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Counterexamples to the $I^{(3)} \subset I^2$ containment

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The purpose of this short note is to show that there is in general no containment

$$I^{(3)} \subset I^2$$

for an ideal I of points in $\mathbb{P}^2$. This answers in the negative a question asked by Huneke and generalized by Harbourne. The sets of points constituting counterexamples come from the dual of the Hesse configuration and more generally from Fermat arrangements.

1. Introduction

Let $\mathcal{I} \subset S = \mathbb{C}[x_0, \ldots, x_n]$ be a homogeneous ideal in the graded ring of polynomials. The m-th *symbolic power* $\mathcal{I}^{(m)}$ of $\mathcal{I}$ is defined as

$$\mathcal{I}^{(m)} = S \cap \left(\bigcap_{\mathfrak{p} \in \mathrm{Ass}(\mathcal{I})} \mathcal{I}^m S_{\mathfrak{p}} \right),$$

where the intersection is taken in the field of fractions of S.

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There has been considerable interest in containment relations between usual and symbolic powers of homogeneous ideals over the last two decades. The most general results in this direction have been obtained with multiplier ideal techniques in characteristic zero by Ein, Lazarsfeld and Smith [7] and using tight closures in positive characteristic by Hochster and Huneke [10]. Applying these results to a homogeneous ideal $\mathcal{I}$ in the coordinate ring S of the projective space we obtain the following containment statement

$$\mathcal{I}^{(nr)} \subset \mathcal{I}^r \quad \text{for all } r \geqslant 0.$$

Quite a number of examples has suggested that the following statement could be true, see [2, Conjecture 8.4.2], [4, Conjecture 1.1], [9, Conjecture 4.1.1].

Conjecture 1.1. *Let $\mathcal{I} \subset \mathbb{C}[\mathbb{P}^n]$ be a homogeneous ideal. For $m \geqslant rn - (n-1)$ there is the containment*

$$\mathcal{I}^{(m)} \subset \mathcal{I}^r.$$

This conjecture asserts in particular that an earlier question raised by Huneke has a positive answer; see [11, Problem 0.4], see also [3, page 400] and [9, Section 4.1]. The referee has kindly informed us that Huneke had been raising this question verbally for a few years before 2006 but [11, Problem 0.4] seems to be its first occurrence in print.

Question 1.2 *(Huneke).* Let $\mathcal{I}$ be a homogeneous radical ideal of points in the projective plane. Is there then the containment

$$\mathcal{I}^{(3)} \subset \mathcal{I}^2?$$

This question has been affirmatively answered for general points, see [3], star configurations, see [9], complete intersections and some special configurations of points, see [5].

We show here that the containment in Question 1.2 fails in general. This implies also that Conjecture 1.1 is false. There is quite a number of closely related yet distinct conjectures concerning containment relations between symbolic and usual powers of ideals in the literature. For the convenience of the reader we mention here that the results of this note show that the following conjectures are false: [9, Conjecture 4.1.1 and Conjecture 4.1.5] (for $N = r = 2$), [2, Conjecture 8.4.2], [4, Conjecture 1.1], [5, Conjecture 3.9] (for $N = t = 2$ and $m = 1$), also Questions 4.2.2 and 4.2.3 in [5] have a negative answer.

2. The dual Hesse configuration

We begin with an explicit realization of the dual Hesse configuration. Note that up to projective change of coordinates there is a unique configuration of that type [12, Example 7.3].

Let ε be a primitive root of 1 of order 3. We consider the radical ideal $\mathcal{I}$ of the following set of 12 points in $\mathbb{P}^2$:

$$\begin{array}{lll}
P_1 = (1:0:0), & P_2 = (0:1:0), & P_3 = (0:0:1), \\
P_4 = (1:1:1), & P_5 = \left(1:\varepsilon:\varepsilon^2\right), & P_6 = \left(1:\varepsilon^2:\varepsilon\right), \\
P_7 = (\varepsilon:1:1), & P_8 = (1:\varepsilon:1), & P_9 = (1:1:\varepsilon), \\
P_{10} = \left(\varepsilon^2:1:1\right), & P_{11} = \left(1:\varepsilon^2:1\right), & P_{12} = \left(1:1:\varepsilon^2\right).
\end{array}$$

These points form a $12_3 9_4$ configuration, i.e. there are 9 lines

$$\begin{array}{lll} L_1\colon\ x-y, & L_2\colon\ y-z, & L_3\colon\ z-x, \\ L_4\colon\ x-\varepsilon y, & L_5\colon\ y-\varepsilon z, & L_6\colon\ z-\varepsilon x, \\ L_7\colon\ x-\varepsilon^2 y, & L_8\colon\ y-\varepsilon^2 z, & L_9\colon\ z-\varepsilon^2 x, \end{array}$$

such that exactly 3 configuration lines pass through each of configuration points and exactly 4 points lie on a configuration line. This is the dual of the well known Hesse configuration, see [1] for a lot more on this beautiful subject.

Turning back to the ideal $\mathcal{I}$, we exhibit first its generators.

Lemma 2.1. *The ideal $\mathcal{I}$ is generated by polynomials*

$$f_1 := z\bigl(x^3-y^3\bigr), \qquad f_2 := x\bigl(y^3-z^3\bigr) \quad \textit{and} \quad f_3 = y\bigl(z^3-x^3\bigr).$$

Proof. We have obviously $(f_1, f_2, f_3) \subset \mathcal{I}$, so it remains to check the opposite inclusion. To this end let first $\mathcal{J}$ be the radical ideal of points $P_4, \ldots, P_{12}$.

Claim. *Polynomials*

$$g_1 = z^3 - x^3, \quad \textit{and} \quad g_2 = y^3 - z^3$$

generate $\mathcal{J}$.

Let $g \in \mathcal{J}$ be a homogeneous element. Using identities

$$x^3 = -\bigl(z^3-x^3\bigr) + z^3, \qquad y^3 = \bigl(y^3-z^3\bigr) + z^3$$

we can write $g = g' + g''$ with some homogeneous g', g'' satisfying $g'' \in (g_1, g_2)$ and g' such that $\deg_x g' \leqslant 2$ and $\deg_y g' \leqslant 2$. Note that $g' = g - g'' \in \mathcal{J}$.

Now we set $h(x,y) = g'(x,y,1)$ and let $\mathcal{K}$ be the radical ideal of points $P_1, \ldots, P_{12}$ in the affine chart $z=1$. Then $h \in \mathcal{K}$. Note that h is supported on the following set of monomials

$$\operatorname{supp}(h) \subset Q = \bigl\{1, x, y, x^2, xy, y^2, x^2y, xy^2, x^2y^2\bigr\}.$$

Observe that the set Q is a monomial basis for the algebra $\mathbb{C}[x,y]/\mathcal{K}$. Indeed, since $(x-1)(x-\varepsilon)(x-\varepsilon^2)$ and $(y-1)(y-\varepsilon)(y-\varepsilon^2)$ belong to $\mathcal{K}$ the set Q generates $\mathbb{C}[x,y]/\mathcal{K}$. Since $\dim_{\mathbb{C}} \mathbb{C}[x,y]/\mathcal{K} = 9$ (we have 9 points), Q is a basis.

Now the inclusion $h \in \mathcal{K}$ with h supported on Q implies $h = 0$, thus $g' = 0$ and consequently $g = g'' \in (g_1, g_2)$. This exactly means that $\mathcal{J} = (g_1, g_2)$.

Now we turn back to the inclusion

$$\mathcal{I} \subset (f_1, f_2, f_3).$$

All polynomials in the subsequent argument are supposed to be homogeneous. Let $g \in \mathcal{I}$ be an arbitrary element. Of course $g \in \mathcal{J}$, hence

$$g = h_1(x,y,z) \cdot \bigl(z^3-x^3\bigr) + h_2(x,y,z) \cdot \bigl(y^3-z^3\bigr)$$

for some polynomials h_1 and h_2. We can split

$$h_1(x, y, z) = zh_3(x, y, z) + h_4(x, y)$$

into monomials containing z and those depending only on x and y. Hence we can also write

$$g = h_4(x, y) \cdot \left(z^3 - x^3\right) + \left(h_2(x, y, z) - zh_3(x, y, z)\right) \cdot \left(y^3 - z^3\right) - h_3(x, y, z) \cdot z\left(x^3 - y^3\right).$$

Gathering together terms divisible by $y(z^3 - x^3)$, $x(y^3 - z^3)$ and $z(x^3 - y^3)$ we can write

$$g = h_5(x)\left(z^3 - x^3\right) + h_6(y, z)\left(y^3 - z^3\right) + h_7(x, y, z)$$

for some $h_7 \in (f_1, f_2, f_3)$. Obviously

$$h_5(x)\left(z^3 - x^3\right) + h_6(y, z)\left(y^3 - z^3\right)$$

vanishes at P_1, P_2, P_3. This implies

$$h_5(x) = 0 \quad \text{and consequently} \quad yz \text{ divides } h_6(y, z).$$

Since

$$yz\left(y^3 - z^3\right) = -z \cdot y\left(z^3 - x^3\right) - y \cdot z\left(x^3 - y^3\right) \in (f_1, f_2, f_3)$$

this completes the proof. □

We have the following two relations between the generators of $\mathcal{J}$:

$$xyf_1 + yzf_2 + zxf_3 = 0 \quad \text{and} \quad z^2 f_1 + x^2 f_2 + y^2 f_3 = 0.$$

It is easy to check that these relations determine the minimal resolution of $\mathcal{J}$:

$$0 \to \bigoplus^2 S(-6) \to \bigoplus^3 S(-4) \to \mathcal{J} \to 0.$$

Hence the Castelnuovo–Mumford regularity of $\mathcal{J}$ is $\operatorname{reg}(\mathcal{J}) = 5$. Then [8, Theorem 1.1] implies that

$$\left(\mathcal{J}^{(2)}\right)_t = \left(\mathcal{J}^2\right)_t \quad \text{for } t \geqslant 10$$

and hence

$$\left(\mathcal{J}^{(3)}\right)_t \subset \left(\mathcal{J}^{(2)}\right)_t = \left(\mathcal{J}^2\right)_t \quad \text{for } t \geqslant 10.$$

Thus the containment problem $\mathcal{J}^{(3)} \subset \mathcal{J}^2$ reduces to finding an element of degree less than 10 in $\mathcal{J}^{(3)}$ which is not in I^2.

Theorem 2.2. *The polynomial*

$$f = L_1 \cdot \dots \cdot L_9 = x^3y^6 - x^6y^3 + y^3z^6 - y^6z^3 + z^3x^6 - z^6x^3 = \left(x^3 - y^3\right)\left(y^3 - z^3\right)\left(z^3 - x^3\right)$$

is contained in $\mathcal{J}^{(3)}$ but it is not contained in $\mathcal{J}^2$.

Proof. The geometry of the configuration implies that $f \in \mathcal{I}^{(3)}$. In fact this is the only nonzero element (up to a multiplicative constant) of degree 9 in the third symbolic power of $\mathcal{I}$, which can be easily checked by Bezout's theorem.

For the second claim we assume to the contrary that $f \in \mathcal{I}^2$. Using Lemma 2.1 we can write

$$f = (ax + by + cz) f_1^2 + \text{other terms divisible by } xy \text{ or } z^4. \tag{1}$$

Substituting $x = 0$ in (1) we get

$$y^3 z^6 - y^6 z^3 = (by + cz) y^6 z^2 + \text{terms divisible by } z^4.$$

Comparing coefficients at $y^6 z^3$ we obtain $c = -1$.

Substituting in turn $y = 0$ in (1) we get

$$z^3 x^6 - z^6 x^3 = (ax + cz) x^6 z^2 + \text{terms divisible by } z^4,$$

which comparing again coefficients at $x^6 z^3$ gives $c = 1$, a contradiction. □

One can use the following Singular [6] script in order to verify all above claims, in particular that $f \notin \mathcal{I}^2$.

```
ring R=(0,e),(x,y,z),dp; option(redSB);
minpoly=e2+e+1;
ideal P1=y,z; ideal P2=x,z; ideal P3=x,y;
ideal P4=x-z,y-z; ideal P5=y-e*x,z-e^2*x; ideal P6=y-e^2*x,z-e*x;
ideal P7=x-e*z,y-z; ideal P8=x-z,y-e*z; ideal P9=y-x,z-e*x;
ideal P10=x-e^2*z,y-z; ideal P11=x-z,y-e^2*z; ideal P12=y-x,z-e^2*x;
ideal I=intersect(P1,P2,P3,P4,P5,P6,P7,P8,P9,P10,P11,P12);
regularity(mres(I,0));
ideal I3=intersect(P1^3,P2^3,P3^3,P4^3,P5^3,
                   P6^3,P7^3,P8^3,P9^3,P10^3,P11^3,P12^3);
poly F=I3[1];
reduce(F,std(I^2));
quit;
```

3. Fermat arrangements

The dual Hesse configuration from Section 2 is a special case of Fermat arrangements, see [13, Example II.6]. More specifically, the $3d$ lines in a Fermat arrangement are defined as the zero locus of the polynomial

$$f_d = \left(x^d - y^d\right)\left(y^d - z^d\right)\left(z^d - x^d\right).$$

If η is a primitive root of 1 of order d, then the intersection points of these lines are $Q_{a,b} = (1 : \eta^a : \eta^b)$ for $a, b = 1, \ldots, d$ and the three coordinate points $P_1 = (1 : 0 : 0)$, $P_2 = (0 : 1 : 0)$ and $P_3 = (0 : 0 : 1)$. There are exactly d lines meeting in a coordinate point and exactly 3 lines passing through every point $Q_{a,b}$. Taking $\mathcal{I}_d$ as the radical ideal of the union of all points $Q_{a,b}$ and P_i, it follows that $f_d \in \mathcal{I}^{(3)}$. It turns out that f_d is not an element of $\mathcal{I}^2$. So also in this case we have

$$\mathcal{I}^{(3)} \not\subset \mathcal{I}^2.$$

The arguments are similar as in Section 2 and we don't pursue any exact proofs here. Note that for $d = 3$ we recover exactly the dual Hesse configuration.

It is natural to wonder if Conjecture 1.1 could be true after some modifications. The constructions carried out in this note do not exclude the following variant of Conjecture 1.1.

Question 3.1. Let $\mathcal{J}$ be a homogeneous radical ideal of points in the projective plane. Is there then the containment

$$\mathcal{J}^{(m)} \subset \mathcal{J}^r$$

for $m \geqslant 2r - 1$ and $r \geqslant 3$?

The first problem to decide would be the containment $\mathcal{J}^{(5)} \subset \mathcal{J}^3$.

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