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Ext-quivers of hearts of A-type and the orientations of associahedrons

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ABSTRACT

We classify the Ext-quivers of hearts in the bounded derived category $\mathcal{D}(A_n)$ and the finite-dimensional derived category $\mathcal{D}(\Gamma_N A_n)$ of the Calabi–Yau-N Ginzburg algebra $\Gamma_N A_n$. This provides the classification for Buan–Thomas' colored quivers for higher clusters of A-type. We also give an explicit combinatorial constructions from a binary tree with $n+2$ leaves to a torsion pair in $\operatorname{mod}\mathbf{k}\overrightarrow{A_n}$ and a cluster tilting set in the corresponding cluster category, for the straight oriented A-type quiver $\overrightarrow{A_n}$. As an application, we show that the orientation of the n-dimensional associahedron induced by poset structure of binary trees coincides with the orientation induced by poset structure of torsion pairs in $\operatorname{mod}\mathbf{k}\overrightarrow{A_n}$ (under the correspondence above).

Summary

Assem and Happel [1] gave a classification of repeatedly tilted algebras of A-type using tilting theory decades ago. In the first part of the paper (Section 1 and Section 2), we generalize their result to classify (Theorem 2.11) the Ext-quivers of hearts of A-type (i.e. in $\mathcal{D}^b(A_n)$), in terms of graded gentle trees. As an application, we describe (Corollary 2.12) the Ext-quivers of hearts in $\mathcal{D}(\Gamma_N A_n)$, the finite-dimensional derived category of the Calabi–Yau-N Ginzburg algebra $\Gamma_N A_n$, which correspond (cf. [10, Theorem 8.6]) to colored quivers for $(N-1)$-clusters of A-type, in the sense of Buan–Thomas [3].

In the second part of the paper (Section 3), we give explicit combinatorial constructions (Proposition 3.2 and Proposition 3.3), from a binary tree with $n+2$ leaves (for parenthesizing a word with $n+2$ letters) to a torsion pair in $\operatorname{mod}\mathbf{k}\overrightarrow{A_n}$ and a cluster tilting sets in the (normal) cluster category $\mathcal{C}(A_n)$, where $\overrightarrow{A_n}$ is a straight oriented A_n quiver. Thus, we obtain the bijections between these sets. As an application, we show (Theorem 3.5) that under the bijection above, the orientation of the

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n-dimensional associahedron induced by poset structure of binary trees (cf. [11]) coincides with the orientation induced by poset structure of torsion pairs (or hearts, in the sense of King–Qiu [10]).

Note that there are many potential orientations for the n-dimensional associahedron, arising from the representation theory of quivers, cf. [10, Fig. 4 and Theorem 9.6]. These orientations are also of interest in physics (see [4]), as they are related to wall crossing formula, quantum dilogarithm identities and Bridgeland's stability conditions (cf. [7] and [12]).

1. Preliminaries

1.1. Derived category and cluster category

Let Q be a quiver of A-type with n vertices and $\mathbf{k}$ a fixed algebraic closed field. Let $\mathbf{k}Q$ be the path algebra, $\mathcal{H}_Q = \operatorname{mod} \mathbf{k}Q$ its module category and let $\mathcal{D}(Q) = \mathcal{D}^b(\mathcal{H}_Q)$ be the bounded derived category. Note that $\mathcal{D}(Q)$ is independent of the orientation of Q and we will write A_n for Q sometimes.

Denote by τ the *AR-functor* (cf. [2, Chapter IV]). Let $\mathcal{C}(A_n)$ be the *cluster category* of $\mathcal{D}(A_n)$, that is the orbit category of $\mathcal{D}(A_n)$ quotiented by $[-1] \circ \tau$. Denote by π_n be the quotient map

$$\pi_n : \mathcal{D}(A_n) \to \mathcal{C}(A_n).$$

1.2. Calabi–Yau category

Let $N > 1$ be an integer. Denote by $\Gamma_N Q$ the *Calabi–Yau-N Ginzburg* (*dg*) *algebra* associated to Q, which is constructed as follows (cf. e.g. [10, Section 7]):

- Let Q^N be the graded quiver whose vertex set is Q_0 and whose arrows are: the arrows in Q with degree 0; an arrow $a^* : j \to i$ with degree $2 - N$ for each arrow $a : i \to j$ in Q; a loop $e^* : i \to i$ with degree $1 - N$ for each vertex e in Q.
- The underlying graded algebra of $\Gamma_N Q$ is the completion of the graded path algebra $\mathbf{k}Q^N$ in the category of graded vector spaces with respect to the ideal generated by the arrow of Q^N.
- The differential of $\Gamma_N Q$ is the unique continuous linear endomorphism homogeneous of degree 1 which satisfies the Leibniz rule and takes the following values on the arrow of Q^N

$$\mathrm{d} \sum_{e \in Q_0} e^* = \sum_{a \in Q_1} [a, a^*].$$

Write $\mathcal{D}(\Gamma_N Q)$ for $\mathcal{D}_{fd}(\operatorname{mod} \Gamma_N Q)$, the *finite-dimensional derived category* of $\Gamma_N Q$ and $\mathcal{H}_\Gamma$ its canonical heart.

Notice that the derived categories are always triangulated. Again, since $\mathcal{D}(\Gamma_N Q)$ is independent of the orientation of Q, we will write $\Gamma_N A_n$ for $\Gamma_N Q$.

1.3. Hearts of triangulated categories

A *torsion pair* in an abelian category $\mathcal{C}$ is a pair of full subcategories $\langle \mathcal{F}, \mathcal{T} \rangle$ of $\mathcal{C}$, such that $\operatorname{Hom}(\mathcal{T}, \mathcal{F}) = 0$ and furthermore every object $E \in \mathcal{C}$ fits into a short exact sequence $0 \to E^{\mathcal{T}} \to E \to E^{\mathcal{F}} \to 0$ for some objects $E^{\mathcal{T}} \in \mathcal{T}$ and $E^{\mathcal{F}} \in \mathcal{F}$.

A *t-structure* on a triangulated category $\mathcal{D}$ is a full subcategory $\mathcal{P} \subset \mathcal{D}$ with $\mathcal{P}[1] \subset \mathcal{P}$ such that, if one defines $\mathcal{P}^\perp = \{G \in \mathcal{D} \colon \operatorname{Hom}_{\mathcal{D}}(F, G) = 0,\ \forall F \in \mathcal{P}\}$, then, for every object $E \in \mathcal{D}$, there is a unique triangle $F \to E \to G \to F[1]$ in $\mathcal{D}$ with $F \in \mathcal{P}$ and $G \in \mathcal{P}^\perp$. A t-structure $\mathcal{P}$ is *bounded* if

$$\mathcal{D} = \bigcup_{i,j \in \mathbb{Z}} \mathcal{P}^\perp[i] \cap \mathcal{P}[j].$$

The *heart* of a t-structure $\mathcal{P}$ is the full subcategory

$$\mathcal{H} = \mathcal{P}^{\perp}[1] \cap \mathcal{P}$$

and any bounded t-structure is determined by its heart. In this paper, we only consider bounded t-structures and their hearts.

Recall that we can forward/backward tilt a heart $\mathcal{H}$ to get a new one, with respect to any torsion pair in $\mathcal{H}$ in the sense of Happel–Reiten–Smalø ([5], see also [10, Proposition 3.2]). Further, all forward/backward tilts with respect to torsion pairs in $\mathcal{H}$, correspond one–one to all hearts between $\mathcal{H}$ and $\mathcal{H}[\pm 1]$ (in the sense of King–Qiu [10]). In particular there is a special kind of tilting which is called simple tilting (cf. [12, Definition 3.6]). We denote by $\mathcal{H}_S^{\sharp}$ and $\mathcal{H}_S^{\flat}$, respectively, the simple forward/backward tilts of a heart $\mathcal{H}$, with respect to a simple S.

We define the *exchange graph* of a triangulated category $\mathcal{D}$ to be the oriented graph, whose vertices are all hearts in $\mathcal{D}$ and whose edges correspond to the simple forward titling between them. Denote by $\mathrm{EG}(A_n)$ the exchange graph of $\mathcal{D}(A_n)$, and $\mathrm{EG}^{\circ}(\Gamma_N A_n)$ the principal component of the exchange graph of $\mathcal{D}(\Gamma_N A_n)$, that is, the connected component containing $\mathcal{H}_\Gamma$.

2. Ext-quivers of A-type

2.1. Graded gentle tree

In [1], there is a complete description of all repeatedly tilted algebras of type A_n, namely:

Definition 2.1. (See [2].) Let A be a quiver algebra with acyclic quiver T_A. The algebra $A \cong \mathbf{k}T_A/\mathcal{I}$ is called *gentle* if the bound quiver $(T_A, \mathcal{I})$ has the following properties:

1°. Each vertex of T_A is the source and the target of at most two arrows.
2°. For each arrow $\alpha \in (T_A)_1$, there is at most one arrow β and one arrow γ such that $\alpha\beta \notin \mathcal{I}$ and $\gamma\alpha \notin \mathcal{I}$.
3°. For each arrow $\alpha \in (T_A)_1$, there is at most one arrow ξ and one arrow ζ such that $\alpha\xi \in \mathcal{I}$ and $\zeta\alpha \in \mathcal{I}$.
4°. The ideal $\mathcal{I}$ is generated by the paths in 3°.

If T_A is a tree, the gentle algebra $A \cong \mathbf{k}T_A/\mathcal{I}$ is called a *gentle tree algebra*.

Theorem 2.2. *Let A be a quiver algebra with bound quiver $(T_A, \mathcal{I})$. Then A is (repeatedly) tilted algebras of type A_n if and only if $(T_A, \mathcal{I})$ is a gentle tree algebra* (cf. [1], also [2]).

Considering the special properties of T_A, we can color the arrows of T_A with two colors, such that any two neighbor arrows α, β has the same color if and only if $\alpha\beta \in \mathcal{I}$ or $\beta\alpha \in \mathcal{I}$. Alternatively, we can also color it with two colors, such that any two neighbor arrows α, β has the different colors if and only if $\alpha\beta \in \mathcal{I}$ or $\beta\alpha \in \mathcal{I}$. By the properties above, either coloring is unique up to swapping colors. Hence we have another way to characterize gentle tree algebra as follows.

Definition 2.3. A *gentle tree* is a quiver T with a 2-coloring, such that each vertex has at most one arrow of each color incoming or outgoing.

For a colored quiver T, there are two natural ideals

$$\mathcal{I}_T^{+}: \quad \text{generated by all unicolor-paths of length two;}$$

$$\mathcal{I}_T^{-}: \quad \text{generated by all alternating color paths of length two.}$$

Proposition 2.4. *Let $A = \mathbf{k}T/\mathcal{I}$ be a bound quiver algebra. We have the following equivalent statement*:

- *A is a gentle tree algebra.*
- *T is some gentle tree with $\mathcal{I} = \mathcal{I}_T^+$ or $\mathcal{I} = \mathcal{I}_T^-$.*

Proof. By the one of two ways of coloring, the relations in the ideal and the coloring of the gentle tree can be determined uniquely by each other. $\square$

Remark 2.5. In fact, there is an interesting result (but not used in this paper) for a gentle tree T, that $\mathbf{k}T/\mathcal{I}_T^+$ and $\mathbf{k}T/\mathcal{I}_T^-$ are Koszul dual to each other.

We are going to generalize Theorem 2.2 to describe all hearts in $\mathcal{D}(A_n)$.

2.2. Ext-quivers of hearts

Recall that a heart $\mathcal{H}$ is finite, if the set of its simples, denoted by $\operatorname{Sim}\mathcal{H}$, is finite and generates $\mathcal{H}$ by means of extensions,

Definition 2.6. Let $\mathcal{H}$ be a finite heart in a triangulated category $\mathcal{D}$ and $\mathbf{S} = \bigoplus_{S \in \operatorname{Sim}\mathcal{H}} S$. The Ext-quiver $\mathcal{Q}(\mathcal{H})$ is the (positively) graded quiver whose vertices are the simples of $\mathcal{H}$ and whose graded edges correspond to a basis of $\operatorname{End}^\bullet(\mathbf{S}, \mathbf{S})$ (where the grading is given by the degree of Hom).

Note that, by [10], $\mathcal{H}$ is finite, rigid and strongly monochromatic for any $\mathcal{H}$ in $\mathcal{D}(A_n)$. By [10, Lemma 3.3], we know that there is at most one arrow between any two vertices in $\mathcal{Q}(\mathcal{H})$.

Definition 2.7. A *graded gentle tree* $\mathcal{G}$ is a gentle tree with a positive grading for each arrow. The associated quiver $\mathcal{Q}(\mathcal{G})$ of $\mathcal{G}$, is a graded quiver with the same vertex set and an arrow $a: i \to j$ for each unicolored path $p: i \to j$ in $\mathcal{G}$, with the natural grading of p.

Define a mutation μ on graded gentle tree as follow.

Definition 2.8. For a graded gentle tree $\mathcal{G}$, let V be a vertex with neighborhood

$$\begin{tikzcd}
\mathfrak{R}_1 \arrow[dr, "\gamma_1"] & & \mathfrak{B}_2 \\
& V \arrow[ur, squiggly, "\delta_2"] \arrow[dr, "\gamma_2"] & \\
\mathfrak{B}_1 \arrow[ur, squiggly, "\delta_1"] & & \mathfrak{R}_2
\end{tikzcd}$$

where $\mathfrak{B}_i, \mathfrak{R}_i$ are the sub trees and γ_i, δ_i are degrees of $\mathcal{G}$, $i = 1, 2$. The straight line represents one color and the curly line represent the other color. Define the *forward mutation* μ_V at vertex V (on $\mathcal{G}$) as follows:

- if $\delta_1 \geqslant 1$, μ_V on the lower part of the quiver is:

$$\begin{tikzcd}
& V \arrow[dr, "\gamma_2"] & & & & V \arrow[dr, "\gamma_2+1"] & \\
\mathfrak{B}_1 \arrow[ur, squiggly, "\delta_1"] & & \mathfrak{R}_2 \arrow[rr, Rightarrow, bend left, "\mu_i"] & & \mathfrak{B}_1 \arrow[ur, squiggly, "\delta_1-1"] & & \mathfrak{R}_2
\end{tikzcd}$$

- if $\delta_1 = 1$, denote

$$\mathfrak{B}_1 \quad = \quad \begin{tikzcd} \mathfrak{E}_1 \arrow[dr, "\theta_1"] & & \\ & W \arrow[dr, "\theta_2"] & \\ \mathfrak{L}_1 \arrow[ur, rightsquigarrow, "\beta_1"] & & \mathfrak{E}_2 \end{tikzcd}$$

  and μ_V on the lower part of the quiver is:

$$\begin{tikzcd} \mathfrak{E}_1 \arrow[dr, "\theta_1"] & & V \arrow[dr, "\gamma_2"] & \\ & W \arrow[ur, rightsquigarrow, "1"] \arrow[dr, "\theta_2"] & & \mathfrak{R}_2 \\ \mathfrak{L}_1 \arrow[ur, rightsquigarrow, "\beta"] & & \mathfrak{E}_2 & \end{tikzcd} \quad \overset{\mu_i}{\curvearrowright} \quad \begin{tikzcd} & V \arrow[dr, "1"] & & \mathfrak{E}_2^{\times} \\ \mathfrak{L}_1 \arrow[ur, rightsquigarrow, "\beta"] & & W \arrow[ur, rightsquigarrow, "\theta_2"] \arrow[dr, "\gamma_2"] & \\ & \mathfrak{E}_1^{\times} \arrow[ur, rightsquigarrow, "\theta_1"] & & \mathfrak{R}_2 \end{tikzcd} \tag{2.1}$$

  where $\mathfrak{X}^{\times}$ is the operation of swapping colors on a graded gentle trees $\mathfrak{X}$.
- μ_V on the upper part follows the mirror of the lower part.

Dually, define the backward mutation μ_V^{-1} to be the reverse of μ_V (which follows a similar rule).

Clearly, the set of all graded gentle trees with n vertexes is closed under such mutation. In fact, this set is also connected under (forward/backward) mutation.

Lemma 2.9. *Any graded gentle tree with n vertices can be repeatedly mutated from another graded gentle tree with n vertices.*

Proof. Use induction, starting from the trivial case when $n = 1$. Suppose that the lemma holds for $n = m$ and consider the case for $n = m + 1$. We only need to show that any graded gentle tree $\mathcal{G}$ with $m + 1$ vertices can be repeatedly mutated from a unicolor graded gentle tree with all degrees equal one. Let V be a sink in $\mathcal{G}$ and the subtree of $\mathcal{G}$ by deleting V is $\mathcal{G}'$ while the connecting arrow from $\mathcal{G}'$ to V has degree d. By backward mutating on V, we can increase d as large as possible without changing $\mathcal{G}'$. Then the mutation at a vertex other than V on $\mathcal{G}$ restricted to $\mathcal{G}'$ will be the same as mutating at that vertex on $\mathcal{G}'$. Thus, by the induction assumption, we can mutate $\mathcal{G}$ such that $\mathcal{G}'$ becomes unicolor with all degrees equal zero. Then, repeatedly forward mutating many times on V will turn $\mathcal{G}$ into unicolor with all degrees equal zero. $\square$

Using Lemma A.4, a direct calculation gives the following proposition.

Proposition 2.10. *Let $\mathcal{G}$ be a graded gentle tree and $\mathcal{H}$ be a heart in $\mathcal{D}(A_n)$. If $\mathcal{Q}(\mathcal{G}) = \mathcal{Q}(\mathcal{H})$ with vertex V in $\mathcal{G}$ corresponding to the simple S in $\mathcal{H}$, then*

$$\mathcal{Q}\big(\mathcal{H}_S^{\sharp}\big) = \mathcal{Q}(\mu_V \mathcal{G}), \qquad \mathcal{Q}\big(\mathcal{H}_S^{\flat}\big) = \mathcal{Q}\big(\mu_V^{-1} \mathcal{G}\big). \tag{2.2}$$

Now we can describe all Ext-quiver of hearts of A-type.

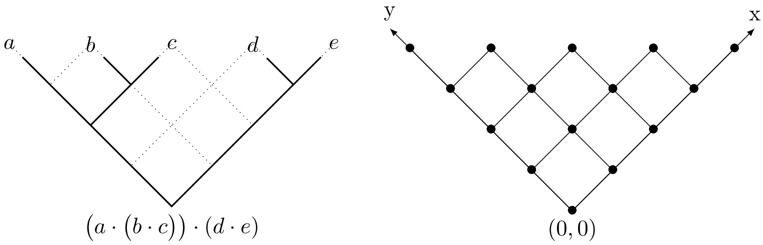

Fig. 1. A parenthesizing of four words on the left and G_4 on the right.

Theorem 2.11. *The Ext-quivers of hearts in $\mathcal{D}(A_n)$ are precisely the associated quivers of graded gentle trees with n vertices.*

Proof. Note that any heart in $\mathcal{D}(A_n)$ can be repeatedly tilted from the standard heart $\mathcal{H}_Q$, by [9]. Without loss of generality, let Q have straight orientation. Then $\mathcal{Q}(\mathcal{H}_Q)$ certainly is the associated quiver for the graded gentle tree $\mathcal{G}_Q$ with the same orientation and alternating colored arrows. Then, inducting from $\mathcal{H}_Q$ and using (2.2), we deduce that the Ext-quiver of any heart in $\mathcal{D}(A_n)$ is the associated quivers of some graded gentle tree with n vertices. On the other hand, the set of graded gentle trees with n vertices is connected (Lemma 2.9). Then, also by induction, we deduce that the associated quiver of any graded gentle tree with n vertices is the Ext-quiver of some heart, because (2.2) and the fact that we can forward/backward tilt any simples in any heart in $\mathcal{D}(A_n)$ [10, Theorem 5.7]. $\square$

Recall that we can Calabi–Yau-N (CY-N) double a graded quiver in the sense of [10, Definition 6.2]. Then we have the following corollary.

Corollary 2.12. *The Ext-quivers of hearts in* $\mathrm{EG}^\circ(\Gamma_N A_n)$ *are precisely the CY-N double of the associated quivers of graded gentle trees with n vertices.*

Proof. By [10, Corollary 8.3], any heart $\mathcal{H}$ in $\mathrm{EG}^\circ(\Gamma_N A_n)$ is induced from some heart $\mathcal{H}'$ in $\mathcal{D}(A_n)$, while Ext-quiver $\mathcal{Q}(\mathcal{H})$ is the CY-N double of $\mathcal{Q}(\mathcal{H}')$ by [10, Proposition 7.5]. Thus the corollary follows from Theorem 2.11. $\square$

By [10, Proposition 8.6], the augmented graded quivers of colored quivers for $(N-1)$-clusters (cf. [10, Definition 6.1] and [3]) of type A_n are also precisely the CY-N double of the associated quivers of graded gentle trees.

3. Associahedron

3.1. Binary trees

Let BT_m be the set of binary trees with $m+1$ leaves (and hence with m internal vertices), which can be used to parenthesize a word with $m+1$ letters (see Fig. 1 and cf. [11]). Let G_m be the full subgraph of the grid $\mathbb{Z}^2$ induced by

$$G_m = \{(x, y) \mid x \geqslant 0,\ y \geqslant 0,\ x + y \leqslant m\} \subset \mathbb{Z}^2.$$

It is well-known that a binary tree with $m+1$ leaves has a normal form as a subgraph of G_m, such that the leaves are $\{(x, m-x)\}_{x=0}^{m}$, and we will identify the binary tree with such normal form (see Fig. 1).

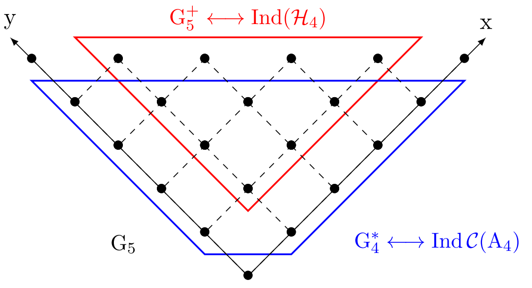

Fig. 2. G_5^+ (red) and G_4^* (blue) sit inside G_5. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Example 3.1. Let

$$G_m^+ = G_m \cap \{(x, y) \mid xy > 0\}, \qquad G_m^* = G_m - \{(0,0)\}.$$

Consider the A_n-quiver $\overrightarrow{A_n}: n \to \cdots \to 1$ and let $\mathcal{H}_n = \mathrm{mod}\,\mathbf{k}\overrightarrow{A_n}$ with corresponding simples $S_1, \ldots, S_n$. Then, there are canonical bijections (cf. Fig. 2)

$$\xi_n : G_{n+1}^+ \to \mathrm{Ind}(\mathcal{H}_n),$$
$$\varsigma_n : G_n^* \to \mathrm{Ind}\,\mathcal{H}_n \cup \mathrm{Proj}\,\mathcal{H}_n[1]$$

satisfying $\xi_n(i, j) = \varsigma_n(i-1, j) = M_{i,j}$, where $M_{i,j} \in \mathrm{Ind}\,\mathcal{H}_n$ is determined by

$$[M_{i,j}] = \sum_{i}^{n+1-j} [S_k]. \tag{3.1}$$

Let $\zeta_n = \pi_n \circ \varsigma_n : G_n^* \to \mathrm{Ind}\,\mathcal{C}(A_n)$.

It is known that the following sets (see [6] for more possible sets) can parameterize the vertex set of an n-dimensional associahedron:

1°. the set BT_{n+1} of binary trees with $n+2$ leaves;
2°. the set of triangulations of regular $(n+3)$-gon;
3°. the set $\mathrm{CEG}(A_n)$ of (2-)cluster tilting sets in $\mathcal{C}(A_n)$;
4°. the set $\mathrm{TP}(\overrightarrow{A_n})$ of torsion pairs in $\mathcal{H}_n$ (cf. [10] and [5]);
5°. the set $\mathrm{EG}(\mathcal{H}_n, \mathcal{H}_n[1])$ of hearts in $\mathcal{D}(A_n)$ between $\mathcal{H}_n$ and $\mathcal{H}_n[1]$ (in the sense of King–Qiu, [10]).

There are natural bijections between these sets (cf. [10,6,8]).

Furthermore, by [10, Section 9], the poset structure of torsion pairs (hearts) gives an orientation O_t of the n-dimensional associahedron, i.e. the orientation of $\mathrm{EG}(\mathcal{H}_n, \mathcal{H}_n[1])$ (considered as a subgraph of $\mathrm{EG}(A_n)$).

On the other hand, there is a natural poset structure on binary trees, inducing by locally flipping a binary tree (as shown in Fig. 3), or equivalently, changing the corresponding parenthesizing of words from $(A \cdot B) \cdot C$ to $A \cdot (B \cdot C)$ (see [11] for details). This poset structure also gives an orientation O_p for the associahedron. We aim to prove $O_t = O_p$ this section.

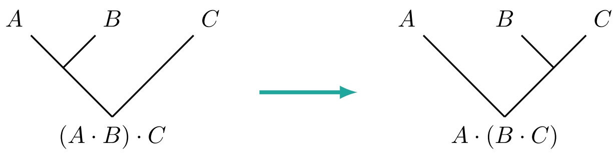

Fig. 3. A local flip of a binary tree (at the word B).

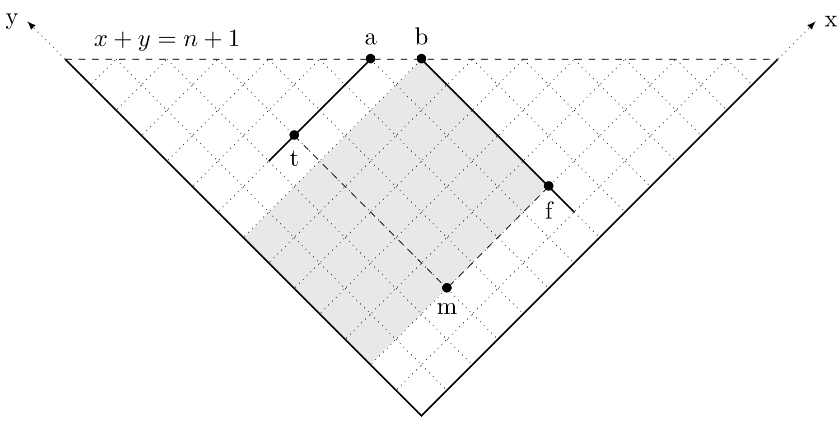

Fig. 4. A short exact sequence in G_{n+1}.

3.2. Combinatorial constructions

First, we give explicit construction of torsion pairs from binary trees. For any $p \in \mathbb{Z}^2$ with coordinate (x_p, y_p), let $L(p)$ be the edge connecting $(x_p - 1, y_p)$ and p and $R(p)$ be the edge connecting $(x_p, y_p - 1)$ and p. Given a tree $\mathbf{b}$ in BT_{n+1}, define

$$\mathcal{T}(\mathbf{b}) = \left\langle \xi_n(p) \;\middle|\; p \in \mathsf{G}^+_{n+1}, L(p) \in \mathbf{b} \right\rangle, \qquad \mathcal{F}(\mathbf{b}) = \left\langle \xi_n(p) \;\middle|\; p \in \mathsf{G}^+_{n+1}, R(p) \in \mathbf{b} \right\rangle, \tag{3.2}$$

where $\langle\,\rangle$ means generating by extension.

Proposition 3.2. *There is a bijection* $\Theta_n : \mathrm{BT}_{n+1} \to \mathrm{TP}(\overrightarrow{A_n})$, *sending* $\mathbf{b} \in \mathrm{BT}_{n+1}$ *to* $\langle \mathcal{T}(\mathbf{b}), \mathcal{F}(\mathbf{b}) \rangle$.

Proof. We only need to show that $\Theta_n : \mathbf{b} \mapsto \langle \mathcal{T}(\mathbf{b}), \mathcal{F}(\mathbf{b}) \rangle$ is well-defined (and obviously injective) and hence bijective since both sets have the nth Catalan number many elements.

To do so, we first show that any object $M \in \mathcal{H}_n$ admits a short exact sequence

$$0 \to T \to M \to F \to 0 \tag{3.3}$$

for some $T \in \mathcal{T}(\mathbf{b})$ and $F \in \mathcal{F}(\mathbf{b})$. Let $m = \xi_n^{-1}(M) \in \mathsf{G}^+_{n+1}$. If $m \in \mathbf{b}$ then $M \in \mathcal{T}(\mathbf{b}) \cup \mathcal{F}(\mathbf{b})$ and we have a trivial short exact sequence (3.3). If $m \notin \mathbf{b}$, let t be the vertex in $\mathbf{b} \cap \{(x_m, j) \mid j \geqslant y_m\}$ with minimal y-coordinate and f be the vertex in $\mathbf{b} \cap \{(i, y_m) \mid i \geqslant x_m\}$ with minimal x-coordinate; let a and b be the vertices with coordinates $(n+1-y_t, y_t)$ and $(x_f, n+1-x_f)$, see Fig. 4. By construction and the property of the binary tree, we know that

- edges in the line segments, from m to t and from m to f, are not in $\mathbf{b}$;
- edges in the line segments, from a to t and from b to f, are in $\mathbf{b}$;
- $(x_a, y_a) + (1, -1) = (x_b, y_b)$, i.e. a, b are neighbors in the line $x + y = n + 1$;
- $L(t)$ and $R(f)$ are in $\mathbf{b}$.

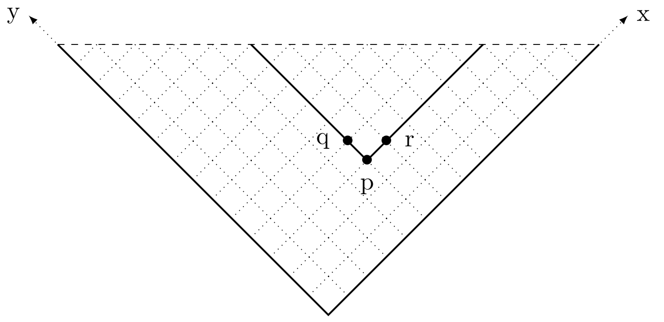

Fig. 5. An interval vertex of a binary tree in G_{n+1}.

Thus $T = \xi_n(t) \in \mathcal{T}(\mathbf{b})$ and $F = \xi_n(f) \in \mathcal{F}(\mathbf{b})$. By (3.1), a direct calculation shows that $[M] = [T] + [F]$, which implies we have (3.3), by Lemma A.1, as required.

To finish, we need to show that $\operatorname{Hom}(\mathcal{T}(\mathbf{b}), \mathcal{F}(\mathbf{b})) = 0$. Let $F = \xi_n(f) \in \mathcal{F}(\mathbf{b})$. As above, edges in the line segments from b to f are in $\mathbf{b}$. By the property of binary tree, the horizontal edges (i.e. parallel to x-axis) in the shaded area in Fig. 4 are not in $\mathbf{b}$, which implies, by Lemma A.2, that the modules in $\mathcal{H}_n$ that have nonzero maps to F are not in $\mathcal{T}(\mathbf{b})$, as required. $\square$

Next, we identify cluster tilting sets from binary trees via ζ_n. For any $\mathbf{b} \in \mathrm{BT}_{n+1}$, let $\mathrm{iv}(\mathbf{b})$ be set of the internal vertices expect $(0,0)$ so that $\#\mathrm{iv}(\mathbf{b}) = n$. Denote by $\operatorname{Proj}\mathcal{H}$ a complete set of indecomposable projectives of a heart $\mathcal{H}$. Recall [10, Section 2] that

$$P \in \operatorname{Proj}\mathcal{H} \iff P \in \operatorname{Ind}\big(\mathcal{P} \cap \tau^{-1}\mathcal{P}^{\perp}\big), \tag{3.4}$$

where $\mathcal{P}$ is the t-structure corresponding to $\mathcal{H}$.

Proposition 3.3. *Let $\mathbf{b} \in \mathrm{BT}_{n+1}$ and $\mathcal{H}(\mathbf{b})$ be the heart corresponding to the torsion pair $\Theta_n(\mathbf{b})$ in $\mathcal{H}_n$. Then we have $\operatorname{Proj}\mathcal{H}(\mathbf{b}) = \varsigma_n(\mathrm{iv}(\mathbf{b}))$ and there is a bijection $\zeta_n \circ \mathrm{iv} : \mathrm{BT}_{n+1} \to \mathrm{CEG}(A_n)$.*

Proof. By [10, Corollary 5.12], we know that $\pi_n \operatorname{Proj}\mathcal{H}(\mathbf{b}) \in \mathrm{CEG}(A_n)$ and hence the second claim follows immediately from the first one.

Let $p \in \mathrm{iv}(\mathbf{b})$, which is the intersection of the edges $L(r)$ and $R(q)$, where q, r be the points with coordinates $(x_p, y_p + 1)$ and $(x_p + 1, y_p)$ (see Fig. 5). Note that p is not in the line $x_p + y_p = n$ and thus $q, r \in \mathrm{G}_{n+1}$. Let $\mathcal{P}(\mathbf{b})$ be the t-structure corresponding to $\mathcal{H}(\mathbf{b})$. Note that

$$\mathcal{P}(\mathbf{b}) = \mathcal{T}(\mathbf{b}) \cup \bigcup_{j>0} \mathcal{H}_n[j], \qquad \mathcal{P}(\mathbf{b})^{\perp} = \mathcal{F}(\mathbf{b}) \cup \bigcup_{j<0} \mathcal{H}_n[j].$$

If $r \in \mathrm{G}^{+}_{n+1}$, then $P = \varsigma_n(p) = \xi_n(r)$ is in $\mathcal{T}(\mathbf{b})$; otherwise, $y_p = 0$ and then $P \in \mathcal{H}_n[1]$. Either way, $P \in \mathcal{P}(\mathbf{b})$. Similarly, if $q \in \mathrm{G}^{+}_{n+1}$, then $\tau P = \xi_n(q)$ is in $\mathcal{F}(\mathbf{b})$; otherwise, $x_p = 0$ and then $\tau P \in \mathcal{H}_n[-1]$. Either way, $\tau P \in \mathcal{P}(\mathbf{b})^{\perp}$. Therefore $P \in \operatorname{Proj}\mathcal{H}(\mathbf{b})$ by (3.4). Thus $\operatorname{Proj}\mathcal{H}(\mathbf{b})$ contains, and hence equals $\varsigma_n(\mathrm{iv}(\mathbf{b}))$ as required, noticing that $\#\operatorname{Proj}\mathcal{H}(\mathbf{b}) = n = \#\mathrm{iv}(\mathbf{b})$.

Example 3.4. Keep the notation in Example 3.1. Then the binary tree in Fig. 1 corresponds to the torsion pair $\mathcal{T} = \langle M_{2,2}, M_{1,2}\rangle$, $\mathcal{F} = \langle M_{1,3}, M_{3,1}\rangle$ and the cluster tilting set $\{M_{1,2}, M_{2,2}, M_{1,1}[1]\}$.

3.3. The orientation

Now we apply the constructions above to show that $O_t = O_p$.

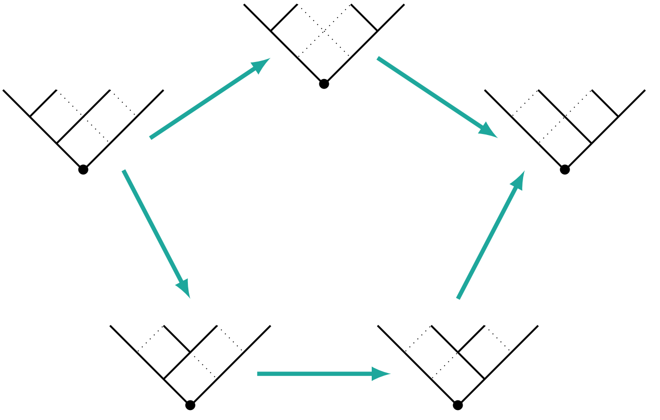

Fig. 6. The orientation of the 2-dimensional associahedron.

Theorem 3.5. *Under the bijection Θ_n in Proposition 3.2, the orientations O_t and O_p of the n-dimensional associahedron coincide.*

Proof. Consider an edge $e : \mathbf{b}_1 \to \mathbf{b}_2$ in BT_{n+1}, which corresponds to a local flip as in Fig. 3. Let $\mathcal{H}(\mathbf{b}_i)$ forward tilt of $\mathcal{H}_n$ with respect to $\Theta_n(\mathbf{b}_i)$. We only need to show that $\mathcal{H}(\mathbf{b}_2)$ is a simple forward tilt of $\mathcal{H}(\mathbf{b}_1)$.

By Proposition 3.3, we know that $\operatorname{Proj}\mathcal{H}(\mathbf{b}_i) = \varsigma_n(\mathrm{iv}(\mathbf{b}_i))$, for $i = 1, 2$, differ by one object. Denote by $P_i \in \operatorname{Proj}\mathcal{H}(\mathbf{b}_i)$ the different objects. Thus, $\pi_n \operatorname{Proj}\mathcal{H}(\mathbf{b}_i) \in \mathrm{CEG}(A_n)$ are related by one mutation, which implies $\mathcal{H}(\mathbf{b}_i)$ are related by a single simple tilting, by [10, Corollary 5.12], and P_1, P_2 are related by some triangle

$$P_j \to M \to P_k \to P_j[1]$$

in $\mathcal{D}(A_n)$ for some ordering $\{j, k\} = \{1, 2\}$. By Lemma A.3, P_j is a predecessor of P_k. But, from the flip we know that P_1 is the predecessor of P_2, which implies $j = 1$ and $k = 2$. Thus the forward simple tilting is from $\mathcal{H}(\mathbf{b}_1)$ to $\mathcal{H}(\mathbf{b}_2)$ as required. □

Example 3.6. Fig. 6 is the orientation of the 2-dimensional associahedron, induced by poset structure of binary trees, which is the oriented pentagon in [10, Fig. 3] and [12, (3.5)], cf. also [7, Fig. 5].

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Appendix A. Maps and triangles in $\mathcal{D}(A_n)$

In this appendix, we collect several facts about the maps and triangles in $\mathcal{D}(A_n)$. See [2, Chapter IX] for the proofs of the first three lemmas.

Recall there are notions of sectional paths and predecessors in $\mathcal{D}(A_n)$ cf. [12, Section 2.2].

Lemma A.1. *Let $M, A, B \in \operatorname{Ind}\mathcal{D}(A_n)$ such that $A \in \mathrm{Ps}^{-1}(M)$ and $B \in \mathrm{Ps}(M) - \mathrm{Ps}(A)$. Then there is a short exact sequence $0 \to A \to M \to B \to 0$ if and only if $[M] = [A] + [B]$.*

Lemma A.2. *Let $M, L \in \operatorname{Ind}\mathcal{D}(A_n)$. Then $\operatorname{Hom}(M, L) \neq 0$ if and only if*

$$L \in \left[\mathrm{Ps}(M), \mathrm{Ps}^{-1}\left(\tau\left(M[1]\right)\right)\right], \qquad M \in \left[\mathrm{Ps}\left(\tau^{-1}\left(L[-1]\right)\right), \mathrm{Ps}^{-1}(L)\right].$$

Lemma A.3. *If* $\operatorname{Hom}(L, M[1]) \neq 0$ *for some* M *and* L *in* $\operatorname{Ind}\mathcal{D}(A_n)$, *then* M *is a predecessor of* L. *Any two non-isomorphic indecomposables in* $\mathcal{D}(A_n)$ *cannot be predecessors of each other.*

Lemma A.4. *Let* $\mathcal{H}$ *be a heart in* $\mathcal{D}(A_n)$. *For any the following sub-quivers*

$$\begin{tikzcd} & S \arrow[dr, "a"] & \\ T \arrow[ur, "1"] \arrow[rr, "a+1"'] & & A \end{tikzcd} \quad \begin{tikzcd} & S \arrow[dr, "b"] & \\ T \arrow[ur, "1"] & & B \end{tikzcd} \quad \begin{tikzcd} & S & \\ T \arrow[ur, "1"] & & C \arrow[ul, "c+1"'] \arrow[ll, "c"'] \end{tikzcd} \quad \begin{tikzcd} & S & \\ T \arrow[ur, "1"] & & D \arrow[ul, "d+1"'] \end{tikzcd}$$

in the Ext-quiver $\mathcal{Q}(\mathcal{H})$ *for some* $S, T, A, B, C, D \in \operatorname{Sim}\mathcal{H}$ *and positive integers a, b, c, d, they become*

$$\begin{tikzcd} & S[1] \arrow[dl, "1"'] \arrow[dr, "a+1"] & \\ R & & A \end{tikzcd} \quad \begin{tikzcd} & S[1] \arrow[dl, "1"'] \arrow[dr, "b+1"] & \\ R \arrow[rr, "b"'] & & B \end{tikzcd} \quad \begin{tikzcd} & S[1] \arrow[dl, "1"'] & \\ R & & C \arrow[ul, "c"'] \end{tikzcd} \quad \begin{tikzcd} & S[1] \arrow[dl, "1"'] & \\ R & & D \arrow[ul, "d"'] \arrow[ll, "d+1"] \end{tikzcd}$$

in the Ext-quiver $\mathcal{Q}(\mathcal{H}_S^\sharp)$, *where* R *is the nontrivial extension of* T *on top of* S.

Proof. We only prove the first case while the other cases are similar. By [10, Theorem 5,7], we know that the simples in $\mathcal{H}_S^\sharp$ corresponding to S, T and A are $S[1]$, R and A. By [12, Lemma 3.3], we have an isomorphism $\operatorname{Hom}^1(T, S) \otimes \operatorname{Hom}^a(S, A) \to \operatorname{Hom}^{a+1}(T, A)$. Thus, applying $\operatorname{Hom}(-, A)$ to the triangle $S \to R \to T \to S[1]$ gives $\operatorname{Hom}^\bullet(R, A) = 0$. Similarly, a direct calculation of other $\operatorname{Hom}^\bullet$ between $S[1]$, R, A shows the new sub-quiver is as required. $\square$

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