gamma v_t^* \equiv c_{s,t}^\gamma \pmod{\bar{A}^\gamma}.$$

- (2) $\{v_s y_\gamma v_t^*: \gamma \in \Gamma \text{ and } s, t \in \mathcal{T}(\gamma)\}$ is a cellular basis of A equivalent to the original cellular basis C of A .

Proof. Point (1) holds because both $v_s y_\gamma v_t^*$ and $c_{s,t}^\gamma$ are liftings in A^γ of $\alpha_\gamma^{-1}(c_s^\gamma \otimes (c_t^\gamma)^*)$. Point (2) follows from point (1). $\square$

We record a version of [Lemma 2.3](#) that is adapted to the context of cyclic cellular algebras.

Lemma 2.9. For each $\gamma \in \Gamma$, let $\{b_t: t \in \mathcal{T}(\gamma)\}$ be an R -basis of the cell module Δ^γ . For $t \in \mathcal{T}(\gamma)$, choose $v_t' \in A$ such that $b_t = v_t' \delta^\gamma$. For $s, t \in \mathcal{T}(\gamma)$, let $b_{s,t}^\gamma = v_s' y_\gamma (v_t')^*$. Then $\mathcal{B} = \{b_{s,t}^\gamma: \gamma \in \Gamma \text{ and } s, t \in \mathcal{T}(\gamma)\}$ is a cellular basis of A equivalent to the original cellular basis $\mathcal{C}$.

Proof. For each $\gamma \in \Gamma$ and each $s, t \in \mathcal{T}(\gamma)$, $b_{s,t}^\gamma$ is a lifting in A^γ of $\alpha_\gamma^{-1}(b_s \otimes (b_t)^*)$, so this follows at once from [Lemma 2.3](#). $\square$

2.5. Abelian cyclic cellular algebras

Lemma 2.10. Let A be an abelian cellular algebra with cellular basis

$$\mathcal{C} = \{c_{s,t}^\gamma: \gamma \in \Gamma \text{ and } s, t \in \mathcal{T}(\gamma)\}.$$

- (1) For every $a \in A$ and $\gamma \in \Gamma$, there exists $r(a) \in R$ such that for all $x \in \Delta^\gamma$, $ax = r(a)x$. Moreover, $a \mapsto r(a)$ is an R -algebra homomorphism from A to R .
- (2) If $|\mathcal{T}(\gamma)| > 1$ for some $\gamma \in \Gamma$, then $(A^\gamma)^2 \subseteq \bar{A}^\gamma$.

Proof. Let $a \in A$, $\gamma \in \Gamma$, and $s, t \in \mathcal{T}(\gamma)$. From [Definition 2.1\(2\)](#), we have

$$ac_{s,t}^\gamma \equiv \sum_v r_v^s(a) c_{v,t}^\gamma \pmod{\bar{A}^\gamma},$$

where the coefficients $r_v^s(a)$ do not depend on $t \in \mathcal{T}(\gamma)$. On the other hand, from [Eq. \(2.1\)](#), we have

$$ac_{s,t}^\gamma = c_{s,t}^\gamma a \equiv \sum_w r_w^t(a^*) c_{s,w}^\gamma \pmod{\bar{A}^\gamma},$$

where the coefficients $r_w^t(a^*)$ do not depend on $s \in \mathcal{T}(\gamma)$. Comparing these expansions, we have $r_v^s(a) = 0$ if $s \neq v$. Moreover, we have

$$ac_{s,t}^\gamma \equiv r_s^s(a) c_{s,t}^\gamma \equiv r_t^t(a^*) c_{s,t}^\gamma \pmod{\bar{A}^\gamma}.$$

It follows from this that the coefficient $r_s^s(a)$ is actually independent of $s \in \mathcal{T}(\gamma)$. This implies statement (1) of the lemma.

Suppose that $|\mathcal{T}(\gamma)| > 1$ for some $\gamma \in \Gamma$. Let $a, b, t \in \mathcal{T}(\gamma)$, with $a \neq b$. It follows from $c_{a,t}^\gamma c_{b,t}^\gamma = c_{b,t}^\gamma c_{a,t}^\gamma$ that

$$\langle c_t^\gamma, c_b^\gamma \rangle c_{a,t}^\gamma \equiv \langle c_t^\gamma, c_a^\gamma \rangle c_{b,t}^\gamma \pmod{\bar{A}^\gamma}.$$

This implies that $\langle c_t^\gamma, c_b^\gamma \rangle = 0$. Since b, t are arbitrary elements of $\mathcal{T}(\gamma)$, this means that the bilinear form $\langle \cdot, \cdot \rangle$ on Δ^γ is identically zero. Statement (2) of the lemma follows. $\square$

Proposition 2.11. Let A be an abelian cellular algebra over an integral domain R , with cellular basis

$$\mathcal{C} = \{c_{s,t}^\gamma : \gamma \in \Gamma \text{ and } s, t \in \mathcal{T}(\gamma)\}.$$

Consider the following conditions.

- (1) A is cyclic cellular.
- (2) $|\mathcal{T}(\gamma)| = 1$ for all $\gamma \in \Gamma$.
- (3) The involution $*$ induces the identity map on $A^\gamma / \bar{A}^\gamma$ for all $\gamma \in \Gamma$.
- (4) The involution $*$ is trivial, i.e. $a^* = a$ for all $a \in A$.

The implications

$$(1) \iff (2) \iff (3) \iff (4)$$

hold. If A is strictly cellular, then all the conditions are equivalent. If $\text{char}(R) \neq 2$, then all the conditions are equivalent; and if they hold, then A is strictly cellular.

Proof. The following implications are evident:

$$(1) \iff (2) \iff (3) \iff (4).$$

If A is cyclic cellular, then for all $\gamma \in \Gamma$ and $s, t \in \mathcal{T}(\gamma)$, there exists $r \in R$ such that

$$c_{s,t}^\gamma = v_s^\gamma y_\gamma (v_t^\gamma)^* \equiv r y_\gamma \pmod{\bar{A}^\gamma},$$

by Lemma 2.10, part (1). Thus $\mathcal{T}(\gamma)$ is a singleton for all $\gamma \in \Gamma$, and we have $(1) \implies (2)$. If A is strictly cellular, the implication $(2) \implies (4)$ is evident, so all the conditions are equivalent.

Finally, let us suppose that $\text{char}(R) \neq 2$ and that $\mathcal{T}(\gamma)$ is a singleton for all $\gamma \in \Gamma$. Let F denote the field of fractions of R . Because 2 is invertible in F , the cellular algebra $A^F = A \otimes_R F$ is strictly cellular by Remark 2.4 in [11]. Therefore the involution $*$ on A^F is trivial. But this implies that the involution $*$ is trivial on A . Since the involution on A is trivial, in fact A is strictly cellular. $\square$

The following example is due to Johannes Hahn, and we are grateful to him for permission to include it here.

Example 2.12. Let $R = \mathbb{Z}/2\mathbb{Z}$ and let $A = R[x]/(x^3)$. For $p(x) \in R[x]$ write $[p(x)]$ for the image of $p(x)$ in A . Define an involution on A by $[1]^* = [1]$, $[x]^* = [x] + [x^2]$ and $[x^2]^* = [x^2]$. (Note here that the verification that $*$ is an involution depends on R being of characteristic 2; namely $([x]^*)^* = [x] + 2[x^2] = [x]$, because $2 = 0$ in R .)

Let $\Gamma = \{2 > 1 > 0\}$, and let $\mathcal{T}(i)$ be a singleton for each $i \in \Gamma$. Then $\{[x^i] : i \in \Gamma\}$ is a cellular basis with respect to the involution $*$, each $\mathcal{T}(i)$ is a singleton, but the involution on A is non-trivial.

This provides an example of the failure of the implication $(2) \implies (4)$ in Proposition 2.11, in case the characteristic of the ground ring is 2. It also provides an example of a cellular algebra which is not strictly cellular, i.e. has no equivalent cellular basis which is strictly cellular.

Example 2.13. Here is an example of an abelian cellular algebra that is not cyclic cellular. Let Γ be the two-element set $\{1, 2\}$, with total order, $2 > 1$. Let $\mathcal{T}(1) = \{1\}$ and $\mathcal{T}(2) = \{1, 2\}$. Let A be the free $\mathbb{Z}$ -module with basis consisting of $1 = c_{1,1}^1$ and elements $c_{s,t}^2$ for $1 \leq s, t \leq 2$. Declare $(c_{s,t}^\gamma)^* = c_{t,s}^\gamma$ for all $\gamma \in \Gamma$ and $s, t \in \mathcal{T}(\gamma)$. Declare $1x = x1 = x$ for all basis elements x , and all other products of basis elements to be zero. Then A is an abelian cellular algebra, and A is not cyclic cellular, since $\mathcal{T}(2)$ is not a singleton.

2.6. Examples of cyclic cellular algebras

Many of the most important examples of cellular algebras are cyclic cellular.

Example 2.14 (*The Hecke algebra of type A_{n-1}*). Fix $n \geq 1$. Let R be an integral domain and q an invertible element of R . The Hecke algebra $H_n = H_{n,R}(q)$ is the unital R -algebra with generators $T_1, \dots, T_{n-1}$ satisfying the braid relations $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$ for $i \leq n-2$ and $T_i T_j = T_j T_i$ when $|i-j| \geq 2$, and the quadratic relation $(T_i - q)(T_i + 1) = 0$ for all i .

Let s_i denote the transposition $(i, i+1)$ in the symmetric group $\mathfrak{S}_n$. Let $\pi \in \mathfrak{S}_n$ and let $\pi = s_{i_1} \cdots s_{i_\ell}$ be a reduced expression for π ; the element $T_{i_1} \cdots T_{i_\ell}$ in H_n is independent of the choice of the reduced expression and is denoted by T_π . The set of T_π as π varies over $\mathfrak{S}_n$ is a linear basis of H_n . Because of the symmetry of the defining relations, H_n has a unique R -linear algebra involution $*$ such that $T_i^* = T_i$ for each i ; we have $T_\pi^* = T_{\pi^{-1}}$.

Murphy [22] described a cellular basis of H_n ; for an alternative and original treatment of Murphy's theory, see [20, Chapter 3]. It is evident from the description of Murphy's cellular basis that the cell modules are cyclic.

Example 2.15 (*The cyclotomic Hecke algebras*). Fix $n \geq 1$ and $r \geq 1$. Let R be an integral domain and $q, Q_1, \dots, Q_r$ invertible elements of R . The cyclotomic Hecke algebra $\mathcal{H}_n = \mathcal{H}_{n,R}(q, Q_1, \dots, Q_r)$ [2,1,7] is the unital R -algebra with generators $T_0, T_1, \dots, T_{n-1}$ such that

- (1) $T_1, \dots, T_{n-1}$ satisfy the relations of the Hecke algebra $H_{n,R}(q)$.
- (2) $T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0$, and T_0 commutes with T_i for $i \geq 2$.
- (3) $\prod_{j=1}^r (T_0 - Q_j) = 0$.

In [7] a cellular basis of $\mathcal{H}_n$ is described that is analogous to the Murphy basis of the Hecke algebra H_n . The cell modules defined via this basis are manifestly cyclic.

Example 2.16 (*The q -Schur algebras*). There are a number of different descriptions of the q -Schur algebras in the literature; here we follow the approach in [20, Chapter 4]. Let R be an integral domain and q an invertible element in R . Write H_n for the Hecke algebra over R with parameter q as in Example 2.14. To each composition μ of n , one associates a “permutation module” M^μ of H_n . Fix $d \leq n$; let $C(d, n)$ denote the compositions of n with no more than d parts. The q -Schur algebra $\mathcal{S} = \mathcal{S}(d, n)$ is

$$\mathcal{S} = \text{End}_{H_n} \left(\bigoplus_{\mu \in C(d, n)} M^\mu \right).$$

The cellular structure of the q -Schur algebra is described in detail in [20, Chapter 4], and one can easily see from the description that the cell modules are cyclic.

Example 2.17 (*Algebras containing a Jones basic construction*). In [11], a framework was developed for proving cellularity of a tower $(A_n)_{n \geq 0}$ of algebras which is obtained “by repeated Jones basic constructions” from a tower $(Q_n)_{n \geq 0}$ of cellular algebras. In [9], the framework was refined by adding the assumption that the algebras Q_n are cyclic cellular; the refinement of the main theorem from [11] then gives that the algebras A_n are cyclic cellular. This theory applies in particular to the following examples: Brauer algebras, BMW algebras, Jones–Temperley–Lieb algebras, and partition algebras. The theory yields that all of these algebras are cyclic cellular.

It should be noted that the cellularity of these algebras was known previously, and that in general one can verify from previous proofs of cellularity that the algebras are cyclic cellular. We refer the reader to the relevant citations of the literature in [11,9].

3. Preliminaries

3.1. Combinatorial preliminaries

A *composition* is a finite sequence of nonnegative integers, and a *partition* is a composition whose entries are non-increasing. The entries of a composition are called its *parts*. For composition α , we denote by $|\alpha|$ the sum of the parts of α . Let C_n denote the set of compositions α with $|\alpha| = n$ and Λ_n the set of partitions λ with $|\lambda| = n$. We also say that α is a composition of n if $\alpha \in C_n$ and that λ is a partition of n if $\lambda \in \Lambda_n$.

The *Young diagram* of a composition $\alpha = (\alpha_i)$ is the set

$$\{(i, j): i \geq 1 \text{ and } j \leq \alpha_i\},$$

which is regarded as a left justified array of nodes or boxes in the plane, as usual. It is convenient to identify a composition with its Young diagram.

Let $(\Gamma, \geq)$ be a finite partially ordered set of cardinality r . Let C_n^Γ denote the set of maps λ from Γ to the set of compositions such that $\sum_{\gamma \in \Gamma} |\lambda(\gamma)| = n$. Let $\Lambda_n^\Gamma \subseteq C_n^\Gamma$ denote the set of such maps such that each $\lambda(\gamma)$ is a partition. Fix a listing of Γ ,

$$\Gamma = (\gamma(1), \dots, \gamma(r)). \quad (3.1)$$

We suppose (although this is not essential) that the listing is consistent with the partial order of Γ , in the sense that

$$\gamma(i) \geq \gamma(j) \implies i \leq j.$$

Using this fixed listing, we identify C_n^Γ with *multicompositions* (that is, sequences $(\lambda^{(1)}, \dots, \lambda^{(r)})$ of compositions) via

$$\lambda^{(i)} = \lambda(\gamma(i)) \quad (1 \leq i \leq r).$$

To each multicomposition λ we associate the composition $\alpha(\lambda)$ with parts $\alpha(\lambda)_i = |\lambda^{(i)}|$. The Young diagram of a multicomposition is the sequence of Young diagrams of its components. We will identify a multicomposition with its Young diagram.

For $\lambda \in C_n^\Gamma$, a λ -*tableau* $t = (t^{(1)}, \dots, t^{(r)})$ is a filling of the nodes of λ with the numbers $\{1, \dots, n\}$ without repetition. The $\lambda^{(k)}$ -tableaux $t^{(k)}$ are called the components of t . A λ -tableau is called *row standard* if the entries in each row of each component increase from left to right. A λ -tableau t is called *standard* if $\lambda \in \Lambda_n^\Gamma$, t is row standard, and the entries in each column of each component increase from top to bottom.

Let $\lambda \in C_n^\Gamma$. The symmetric group $\mathfrak{S}_n$ acts freely and transitively on λ -tableaux by acting on the entries of the tableaux. Let t^λ be the λ -tableau in which the entries $1, 2, \dots, n$ appear in order along the rows of the first component, then along the rows of the second component, and so forth. The row stabilizer of t^λ (that is, the subgroup of permutations leaving invariant the set of entries in each row of each component of t^λ) is the Young subgroup $\mathfrak{S}_\lambda$. For each λ -tableau t , let $d(t)$ be the unique permutation such that $t = d(t)t^\lambda$; then $t \mapsto d(t)$ is a bijection between λ -tableaux and permutations. The tableau t is row standard if and only if $d(t)$ is a distinguished (minimal length) left coset representative of $\mathfrak{S}_\lambda$ in $\mathfrak{S}_n$; see [20, Proposition 3.3].

We define two partial orders on C_n^Γ :

Definition 3.1. Let λ and μ be elements of C_n^Γ .

- (1) Dominance order $\triangleright$ is defined as in [7, Definition 3.11], namely $\lambda \triangleright \mu$ if for all k ($1 \leq k \leq r$) and all $j \geq 0$,

$$\sum_{s < k} |\lambda^{(s)}| + \sum_{i \leq j} \lambda_i^{(k)} \geq \sum_{s < k} |\mu^{(s)}| + \sum_{i \leq j} \mu_i^{(k)}.$$

We write $\lambda \triangleright \mu$ if $\lambda \triangleright \mu$ and $\lambda \neq \mu$.

- (2) Γ -dominance order $\triangleright_\Gamma$ is defined similarly: $\lambda \triangleright_\Gamma \mu$ if for all $\gamma \in \Gamma$, and for all $j \geq 0$,

$$\sum_{\gamma' > \gamma} |\lambda(\gamma')| + \sum_{i \leq j} \lambda(\gamma)_i \geq \sum_{\gamma' > \gamma} |\mu(\gamma')| + \sum_{i \leq j} \mu(\gamma)_i.$$

We write $\lambda \triangleright_\Gamma \mu$ if $\lambda \triangleright_\Gamma \mu$ and $\lambda \neq \mu$.

Example 3.2. Neither of the conditions $\lambda \triangleright \mu$ and $\lambda \triangleright_\Gamma \mu$ implies the other.

- (1) Let $\Gamma = \{a, b\}$, with a and b incomparable. Choose the listing $\Gamma = (a, b)$. Let $\lambda(a) = (2)$, $\lambda(b) = \emptyset$ and $\mu(a) = \mu(b) = (1)$. Then $\lambda \triangleright \mu$ but λ and μ are incomparable in Γ -dominance order.
- (2) Let $\Gamma = \{a, b, c, d\}$ with $a > x$ for each $x \in \{b, c, d\}$ and $\{b, c, d\}$ mutually incomparable. Choose the listing $\Gamma = (a, b, c, d)$. Let $\lambda(a) = (4)$, $\lambda(b) = (2)$, $\lambda(c) = (1)$, $\lambda(d) = (1)$ and let $\mu(a) = (3)$, $\mu(b) = (3)$, $\mu(c) = (2)$, $\mu(d) = (0)$. Then $\lambda \triangleright_\Gamma \mu$ but λ and μ are incomparable in dominance order. In this example, λ cannot be obtained from μ by any sequence of “raising operators”, cf. [19, Section I.1].

We extend these partial orders to partial orders on the set of row standard tableaux. If t is a row standard λ -tableau, then for each $m \leq n$, the entries $1, 2, \dots, m$ in t occupy the diagram of a multicomposition, say μ ; the μ -tableau obtained by removing the nodes containing $m+1, \dots, n$ from t is denoted by $t \downarrow m$; we also denote μ by $[t \downarrow m]$.

Definition 3.3. Let λ and μ be elements of C_n^Γ , and let t be a row standard λ -tableau, and s a row standard μ -tableau.

- (1) We say that t dominates s and write $t \triangleright s$, if $[t \downarrow m] \triangleright [s \downarrow m]$ for each m . We write $t \triangleright s$ if $t \triangleright s$ and $t \neq s$.
- (2) Let t be a row standard λ -tableau, and s a row standard μ -tableau. We say that t Γ -dominates s and write $t \triangleright_\Gamma s$, if $[t \downarrow m] \triangleright_\Gamma [s \downarrow m]$ for each m . We write $t \triangleright_\Gamma s$ if $t \triangleright_\Gamma s$ and $t \neq s$.

Definition 3.4. Let $\lambda \in C_n^\Gamma$ and let t be a λ -tableau. Define the component function of t by $\text{comp}_t(j) = k$ if j appears in the k -th component of t . For a λ -tableau t and a μ -tableau s , write $\text{comp}_t = \text{comp}_s$ if $\text{comp}_t(j) = \text{comp}_s(j)$ for all j .

Definition 3.5. A λ -tableau t is of the initial kind if $\text{comp}_t = \text{comp}_{\lambda}$. Equivalently, $d(t) \in \mathfrak{S}_{\alpha(\lambda)}$.

Remark 3.6. If λ and μ satisfy $|\lambda(\gamma)| = |\mu(\gamma)|$ for all γ (i.e. $\alpha(\lambda) = \alpha(\mu)$), then

$$\lambda \triangleright \mu \iff \lambda \triangleright_\Gamma \mu \iff \lambda(\gamma) \triangleright \mu(\gamma) \text{ for all } \gamma.$$

Suppose that t is a λ -tableau and s a μ -tableau, with $\text{comp}_t = \text{comp}_s$. It follows that $\alpha([t \downarrow m]) = \alpha([s \downarrow m])$ for all m . Therefore

$$t \triangleright s \iff t \triangleright_\Gamma s \iff t^{(k)} \triangleright s^{(k)} \text{ for all } k.$$

Remark 3.7. All of the definitions of this section apply in particular when $r = 1$. Thus we have defined λ -tableaux and row standard λ -tableaux for $\lambda \in C_n$, standard λ -tableaux for $\lambda \in A_n$, dominance order on compositions and on row standard tableaux, and the “superstandard” tableau t^λ for $\lambda \in C_n$.

3.2. Cellularity of tensor products

Let $A^{(i)}$ ($1 \leq i \leq K$) be cellular algebras over an integral domain R , with algebra involutions denoted by $*$, partially ordered sets $(\Gamma_i, \geq)$, index sets $\mathcal{T}(\gamma)$ for $\gamma \in \Gamma_i$, and cellular bases $\{a_{s,t}^\gamma: \gamma \in \Gamma_i, s, t \in \mathcal{T}(\gamma)\}$. Then $A^{(1)} \otimes \cdots \otimes A^{(K)}$ is also a cellular algebra. The involution on the tensor product algebra is the tensor product of the involutions on each $A^{(i)}$. The partially ordered set in the cellular structure for the tensor product algebra is the cartesian product $\Pi = \Gamma_1 \times \cdots \times \Gamma_K$, with the product partial order, namely $(\gamma_1, \dots, \gamma_K) \geq (\gamma'_1, \dots, \gamma'_K)$ if $\gamma_i \geq \gamma'_i$ for all i . For each $\gamma = (\gamma_1, \dots, \gamma_K) \in \Pi$, the index set $\mathcal{T}(\gamma)$ is $\mathcal{T}(\gamma_1) \times \cdots \times \mathcal{T}(\gamma_K)$. The cellular basis is the set of simple tensors

$$a_{s,t}^\gamma = a_{s_1,t_1}^{\gamma_1} \otimes \cdots \otimes a_{s_K,t_K}^{\gamma_K},$$

for $\gamma = (\gamma_1, \dots, \gamma_K) \in \Pi$, and $s = (s_1, \dots, s_K)$, $t = (t_1, \dots, t_K) \in \mathcal{T}(\gamma)$. If Δ^γ ($\gamma \in \Gamma_i$) denote the cell modules of each $A^{(i)}$, then the cell modules of the tensor product algebra are

$$\Delta^\gamma = \Delta^{\gamma_1} \otimes \cdots \otimes \Delta^{\gamma_K},$$

for $\gamma = (\gamma_1, \dots, \gamma_K) \in \Pi$.

Suppose that each $A^{(i)}$ is cyclic cellular with data

$$\delta^\gamma, V^\gamma = \{v_s: s \in \mathcal{T}(\gamma)\} \text{ and } y_\gamma,$$

for each $\gamma \in \Gamma_i$, as in [Notation 2.6](#). For $\gamma = (\gamma_1, \dots, \gamma_K) \in \Pi$ and $s = (s_1, \dots, s_K) \in \mathcal{T}(\gamma)$, define

$$v_s = v_{s_1} \otimes \cdots \otimes v_{s_K}.$$

Then the tensor product algebra is cyclic cellular as well, with

$$\delta^\gamma = \delta^{\gamma_1} \otimes \cdots \otimes \delta^{\gamma_K}, \quad V^\gamma = \{v_s: s \in \mathcal{T}(\gamma)\}, \quad \text{and} \quad y_\gamma = y_{\gamma_1} \otimes \cdots \otimes y_{\gamma_K}.$$

3.3. The Murphy basis of $R\mathfrak{S}_n$

Let R be an integral domain. Fix $n \geq 1$. Consider the set Λ_n of partitions of n with dominance order. For each $\lambda \in \Lambda_n$, let $\mathcal{T}(\lambda)$ be the set of standard λ -tableaux. Let $*$ be the algebra involution of $R\mathfrak{S}_n$ such that $\pi^* = \pi^{-1}$ for $\pi \in \mathfrak{S}_n$. For $\lambda \in \Lambda_n$, let $x_\lambda = \sum_{w \in \mathfrak{S}_\lambda} w$, where $\mathfrak{S}_\lambda$ is the row stabilizer of t^λ . For $s, t \in \mathcal{T}(\lambda)$, define

$$m_{s,t}^\lambda = d(s)x_\lambda d(t)^*.$$

The symmetric group algebra is the special case of the Hecke algebra $H_{n,R}(q)$ with $q = 1$, and the collection of elements $m_{s,t}^\lambda$ is the specialization of the Murphy basis of the Hecke algebra, see [Example 2.14](#). Thus $\{m_{s,t}^\lambda: \lambda \in \Lambda_n, s, t \in \mathcal{T}(\lambda)\}$ is a cellular basis of $R\mathfrak{S}_n$, and $R\mathfrak{S}_n$ is cyclic cellular.

3.4. The cellular structure of $R\mathfrak{S}_\alpha$

Let R be an integral domain. Let $\alpha = (\alpha_1, \dots, \alpha_r)$ be a composition of an integer n , with r parts. Let $\mathfrak{S}_\alpha \subseteq \mathfrak{S}_n$ be the Young subgroup, the row stabilizer of t^α . Let $n_k = \alpha_1 + \dots + \alpha_{k-1}$ for $1 \leq k \leq r$ and for each k realize $\mathfrak{S}_{\alpha_k}$ as the symmetric group of the set $\{n_k + 1, \dots, n_{k+1}\}$. Then $\mathfrak{S}_{\alpha_1} \times \dots \times \mathfrak{S}_{\alpha_r} \cong \mathfrak{S}_\alpha$ by $(\pi_1, \dots, \pi_r) \mapsto \pi_1 \pi_2 \dots \pi_r$, and $R\mathfrak{S}_{\alpha_1} \otimes \dots \otimes R\mathfrak{S}_{\alpha_r} \cong R\mathfrak{S}_\alpha$ by

$$\pi_1 \otimes \dots \otimes \pi_r \mapsto \pi_1 \pi_2 \dots \pi_r. \quad (3.2)$$

By the discussion in Sections 3.2 and 3.3, the tensor product algebra $R\mathfrak{S}_{\alpha_1} \otimes \dots \otimes R\mathfrak{S}_{\alpha_r}$ is cellular; the partially ordered set for the cellular structure is $\Lambda_{\alpha_1} \times \dots \times \Lambda_{\alpha_r}$ with the product of dominance partial order on each component; but this is the same as the set Λ_α of multipartitions λ with $\alpha(\lambda) = \alpha$, with dominance order of multipartitions, by Remark 3.6. For $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)}) \in \Lambda_\alpha$, the index set $\mathcal{T}(\lambda)$ is $\mathcal{T}(\lambda^{(1)}) \times \dots \times \mathcal{T}(\lambda^{(r)})$; but this can be identified with the set of standard λ -tableaux of the initial kind.

The cellular basis is the tensor product of the Murphy cellular basis of each $R\mathfrak{S}_{\alpha_i}$. Under the isomorphism (3.2), this basis is carried to the set of elements

$$m_{\mathfrak{s}, \mathfrak{t}}^\lambda = d(\mathfrak{s}) x_\lambda d(\mathfrak{t})^*,$$

where $\lambda \in \Lambda_\alpha$, and $\mathfrak{s}$ and $\mathfrak{t}$ are standard λ -tableaux of the initial kind. The cell modules for the tensor product algebra $R\mathfrak{S}_{\alpha_1} \otimes \dots \otimes R\mathfrak{S}_{\alpha_r}$ are the modules

$$\Delta_R^\lambda = \Delta_r^{\lambda^{(1)}} \otimes \dots \otimes \Delta_R^{\lambda^{(r)}},$$

where $\lambda \in \Lambda_\alpha$ and $\Delta_R^{\lambda^{(i)}}$ denotes a cell module for $R\mathfrak{S}_{\alpha_i}$.

We are going to write J^λ for $\overline{R\mathfrak{S}_\alpha}^\lambda = \text{span}\{m_{\mathfrak{u}, \mathfrak{v}}^\mu : \mu \triangleright \lambda \text{ and } \mathfrak{u}, \mathfrak{v} \in \mathcal{T}(\mu)\}$. Under the identification of the tensor product algebra with $R\mathfrak{S}_\alpha$, the cell module Δ_R^λ can be identified with

$$\text{span}\{m_{\mathfrak{s}, \mathfrak{t}^\lambda}^\lambda + J^\lambda\},$$

where $\mathfrak{s}$ varies over standard λ -tableaux of the initial kind.

4. Wreath products

Let A be an R -algebra. The symmetric group $\mathfrak{S}_n$ acts by algebra automorphisms on the tensor product algebra $A^{\otimes n}$, by place permutations,

$$\pi(a_1 \otimes \dots \otimes a_n) = a_{\pi^{-1}(1)} \otimes \dots \otimes a_{\pi^{-1}(n)}.$$

The wreath product $A \wr \mathfrak{S}_n$ is the crossed product algebra $A^{\otimes n} \rtimes \mathfrak{S}_n$. If A is an algebra with involution $*$, then $\mathfrak{S}_n$ acts by $*$ -preserving automorphisms and the wreath product is also an algebra with involution determined by

$$\begin{aligned} ((a_1 \otimes \dots \otimes a_n)\pi)^* &= \pi^{-1}(a_1^* \otimes \dots \otimes a_n^*) \\ &= \pi^{-1}(a_1^* \otimes \dots \otimes a_n^*)\pi^{-1}. \end{aligned}$$

An R -module M is a module for the wreath product $A \wr \mathfrak{S}_n$ if and only if M is both an $A^{\otimes n}$ -module and a $\mathfrak{S}_n$ -module, and the following covariance condition is satisfied:

$$\pi(am) = {}^\pi a(\pi m),$$

for $\pi \in \mathfrak{S}_n$, $a \in A^{\otimes n}$, and $m \in M$.

4.1. Cellularity of wreath products

In this section, we will prove the following theorem:

Theorem 4.1. *Let A be a cyclic cellular algebra. Then for all $n \geq 1$, the wreath product algebra $A \wr \mathfrak{S}_n$ is a cyclic cellular algebra.*

In Dipper, James, and Mathas [7, Section 3], it is shown that the cyclotomic Hecke algebras are cellular. This theorem contains as a special case the statement that the wreath product algebra $RC_r \wr \mathfrak{S}_n$ is cellular, where C_r denotes the cyclic group with r elements, and R is assumed to contain a primitive r -th root of unity. In proving Theorem 4.1, we adapt the proof of [7, Theorem 3.26].

Remark 4.2. In Theorem 4.1, if A is strictly cellular then so is $A \wr \mathfrak{S}_n$.

Let A be a cyclic cellular algebra, over an integral domain R , with cell datum $(A, *, \Gamma, \geq, \mathcal{T}, C)$. For $\gamma \in \Gamma$, let

$$\delta^\gamma, V^\gamma = \{v_j : j \in \mathcal{T}(\gamma)\}, \text{ and } y_\gamma$$

be as described in Notation 2.6.

First we are going to define some elements of $A \wr \mathfrak{S}_n$ which are analogues of the Murphy elements $m_{s,t}^\lambda$ (in the Hecke algebra or in the symmetric group algebra) associated to a composition λ and a pair of row standard λ -tableaux, see [22, Section 4].

Let r denote the cardinality of Γ , and let $\Gamma = (\gamma(1), \dots, \gamma(r))$ be a listing of Γ such that $\gamma(i) \geq \gamma(j)$ implies $i \leq j$.

Given $\lambda \in C_n^\Gamma$, let $\alpha = \alpha(\lambda)$. For $1 \leq k \leq r$, let $n_k = \alpha_1 + \dots + \alpha_{k-1}$. Let V^α be the set of simple tensors $w = w_1 \otimes \dots \otimes w_n$ in $A^{\otimes n}$ such that $w_i \in V^{\gamma(k)}$ if $n_k + 1 \leq i \leq n_{k+1}$; that is, the first α_1 tensorands belong to $V^{\gamma(1)}$, the next α_2 tensorands belong to $V^{\gamma(2)}$, and so forth. Note that ${}^\pi(V^\alpha) = V^\alpha$ for all $\pi \in \mathfrak{S}_\alpha$. Set

$$y^\alpha = y_{\gamma(1)}^{\otimes \alpha_1} \otimes \dots \otimes y_{\gamma(r)}^{\otimes \alpha_r}.$$

Observe that ${}^\pi(y^\alpha) = y^\alpha$ for $\pi \in \mathfrak{S}_\alpha$.

Remark 4.3. If we let $\gamma(\alpha) \in \Gamma^n$ be the sequence whose first α_1 entries are equal to $\gamma(1)$, whose next α_2 entries are equal to $\gamma(2)$, and so forth, then in the notation of Section 3.2, $V^\alpha = V^{\gamma(\alpha)}$ and $y^\alpha = y_{\gamma(\alpha)}$.

Define

$$x_\lambda = \sum_{w \in \mathfrak{S}_\lambda} w \in R\mathfrak{S}_n.$$

Note that $x_\lambda^* = x_\lambda$ and $\pi x_\lambda = x_\lambda \pi = x_\lambda$ for $\pi \in \mathfrak{S}_\lambda$. Since elements of $\mathfrak{S}_\alpha$ commute with y^α , and $\mathfrak{S}_\lambda \subseteq \mathfrak{S}_\alpha$ we have $x_\lambda y^\alpha = y^\alpha x_\lambda$.

We consider the following elements of $A \wr \mathfrak{S}_n$,

$$m_{(s,v), (t,w)}^\lambda = d(s) v y^\alpha x_\lambda w^* d(t)^*,$$

where $\lambda \in C_n^\Gamma$, $\alpha = \alpha(\lambda)$, s, t are row standard λ -tableaux, and $v, w \in V^\alpha$. We will also need to consider the elements

$$m_{(s,v), (t,1)}^\lambda = d(s)vy^\alpha x_\lambda d(t)^*,$$

where $w \in V^\alpha$ has been replaced by the multiplicative identity. In order to facilitate statements that apply to both types of elements, we write $V_1^\alpha = V^\alpha \cup \{1\}$.

Lemma 4.4. *Let $\lambda \in C_n^\Gamma$ and $\alpha = \alpha(\lambda)$. Let s, t be row standard λ -tableaux, and let $v \in V^\alpha$ and $w \in V_1^\alpha$. For $\pi \in \mathfrak{S}_n$,*

$$\pi m_{(s,v), (t,w)}^\lambda = m_{(s',v'), (t,w)}^\lambda,$$

for some row standard λ -tableau s' and some $v' \in V^\alpha$. Similarly, for s, t as above and $v, w \in V^\alpha$,

$$m_{(s,v), (t,w)}^\lambda \pi = m_{(s,v), (t',w')}^\lambda,$$

for some row standard λ -tableau t' and some $w' \in V^\alpha$.

Proof. $\pi d(s) = d(s')\pi'$, where $\pi' \in \mathfrak{S}_\lambda$ and $d(s')$ is a distinguished left coset representative of $\mathfrak{S}_\lambda$ in $\mathfrak{S}_n$. Also, $\pi'vy^\alpha x_\lambda = \pi'v\pi'^\alpha y^\alpha \pi'^\alpha x_\lambda = \pi'vy^\alpha x_\lambda$, since y^α is fixed by $\pi' \in \mathfrak{S}_\lambda$ and $\pi'x_\lambda = x_\lambda$.

Thus

$$\begin{aligned} \pi m_{(s,v), (t,w)}^\lambda &= \pi d(s)vy^\alpha x_\lambda w^* d(t)^* \\ &= d(s')\pi'vy^\alpha x_\lambda w^* d(t)^* \\ &= d(s')v'y^\alpha x_\lambda w^* d(t)^* = m_{(s',v'), (t,w)}^\lambda, \end{aligned}$$

where $v' = \pi'v$. The statement concerning multiplication by π on the right follows by a similar computation. $\square$

Notation 4.5. For $\lambda \in C_n^\Gamma$, let $\mathcal{T}(\lambda)$ denote the set of pairs (s, v) , where s is a standard λ -tableau and $v \in V^{\alpha(\lambda)}$.

The statement and proof of the following result are adapted from [7, Proposition 3.18], which in turn generalizes a theorem of Murphy, [22, Theorem 4.18].

Proposition 4.6. *Let $\lambda \in C_n^\Gamma$, and let $\alpha = \alpha(\lambda)$. Suppose that s, t are row standard λ -tableaux and $v, w \in V^\alpha$. Then $m_{(s,v), (t,w)}^\lambda$ is a linear combination of elements $m_{(u,v'), (v,w')}^\mu$, where:*

- (1) $\mu \in \Lambda_n^\Gamma$ with $\alpha(\mu) = \alpha$, and $\mu \triangleright_\Gamma \lambda$.
- (2) (u, v') and (v, w') are elements of $\mathcal{T}(\mu)$.
- (3) $u \triangleright_\Gamma s$, and $\text{comp}_u = \text{comp}_s$.
- (4) $v \triangleright_\Gamma t$ and $\text{comp}_v = \text{comp}_t$.

Proof. Consider first the special case that s and t are of the initial kind for λ (see Definition 3.5). Thus $d(s), d(t) \in \mathfrak{S}_\alpha$, and

$$\begin{aligned}
 m_{(\mathfrak{s}, \mathfrak{v}), (\mathfrak{t}, \mathfrak{w})}^{\lambda} &= d(\mathfrak{s}) v y^{\alpha} x_{\lambda} w^* d(\mathfrak{t})^* \\
 &= (d(\mathfrak{s}) v) y^{\alpha} d(\mathfrak{s}) x_{\lambda} d(\mathfrak{t})^* (d(\mathfrak{t}) w^*).
 \end{aligned} \tag{4.1}$$

Recall the notation $n_1 = 0$, and $n_k = \alpha_1 + \dots + \alpha_{k-1}$ for $k \geq 2$. We can write $d(\mathfrak{s}) x_{\lambda} d(\mathfrak{t})^*$ as the product of r commuting factors, say $d(\mathfrak{s}) x_{\lambda} d(\mathfrak{t})^* = x_1 \dots x_r$, where for each k , x_k is contained in the group algebra of the symmetric group on the letters $\{n_k + 1, \dots, n_{k+1}\}$. Indeed, $x_k = d(\mathfrak{s}^{(k)}) x_{\lambda^{(k)}} d(\mathfrak{t}^{(k)})^*$, where $\lambda^{(k)}$ is the k -th component of λ , $\mathfrak{s}^{(k)}$ the k -th component of $\mathfrak{s}$, and $\mathfrak{t}^{(k)}$ the k -th component of $\mathfrak{t}$.

Applying the theorem of Murphy, [22, Theorem 4.18], to the symmetric group algebra on the letters $\{n_k + 1, \dots, n_{k+1}\}$, we can write each x_k as a linear combination of elements $x'_k = d(\mathfrak{u}^{(k)}) x_{\mu^{(k)}} d(\mathfrak{v}^{(k)})^*$, where $\mu^{(k)}$ is a partition of size α_k , $\mathfrak{u}^{(k)}$ and $\mathfrak{v}^{(k)}$ are standard $\mu^{(k)}$ -tableaux (but with entries in $\{n_k + 1, \dots, n_{k+1}\}$), and $\mathfrak{u}^{(k)} \triangleright \mathfrak{s}^{(k)}$ and $\mathfrak{v}^{(k)} \triangleright \mathfrak{t}^{(k)}$.

For any choice of $x'_k = d(\mathfrak{u}^{(k)}) x_{\mu^{(k)}} d(\mathfrak{v}^{(k)})^*$, for $1 \leq k \leq r$, let $x' = x'_1 x'_2 \dots x'_r$ and let μ be the multi-partition whose k -th component is $\mu^{(k)}$ for each k , $\mathfrak{u}$ the μ -tableau whose k -th component is $\mathfrak{u}^{(k)}$ for each k , and $\mathfrak{v}$ the μ -tableau whose k -th component is $\mathfrak{v}^{(k)}$ for each k . Then $x' = d(\mathfrak{u}) x_{\mu} d(\mathfrak{v})^*$. Since $\mathfrak{u}$ and $\mathfrak{v}$ are of the initial kind (for λ) and $\mathfrak{u}^{(k)} \triangleright \mathfrak{s}^{(k)}$ and $\mathfrak{v}^{(k)} \triangleright \mathfrak{t}^{(k)}$, we have $\mathfrak{u} \triangleright \mathfrak{s}$ and $\mathfrak{v} \triangleright \mathfrak{t}$, by Remark 3.6. (Equivalently, by the same remark, $\mathfrak{u} \triangleright_{\Gamma} \mathfrak{s}$ and $\mathfrak{v} \triangleright_{\Gamma} \mathfrak{t}$.) It follows that $d(\mathfrak{s}) x_{\lambda} d(\mathfrak{t})^*$ is a linear combination of terms $d(\mathfrak{u}) x_{\mu} d(\mathfrak{v})^*$, where μ , $\mathfrak{u}$ and $\mathfrak{v}$ are as in the statement of the lemma. Thus from Eq. (4.1), we have that $m_{(\mathfrak{s}, \mathfrak{v}), (\mathfrak{t}, \mathfrak{w})}^{\lambda}$ is a linear combination of terms

$$\begin{aligned}
 (d(\mathfrak{s}) v) y^{\alpha} d(\mathfrak{u}) x_{\mu} d(\mathfrak{v})^* (d(\mathfrak{t}) w^*) &= d(\mathfrak{u}) (d(\mathfrak{u})^* d(\mathfrak{s}) v) y^{\alpha} x_{\mu} (d(\mathfrak{v})^* d(\mathfrak{t}) w^*) d(\mathfrak{v})^* \\
 &= d(\mathfrak{u}) v' y^{\alpha} x_{\mu} (w')^* d(\mathfrak{v})^* \\
 &= m_{(\mathfrak{u}, \mathfrak{v}'), (\mathfrak{v}, \mathfrak{w}')}^{\mu}.
 \end{aligned} \tag{4.2}$$

This completes the proof in the special case that $\mathfrak{s}$ and $\mathfrak{t}$ are of the initial kind for λ .

We pass to the general case, where the component functions $\text{comp}_{\mathfrak{s}}$ and $\text{comp}_{\mathfrak{t}}$ are arbitrary. We can write $\mathfrak{s} = \sigma \mathfrak{s}'$ and $\mathfrak{t} = \tau \mathfrak{t}'$, where $\mathfrak{s}'$ and $\mathfrak{t}'$ are row standard λ -tableaux of the initial kind, and σ, τ are distinguished left coset representatives of $\mathfrak{S}_{\alpha}$ in $\mathfrak{S}_n$. Furthermore, $d(\mathfrak{s}) = \sigma d(\mathfrak{s}')$ and $d(\mathfrak{t}) = \tau d(\mathfrak{t}')$. It follows that

$$m_{(\mathfrak{s}, \mathfrak{v}), (\mathfrak{t}, \mathfrak{w})}^{\lambda} = \sigma m_{(\mathfrak{s}', \mathfrak{v}), (\mathfrak{t}', \mathfrak{w})}^{\lambda} \tau^*.$$

Applying the result for the special case above to $m_{(\mathfrak{s}', \mathfrak{v}), (\mathfrak{t}', \mathfrak{w})}^{\lambda}$, we have that $m_{(\mathfrak{s}, \mathfrak{v}), (\mathfrak{t}, \mathfrak{w})}^{\lambda}$ is a linear combination of terms

$$\sigma m_{(\mathfrak{u}', \mathfrak{v}'), (\mathfrak{v}', \mathfrak{w}')}^{\mu} \tau^* = \sigma d(\mathfrak{u}') v' y^{\alpha} x_{\mu} (w')^* d(\mathfrak{v}') \tau^*, \tag{4.3}$$

where: $\mu \in \Lambda_n^{\Gamma}$, $\alpha(\mu) = \alpha$, and $\mu \triangleright \lambda$; $\mathfrak{u}', \mathfrak{v}'$ are standard μ -tableaux of the initial kind such that $\mathfrak{u}' \triangleright \mathfrak{s}'$ and $\mathfrak{v}' \triangleright \mathfrak{t}'$; and $\mathfrak{v}', \mathfrak{w}' \in V^{\alpha}$. For each such term, let $\mathfrak{u} = \sigma \mathfrak{u}'$ and $\mathfrak{v} = \tau \mathfrak{v}'$. Then $\mathfrak{u}$ and $\mathfrak{v}$ are standard μ -tableaux with $\text{comp}_{\mathfrak{u}} = \text{comp}_{\mathfrak{s}}$ and $\text{comp}_{\mathfrak{v}} = \text{comp}_{\mathfrak{t}}$. We have $\sigma d(\mathfrak{u}') = d(\mathfrak{u})$ and $\tau d(\mathfrak{v}') = d(\mathfrak{v})$, so the term $\sigma m_{(\mathfrak{u}', \mathfrak{v}'), (\mathfrak{v}', \mathfrak{w}')}^{\mu} \tau^*$ from Eq. (4.3) is equal to

$$d(\mathfrak{u}) v' y^{\alpha} x_{\mu} (w')^* d(\mathfrak{v}) = m_{(\mathfrak{u}, \mathfrak{v}'), (\mathfrak{v}, \mathfrak{w}')}^{\mu}.$$

By [7, Lemma 3.17, part (iii)], we have $\mathfrak{u} \triangleright \mathfrak{s}$ and $\mathfrak{v} \triangleright \mathfrak{t}$. But since $\text{comp}_{\mathfrak{u}} = \text{comp}_{\mathfrak{s}}$ and $\text{comp}_{\mathfrak{v}} = \text{comp}_{\mathfrak{t}}$, this is equivalent to $\mathfrak{u} \triangleright_{\Gamma} \mathfrak{s}$ and $\mathfrak{v} \triangleright_{\Gamma} \mathfrak{t}$, using Remark 3.6 again.

Thus $m_{(\mathfrak{s}, \mathfrak{v}), (\mathfrak{t}, \mathfrak{w})}^{\lambda}$ is a linear combination of terms of the required type. $\square$

Corollary 4.7. Let $\lambda \in C_n^\Gamma$, and write $\alpha = \alpha(\lambda)$. Suppose that s, t are row standard λ -tableaux and $v \in V^\alpha$. Then $m_{(s,v), (t,1)}^\lambda$ is a linear combination of elements $m_{(u,v'), (v,1)}^\mu$, where μ, u, v , and v' are as in the statement of Proposition 4.6.

Proof. Trace through the proof of Proposition 4.6 with $w \in V^\alpha$ replaced by the multiplicative identity. $\square$

Corollary 4.8. Let $\lambda \in C_n^\Gamma$, and write $\alpha = \alpha(\lambda)$. Suppose that s, t are row standard λ -tableaux, and $v, w \in V^\alpha$. Let $\pi \in \mathfrak{S}_n$.

- (1) $\pi m_{(s,v), (t,w)}^\lambda$ is a linear combination of terms $m_{(u,v'), (v,w')}^\mu$, where:
 - (a) $\mu \in \Lambda_n^\Gamma$, $\alpha(\mu) = \alpha$, and $\mu \triangleright_\Gamma \lambda$.
 - (b) (u, v') and (v, w') are elements of $\mathcal{T}(\mu)$.
 - (c) $v \triangleright_\Gamma t$ and $\text{comp}_v = \text{comp}_t$.
- (2) $m_{(s,v), (t,w)}^\lambda \pi$ is a linear combination of terms $m_{(u,v'), (v,w')}^\mu$, where $\mu, (u, v')$, and (v, w') satisfy conditions (a) and (b) of part (1) as well as:
 - (c') $u \triangleright_\Gamma s$ and $\text{comp}_u = \text{comp}_s$.
- (3) For $\pi \in \mathfrak{S}_n$, $\pi m_{(s,v), (t,1)}^\lambda$ is a linear combination of terms $m_{(u,v'), (v,1)}^\mu$, where μ, u, v , and v' are as in part (1).

Proof. Parts (1) and (2) result from combining Lemma 4.4 and Proposition 4.6. Part (2) uses Corollary 4.7 as well. $\square$

We define certain R -submodules of the wreath product algebra $A \wr \mathfrak{S}_n$. We will show later that these are two-sided ideals of $A \wr \mathfrak{S}_n$.

Definition 4.9. Let $\lambda \in C_n^\Gamma$.

- (1) Let N^λ be the R -submodule of $A \wr \mathfrak{S}_n$ spanned by the set of $m_{(s,v), (t,w)}^\mu$ where $\mu \in \Lambda_n^\Gamma$, $\mu \triangleright_\Gamma \lambda$, and $(s, v), (t, w) \in \mathcal{T}(\mu)$.
- (2) Let $\bar{N}^\lambda$ be the R -submodule of $A \wr \mathfrak{S}_n$ spanned by the set of $m_{(s,v), (t,w)}^\mu$ where $\mu \in \Lambda_n^\Gamma$, $\mu \triangleright_\Gamma \lambda$, and $(s, v), (t, w) \in \mathcal{T}(\mu)$.

Corollary 4.10. Let $\lambda \in \Lambda_n^\Gamma$ and write $\alpha = \alpha(\lambda)$. Let s be a standard λ -tableau and $v \in V^\alpha$. Then for $\pi \in \mathfrak{S}_n$,

$$\pi m_{(s,v), (t^\lambda, 1)}^\lambda = x_1 + x_2,$$

where x_1 is a linear combination of elements $m_{(u,v'), (t^\lambda, 1)}^\lambda$, with $(u, v') \in \mathcal{T}(\lambda)$, and $x_2 \in \bar{N}^\lambda$.

Proof. It follows from Corollary 4.8, part (3), that $\pi m_{(s,v), (t^\lambda, 1)}^\lambda = x_1 + x_2$ where $x_2 \in \bar{N}^\lambda$, and x_1 is a linear combination of terms $m_{(u,v'), (v,1)}^\lambda$ with u, v standard λ -tableaux, $v \triangleright_\Gamma t^\lambda$, $\text{comp}_v = \text{comp}_{t^\lambda}$, and $v' \in V^\alpha$. By Remark 3.6, we have $v \triangleright_\Gamma t^\lambda$. Since t^λ is maximum among standard λ -tableaux with respect to dominance order, all the terms featuring in the expansion of x_1 have $v = t^\lambda$. $\square$

In the following discussion, recall that $\gamma(\alpha)$ denotes the sequence in Γ^n whose first α_1 entries are equal to $\gamma(1)$, next α_2 entries are equal to $\gamma(2)$, and so forth.

Lemma 4.11. Let $\lambda \in \Lambda_n^\Gamma$ and let $\alpha = \alpha(\lambda)$. If λ is maximal in Λ_n^Γ with respect to Γ -dominance order $\triangleright_\Gamma$, then $\gamma(\alpha)$ is maximal in Γ^n with respect to the product order.

Proof. Suppose that λ is maximal in Λ_n^Γ . Then for all $\gamma \in \Gamma$, $\lambda(\gamma) = \emptyset$ unless γ is maximal in Γ . For if γ is not maximal, and $\lambda(\gamma)$ is a non-empty partition, then there exists a $\gamma' > \gamma$, and one can produce a $\lambda' \triangleright \lambda$ by removing a box from the γ component of λ and appending it as a new row to the γ' component. This means that $\alpha_i = 0$ unless $\gamma(i)$ is maximal, so only maximal $\gamma(i)$'s appear in the sequence $\boldsymbol{\gamma}(\alpha)$. Thus $\boldsymbol{\gamma}(\alpha)$ is maximal in the product order. $\square$

Corollary 4.12. Suppose $\lambda \in \Lambda_n^\Gamma$ is maximal with respect to $\triangleright_\Gamma$. Let $\alpha = \alpha(\lambda)$. Then for all $a \in A^{\otimes n}$, ay^α is a linear combination of elements $v'y^\alpha$ with $v' \in V^\alpha$. Similarly, $y^\alpha a$ is a linear combination of elements $y_\alpha(v')^*$ with $v' \in V^\alpha$.

Proof. Since $A^{\otimes n}$ is cyclic cellular, for any $\boldsymbol{\gamma} \in \Gamma^n$,

$$A^{\otimes n}y_{\boldsymbol{\gamma}} + \overline{A^{\otimes n}}^{\boldsymbol{\gamma}} = \text{span}\{vy_{\boldsymbol{\gamma}} : v \in V^{\boldsymbol{\gamma}}\} + \overline{A^{\otimes n}}^{\boldsymbol{\gamma}}$$

by Lemma 2.7. If $\boldsymbol{\gamma}$ is maximal in Γ^n , then

$$A^{\otimes n}y_{\boldsymbol{\gamma}} = \text{span}\{vy_{\boldsymbol{\gamma}} : v \in V^{\boldsymbol{\gamma}}\}.$$

If λ is maximal, then $\boldsymbol{\gamma}(\alpha)$ is maximal in Γ^n by Lemma 4.11. Moreover, $y^\alpha = y_{\boldsymbol{\gamma}(\alpha)}$ and $V^\alpha = V^{\boldsymbol{\gamma}(\alpha)}$ by Remark 4.3. Thus,

$$ay^\alpha \in \text{span}\{vy^\alpha : v \in V^\alpha\}.$$

The statement regarding multiplication by a on the right follows similarly. $\square$

Corollary 4.13. Suppose $\lambda \in \Lambda_n^\Gamma$ is maximal with respect to $\triangleright_\Gamma$. Let $\alpha = \alpha(\lambda)$. Then $y^\alpha x_\lambda$ is in the span of $\{vy^\alpha x_\lambda w^* : v, w \in V^\alpha\}$.

Proof. Applying Lemma 2.7 to the cyclic cellular algebra $A^{\otimes n}$, and $y^\alpha = y_{\boldsymbol{\gamma}(\alpha)}$ and taking into account that $\boldsymbol{\gamma}(\alpha)$ is maximal, we get that y^α is a linear combination of elements vy^α with $v \in V^\alpha$. Therefore $y^\alpha x_\lambda$ is a linear combination of elements $vy^\alpha x_\lambda = vx_\lambda y^\alpha$. Now apply Lemma 2.7 again on the right to express each of these elements as a linear combination of elements $vx_\lambda y^\alpha w^*$ with $w \in V^\alpha$. $\square$

The following sets of elements of $A^{\otimes n}$ play an essential role at several points in our arguments.

Notation 4.14. Let α be a composition of n with r parts. For $\boldsymbol{\gamma} > \boldsymbol{\gamma}(\alpha)$, define $V^{\boldsymbol{\gamma}, \alpha}$ to be the set of simple tensors $v_1 \otimes \cdots \otimes v_n$, where

$$\begin{aligned} v_j &= 1 & \text{if } \gamma_j &= \boldsymbol{\gamma}(\alpha)_j, \quad \text{and} \\ v_j &\in V^{\gamma_j} & \text{if } \gamma_j &> \boldsymbol{\gamma}(\alpha)_j. \end{aligned}$$

Lemma 4.15. Let $a \in A^{\otimes n}$. Let α be a composition of n with r parts. Then:

- (1) $ay^\alpha = z_1 + z_2$, where z_1 is a linear combination of terms of the form $v'y^\alpha$ with $v' \in V^\alpha$, and z_2 is a linear combination of elements $vy_{\boldsymbol{\gamma}} w^*$ with $\boldsymbol{\gamma} > \boldsymbol{\gamma}(\alpha)$, $v \in V^{\boldsymbol{\gamma}}$, and $w \in V^{\boldsymbol{\gamma}, \alpha}$.
- (2) $y^\alpha a = z_1 + z_2$, where z_1 is a linear combination of terms of the form $y^\alpha(v')^*$ with $v' \in V^\alpha$, and z_2 is a linear combination of elements $vy_{\boldsymbol{\gamma}} w^*$ with $\boldsymbol{\gamma} > \boldsymbol{\gamma}(\alpha)$, $v \in V^{\boldsymbol{\gamma}, \alpha}$, and $w \in V^{\boldsymbol{\gamma}}$.

Proof. We can suppose without loss of generality that $a = a_1 \otimes \cdots \otimes a_n$ is a simple tensor. Then

$$ay^\alpha = a_1 y_1 \otimes \cdots \otimes a_n y_n,$$

where $y_j = y_{\gamma(\alpha)_j}$. By Lemma 2.7, for each j we have $a_j y_j = b_1 + b_2$, where b_1 is a linear combination of terms $v'_j y_j$ with $v'_j \in V^{\gamma(\alpha)_j}$, and b_2 is a linear combination of terms $v_j y_{\gamma(w_j)^*}$, where $\gamma > \gamma(\alpha)_j$, and $v_j, w_j \in V^\gamma$. Statement (1) follows. The proof of statement (2) is similar. $\square$

Lemma 4.16. Let $\lambda \in \Lambda_n^\Gamma$ and let $\alpha = \alpha(\lambda)$. Suppose that for all $\mu \triangleright_\Gamma \lambda$ in Λ_n^Γ , N^μ is a two-sided ideal of $A \wr \mathfrak{S}_n$, and $y^{\alpha(\mu)} x_\mu \in N^\mu$. Suppose that $\gamma > \gamma(\alpha)$ in Γ^n and $v \in V^{\gamma \cdot \alpha}$. Then $y_\gamma v^* x_\lambda \in \bar{N}^\lambda$ and $x_\lambda v y_\gamma \in \bar{N}^\lambda$.

Proof. As usual, write $n_k = \alpha_1 + \cdots + \alpha_{k-1}$ for $1 \leq k \leq n$. Write $\gamma = (\gamma_1, \dots, \gamma_n)$. Since $\gamma > \gamma(\alpha)$, for each k and for $n_k + 1 \leq j \leq n_{k+1}$, we have $\gamma_j \geq \gamma(k)$, with strict inequality for at least one value of k and j .

We partition the set $\{1, 2, \dots, n\}$ into subsets, an integer j being assigned to a subset according to the row of j in t^λ and the value of γ_j and v_j , as follows: For integers k and i , let $P_{k,i}$ be the set of integers j such that j is in the i -th row of the k -th component of t^λ and $\gamma_j = \gamma(k)$, so $v_j = 1$. For integers k, i and ℓ , with $\gamma(\ell) > \gamma(k)$, and for $t \in \mathcal{T}(\gamma(\ell))$, let $P_{k,i,\ell,t}$ be the set of integers j such that j is in the i -th row of the k -th component of t^λ , $\gamma_j = \gamma(\ell)$, and $v_j = v_t$. Then $P = \{P_{k,i}\} \cup \{P_{k,i,\ell,t}\}$ is a set partition of $\{1, 2, \dots, n\}$. Moreover, since $\gamma > \gamma(\alpha)$, it follows that there is at least one index (k, i, ℓ, t) with $P_{k,i,\ell,t} \neq \emptyset$. Note that y_{γ_j} and v_j are constant for j in each block of the set partition P , namely $y_{\gamma_j} = y_{\gamma(k)}$ and $v_j = 1$ for $j \in P_{k,i}$; and $y_{\gamma_j} = y_{\gamma(\ell)}$ and $v_j = v_t$ for $j \in P_{k,i,\ell,t}$.

Let $\mathfrak{S}_P$ be the set of $w \in \mathfrak{S}_n$ such that $w(X) = X$ for each block $X \in P$. Then $\mathfrak{S}_P$ is a subgroup of $\mathfrak{S}_\lambda$. Let x_P denote the corresponding symmetrizer, $x_P = \sum_{w \in \mathfrak{S}_P} w$. We have $x_\lambda = x_P S$, where S is the sum of the elements in a complete family of right coset representatives of $\mathfrak{S}_P$ in $\mathfrak{S}_\lambda$.

Note that x_P commutes with v^* , so that

$$y_\gamma v^* x_\lambda = y_\gamma v^* x_P S = (y_\gamma x_P) v^* S.$$

We will now show that $y_\gamma x_P \in N^\nu$ for some $\nu \triangleright_\Gamma \lambda$.

Let ν be the multipartition such that for each ℓ , $\nu^{(\ell)}$ has parts $\{|P_{\ell,i}\}_i \cup \{|P_{k,i,\ell,t}\}_{k,i,t}$, sorted into decreasing order. Then $\nu \triangleright_\Gamma \lambda$, because ν can be obtained from λ by applying “raising operators”. In fact, ν can be obtained from λ in two stages. First, for each pair k, ℓ with $\gamma(\ell) > \gamma(k)$, and for each i and t such that $P_{k,i,\ell,t}$ is non-empty, move $|P_{k,i,\ell,t}|$ boxes from the i -th row of $\lambda^{(k)}$ to form a new row appended to $\lambda^{(\ell)}$. Then sort the rows of each component of the resulting multicomposition into decreasing order; this involves raising boxes from lower rows to higher rows in each component. Note that $\nu \neq \lambda$ because some $P_{k,i,\ell,t}$ is non-empty.

There exists $\pi \in \mathfrak{S}_n$ such that $y_\gamma x_P = \pi y^{\alpha(\nu)} x_\nu \pi^{-1}$. In fact, π can be chosen so that π^{-1} sends the elements of each non-empty $P_{k,i}$ to the entries in some row of the k -th component of t^ν and the elements of each non-empty $P_{k,i,\ell,t}$ to the entries in some row of the ℓ -th component of t^ν . Since N^ν is assumed to be a two-sided ideal of $A \wr \mathfrak{S}_n$, and $y^{\alpha(\nu)} x_\nu$ is assumed to be an element of N^ν , it follows that $y_\gamma x_P \in N^\nu$, and therefore $y_\gamma v^* x_\lambda = (y_\gamma x_P) v^* S \in N^\nu$. But $N^\nu \subset \bar{N}^\lambda$, since $\nu \triangleright_\Gamma \lambda$. This completes the proof that $y_\gamma v^* x_\lambda \in \bar{N}^\lambda$. The proof that $x_\lambda v y_\gamma \in \bar{N}^\lambda$ is similar. $\square$

Proposition 4.17. Let $\lambda \in \Lambda_n^\Gamma$, and let $\alpha = \alpha(\lambda)$. Let s, t be standard λ -tableaux, and let $a \in A^{\otimes n}$.

- (1) For $v \in V^\alpha$, and $w \in V_1^\alpha$, $am_{(s,v), (t,w)}^\lambda = x_1 + x_2$, where x_1 is a linear combination of terms $m_{(s,v'), (t,w)}^\lambda$, with $v' \in V^\alpha$, and $x_2 \in \bar{N}^\lambda$.
- (2) For $v, w \in V^\alpha$, $m_{(s,v), (t,w)}^\lambda a = x_1 + x_2$, where x_1 is a linear combination of terms $m_{(s,v), (t,w')}^\lambda$, with $w' \in V^\alpha$, and $x_2 \in \bar{N}^\lambda$.

- (3) N^λ and $\bar{N}^\lambda$ are two-sided ideals of $A \wr \mathfrak{S}_n$.
 (4) $y^\alpha x_\lambda \in N^\lambda$.

Proof. We prove all parts together, by induction on the partial order $\triangleright_\Gamma$ of Λ_n^Γ . For parts (1) and (2), we can assume without loss of generality that a is a simple tensor, $a = a_1 \otimes \cdots \otimes a_n$.

First assume that λ is maximal in $(\Lambda_n^\Gamma, \triangleright_\Gamma)$. For $a \in A^{\otimes n}$, we have

$$\begin{aligned} am_{(s,v), (t,w)}^\lambda &= ad(s)vy^\alpha x_\lambda w^*d(t)^* \\ &= d(s)(d(s)^*a)vy^\alpha x_\lambda w^*d(t)^*. \end{aligned}$$

By Corollary 4.12, $(d(s)^*a)vy^\alpha$ is a linear combination of elements $v'y^\alpha$ with $v' \in V^\alpha$, since λ is maximal. Hence $am_{(s,v), (t,w)}^\lambda$ is a linear combination of terms of the form $m_{(s,v'), (t,w)}^\lambda$. This demonstrates statement (1) of the proposition in case λ is maximal. Part (2) follows similarly. We have shown that N^λ is invariant under multiplication from the left or right by elements $a \in A^{\otimes n}$. Corollary 4.8 implies that N^λ is invariant under multiplication from the left or right by elements of $\mathfrak{S}_n$. Since $A \wr \mathfrak{S}_n$ is generated as an algebra by $A^{\otimes n}$ and $\mathfrak{S}_n$, we have that N^λ is a two-sided ideal in $A \wr \mathfrak{S}_n$. Since λ is maximal, $\bar{N}^\lambda = (0)$. Finally, when λ is maximal, $y^\alpha x_\lambda \in N^\lambda$ by Corollary 4.13. This completes the proof of all the statements under the assumption that λ is maximal.

Now take $\lambda \in \Lambda_n^\Gamma$ arbitrary, and assume that all the statements of the proposition holds when λ is replaced by $\mu \in \Lambda_n^\Gamma$ with $\mu \triangleright_\Gamma \lambda$. Since $\bar{N}^\lambda$ is the span of $\{N^\mu : \mu \triangleright_\Gamma \lambda\}$, it already follows that $\bar{N}^\lambda$ is a two-sided ideal. As before, we have

$$am_{(s,v), (t,w)}^\lambda = d(s)(d(s)^*a)vy^\alpha x_\lambda w^*d(t)^*.$$

Write $(d(s)^*a) = a'$. Apply Lemma 4.15(1) to $(a'v)y^\alpha$. It follows that $a'vy^\alpha = z_1 + z_2$, where z_1 is a linear combination of terms of the form $v'y^\alpha$ with $v' \in V^\alpha$, and z_2 is a linear combination of elements $vy_\gamma w^*$ with $\gamma > \gamma(\alpha)$, $v \in V^\gamma$, and $w \in V^{\gamma,\alpha}$. Since N^μ is a two-sided ideal and $y^{\alpha(\mu)}x_\mu \in N^\mu$ for all $\mu \triangleright_\Gamma \lambda$, by the inductive hypothesis, we can apply Lemma 4.16, which tells us that $z_2x_\lambda \in \bar{N}^\lambda$. On the other hand, $d(s)z_1x_\lambda w^*d(t)^*$ is a linear combination of terms of the form $m_{(s,v'), (t,w)}^\lambda$. This proves statement (1). Statement (2) is proven similarly, using Lemma 4.15(2).

This finishes the proof of parts (1) and (2) of the proposition. We have already seen, in the discussion of the case that λ is maximal, that parts (1) and (2) of the proposition together with Corollary 4.8 imply that N^λ is a two-sided ideal.

Finally, we prove part (4). By Lemma 4.15 part (1), we have $y^\alpha = z_1 + z_2$, where z_1 is a linear combination of terms of the form vy^α with $v \in V^\alpha$, and z_2 is a linear combination of elements $vy_\gamma w^*$ with $\gamma > \gamma(\alpha)$, $v \in V^\gamma$, and $w \in V^{\gamma,\alpha}$. Since by the induction hypothesis, $y^{\alpha(\mu)}x_\mu \in N^\mu$ for all $\mu \triangleright_\Gamma \lambda$, and since we know at this point that N^μ is an ideal for all μ , we can apply Lemma 4.16 to obtain that $z_2x_\lambda \in \bar{N}^\lambda$. Thus $y^\alpha x_\lambda$ is congruent modulo $\bar{N}^\lambda$ to a linear combination of terms $vy^\alpha x_\lambda = vx_\lambda y^\alpha$, with $v \in V^\alpha$. Applying Lemma 4.15 part (2) and Lemma 4.16 again to each of these terms gives that $y^\alpha x_\lambda$ is congruent modulo $\bar{N}^\lambda$ to a linear combination of terms $vy^\alpha x_\lambda w^*$ with $v, w \in V^\alpha$. But $vy^\alpha x_\lambda w^* = m_{(t^\lambda, v), (t^\lambda, w)}^\lambda \in N^\lambda$. $\square$

Corollary 4.18. Let $\lambda \in \Lambda_n^\Gamma$, and write $\alpha = \alpha(\lambda)$. We have

$$y^\alpha x_\lambda \in \text{span}\{vy^\alpha x_\lambda : v \in V^\alpha\} + \bar{N}^\lambda.$$

Proof. The proof of Proposition 4.17 part (4) shows that $y^\alpha x_\lambda$ is congruent modulo $\bar{N}^\lambda$ to a linear combination of terms $vy^\alpha x_\lambda$ where $v \in V^\alpha$. $\square$

Proposition 4.19. Suppose that $\lambda \in \Lambda_n^\Gamma$, and $(s, v) \in \mathcal{T}(\lambda)$. Let $x \in A \wr \mathfrak{S}_n$. Then for each $(u, v') \in \mathcal{T}(\lambda)$, there exists $r_{(u, v')} \in R$ such that

$$xm_{(s, v), (t^\lambda, 1)}^\lambda \equiv \sum_{(u, v') \in \mathcal{T}(\lambda)} r_{(u, v')} m_{(u, v'), (t^\lambda, 1)}^\lambda \pmod{\bar{N}^\lambda}. \quad (4.4)$$

Moreover, for all $(t, w) \in \mathcal{T}(\lambda)$,

$$xm_{(s, v), (t, w)}^\lambda \equiv \sum_{(u, v') \in \mathcal{T}(\lambda)} r_{(u, v')} m_{(u, v'), (t, w)}^\lambda \pmod{\bar{N}^\lambda}. \quad (4.5)$$

Proof. Let U be the R -module spanned by the set of $m_{(s', v'), (t^\lambda, 1)}^\lambda$, where $(s', v') \in \mathcal{T}(\lambda)$. By Proposition 4.17(1) and Corollary 4.10, $U + \bar{N}^\lambda$ is invariant under left multiplication by elements $a \in A^{\otimes n}$ and by $\pi \in \mathfrak{S}_n$. Since these elements generate $A \wr \mathfrak{S}_n$ as an algebra, it follows that $U + \bar{N}^\lambda$ is a left ideal. Therefore, there exist coefficients $r_{(u, v')} \in R$ such that Eq. (4.4) holds. Now for any $(t, w) \in \mathcal{T}(\lambda)$, multiply the congruence of Eq. (4.4) on the right by $w^*d(t)^*$. Since $\bar{N}^\lambda$ is an ideal and

$$m_{(u, v'), (t^\lambda, 1)}^\lambda w^*d(t)^* = m_{(u, v'), (t, w)}^\lambda,$$

this yields the congruence of Eq. (4.5). $\square$

Lemma 4.20. Let $\lambda \in \Lambda_n^\Gamma$ and write $\alpha = \alpha(\lambda)$. Let $(s, v), (t, w) \in \mathcal{T}(\lambda)$.

- (1) $(y^\alpha x_\lambda)^* \equiv y^\alpha x_\lambda \pmod{\bar{N}^\lambda}$.
- (2) $(m_{(s, v), (t, w)}^\lambda)^* \equiv m_{(t, w), (s, v)}^\lambda \pmod{\bar{N}^\lambda}$.

Proof. Write $y^\alpha = y_1 \otimes \cdots \otimes y_n$, where for each j , $y_j = y_{\gamma(\alpha)_j}$. By point (2a) of Lemma 2.5, we have that y_j^* is equal to y_j plus a linear combination of elements of the form $v_j y_{\gamma'} w_j$, where $\gamma' > \gamma(\alpha)_j$, and $v_j, w_j \in V^{\gamma'}$. Therefore $(y^\alpha)^*$ is equal to $y^\alpha + z$, where z is a linear combination of terms $v y_\gamma w^*$, where $\gamma > \gamma(\alpha)$ and both v and w are in $V^{\gamma \cdot \alpha}$. Since $x_\lambda^* = x_\lambda$ and x_λ commutes with y^α ,

$$(y^\alpha x_\lambda)^* = (y^\alpha)^* x_\lambda = y^\alpha x_\lambda + z x_\lambda.$$

Since N^μ is a two-sided ideal and $y^{\alpha(\mu)} x_\mu \in N^\mu$, for all $\mu \in \Lambda_n^\Gamma$, we can apply Lemma 4.16, which tells us that $z x_\lambda \in \bar{N}^\lambda$. This proves part (1). Part (2) follows from part (1) together with the factorization

$$m_{(s, v), (t, w)}^\lambda = d(s) v y^\alpha x_\lambda w^* d(t)^*. \quad \square$$

We are now ready to propose a cellular structure for the wreath product $A \wr \mathfrak{S}_n$. The partially ordered set for the cellular structure is $(\Lambda_n^\Gamma, \triangleright_\Gamma)$. For each $\lambda \in \Lambda_n^\Gamma$, we take the index set $\mathcal{T}(\lambda)$ to be one introduced above, namely, the set of pairs (s, v) , where s is a standard λ -tableau and $v \in V^{\alpha(\lambda)}$. The proposed cellular basis is

$$\mathcal{B} = \{m_{(s, v), (t, w)}^\lambda : \lambda \in \Lambda_n^\Gamma, (s, v), (t, w) \in \mathcal{T}(\lambda)\}.$$

For $\lambda \in \Lambda_n^+$, we set

$$y_\lambda = y^{\alpha(\lambda)} x_\lambda.$$

For $\lambda \in \Lambda_n^+$ and $(s, u) \in \mathcal{T}(\lambda)$, we set

$$v_{(s, u)} = d(s)u.$$

Thus,

$$m_{(s, u), (t, w)}^\lambda = v_{(s, u)} y_\lambda (v_{(t, w)})^*.$$

We have already verified that $\mathcal{B}$ satisfies several of the defining properties of a cellular basis. In fact, Proposition 4.19 and Lemma 4.20 give us properties (2) and (3) of Definition 2.1. What remains to show is that $\mathcal{B}$ is an R -basis of $A \wr \mathfrak{S}_n$.

Lemma 4.21. $\mathcal{B}$ spans $A \wr \mathfrak{S}_n$ over R .

Proof. Let U denote the R -span of $\mathcal{B}$. It follows from Proposition 4.19 and Lemma 4.20 that U is a two-sided ideal in $A \wr \mathfrak{S}_n$. Fix a composition α of n and take

$$\lambda = ((1)^{\alpha_1}, (1)^{\alpha_2}, \dots, (1)^{\alpha_r}).$$

Then $x_\lambda = 1$ and for $v, w \in V^\alpha$,

$$m_{(t^\lambda, v), (t^\lambda, w)}^\lambda = v y^\alpha w^*.$$

For $\pi \in \mathfrak{S}_n$,

$$\pi(v y^\alpha w^*) = \pi v y^\alpha w^* \pi^{-1} \in U,$$

since U is a two-sided ideal.

The collection of these elements, as α , v , w , and π vary, constitutes a (cellular) basis of $A^{\otimes n} \subseteq A \wr \mathfrak{S}_n$. In particular, the multiplicative identity 1 is contained in U . $\square$

To show that $\mathcal{B}$ is linearly independent over R , we require an enumerative result.

Let d_i denote the rank of the cell module of A associated with $\gamma_i \in \Gamma$, $d_i = |\mathcal{T}(\gamma_i)|$. Let $d = \sum_{i=1}^r d_i^2$ denote the rank of A .

For $\lambda \in \Lambda_n^+$, let f_λ denote the number of standard λ -tableaux; then we have

$$f_\lambda = \binom{n}{\alpha(\lambda)} \prod_i f_{\lambda^{(i)}},$$

where $\binom{n}{\alpha(\lambda)}$ denotes the multinomial coefficient $\binom{n}{|\lambda^{(1)}|, \dots, |\lambda^{(r)}|}$. The size of $\mathcal{T}(\lambda)$ is

$$|\mathcal{T}(\lambda)| = |V^{\alpha(\lambda)}| f_\lambda = \left(\prod_i d_i^{|\lambda^{(i)}|} \right) f_\lambda.$$

Claim 4.22.

$$\sum_{\lambda \in \Lambda_n^\Gamma} |\mathcal{T}(\lambda)|^2 = d^n n!.$$

To verify this claim, we will use some results about wreath products of semisimple algebras. Let B be a semisimple algebra over the complex numbers $\mathbb{C}$, with simple modules $W_1, \dots, W_r$ of dimensions $\dim(W_i) = d_i$; hence $\dim(B) = \sum_i d_i^2 = d$. The simple modules of $B \wr \mathfrak{S}_n$ are indexed by multipartitions $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)})$. For a multipartition λ , let $\alpha = \alpha(\lambda)$. Let W^α denote the simple $B^{\otimes n}$ module

$$W^\alpha = W_1^{\otimes \alpha_1} \otimes W_2^{\otimes \alpha_2} \otimes \dots \otimes W_r^{\otimes \alpha_r}$$

and let S^λ denote the simple $\mathfrak{S}_\alpha$ module

$$S^\lambda = S^{\lambda^{(1)}} \otimes \dots \otimes S^{\lambda^{(r)}},$$

where for a partition λ of size k , S^λ is the Specht module of $\mathfrak{S}_k$ corresponding to λ . The simple $B \wr \mathfrak{S}_n$ -module corresponding to λ has the form

$$W^\lambda = \text{Ind}_{\mathfrak{S}_\alpha}^{\mathfrak{S}_n} (W^\alpha \otimes S^\lambda).$$

An explanation of this construction of simple modules of $B \wr \mathfrak{S}_n$ can be found in [18] and [5]. The dimension of the module W^λ is

$$\dim(W^\lambda) = [\mathfrak{S}_n : \mathfrak{S}_\alpha] \dim(W^\alpha) \dim(S^\lambda) = \binom{n}{\alpha} \prod_i d_i^{|\lambda^{(i)}|} \prod_i f_{\lambda^{(i)}} = |\mathcal{T}(\lambda)|.$$

Since $\dim(B \wr \mathfrak{S}_n) = d^n n!$, Claim 4.22 is equivalent to the assertion that $B \wr \mathfrak{S}_n$ is semisimple. This follows from a more general Maschke-type lemma:

Lemma 4.23. *If B is a semisimple algebra over a field of characteristic zero and G is a finite group acting on B by automorphisms, then $B \rtimes G$ is semisimple.*

Sketch of a proof. One can reduce to the case that the field is algebraically closed by tensoring with a field extension. The trace Tr of the left regular representation of $B \rtimes G$ satisfies $\text{Tr}(\sum_{g \in G} b_g g) = \text{Tr}(b_e) = |G| \text{Tr}_B(b_e)$, where Tr_B is the trace of the left regular representation of B . It follows from this that Tr is non-degenerate, and therefore $\text{rad}(B \rtimes G)$ is zero, since $\text{rad}(B \rtimes G) \subseteq \text{rad}(\text{Tr})$. $\square$

Returning to our candidate $\mathcal{B}$ for a cellular basis of $A \wr \mathfrak{S}_n$, we have:

Lemma 4.24. *$\mathcal{B}$ is linearly independent over R .*

Proof. We already know that $\mathcal{B}$ spans $A \wr \mathfrak{S}_n$ over R , and the cardinality of $\mathcal{B}$ is $\sum_\lambda |\mathcal{T}(\lambda)|^2$. By Claim 4.22, this equals $d^n n!$. On the other hand, $A \wr \mathfrak{S}_n$ is free of rank $d^n n!$ over R since A is free of rank d . It follows that $\mathcal{B}$ is linearly independent, since the ground ring R is assumed to be an integral domain. $\square$

Completion of the proof of Theorem 4.1. We have shown that $\mathcal{B}$ is a cellular basis of $A \wr \mathfrak{S}_n$. Fix $\lambda \in \Lambda_n^r$ and let $\alpha = \alpha(\lambda)$. Proposition 4.19 shows that

$$\begin{aligned} M &:= \text{span}\{m_{(\mathfrak{s}, u), (\mathfrak{t}^\lambda, 1)}^\lambda + \bar{N}^\lambda : (\mathfrak{s}, u) \in \mathcal{T}(\lambda)\} \\ &= \text{span}\{v_{(\mathfrak{s}, u)} y_\lambda + \bar{N}^\lambda : (\mathfrak{s}, u) \in \mathcal{T}(\lambda)\} \end{aligned}$$

is an $A \wr \mathfrak{S}_n$ module and isomorphic to the cell module Δ^λ . Corollary 4.18 implies that $y_\lambda + \bar{N}^\lambda \in M$, so that Δ^λ is cyclic. $\square$

4.2. The structure of cell modules of $A \wr \mathfrak{S}_n$

Let A be any R -algebra, and let $E_1, \dots, E_r$ be a collection of A -modules. We discuss a construction of $A \wr \mathfrak{S}_n$ -modules based on this family of A -modules. For a thorough discussion of this construction, see [5].

Let $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)})$ be a multipartition of total size n with r parts. Let $\alpha = \alpha(\lambda)$. Recall from Section 3.4 that

$$\Delta_R^\lambda = \Delta_R^{\lambda^{(1)}} \otimes \cdots \otimes \Delta_R^{\lambda^{(r)}}$$

is a cell module for $R\mathfrak{S}_\alpha \cong R\mathfrak{S}_{\alpha_1} \otimes \cdots \otimes R\mathfrak{S}_{\alpha_r}$, and that we identify Δ_R^λ with the R -span of the set of $d(\mathfrak{s})x_\lambda + J^\lambda$, where $\mathfrak{s}$ is a standard λ -tableau of the initial kind, and J^λ denotes $\overline{R\mathfrak{S}_\alpha}^\lambda$.

Let $E^\alpha = E_1^{\otimes \alpha_1} \otimes \cdots \otimes E_r^{\otimes \alpha_r}$. Then E^α is an $A \wr \mathfrak{S}_\alpha$ -module, with $A^{\otimes n}$ acting by the tensor product action and $\mathfrak{S}_\alpha$ acting by place permutations. Moreover, $E^\alpha \otimes \Delta_R^\lambda$ is also an $A \wr \mathfrak{S}_\alpha$ -module, with $a(v \otimes m) = av \otimes m$ and $\pi(v \otimes m) = \pi v \otimes \pi m$, for $a \in A^{\otimes n}$, $\pi \in \mathfrak{S}_\alpha$, $v \in E^\alpha$ and $m \in \Delta_R^\lambda$.

Finally, we obtain an $A \wr \mathfrak{S}_n$ -module by

$$\text{Ind}_{A \wr \mathfrak{S}_\alpha}^{A \wr \mathfrak{S}_n}(E^\alpha \otimes \Delta_R^\lambda) = (A \wr \mathfrak{S}_n) \otimes_{A \wr \mathfrak{S}_\alpha} (E^\alpha \otimes \Delta_R^\lambda).$$

When this construction is applied to the simple modules of A , one obtains the simple modules of the wreath product $A \wr \mathfrak{S}_n$. More exactly, suppose now that A is a finite dimensional algebra over a field K and that K is a splitting field for A . Let $E_1, \dots, E_r$ be a complete list of mutually inequivalent simple A -modules. Assume moreover that $\text{char}(K) = 0$, so that $K\mathfrak{S}_j$ is split semisimple for all $j \geq 0$, and the cell modules Δ_K^λ , for $\lambda \in \Lambda_j$, are the simple $K\mathfrak{S}_j$ -modules.

The following theorem is obtained by adapting the analysis of the representation theory of wreath products of groups [24], and of semidirect products of groups in general [6] to wreath product of algebras. A proof was given in [18]. See also [5].

Theorem 4.25. *Let A be a finite dimensional algebra over a field K of characteristic zero, with K a splitting field for A . Let $E_1, \dots, E_r$ be a complete list of simple A -modules, up to isomorphism. Then the modules*

$$\text{Ind}_{A \wr \mathfrak{S}_\alpha}^{A \wr \mathfrak{S}_n}(E^\alpha \otimes \Delta_K^\lambda),$$

where λ varies over multipartitions of total size n with r parts and $\alpha = \alpha(\lambda)$, are simple and mutually inequivalent. All simple $A \wr \mathfrak{S}_n$ -modules are of this form.

Not surprisingly, when A is a cyclic cellular algebra over an integral domain R , the cell modules of the wreath product $A \wr \mathfrak{S}_n$ are obtained by the same construction. Let A be such an algebra, and adopt the notations of Section 4.1 regarding the cellular structure of A and of $A \wr \mathfrak{S}_n$. In particular $(\Gamma, \geq)$ is the partially ordered set associated with the cellular structure of A , and $\Gamma = (\gamma(1), \dots, \gamma(r))$ is a

listing of Γ consistent with the partial order. In order to avoid confusing overuse of the symbol Δ for the cell modules of various algebras, let E_i denote the cell module of A associated to $\gamma(i) \in \Gamma$, and let C^λ denote the cell module of $A \wr \mathfrak{S}_n$ associated to $\lambda \in \Lambda_n^\Gamma$. Recall that $\gamma(\alpha) \in \Gamma^n$ is the sequence in Γ^n whose first α_1 entries are equal to $\gamma(1)$, next α_2 entries are equal to $\gamma(2)$, and so forth. The $A^{\otimes n}$ -module $E^\alpha = E_1^{\otimes \alpha_1} \otimes \cdots \otimes E_r^{\otimes \alpha_r}$ is the cell module corresponding to $\gamma(\alpha)$. We are going to write I^α for $\overline{A^{\otimes n}}^{\gamma(\alpha)}$, namely the span of elements of the cellular basis of $A^{\otimes n}$ corresponding to $\gamma \in \Gamma^n$ with $\gamma > \gamma(\alpha)$. The cell module E^α has basis $\{vy^\alpha + I^\alpha : v \in V^\alpha\}$.

Lemma 4.26. Let $\lambda \in \Lambda_n^\Gamma$ and let $\alpha = \alpha(\lambda)$. Write J^λ for $\overline{R\mathfrak{S}_\alpha}^\lambda$. We have $y^\alpha J^\lambda \subset \bar{N}^\lambda$.

Proof. J^λ is the span of elements $d(s)x_\mu d(t)^*$ where $\mu \triangleright \lambda$, $\alpha(\mu) = \alpha$, and s, t are standard μ -tableaux of the initial kind. In particular $d(s), d(t) \in \mathfrak{S}_\alpha$. We have $y^\alpha d(s)x_\mu d(t)^* = d(s)y^\alpha x_\mu d(t)^*$. But $y^\alpha x_\mu \in N^\mu \subseteq \bar{N}^\lambda$ by Proposition 4.17(4). $\square$

Theorem 4.27. Let $\lambda \in \Lambda_n^\Gamma$ and let $\alpha = \alpha(\lambda)$. The cell module C^λ of $A \wr \mathfrak{S}_n$ satisfies

$$C^\lambda \cong \text{Ind}_{A \wr \mathfrak{S}_\alpha}^{A \wr \mathfrak{S}_n} (E^\alpha \otimes \Delta_R^\lambda).$$

Proof. Note that

$$\text{Ind}_{A \wr \mathfrak{S}_\alpha}^{A \wr \mathfrak{S}_n} (E^\alpha \otimes \Delta_R^\lambda) = \bigoplus_{\omega} \omega \otimes E^\alpha \otimes \Delta_R^\lambda,$$

where ω varies through the set of shortest left coset representatives of $\mathfrak{S}_\alpha$ in $\mathfrak{S}_n$. Thus the set of elements

$$\omega \otimes (vy^\alpha + I^\alpha) \otimes (d(s)x_\lambda + J^\lambda) \quad (4.6)$$

where ω is such a coset representative, $v \in V^\alpha$, and s is a standard λ -tableau of the initial kind, is a basis of $\text{Ind}_{A \wr \mathfrak{S}_\alpha}^{A \wr \mathfrak{S}_n} (E^\alpha \otimes \Delta_R^\lambda)$. Note that vy^α is the unique lifting of $vy^\alpha + I^\alpha$ to an element of the cellular basis of $A^{\otimes n}$ and likewise $d(s)x_\lambda$ is the unique lifting of $d(s)x_\lambda + J^\lambda$ to an element of the cellular basis of $R\mathfrak{S}_\alpha$. Therefore we have a well-defined R -linear map

$$\varphi : \text{Ind}_{A \wr \mathfrak{S}_\alpha}^{A \wr \mathfrak{S}_n} (E^\alpha \otimes \Delta_R^\lambda) \rightarrow (A \wr \mathfrak{S}_n) / \bar{N}^\lambda$$

determined by

$$\varphi(\omega \otimes (vy^\alpha + I^\alpha) \otimes (d(s)x_\lambda + J^\lambda)) = \omega vy^\alpha d(s)x_\lambda + \bar{N}^\lambda.$$

We have

$$\omega vy^\alpha d(s)x_\lambda + \bar{N}^\lambda = \omega d(s)(d(s)^* v) y^\alpha x_\lambda + \bar{N}^\lambda.$$

Moreover, $\omega d(s) = d(\omega s)$, the λ -tableau ωs is standard, and all standard λ -tableaux can be written uniquely as ωs , where ω is a distinguished left coset representative of $\mathfrak{S}_\alpha$ in $\mathfrak{S}_n$ and s is a standard λ tableau of the initial kind. Thus φ maps the basis of $\text{Ind}_{A \wr \mathfrak{S}_\alpha}^{A \wr \mathfrak{S}_n} (E^\alpha \otimes \Delta_R^\lambda)$ onto a basis of $C^\lambda \subset (A \wr \mathfrak{S}_n) / \bar{N}^\lambda$, and thus φ is a linear isomorphism with range C^λ .

It remains to show that φ is an $A \wr \mathfrak{S}_n$ -module map. If $\pi \in \mathfrak{S}_n$, $\pi \omega$ can be written uniquely as $\omega' \pi'$ for ω' a distinguished left coset representative of $\mathfrak{S}_\alpha$ in $\mathfrak{S}_n$, and $\pi' \in \mathfrak{S}_\alpha$. Then we have

$$\pi(\omega \otimes (vy^\alpha + I^\alpha) \otimes (d(s)x_\lambda + J^\lambda)) = \omega' \otimes (\pi' vy^\alpha + I^\alpha) \otimes \pi'(d(s)x_\lambda + J^\lambda).$$

There exist coefficients $r_t^s \in R$ and an element $h \in J^\lambda$ such that

$$\pi' d(s)x_\lambda = \sum_t r_t^s d(t)x_\lambda + h.$$

Thus

$$\pi(\omega \otimes (vy^\alpha + I^\alpha) \otimes (d(s)x_\lambda + J^\lambda)) = \sum_t r_t^s (\omega' \otimes (\pi' vy^\alpha + I^\alpha) \otimes (d(t)x_\lambda + J^\lambda)),$$

and

$$\varphi(\pi(\omega \otimes (vy^\alpha + I^\alpha) \otimes (d(s)x_\lambda + J^\lambda))) = \sum_t r_t^s \omega' (\pi' v) y^\alpha d(t)x_\lambda + \bar{N}^\lambda. \quad (4.7)$$

On the other hand,

$$\begin{aligned} \pi\varphi(\omega \otimes (vy^\alpha + I^\alpha) \otimes (d(s)x_\lambda + J^\lambda)) &= \pi\omega vy^\alpha d(s)x_\lambda + \bar{N}^\lambda \\ &= \omega' \pi' vy^\alpha d(s)x_\lambda + \bar{N}^\lambda \\ &= \omega' (\pi' v) y^\alpha \pi' d(s)x_\lambda + \bar{N}^\lambda \\ &= \omega' (\pi' v) y^\alpha \left(\sum_t r_t^s d(t)x_\lambda + h \right) + \bar{N}^\lambda \\ &= \sum_t r_t^s \omega' (\pi' v) y^\alpha d(t)x_\lambda + \bar{N}^\lambda, \end{aligned} \quad (4.8)$$

because $y^\alpha h \in \bar{N}^\lambda$, by Lemma 4.26.

Next, for $a \in A^{\otimes n}$, let $a' = \omega^* a$. We have

$$a(\omega \otimes (vy^\alpha + I^\alpha) \otimes (d(s)x_\lambda + J^\lambda)) = \omega \otimes a'(vy^\alpha + I^\alpha) \otimes (d(s)x_\lambda + J^\lambda). \quad (4.9)$$

In the following, let Z^α denote the span of elements $v'y_\gamma w^*$, where $\gamma > \gamma(\alpha)$, $v' \in V^\gamma$, and $w \in V^{\gamma, \alpha}$. ($V^{\gamma, \alpha}$ was defined in Notation 4.14. Note that $Z^\alpha \subseteq I^\alpha$.) By Lemma 4.15, there exist coefficients $r_{v'}^v \in R$ and $z \in Z^\alpha$ such that $a'vy^\alpha = \sum_{v'} r_{v'}^v v'y^\alpha + z$. From Eq. (4.9), we have

$$a(\omega \otimes (vy^\alpha + I^\alpha) \otimes (d(s)x_\lambda + J^\lambda)) = \sum_{v'} r_{v'}^v (\omega \otimes (v'y^\alpha + I^\alpha) \otimes (d(s)x_\lambda + J^\lambda)).$$

Therefore,

$$\varphi(a(\omega \otimes (vy^\alpha + I^\alpha) \otimes (d(s)x_\lambda + J^\lambda))) = \sum_{v'} r_{v'}^v \omega v'y^\alpha d(s)x_\lambda + \bar{N}^\lambda. \quad (4.10)$$

On the other hand,

$$\begin{aligned}
 a\varphi(\omega \otimes (vy^\alpha + I^\alpha) \otimes (d(s)x_\lambda + J^\lambda)) &= a\omega v y^\alpha d(s)x_\lambda + \bar{N}^\lambda \\
 &= \omega a' v y^\alpha d(s)x_\lambda + \bar{N}^\lambda \\
 &= \omega \left(\sum_{v'} r_{v'}^v v' y^\alpha + z \right) d(s)x_\lambda + \bar{N}^\lambda \\
 &= \sum_{v'} r_{v'}^v \omega v' y^\alpha d(s)x_\lambda + \omega z d(s)x_\lambda + \bar{N}^\lambda. \quad (4.11)
 \end{aligned}$$

The set Z^α defined above is invariant under place permutations by elements of $\mathfrak{S}_\alpha$, so $z' = {}^{d(s)*}z$ is also an element of Z^α . By Lemma 4.16, $z'x_\lambda \in \bar{N}^\lambda$. Therefore,

$$\omega z d(s)x_\lambda = \omega d(s)z'x_\lambda \in \bar{N}^\lambda.$$

Hence, from Eq. (4.11), we have

$$a\varphi(\omega \otimes (vy^\alpha + I^\alpha) \otimes (d(s)x_\lambda + J^\lambda)) = \sum_{v'} r_{v'}^v \omega v' y^\alpha d(s)x_\lambda + \bar{N}^\lambda. \quad (4.12)$$

Comparing Eqs. (4.7) and (4.8) and also Eqs. (4.10) and (4.12), we see that φ is both an $\mathfrak{S}_n$ -module map and an $A^{\otimes n}$ -module map, and therefore an $A \wr \mathfrak{S}_n$ -module map. $\square$

5. The A -Brauer algebras

In this section, we describe the construction of the A -Brauer algebra, where A is an algebra with involution $*$ and with a $*$ -invariant trace. The A -Brauer algebra can be regarded as a sort of wreath product of A with the ordinary Brauer algebra, generalizing the wreath product of A with the symmetric group.

The A -Brauer algebras generalize the G -Brauer algebras for abelian groups G defined in [23] as well as the A -Temperley Lieb algebras described in [15] or [11, Section 5.8]. G -Brauer algebras for non-commutative groups are a particular case of the A -Brauer algebra construction; they have not previously appeared in the literature.

5.1. Algebras with involution-invariant trace

The definition of the A -Brauer algebras, in Section 5.4, requires A to be an algebra with an involution $*$ and a $*$ -invariant trace. Moreover, we will show in Section 6 that if A is a cyclic cellular algebra with a $*$ -invariant trace, then the A -Brauer algebras $D_n(A)$ are also cyclic cellular. It is therefore desirable to have examples of cyclic cellular algebras with a (natural) $*$ -invariant trace.

Example 5.1 (Hecke algebras of type A). Let $H_n = H_{n,R}(q)$ denote the Hecke algebra of type A , as in Example 2.14. For any $z \in R$, there exists a unique R -valued trace tr_z defined on $\bigcup_{n \geq 1} H_n$ with the properties: $\text{tr}_z(1) = 1$ and $\text{tr}_z(aT_n b) = z \text{tr}_z(ab)$, for $a, b \in H_n$. See [14, Theorem 5.1]. Such a trace is called a Markov trace. Note that $\text{tr}_z \circ *$ is another trace with the same properties, and therefore, by uniqueness, tr_z is $*$ -invariant. When $z = 0$, one obtains the trace tr_0 such that $\text{tr}_0(T_\pi) = 0$ for $\pi \neq 1$ and $\text{tr}_0(1) = 1$.

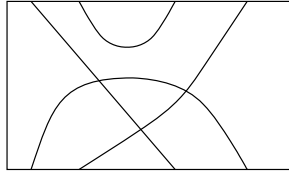


Fig. 5.1. A Brauer diagram.

Example 5.2 (*Jones–Temperley–Lieb, Brauer, and BMW algebras*). Let A denote the n -strand Jones–Temperley–Lieb algebra, Brauer algebra or BMW algebra over an integral domain R with appropriate parameters, and let $\delta_0 \in R$ satisfy $e_i^2 = \delta_0 e_i$ for all i . The algebra A is, or can be realized as, an algebra of diagrams or tangles; see for example Section 5 of [11] for a description of the algebras. The involution $*$ on A acts on diagrams or tangles by flipping over a horizontal line. A has a natural $*$ -invariant trace Tr (the Markov trace) defined by “closing up diagrams”. The trace on the n -strand algebra satisfies $\text{Tr}(1) = \delta_0^n$.

Example 5.3 (*Wreath products*). Suppose that A is an R -algebra with involution $*$ and $*$ -invariant trace tr , with $\text{tr}(1) = \delta_0$. We can define a functional Tr on $A \wr \mathfrak{S}_n$ as follows: Let $a = a_1 \otimes \cdots \otimes a_n$ be a simple tensor in $A^{\otimes n}$. For a $\pi \in \mathfrak{S}_n$, let $\mathbf{o} = (i, \pi(i), \pi^2(i), \dots, \pi^k(i))$ be an orbit for the action of powers of π on $\{1, 2, \dots, n\}$. Let $\mathbf{o}(a) = \text{tr}(a_{\pi^k(i)} \cdots a_{\pi(i)} a_i)$. Define $\text{Tr}(\pi a) = \prod \mathbf{o}(a)$, where the product is over the orbits of powers of π . One can check that Tr is a $*$ -invariant trace on $A \wr \mathfrak{S}_n$ such that $\text{Tr}(1) = \delta_0^n$.

5.2. Ordinary Brauer diagrams and Brauer algebras

We review the construction of the ordinary Brauer algebras [4,25]. Let $\mathcal{R}$ denote the square $[0, 1] \times [0, 1]$. Fix an increasing sequence $(s_i)_{i \geq 1}$ in the open interval $(0, 1)$. Let $\mathbf{i}$ denote $(s_i, 1) \in \mathcal{R}$ and let $\bar{\mathbf{i}}$ denote $(s_i, 0)$.

An (n, n) -Brauer diagram (or n -strand Brauer diagram) is a graph with vertices

$$\{\mathbf{1}, \dots, \mathbf{n}\} \cup \{\bar{\mathbf{1}}, \dots, \bar{\mathbf{n}}\},$$

and with n edges connecting the vertices in pairs; i.e., each vertex is adjacent to exactly one edge, and no loops are allowed. In practice, we draw the edges as curves in $\mathcal{R}$ which intersect the boundary of $\mathcal{R}$ only at their endpoints. Two Brauer diagrams are identified if they are the same as graphs with labeled vertices, i.e. if the edges connect the same pairs of vertices. A typical 4-strand Brauer diagram is given in Fig. 5.1.

The edges in a Brauer diagram will also be referred to as *strands*. The vertices $\mathbf{1}, \dots, \mathbf{n}$ in an n -strand Brauer diagram are called the *top* vertices and the vertices $\bar{\mathbf{1}}, \dots, \bar{\mathbf{n}}$ the *bottom* vertices.

Let R be a commutative ring with identity and let $\delta \in R$. We recall the definition of the n -strand Brauer algebra over R with parameter δ , denoted $D_n = D_{n,R}(\delta)$. As an R -module, D_n is free with basis the set of n -strand Brauer diagrams. The product xy of two Brauer diagrams x and y is a certain power of δ times a third Brauer diagram z , determined as follows: stack y over x and compress the resulting “tangle” vertically to fit in $\mathcal{R}$. The tangle may have some number c of closed loops as well as n strands connecting vertices on the top and bottom edges in pairs. Let z be the Brauer diagram obtained by removing all the closed loops. Then $xy = \delta^c z$. This product extends to a bilinear associative product on D_n . The identity element of the algebra D_n is the Brauer diagram in which $\mathbf{j}$ and $\bar{\mathbf{j}}$ are connected by a strand for each j ($1 \leq j \leq n$). For each Brauer diagram d , let d^* be the diagram obtained by flipping d over a horizontal line. Then $d \mapsto d^*$ extends to an algebra involution of D_n .

A *vertical* or *through* strand in an ordinary Brauer diagram is a strand that connects a top vertex and a bottom vertex. A *horizontal* strand is one that connects two top vertices or two bottom vertices.



Fig. 5.2. Oriented A -labeled Brauer diagram.

A Brauer diagram with only vertical strands is called a *permutation diagram*. The set of permutation diagrams forms a subgroup of the group of invertible elements in D_n , isomorphic to the symmetric group $\mathfrak{S}_n$. For a permutation $\pi \in D_n$, π^* is the inverse permutation.

5.3. Order and orientation of edges

We will need a standard ordering and a standard orientation of the edges of a Brauer diagram. To this end, order the vertices of the n -strand Brauer diagrams by

$$1 < 2 < \cdots < n < \bar{1} < \bar{2} < \cdots < \bar{n}.$$

Now we identify each edge in a Brauer diagram by its pair of vertices, written in increasing order. For example, in the Brauer diagram in Fig. 5.1 the edges are $(1, \bar{3})$, $(2, 3)$, $(4, \bar{2})$, and $(\bar{1}, \bar{4})$. Moreover, we impose a total order on the set of edges of a Brauer diagram by declaring $(v_1, v_2) < (v'_1, v'_2)$ if $v_1 < v'_1$. In the example above, we have

$$(1, \bar{3}) < (2, 3) < (4, \bar{2}) < (\bar{1}, \bar{4}).$$

The *standard orientation* of the edges of a Brauer diagram is from lower labeled vertex to higher labeled vertex.

5.4. Definition of the A -Brauer algebras

Let R be a commutative unital ring with distinguished element δ , as in the description of the Brauer algebras. Let A be a (unital, associative, not necessarily commutative) algebra over R . We suppose that A has an R -linear algebra involution $*$ and a $*$ -invariant R -valued trace tr . We suppose that the trace on A is normalized by $\text{tr}(\mathbf{1}) = \delta$, rather than $\text{tr}(\mathbf{1}) = 1$.

An *oriented A -labeled Brauer diagram* is a Brauer diagram in which each strand is endowed with an orientation (not necessarily the standard orientation) and labeled by an element of A . We identify two such diagrams if one is obtained from the other by reversing the orientation on some strands and simultaneously changing the A -valued label of these strands by applying the involution $*$. For example, the diagrams in Fig. 5.2 are identified. We can omit the label on strands which would be labeled by the multiplicative identity of A ; we also don't have to care about the orientation of such strands because $\mathbf{1}^* = \mathbf{1}$ in A . We will frequently write “ A -Brauer diagram” as a shorthand for “oriented A -labeled Brauer diagram”.

As an R -module, the n -strand A -Brauer algebra $D_n(A) = D_{n,R,\delta}(A)$ is defined to be

$$D_n(A) = A^{\otimes n} \otimes D_n = \bigoplus_d (A^{\otimes n} \otimes d), \quad (5.1)$$

where the direct sum is over the set of n -strand Brauer diagrams. We will now explain an identification of simple tensors in $A^{\otimes n} \otimes D_n$ with A -Brauer diagrams. We identify a simple tensor $a_1 \otimes \cdots \otimes a_n \otimes d$, where d is a Brauer diagram, with the diagram in which the strands of d , taken

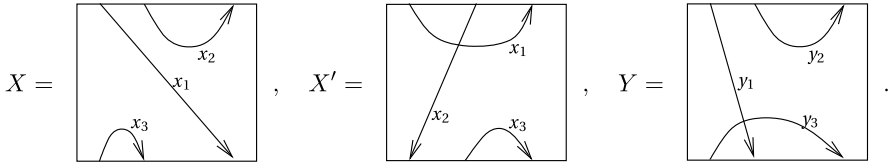


Fig. 5.3. Some A-Brauer diagrams.

in increasing order, are labeled by $a_1, \dots, a_n$, and each strand of d is given the standard orientation. For example, with d the 4-strand Brauer diagram in Fig. 5.1, we identify $a_1 \otimes \dots \otimes a_4 \otimes d$ with the A-Brauer diagram shown in Fig. 5.2.

Lemma 5.4. Let P be any subset of the set of ordinary n -strand Brauer diagrams, and let $\{P_1, \dots, P_k\}$ be any set partition of P . Let M be the span of the A-Brauer diagrams whose underlying Brauer diagram lies in P , and for each i , let M_i be the span of the set of A-Brauer diagrams whose underlying ordinary Brauer diagram lies in P_i . Then $M = \bigoplus_i M_i$ as R -modules.

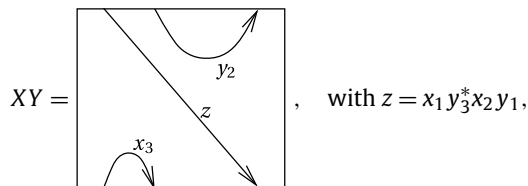
Proof. Using the identification of A-Brauer diagrams with simple tensors, we have $M = \bigoplus_{d \in P} (A^{\otimes n} \otimes d)$ and $M_i = \bigoplus_{d \in P_i} (A^{\otimes n} \otimes d)$. Since $\{P_i\}$ is a set partition of P , the result follows. $\square$

Next we describe the product of two A-Brauer diagrams; the product is another A-Brauer diagram. Given two A-Brauer diagrams X and Y with n strands, to form the product XY , first stack Y over X as for ordinary Brauer diagrams. The resulting “tangle” may contain some closed strands as well as n non-closed strands connecting two vertices; each strand is composed of one or more strands from X and Y . Give each composite strand s an orientation, arbitrarily; for non-closed strands we can take the standard orientation, but for closed strands there is no preferred orientation. Make the orientations of the component strands of s from X and Y agree with the chosen orientation of s , changing labels by applying $*$ whenever the orientation of a component strand is reversed. Label each strand by the product of the labels of its component strands, written from right to left in the order in which the labels are encountered as the strand is traversed according to its chosen orientation. For closed strands, this product is defined only up to cyclic permutation of the factors, since there is no preferred starting point for traversing the strand.

Finally, remove any A-labeled closed strands, but for each such strand multiply the diagram by the R -valued trace of its label; since the trace of a product is invariant under cyclic permutation of the factors, this element in R is well-defined. Moreover, the element of R obtained is independent of the choice of orientation of the closed strand; if the orientation of a closed strand is reversed, its label is replaced by $*$ applied to the old label, and, since the trace is $*$ -invariant, again the factor in R is unchanged.

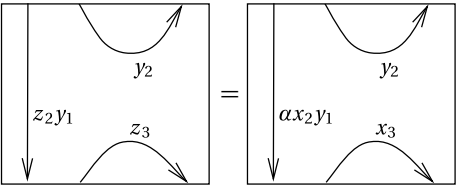
So far the product of two A-Brauer diagrams has been described as a multiple of a third, $XY = \alpha Z$, where $\alpha \in R$, but using the identification of A-Brauer diagrams with simple tensors, the factor α can be absorbed into the A-valued label of any (arbitrarily chosen) strand of Z .

Here are two examples of the product of A-Brauer diagrams. Let X , X' , and Y be as in Fig. 5.3. Then



and

$X'Y = \alpha$



, where $\alpha = \text{tr}(y_3^* x_1) = \text{tr}(x_1^* y_3)$.

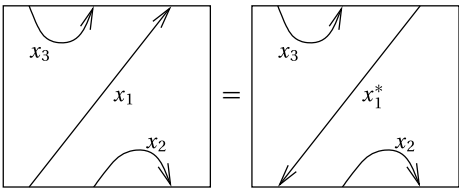
Fix two Brauer diagrams x and y , and let z be the Brauer diagram obtained by stacking y over x and removing closed loops. The procedure described above, applied to A -Brauer diagrams X and Y with underlying Brauer diagrams x and y , defines a multilinear map $A^{2n} \rightarrow A^{\otimes n} \otimes z$, hence a bilinear map

$$(A^{\otimes n} \otimes x) \times (A^{\otimes n} \otimes y) \rightarrow A^{\otimes n} \otimes z.$$

Taking Eq. (5.1) into account, we see that the product extends to a bilinear product $D_n(A) \times D_n(A) \rightarrow D_n(A)$. One can check that this product is associative.

For each oriented A -labeled Brauer diagram X , let X^* be the diagram obtained by flipping X over a horizontal line. For example, with X as in Fig. 5.3, we have

$X^* =$



.

Lemma 5.5. $X \mapsto X^*$ extends to an R -linear algebra involution on $D_n(A)$.

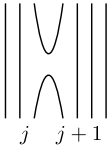
Proof. Straightforward. $\square$

Remark 5.6. It follows from Eq. (5.1), the rule for multiplying A -Brauer diagrams, and the definition of the involution $*$, that the ordinary n -strand Brauer algebra D_n imbeds as a unital $*$ -subalgebra of $D_n(A)$, spanned by unlabeled Brauer diagrams. In particular the symmetric group $\mathfrak{S}_n$ can be identified with the set of unlabeled permutation diagrams in $D_n(A)$. $D_n(A)$ is thus an $\mathfrak{S}_n$ -bimodule, with the left and right actions of the symmetric group given by left and right multiplication by permutation diagrams.

Remark 5.7. $D_1(A) \cong A$ as algebras with involution and trace. It is convenient to define $D_0(A)$ to be R with the trivial involution.

Let e_j and s_j denote the (n, n) -Brauer diagrams:

$e_j =$



,

$s_j =$



.

For $a \in A$ and $1 \leq i \leq n$, let $a^{(i)}$ be the A -Brauer diagram whose underlying Brauer diagram is the identity diagram, with strands oriented from top to bottom, with the i -th strand labeled by a .

Lemma 5.8. $D_n(A)$ is generated as a unital algebra by the ordinary Brauer diagrams e_i and s_i for $1 \leq i \leq n-1$, and by the diagrams $a^{(1)}$ for $a \in A$. One has $e_i^* = e_i$, $s_i^* = s_i$ and $(a^{(1)})^* = (a^*)^{(1)}$.

Proof. Straightforward. $\square$

Remark 5.9. If A is free as an R -module, then $D_n(A) = A^{\otimes n} \otimes D_n$ is also free as an R -module, with basis consisting of n -strand Brauer diagrams whose strands are given the standard orientation and labeled by basis elements of A .

5.5. Filtration by rank, and the embedding of the wreath product $A \wr \mathfrak{S}_n$

The *rank* of an ordinary or A -labeled Brauer diagram is the number of through strands. Note that the rank of an n -Brauer diagram has the same parity as n . It follows immediately from the definition of the multiplication that for two A -Brauer diagrams X and Y , $\text{rank}(XY) \leq \min\{\text{rank}(X), \text{rank}(Y)\}$. Hence if we write W_r^n for the span of A -Brauer diagrams with rank no more than r in $D_n(A)$, then W_r^n is an ideal and

$$(0) \subset W_0^n \subset W_1^n \subset \cdots \subset W_n^n = D_n(A).$$

Write V_s^n for the span of A -Brauer diagrams with rank exactly s in $D_n(A)$.

Lemma 5.10. $D_n(A) = \bigoplus_s V_s^n$, and $W_r^n = \bigoplus_{s \leq r} V_s^n$, as R -modules.

Proof. This follows from Lemma 5.4. $\square$

Lemma 5.11. V_n^n is a $*$ -subalgebra of $D_n(A)$ isomorphic to the wreath product algebra $A \wr \mathfrak{S}_n$, as algebras with involution.

Proof. V_n^n is closed under multiplication and under the involution $*$. Moreover, V_n^n is isomorphic to $A \wr \mathfrak{S}_n$ as R -modules, since $V_n^n = \bigoplus_{\pi} A^{\otimes n} \otimes \pi$, where the direct sum is over permutation diagrams. It is straightforward to check that the isomorphism of R -modules $V_n^n \cong A \wr \mathfrak{S}_n$ also respects multiplication and involution. $\square$

5.6. Examples: the G -Brauer algebras

Throughout this section, let R be a commutative ring with distinguished element δ as above.

Let G be an arbitrary group. There is a canonical algebra involution $*$ on the group algebra $A = RG$, namely the linear extension of the inverse map $g \mapsto g^{-1}$ on G . The R -valued trace functions on $A = RG$ correspond to class functions on G ; a trace is $*$ -invariant when $\text{tr}(g^{-1}) = \text{tr}(g)$ for all $g \in G$. According to our convention, we have to normalize such a class function by $\text{tr}(\mathbf{1}) = \delta$. For any such choice of a trace function, we can form the A -Brauer algebras with $A = RG$. Since A is free as an R -module with basis G , by Remark 5.9, the A -Brauer algebra $D_n(A)$ is free as an R -module with basis the set of G -labeled oriented n -strand Brauer diagrams (with two such diagrams identified if one is obtained from the other by reversing the orientation of some strands and simultaneously inverting the G -valued label of the strand). We call these algebras G -Brauer algebras.

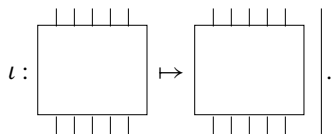
When G is an abelian group, there is another choice for the involution on $A = RG$, namely the identity map. With this choice of involution, any class function on G yields an (involution-invariant) trace on $A = RG$; again, we have to normalize the trace by $\text{tr}(\mathbf{1}) = \delta$. For any choice of the trace, we can form the A -Brauer algebras. Since the involution on A is trivial, the orientation of the strands of A -labeled Brauer diagrams becomes irrelevant. Thus the A -Brauer algebra $D_n(A)$ is free as an R -module with basis the set of *unoriented* G -labeled n -strand Brauer diagrams. These G -Brauer algebras are the same as those defined by Parvathi and Savithri [23].

Thus when G is abelian, we have two slightly different constructions of G -Brauer algebras. The Parvathi–Savithri construction is the more interesting one for this paper, because $A = RG$ is a cyclic cellular algebra with the trivial involution, assuming R has sufficiently many roots of unity, while (assuming $\text{char}(R) \neq 2$) $A = RG$ cannot be cyclic cellular with any non-trivial involution, by [Proposition 2.11](#).

Note that for any G , abelian or not, there is a canonical involution-invariant trace on $A = RG$ given by $\text{tr}(g) = 0$ if $g \neq \mathbf{1}$ and $\text{tr}(\mathbf{1}) = \delta$.

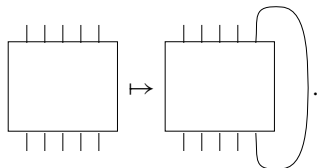
5.7. Inclusion, closure, and trace

For $n \geq 1$, and for X an A -Brauer diagram with n strands, let $\iota(X)$ be the diagram obtained by adding an unlabeled strand to X , connecting $\mathbf{n} + \mathbf{1}$ to $\mathbf{n} + \mathbf{1}$.



Then ι determines a unital algebra homomorphism $\iota : D_n(A) \rightarrow D_{n+1}(A)$.

For $n \geq 1$ define a map cl from oriented A -labeled Brauer diagrams with n strands into $D_{n-1}(A)$ as follows. First “partially close” a given A -Brauer diagram X by adding an additional smooth curve connecting $\mathbf{n}$ to $\bar{\mathbf{n}}$,



The resulting “tangle” contains a closed curve precisely when the original diagram already had a strand connecting $\mathbf{n}$ to $\bar{\mathbf{n}}$; we can suppose that this strand is oriented from $\mathbf{n}$ to $\bar{\mathbf{n}}$ and labeled by $a \in A$; in this case, remove this loop and replace it with a factor of $\text{tr}(a)$. Otherwise, the added curve is part of a composite strand containing two strands from X ; compute the A -valued label of this composite strand as in the rule for multiplying diagrams. The map cl on oriented A -labeled diagrams determines an R -linear map $\text{cl} : D_n(A) \rightarrow D_{n-1}(A)$. Moreover, cl is a (non-unital) $D_{n-1}(A)$ – $D_{n-1}(A)$ bimodule map, and $\text{cl} \circ * = * \circ \text{cl}$ for all n . Note that $\text{cl} : D_1(A) \rightarrow D_0(A) = R$ is just the trace on $D_1(A) \cong A$. For X an A -Brauer diagram with n strands, we have

$$\text{cl}(\iota(X)e_n) = X;$$

it follows that ι is injective, so we can regard $D_n(A)$ as a unital subalgebra of $D_{n+1}(A)$.

We define $\text{Tr} : D_n(A) \rightarrow R$ by $\text{Tr} = \text{cl} \circ \dots \circ \text{cl} \circ \text{cl}$. Then, as for other diagram or tangle algebras, Tr is a trace. It follows from the properties of cl discussed above that Tr is $*$ -invariant.

5.8. The A -Brauer category

We can imbed the n -strand A -Brauer algebras for all n in a category as follows. The category $\mathcal{B}_A$ has as objects the integers $0, 1, 2, \dots$.

To describe the morphisms in the category, first we describe (k, ℓ) -Brauer diagrams, where k and ℓ have the same parity. These are “Brauer diagrams” with k upper vertices $\mathbf{1}, \dots, \mathbf{k}$ and ℓ lower vertices

$\bar{1}, \dots, \bar{\ell}$, with the vertices connected in pairs by edges. We order the vertices of (k, ℓ) -Brauer diagrams by

$$1 < 2 < \dots < k < \bar{1} < \bar{2} < \dots < \bar{\ell},$$

and we use this ordering to define a total ordering of the strands of (k, ℓ) -Brauer diagrams, and a standard orientation of strands, as in Section 5.3.

Let $D_{k,\ell}$ be the free R -module with basis the set of (k, ℓ) -Brauer diagrams.

When k and ℓ do not have the same parity, we take $\text{Hom}(k, \ell) = (0)$. When k and ℓ have the same parity we set

$$D_{k,\ell}(A) = \text{Hom}(k, \ell) = A^{\otimes(k+\ell)/2} \otimes D_{k,\ell} = \bigoplus_d A^{\otimes(k+\ell)/2} \otimes d, \quad (5.2)$$

where the direct sum runs over all (k, ℓ) -Brauer diagrams. We identify simple tensors in $\text{Hom}(k, \ell) = A^{\otimes(k+\ell)/2} \otimes D_{k,\ell}$ with oriented A -labeled (k, ℓ) -Brauer diagrams. This is done in the same way as was described previously in the case $k = \ell$, in the description of the k -strand A -Brauer algebra.

Remark 5.12. As in Remark 5.9, if A is free as an R -module, then each Hom space $D_{k,\ell}(A) = \text{Hom}(k, \ell)$ is also a free R -module with basis consisting of (k, ℓ) -Brauer diagrams with strands endowed with the standard orientation and labeled by basis elements of A .

If $X \in \text{Hom}(\ell, m)$ and $Y \in \text{Hom}(k, \ell)$ are A -Brauer diagrams, then we can form the product XY , which is an oriented A -labeled (k, m) -Brauer diagram. The definition of the product is essentially the same as the definition of the product in $D_n(A)$ given previously. The product extends to a bilinear product $\text{Hom}(\ell, m) \times \text{Hom}(k, \ell) \rightarrow \text{Hom}(k, m)$. Moreover, the product is associative; i.e., if Z is an A -Brauer diagram in $\text{Hom}(j, k)$, then $(XY)Z = X(YZ)$ in $\text{Hom}(j, m)$. Thus $\mathcal{B}_A$ is a category whose Hom spaces are R -modules, with bilinear composition of morphisms. Note that the algebra $\text{End}(n)$ is isomorphic to $D_n(A)$.

For an oriented A -labeled (k, ℓ) -Brauer diagram X , let X^* be the oriented A -labeled (ℓ, k) -Brauer diagram obtained by flipping X over a horizontal axis. Then $*$ determines a linear map from $\text{Hom}(k, \ell)$ to $\text{Hom}(\ell, k)$ for each k and ℓ ; moreover, $*$ is a contravariant functor from $\mathcal{B}_A$ to itself; that is, $(XY)^* = Y^*X^*$, whenever the product XY is defined.

The category $\mathcal{B}_A$ has a tensor or monoidal structure, described as follows. The tensor product of objects is $n \odot m = n + m$. Given oriented A -labeled Brauer diagrams $X \in \text{Hom}(k, \ell)$ and $Y \in \text{Hom}(m, n)$, their tensor product $X \odot Y \in \text{Hom}(k + m, \ell + n)$ is given by horizontal juxtaposition of diagrams. Fix ordinary Brauer diagrams $x \in D_{k,\ell}$ and $y \in D_{m,n}$ and let z be the $(k + m, \ell + n)$ -Brauer diagram obtained by horizontal juxtaposition of x and y . Then $(X, Y) \mapsto X \odot Y$, applied to A -Brauer diagrams X and Y with underlying Brauer diagrams x and y , defines a multilinear map from

$$A^{(k+\ell+m+n)/2} \rightarrow A^{\otimes(k+\ell+m+n)/2} \otimes z,$$

hence a bilinear map

$$(A^{\otimes(k+\ell)/2} \otimes x) \times (A^{\otimes(m+n)/2} \otimes y) \rightarrow A^{\otimes(k+\ell+m+n)/2} \otimes z.$$

Taking Eq. (5.2) into account, we see that the product extends to a bilinear product $\text{Hom}(k, \ell) \times \text{Hom}(m, n) \rightarrow \text{Hom}(k + m, \ell + n)$. One can check that $\mathcal{B}_A$ satisfies the axioms of a monoidal tensor category (see for example, [3, p. 12]). We will not need this, so we will not go into more detail here.

The bilinear tensor product operation

$$\odot : \text{Hom}(k, \ell) \times \text{Hom}(m, n) \rightarrow \text{Hom}(k + m, \ell + n)$$

determines a linear map

$$\odot : \text{Hom}(k, \ell) \otimes \text{Hom}(m, n) \rightarrow \text{Hom}(k + m, \ell + n).$$

In particular, identifying $D_n(A)$ with $\text{End}(n)$, we have a linear map

$$\odot : D_n(A) \otimes D_m(A) \rightarrow D_{n+m}(A).$$

Lemma 5.13. *The linear map $\odot : D_n(A) \otimes D_m(A) \rightarrow D_{n+m}(A)$ is injective.*

Proof. Note this is obvious if A is free as an R -module, which is the only case we will actually need. In this, case, the Hom spaces are also free, as described in Remark 5.12, and the map takes basis elements to basis elements.

For the general case, fix an ordinary m -Brauer diagram x and an ordinary n -Brauer diagram y , and let z be the ordinary $(m + n)$ -Brauer diagram obtained by horizontal juxtaposition of x and y . Then because of Eq. (5.1), we only have to check injectivity of the restriction $\odot : (A^{\otimes m} \otimes x) \otimes (A^{\otimes n} \otimes y) \rightarrow A^{\otimes(m+n)} \otimes z$. But this map can be identified with a linear map $A^{\otimes m} \otimes A^{\otimes n} \rightarrow A^{\otimes(m+n)}$, and this linear map is simply a permutation of tensor places. The following example will clarify this. $\square$

Example 5.14. Consider the A -Brauer diagrams X and Y from Fig. 5.3. Let x and y be the underlying Brauer diagrams of X and Y , and let z be the underlying Brauer diagram of $X \odot Y$. Then under the identification of oriented A -labeled diagrams with simple tensors, X is identified with $(x_1 \otimes x_2 \otimes x_3) \otimes x$, Y with $(y_1 \otimes y_2 \otimes y_3) \otimes y$, and $X \odot Y$ with $(x_1 \otimes x_2 \otimes y_1 \otimes y_2 \otimes x_3 \otimes y_3) \otimes z$.

Recall that V_0^{2f} denotes the span of A -Brauer diagrams of rank zero in $D_{2f}(A)$. Any A -Brauer diagram Z of rank zero evidently has a factorization $Z = XY^*$, where X and Y are oriented A -labeled $(0, 2f)$ -Brauer diagrams. X and Y^* are simply the bottom and top halves of Z .

Lemma 5.15. $X \otimes Y \mapsto XY^*$ determines an isomorphism

$$\text{Hom}(0, 2f) \otimes \text{Hom}(0, 2f) \rightarrow V_0^{2f}.$$

Proof. Surjectivity follows from the remark preceding the statement of the lemma, and the proof of injectivity is similar to the proof of Lemma 5.13. $\square$

5.9. Bases of A -Brauer algebras

Definition 5.16. Let $0 \leq k \leq n$. A $(k, n - k)$ -shuffle is a permutation $\pi \in \mathfrak{S}_n$ such that $\pi(i) < \pi(j)$ whenever $1 \leq i < j \leq k$ or $k + 1 \leq i < j \leq n$.

Remark 5.17. If $0 < k < n - k$, the $(k, n - k)$ -shuffles are just the distinguished left coset representatives of $\mathfrak{S}_k \times \mathfrak{S}_{n-k}$ in $\mathfrak{S}_n$. If $k = 0$ or $k = n$ the identity permutation is the unique $(k, n - k)$ -shuffle.

Lemma 5.18. *Let X be an n -strand A -Brauer diagram of rank $s = n - 2f$. Then X has a unique factorization*

$$X = \alpha(X_0 \odot X_1)\beta^*,$$

where α and β are $(2f, s)$ -shuffles, X_0 is a $2f$ -strand A -Brauer diagram of rank 0, and X_1 is an s -strand A -labeled permutation diagram.

Proof. Let $i_1 < \dots < i_{2f}$ be the top vertices adjacent to horizontal strands of X and $j_1 < \dots < j_s$ the top vertices adjacent to vertical strands of X . Likewise, let $\bar{k}_1 < \dots < \bar{k}_{2f}$ be the bottom vertices adjacent to horizontal strands of X and $\bar{l}_1 < \dots < \bar{l}_s$ the bottom vertices adjacent to vertical strands of X . Set

$$\alpha = \begin{pmatrix} 1 & \dots & 2f & 2f+1 & \dots & n \\ k_1 & \dots & k_{2f} & l_1 & \dots & l_s \end{pmatrix}$$

and

$$\beta = \begin{pmatrix} 1 & \dots & 2f & 2f+1 & \dots & n \\ i_1 & \dots & i_{2f} & j_1 & \dots & j_s \end{pmatrix}.$$

Consider $X' = \alpha^* X \beta$. Then X' has horizontal strands connecting $1, \dots, 2f$ and $\bar{1}, \dots, \bar{2f}$ in pairs and vertical strands connecting $2f+1, \dots, n$ and $\bar{2f}+1, \dots, \bar{n}$ in pairs. Hence $X' = X_0 \odot X_1$, where X_0 and X_1 are as in the statement of the lemma. Thus

$$X = \alpha X' \beta^* = \alpha(X_0 \odot X_1) \beta^*,$$

as required.

The factorization is unique because specifying the shuffles α and β is equivalent to specifying the set of vertices adjacent to horizontal and to vertical strands, and X together with α and β determine X_0 and X_1 . $\square$

Remark 5.19. If d is an ordinary n -strand Brauer diagram of rank s , then d has a unique factorization

$$d = \alpha(d_0 \odot d_1) \beta^*,$$

where α and β are $(2f, s)$ -shuffles, d_0 is an ordinary $2f$ -strand Brauer diagram of rank 0, and d_1 is an ordinary s -strand permutation diagram. Moreover, if X is an A -Brauer diagram with underlying ordinary Brauer diagram d , and $X = \alpha(X_0 \odot X_1) \beta^*$ is the factorization of X , then $d = \alpha(d_0 \odot d_1) \beta^*$, where d_0 and d_1 are the underlying ordinary Brauer diagrams of X_0 and X_1 .

Fix $s \leq n$ with $n-s$ even and let $f = (n-s)/2$. Fix $(2f, s)$ -shuffles α and β . Let $P_{\alpha, \beta}$ be the set of ordinary Brauer diagrams d of rank s with factorization $d = \alpha(d_0 \odot d_1) \beta^*$. The sets $P_{\alpha, \beta}$ for different pairs (α, β) are mutually disjoint. Let

$$V_{\alpha, \beta} = \alpha(V_0^{2f} \odot V_s^s) \beta^*.$$

Then $V_{\alpha, \beta}$ is the span of those A -Brauer diagrams of rank s whose underlying ordinary Brauer diagram lies in $P_{\alpha, \beta}$.

Lemma 5.20. $V_s^n = \bigoplus_{\alpha, \beta} V_{\alpha, \beta}$.

Proof. This follows from the discussion above and Lemma 5.4. $\square$

For the rest of this section, assume that A is a free R -module, so that each $D_{k, \ell}(A)$ is also a free R -module, by Remark 5.12.

For each s with $s \leq n$ and $n-s$ even, let $f = (n-s)/2$. Let $\mathbb{B}_s$ be any R -basis of V_s^s . Let $\mathbb{A}_f$ be the basis of $D_{0,2f}(A)$ consisting of $(0, 2f)$ -Brauer diagrams endowed with the standard orientation and labeled by basis elements of A . Define

$$\mathcal{B}_s^n = \{\alpha(XY^* \odot b)\beta^*: \alpha \text{ and } \beta \text{ are } (2f, s)\text{-shuffles, } X, Y \in \mathbb{A}_f, \text{ and } b \in \mathbb{B}_s\}.$$

Lemma 5.21. $\mathcal{B}_s^n$ is an R -basis of V_s^n and $\bigcup_s \mathcal{B}_s^n$ is an R -basis of $D_n(A)$.

Proof. By Lemmas 5.13 and 5.15,

$$\{XY^* \odot b: X, Y \in \mathbb{A}_f, \text{ and } b \in \mathbb{B}_s\}$$

is an R -basis of $V_0^{2f} \odot V_s^s$. Since, for fixed α and β , $Z \mapsto \alpha Z \beta^*$ is a linear isomorphism from $V_0^{2f} \odot V_s^s$ to $V_{\alpha, \beta}$, it follows that

$$\{\alpha(XY^* \odot b)\beta^*: X, Y \in \mathbb{A}_f, \text{ and } b \in \mathbb{B}_s\}$$

is an R -basis of $V_{\alpha, \beta}$. Thus, by Lemmas 5.10 and 5.20, it follows that $\mathcal{B}_s^n$ is a basis of V_s^n and that $\bigcup_s \mathcal{B}_s^n$ is a basis of $D_n(A)$. $\square$

6. Cellularity of A -Brauer algebras

In this section, we will prove the following theorem:

Theorem 6.1. Let A be an algebra with an algebra involution $*$ and a $*$ -invariant trace tr . Suppose that A is cyclic cellular. Then for all $n \geq 1$, the A -Brauer algebra $D_n(A)$ is a cyclic cellular algebra.

The proof of this theorem is similar to proofs in the literature for the cellularity of Brauer or BMW algebras [8,26]. We will base our computations on some arguments from a well-known, but unpublished paper [21] of Morton and Wassermann on bases of the BMW algebras. A similar but necessarily more complicated proof of the cellularity of cyclotomic BMW algebras was given in [10]. It would be possible here to use the method of iterated inflations of Koenig and Xi [17], but we prefer not to take this approach, as we can exhibit explicit cellular bases of the A -Brauer algebras, based on our cellular basis of the wreath product algebras $A \wr \mathfrak{S}_s$.

Remark 6.2. In Theorem 6.1, if A is strictly cellular, then so is the A -Brauer algebra $D_n(A)$.

Assume that A is a cyclic cellular algebra over an integral domain R , and that A has an involution-invariant R -valued trace. We adopt the notation of Section 4 regarding the cellular structure of A ; in particular, the partially ordered set in the cell datum of A is denoted by $(\Gamma, \geq)$. By Theorem 4.1, for all $s \geq 1$, $A \wr \mathfrak{S}_s$ is cyclic cellular. The partially ordered set in the cell datum for $A \wr \mathfrak{S}_s$ is $(\Lambda_s^\Gamma, \geq_\Gamma)$. The cellular basis of $A \wr \mathfrak{S}_s$ is

$$\mathcal{B} = \{m_{(\mathfrak{s}, v), (t, w)}^\lambda: \lambda \in \Lambda_n^\Gamma, \text{ and } (\mathfrak{s}, v), (t, w) \in \mathcal{T}(\lambda)\},$$

where $\mathcal{T}(\lambda)$ denotes the set of pairs $(\mathfrak{s}, v)$ with $\mathfrak{s}$ a standard λ -tableau and $v \in V^{\alpha(\lambda)}$. For fixed s and λ , define the ideals N^λ and $\bar{N}^\lambda$ in $A \wr \mathfrak{S}_s$, as in Definition 4.9.

Let $n \geq 1$. We will now propose a cell datum for $D_n(A)$. Define

$$\Lambda = \{(s, \lambda): s \leq n \text{ and } n-s \text{ even, and } \lambda \in \Lambda_s^\Gamma\},$$

with partial order $\triangleright_\Gamma$ defined by $(s, \lambda) \triangleright_\Gamma (s', \lambda')$ if $s < s'$ or if $s = s'$ and $\lambda \triangleright_\Gamma \lambda'$. For $(s, \lambda) \in \Lambda$, let $f = (n - s)/2$. Let $\mathbb{A}_f$ be the basis of $D_{0,2f}(A)$ consisting of $(0, 2f)$ -Brauer diagrams, endowed with the standard orientation, with strands labeled by elements of the cellular basis of A . Define $\mathcal{T}(s, \lambda)$ to be the set of all (α, X, s, v) where α is a $(2f, s)$ -shuffle, $X \in \mathbb{A}_f$, and $(s, v) \in \mathcal{T}(\lambda)$. For (α, X, s, v) and (β, Y, t, w) in $\mathcal{T}(s, \lambda)$, set

$$m_{(\alpha, X, s, v), (\beta, Y, t, w)}^{(s, \lambda)} = \alpha(XY^* \odot m_{(s, v), (t, w)}^\lambda) \beta^*,$$

where we have identified V_s^s with $A \wr \mathfrak{S}_s$, using Lemma 5.11. Let $\mathcal{D}$ denote the set of all $m_{(\alpha, X, s, v), (\beta, Y, t, w)}^{(s, \lambda)}$, as (s, λ) varies in Λ , and (α, X, s, v) and (β, Y, t, w) vary in $\mathcal{T}(s, \lambda)$. Our claim is that $\mathcal{D}$ is a cellular basis of $D_n(A)$.

Lemma 6.3. $\mathcal{D}$ is an R -basis of $D_n(A)$.

Proof. This follows immediately from Lemma 5.21. $\square$

For fixed $(s, \lambda) \in \Lambda$, define $M^{(s, \lambda)}$ to be the span of basis elements $m_{(\alpha, X, s, v), (\beta, Y, t, w)}^{(s', \mu)}$ with $(s', \mu) \triangleright_\Gamma (s, \lambda)$, and $\overline{M}^{(s, \lambda)}$ to be the span of such basis elements with $(s', \mu) \triangleright_\Gamma (s, \lambda)$. We will use the following observation several times.

Lemma 6.4. For $(s, \lambda) \in \Lambda$ and $f = (n - s)/2$, we have

$$V_0^{2f} \odot \overline{N}^\lambda \subseteq \overline{M}^{(s, \lambda)}.$$

Proof. Evident. $\square$

Fix $s \leq n$ with $n - s$ even and let $f = (n - s)/2$. Let X_0 be the ordinary $(0, 2f)$ -Brauer diagram with strands $(\bar{1}, \bar{2}), (\bar{3}, \bar{4}), \dots, (\overline{2f-1}, \overline{2f})$. Note that $X_0 X_0^* = e_1 e_3 \cdots e_{2f-1}$. For $X \in \mathbb{A}_f$, let

$$(\bar{i}_1, \bar{j}_1) < (\bar{i}_2, \bar{j}_2) < \cdots < (\bar{i}_f, \bar{j}_f)$$

be the edges of X and let $a_1, \dots, a_f$ be the elements of the cellular basis of A labeling these edges. Let $\pi(X) \in \mathfrak{S}_{2f}$ denote the permutation

$$\pi(X) = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & 2f-1 & 2f \\ i_1 & j_1 & i_2 & j_2 & \dots & i_f & j_f \end{pmatrix},$$

and let $a(X) = \prod_{j=1}^f a_j^{(2j)} \in A^{\otimes 2f} \subset A \wr \mathfrak{S}_{2f} \subset D_{2f}(A)$.

Lemma 6.5. Let $X \in \mathbb{A}_f$. Then $X = \pi(X)a(X)X_0$.

Proof. It is straightforward to check that the oriented A -labeled $(0, 2f)$ -Brauer diagram given in factored form on the right hand side has the same oriented labeled edges as does X . $\square$

Set

$$y_{(s, \lambda)} = (e_1 e_3 \cdots e_{2f-1}) \odot y^{\alpha(\lambda)} x_\lambda,$$

and for $(\alpha, X, s, v) \in \mathcal{T}(s, \lambda)$, set

$$v_{(\alpha, X, s, v)} = \alpha(\pi(X)a(X) \odot d(s)v).$$

Combining Lemma 6.5 with the factorization $m_{(s, v), (t, w)}^\lambda = d(s)v y^{\alpha(\lambda)} x_\lambda w^* d(t)^*$, we have the following factorization for the elements of the cellular basis of $D_n(A)$.

Corollary 6.6. For any (α, X, s, u) and $(\beta, Y, t, w) \in \mathcal{T}(s, \lambda)$,

$$m_{(\alpha, X, s, u), (\beta, Y, t, w)}^{(s, \lambda)} = v_{(\alpha, X, s, u)} y_{(s, \lambda)} (v_{(\beta, Y, t, w)})^*. \quad (6.1)$$

Lemma 6.7.

- (1) $(y_{(s, \lambda)})^* \equiv y_{(s, \lambda)} \pmod{\overline{M}^{(s, \lambda)}}$.
- (2) $(m_{(\alpha, X, s, v), (\beta, Y, t, w)}^{(s, \lambda)})^* \equiv m_{(\beta, Y, t, w), (\alpha, X, s, v)}^{(s, \lambda)} \pmod{\overline{M}^{(s, \lambda)}}$.

Proof. Part (1) follows from Lemma 4.20 and Lemma 6.4. Part (2) follows from part (1) and Eq. (6.1). $\square$

In order to show that $\mathcal{D}$ is a cellular basis of $D_n(A)$, we examine the action of $D_n(A)$ on elements of the form

$$v_{(\alpha, X, s, u)} y_{(s, \lambda)} = \alpha(X X_0^* \odot m_{(s, u), (t^\lambda, 1)}^\lambda).$$

Lemma 6.8. Let $S \odot T \in D_{2f}(A) \odot D_s(A)$. For $X \in \mathbb{A}_f$ and $(s, v) \in \mathcal{T}(\lambda)$,

$$(S \odot T) v_{(1, X, s, u)} y_{(s, \lambda)}$$

is congruent modulo $\overline{M}^{(s, \lambda)}$ to a linear combination of elements

$$v_{(1, X', s', u')} y_{(s, \lambda)}$$

with $X' \in \mathbb{A}_f$ and $(s', u') \in \mathcal{T}(\lambda)$.

Proof. Write

$$Z = (S \odot T) v_{(1, X, s, u)} y_{(s, \lambda)} = S X X_0^* \odot T m_{(s, u), (t^\lambda, 1)}^\lambda.$$

If $\text{rank}(T) < s$, then $\text{rank}(Z) < s$, so $Z \in \bigoplus_{r < s} V_r^n$. It follows that Z is in the span of $M^{(r, \mu)}$ with $r < s$, and in particular $Z \in \overline{M}^{(s, \lambda)}$. Thus, we can suppose that $\text{rank}(T) = s$, i.e. $T \in V_s^s \cong A \wr \mathfrak{S}_s$. By Proposition 4.19, $T m_{(s, u), (t^\lambda, 1)}^\lambda$ is congruent modulo $\overline{N}^\lambda$ to a linear combination of elements $m_{(s', u'), (t^\lambda, 1)}^\lambda$. Moreover, $SX \in D_{0, 2f}(A)$, so is a linear combination of basis elements $X' \in \mathbb{A}_f$. Therefore Z is congruent to a linear combination of elements

$$X' X_0^* \odot m_{(s', u'), (t^\lambda, 1)}^\lambda = v_{(1, X', s', u')} y_{(s, \lambda)},$$

modulo $V_0^{2f} \odot \overline{N}^\lambda \subseteq \overline{M}^{(s, \lambda)}$. $\square$

Lemma 6.9. Let $T \in D_n(A)$ and $(\alpha, X, s, u) \in \mathcal{T}(s, \lambda)$. Then $T v_{(\alpha, X, s, u)} y_{(s, \lambda)}$ is congruent modulo $\overline{M}^{(s, \lambda)}$ to a linear combination of elements $v_{(\alpha', X', s', u')} y_{(s, \lambda)}$.

Proof. It suffices to prove this when T is one of the generators $a^{(1)}$, g_i or e_i of $D_n(A)$, see Lemma 5.8.

Case 1. $T = a^{(1)}$. Write $j = \alpha^{-1}(1)$. We have

$$\begin{aligned} a^{(1)} v_{(\alpha, X, s, u)} y_{(s, \lambda)} &= a^{(1)} \alpha(X X_0^* \odot m_{(s, u), (t^\lambda, 1)}^\lambda) \\ &= \alpha a^{(j)}(X X_0^* \odot m_{(s, u), (t^\lambda, 1)}^\lambda). \end{aligned}$$

Since $a^{(j)} \in D_{2f}(A) \odot D_s(A)$, the conclusion follows from Lemma 6.8.

Case 2. $T = g_i$. More generally, if $T = \pi \in \mathfrak{S}_n$, then $\pi \alpha = \alpha'(\pi_1 \odot \pi_2)$, where α' is a $(2f, s)$ -shuffle and $\pi_1 \odot \pi_2 \in \mathfrak{S}_{2f} \times \mathfrak{S}_s$. Thus

$$\begin{aligned} \pi v_{(\alpha, X, s, u)} y_{(s, \lambda)} &= \pi \alpha(X X_0^* \odot m_{(s, u), (t^\lambda, 1)}^\lambda) \\ &= \alpha'(\pi_1 \odot \pi_2)(X X_0^* \odot m_{(s, u), (t^\lambda, 1)}^\lambda), \end{aligned}$$

and again the conclusion follows from Lemma 6.8.

Next, we consider $T = e_i$. There are two cases, depending on the values of $\alpha^{-1}(i)$ and $\alpha^{-1}(i+1)$.

Case 3. $T = e_i$, and both $\alpha^{-1}(i)$ and $\alpha^{-1}(i+1)$ are $\leq 2f$, or both are $\geq 2f+1$. Because α is a $(2f, s)$ -shuffle, we have $\alpha^{-1}(i+1) = \alpha^{-1}(i) + 1$. Therefore, $e_i \alpha = \alpha e_j$, where $j = \alpha^{-1}(i)$. Note that $e_j \in D_{2f}(A) \odot D_s(A)$. The conclusion follows by applying Lemma 6.8, as in the previous cases.

Case 4. $T = e_i$, and one of $\alpha^{-1}(i)$ and $\alpha^{-1}(i+1)$ is $\leq 2f$ while the other is $\geq 2f+1$. We have

$$\begin{aligned} e_i v_{(\alpha, X, s, u)} y_{(s, \lambda)} &= e_i \alpha(X X_0^* \odot m_{(s, u), (t^\lambda, 1)}^\lambda) \\ &= e_i \alpha(X \odot 1)(X_0^* \odot m_{(s, u), (t^\lambda, 1)}^\lambda). \end{aligned}$$

It's easy to see that the A -Brauer diagram $e_i \alpha(X \odot 1) \in D_{s, n}(A)$ has rank s , so using the procedure of Lemma 5.18, it can be written as $\alpha'(X_1 \odot X_2)$, where α' is a $(2f, s)$ -shuffle, $X_1 \in D_{0, 2f}(A)$ and $X_2 \in V_s^s$, and thus as a linear combination of terms $\alpha'(X' \odot X_2)$, where $X' \in \mathbb{A}_f$. Thus $e_i v_{(\alpha, X, s, u)} y_{(s, \lambda)}$ is a linear combination of terms

$$\alpha'(X' \odot X_2)(X_0^* \odot m_{(s, v), (t^\lambda, 1)}^\lambda) = \alpha'(X' X_0^* \odot X_2 m_{(s, v), (t^\lambda, 1)}^\lambda).$$

Now the result follows by using Proposition 4.19, and Lemma 6.4. $\square$

Corollary 6.10. Let $T \in D_n(A)$ and $m_{(\alpha, X, s, u), (\beta, Y, t, w)}^{(s, \lambda)} \in \mathcal{D}$. Then $T m_{(\alpha, X, s, u), (\beta, Y, t, w)}^{(s, \lambda)}$ is congruent modulo $\overline{M}^{(s, \lambda)}$ to a linear combination of basis elements $m_{(\alpha', X', s', u'), (\beta, Y, t, w)}^{(s, \lambda)}$, with the coefficients of the linear combination independent of (β, Y, t, w) .

Proof. We have that $Tv_{(\alpha, X, s, u)}y_{(s, \lambda)}$ is congruent modulo $\overline{M}^{(s, \lambda)}$ to a linear combination of elements $v_{(\alpha', X', s', u')}y_{(s, \lambda)}$. Multiplying this congruence on the right by $(v_{(\beta, Y, t, w)})^*$ yields the result. $\square$

Proof of Theorem 6.1. By Lemma 6.3, $\mathcal{D}$ is an R -basis of $D_n(A)$. By Lemma 6.7 and Corollary 6.10, $\mathcal{D}$ is a cellular basis. Fix $(s, \lambda) \in \Lambda$. Comparing Lemma 6.9 and Corollary 6.10, we see that

$$M := \text{span}\{v_{(\alpha, X, s, u)}y_{(s, \lambda)} + \overline{M}^{(s, \lambda)} : (\alpha, X, s, u) \in \mathcal{T}((s, \lambda))\}$$

is a $D_n(A)$ module isomorphic to the cell module $\Delta^{(s, \lambda)}$. It follows from Corollary 4.18 and Lemma 6.4 that $y_{(s, \lambda)} \in M$, and hence $\Delta^{(s, \lambda)}$ is cyclic. $\square$

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