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Associated primes of local cohomology of flat extensions with regular fibers and Σ-finite D-modules

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ABSTRACT

In this manuscript, we study the following question raised by Mel Hochster: Let (R, m, K) be a local ring and S be a flat extension with regular closed fiber. Is $\mathcal{V}(mS) \cap \operatorname{Ass}_S H^i_I(S)$ finite for every ideal $I \subset S$ and $i \in \mathbb{N}$? We prove that the answer is positive when S is either a polynomial or a power series ring over R and $\dim(R/I \cap R) \leqslant 1$. In addition, we analyze when this question can be reduced to the case where S is a power series ring over R. An important tool for our proof is the use of Σ-finite D-modules, which are not necessarily finitely generated as D-modules, but whose associated primes are finite. We give examples of this class of D-modules and applications to local cohomology.

1. Introduction

Throughout this manuscript A, R and S will always denote commutative Noetherian rings with unit. If M is an S-module and $I \subset S$ is an ideal, we denote the i-th local cohomology of M with support in I by $H^i_I(M)$. The structure of these modules has been widely studied by several authors [7,12,14–16,18,20–22]. Among the results obtained is that the set of associated primes of $H^i_I(R)$ is finite for certain regular rings. Huneke and Sharp proved this for characteristic $p > 0$ [8]. Lyubeznik showed this finiteness property for regular local rings of equal characteristic zero and finitely generated regular algebras over a field of characteristic zero [11]. Recently Bhatt, Blickle, Lyubeznik, Singh, and Zhang proved that the local cohomology modules of a smooth $\mathbb{Z}$-algebra have finitely many associated primes [1]. We point out that this property does not necessarily hold for rings that are not regular [10,24,25]. Motivated by these finiteness results, Mel Hochster raised the following related questions:

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Question 1.1. Let (R, m, K) be a local ring and S be a flat extension with regular closed fiber. Is

$$\operatorname{Ass}_S H^0_{mS}\big(H^i_I(S)\big) = \mathcal{V}(mS) \cap \operatorname{Ass}_S H^i_I(S)$$

finite for every ideal $I \subset S$ and $i \in \mathbb{N}$?

Question 1.2. Let (R, m, K) be a local ring and S denote either $R[x_1, \ldots, x_n]$ or $R[[x_1, \ldots, x_n]]$. Is

$$\operatorname{Ass}_S H^0_{mS}\big(H^i_I(S)\big) = \mathcal{V}(mS) \cap \operatorname{Ass}_S H^i_I(S)$$

finite for every ideal $I \subset S$ and $i \in \mathbb{N}$?

It is clear that Question 1.2 is a particular case of Question 1.1. In Proposition 6.2, we show that under minor additional hypotheses these questions are equivalent. Question 1.2 has a positive answer when R is a ring of dimension 0 or 1 of any characteristic [19]. In her thesis [23], Robbins answered Question 1.2 positively for certain algebras of dimension smaller than or equal to 3 in characteristic 0. In addition, several of her results can be obtained in characteristic $p > 0$, by working in the category $C(S, R)$ (see the discussion after Remark 2.1).

A positive answer for Question 1.1 would help to prove the finiteness of the associated primes of local cohomology modules, $H^i_I(R)$, over certain regular local rings of mixed characteristic, R. For example for

$$\frac{V[[x, y, z_1, \ldots, z_n]]}{(\pi - xy)V[[x, y, z_1, \ldots, z_n]]} = \left(\frac{V[[x, y]]}{(\pi - xy)V[[x, y]]}\right)[[z_1, \ldots, z_n]],$$

where $(V, \pi V, K)$ is a complete DVR of mixed characteristic. This is, to the best of our knowledge, the simplest example of a regular local ring of ramified mixed characteristic in which the finiteness of $\operatorname{Ass}_R H^i_I(R)$ is unknown.

In this manuscript, we give a partial positive answer for Question 1.1 and 1.2. Namely:

Theorem 1.3. *Let (R, m, K) be any local ring. Let S be either $R[x_1, \ldots, x_n]$ or $R[[x_1, \ldots, x_n]]$. Then, $\operatorname{Ass}_S H^0_{mS} H^i_I(S)$ is finite for every ideal $I \subset S$ such that $\dim R/I \cap R \leqslant 1$ and every $i \in \mathbb{N}$. Moreover, if $mS \subset \sqrt{I}$,*

$$\operatorname{Ass}_S H^{j_1}_{J_1} \cdots H^{j_\ell}_{J_\ell} H^i_I(S)$$

is finite for all ideals $J_1, \ldots, J_\ell \subset S$ and integers $j_1, \ldots, j_\ell \in \mathbb{N}$.

Theorem 1.4. *Let $(R, m, K) \to (S, \eta, L)$ be a flat extension of local rings with regular closed fiber such that R contains a field. Let $I \subset S$ be an ideal such that $\dim R/I \cap R \leqslant 1$. Suppose that the morphism induced in the completions $\widehat{R} \to \widehat{S}$ maps a coefficient field of R into a coefficient field of S. Then,*

$$\operatorname{Ass}_S H^0_m H^i_I(S)$$

is finite for every $i \in \mathbb{N}$.

In Theorem 1.4, the hypothesis that $\widehat{\varphi}$ maps a coefficient field of $\widehat{R}$ to a coefficient field of $\widehat{S}$ is not very restrictive. For instance, it is satisfied when L is a separable extension of K (see Remark 6.3). In particular, this holds when K is a field of characteristic 0 or a perfect field of characteristic $p > 0$.

A key part of the proof of Theorem 1.3 is the use of Σ-finite D-modules, which are directed unions of finite length D-modules that satisfy certain conditions (see Definition 3.3). One of the main properties that a Σ-finite D-module satisfies is that its set of associated primes is finite. In addition, the local cohomology of a Σ-finite D-module is again Σ-finite. Proving that the local cohomology modules supported on $H^i_{mS}H^j_I(S)$ are Σ-finite D-modules would answer Question 1.1.

This manuscript is organized as follows. In Section 2, we recall some definitions and properties of local cohomology and D-modules. Later, in Section 3, we define Σ-finite D-modules and give their first properties. In Section 4, we prove that certain local cohomology modules are Σ-finite; as a consequence, we give a proof of Theorem 1.3. Later, in Section 5, we give several examples of Σ-finite D-modules. Finally, in Section 6, we show that under the certain hypotheses Question 1.1 and 1.2 are equivalent, and we prove Theorem 1.4.

2. Preliminaries

2.1. Local cohomology

For the sake of completeness we recall a few definitions and properties of local cohomology. We refer to [9] for details. Let R be a ring, $\underline{f} = f_1, \dots, f_\ell \in R$ elements of R an ideal, and M an R-module. We define the Čech complex of M with respect to $\underline{f}$ by

$$\check{C}^\bullet(\underline{f}; M) = 0 \to M \to \bigoplus_j M_{f_j} \to \cdots \to M_{f_1 \cdots f_\ell} \to 0.$$

If I is generated by $f_1, \dots, f_\ell \in R$, the local cohomology group of M is defined by

$$H^i_I(M) = H^i\big(\check{C}^\bullet(\underline{f}; M)\big) = \frac{\operatorname{Ker}(\check{C}^i(\underline{f}; M) \to \check{C}^{i+1}(\underline{f}; M))}{\operatorname{Im}(\check{C}^{i-1}(\underline{f}; M) \to \check{C}^i(\underline{f}; M))}.$$

We point out that these modules do not depend on the generator chosen; moreover, $H^i_I(M)$ depends only on the radical of I.

Let $\mathcal{K}^\bullet(f_1, \dots, f_s; M)$ denote the Koszul complex of M associated to the sequence $\underline{f} = f_1, \dots, f_\ell$. Let $\underline{f}^t$ denote the sequence $f_1^t, \dots, f_s^t$. We have that the Čech complex is can be obtained from an inductive direct system of Koszul complexes,

$$\check{C}^\bullet(\underline{f}; M) = \lim_{\to t} \mathcal{K}^\bullet\big(f_1^t; M\big) \otimes_S \cdots \otimes_S \mathcal{K}^\bullet\big(f_s^t; M\big).$$

We define the *cohomological dimension of* I by

$$\operatorname{cd}_R I = \operatorname{Max}\big\{i \;\big|\; H^i_I(R) \neq 0\big\}.$$

2.2. D-modules

Given two commutative rings R and S such that $R \subset S$, we define the *ring of R-linear differential operators of* S, $D(S, R)$, as the subring of $\operatorname{Hom}_R(S, S)$ obtained inductively as follows. The differential operators of order zero are morphisms induced by multiplication by elements in S ($\operatorname{Hom}_S(S, S) = S$). An element $\theta \in \operatorname{Hom}_R(S, S)$ is a differential operator of order less than or equal to $k+1$ if $\theta \cdot r - r \cdot \theta$ is a differential operator of order less than or equal to k for every $r \in S$.

We recall that if M is a $D(S, R)$-module, then M_f has the structure of a $D(S, R)$-module such that, for every $f \in S$, the natural morphism $M \to M_f$ is a morphism of $D(S, R)$-modules. As a consequence, $H^{i_1}_{I_1} \cdots H^{i_\ell}_{I_\ell}(S)$ is also a $D(S, R)$-module [11, Examples 2.1].

Remark 2.1. If (R, m, K) is a local ring and S is either $R[x_1, \dots, x_n]$ or $R[[x_1, \dots, x_n]]$, then

$$D(S,R) = S\left[\frac{1}{t!}\frac{\partial^t}{\partial x_i^t} \,\middle|\, t \in \mathbb{N},\ 1 \leqslant i \leqslant n\right] \subset \operatorname{Hom}_R(S,S)$$

(see [5, Theorem 16.12.1]).

There is a natural surjection $\rho : D(S,R) \to D(S/IS, R/IR)$ for every ideal $I \subset R$. If M is a $D(S,R)$-module, then IM is a $D(S,R)$-submodule and the structure of M/IM as a $D(S,R)$-module is given by ρ, i.e., $\delta \cdot v = \rho(\delta) \cdot v$ for all $\delta \in D(S,R)$ and $v \in M/IM$ by the description of $D(S,R)$ above (cf. [5, Theorem 16.12.1]).

Assume that R contains the rational numbers. In this case, $D(S,R) = R[\frac{\partial^t}{\partial x_1^t}, \ldots, \frac{\partial^t}{\partial x_n^t}]$. Let $\Gamma_i = \{\delta \in D(S,R) \mid \operatorname{ord}(\delta) \leqslant i\}$. We have that $\operatorname{gr}^\Gamma D = S[y_1, \ldots, y_n]$, which is Noetherian and then so is D [2, Proposition 6.1, p. 69].

We recall a subcategory of $D(S,R)$-modules introduced by Lyubeznik [13]. We denote by $C(S,R)$ the smallest subcategory of $D(S,R)$-modules that contains S_f for all $f \in S$ and that is closed under subobjects, extensions and quotients. In particular, the kernel, image and cokernel of a morphism of $D(S,R)$-modules that belongs to $C(S,R)$ are also objects in $C(S,R)$. We note that if M is an object in $C(S,R)$, then $H^{i_1}_{I_1} \cdots H^{i_\ell}_{I_\ell}(M)$ is also an object in this subcategory; in particular, $H^{i_1}_{I_1} \cdots H^{i_\ell}_{I_\ell}(S)$ belongs to $C(S,R)$ [13, Lemma 5].

A $D(S,R)$-module, M, is *simple* if its only $D(S,R)$-submodules are 0 and M. We say that a $D(S,R)$-module, M, has *finite length* if there is a strictly ascending chain of $D(S,R)$-modules, $0 \subset M_0 \subset M_1 \subset \cdots \subset M_h = M$, called *a composition series*, such that M_{i+1}/M_i is a nonzero simple $D(S,R)$-module for every $i = 0, \ldots, h$. In this case, h is independent of the filtration and it is called the *length* of M. Moreover, the *composition factors*, M_{i+1}/M_i, are the same, up to permutation and isomorphism, for every filtration.

Notation 2.2. If M is a $D(S,R)$-module of finite length, we denote the set of its composition factors by $\mathcal{C}(M)$.

Remark 2.3. If M is a nonzero simple $D(S,R)$-module, then M has only one associated prime. This is because $H^0_P(M)$ is a $D(S,R)$-submodule of M for every prime ideal $P \subset S$. As a consequence, if M is a $D(S,R)$-module of finite length, then $\operatorname{Ass}_S M \subset \bigcup_{N \in \mathcal{C}(M)} \operatorname{Ass}_S N$, which is finite.

Remark 2.4. If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of $D(S,R)$-modules of finite length, then $\mathcal{C}(M) = \mathcal{C}(M') \cup \mathcal{C}(M'')$.

3. Σ-finite D-modules

Notation 3.1. Thorough this section (R, m, K) denotes a local ring and S denotes either $R[x_1, \ldots, x_n]$ or $R[[x_1, \ldots, x_n]]$. In addition, D denotes $D(S,R)$.

Definition 3.2. For a D-module, M, we denote by $\operatorname{Fin}(M)$ the set of all D-submodules of M that have finite length.

Definition 3.3. Let M be a D-module supported at mS. We say that M is *Σ-finite* if:

(i) $\bigcup_{N \in \operatorname{Fin}(M)} N = M$,
(ii) $\bigcup_{N \in \operatorname{Fin}(M)} \mathcal{C}(N)$ is finite, and
(iii) For every $N \in \operatorname{Fin}(M)$ and $L \in \mathcal{C}(N)$, $L \in C(S/mS, R/mR)$.

We denote the set of *composition factors of* M, $\bigcup_{N \in \operatorname{Fin}(M)} \mathcal{C}(N)$, by $\mathcal{C}(M)$.

Remark 3.4. We have that

$$\operatorname{Ass}_S M \subset \bigcup_{N\in\mathcal{C}(M)} \operatorname{Ass}_S N$$

for every Σ-finite D-module, M. In particular, $\operatorname{Ass}_S M$ is finite.

Lemma 3.5. *Let M be a Σ-finite D-module and N be a D-submodule of M. Then, N has finite length as a D-module if and only if N is finitely generated as a D-module.*

Proof. Suppose that N is finitely generated. Let $v_1,\dots,v_\ell$ be a set the generators of N. Since $\bigcup_{L\in\operatorname{Fin}(M)} L = M$, there exists a finite length module N_i that contains v_i. Then, $N\subset N_1+\cdots+N_\ell$ and it has finite length. It is clear that if N has finite length then it is finitely generated. $\square$

Proposition 3.6. *Let $0\to M'\to M\to M''\to 0$ be a short exact sequence of D-modules. If M is Σ-finite, then M' and M'' are Σ-finite. Moreover, $\mathcal{C}(M)=\mathcal{C}(M')\cup\mathcal{C}(M')$.*

Proof. We first assume that M is Σ-finite. We have that

$$M' = \bigcup_{N\in\operatorname{Fin}(M)} N\cap M' = \bigcup_{N'\in\operatorname{Fin}(M)} N'.$$

Then M' is Σ-finite by Remarks 2.3 and 2.4. Let ρ denote the morphism $M\to M''$ and $N''\in\operatorname{Fin}(M'')$ and $\ell=\operatorname{length}_D N''$. There are $v_1,\dots,v_\ell\in N''$ such that $N''=D\cdot v_1+\cdots+D\cdot v_\ell$. Let w_j be a preimage of v_j and N be the D-module generated by $w_1,\dots,w_\ell$. We have that $N\to N''$ is a surjection, and that N has finite length by Lemma 3.5. Therefore, $M''=\bigcup_{N\in\operatorname{Fin}(M)}\rho(N)=\bigcup_{N''\in\operatorname{Fin}(M'')} N''$ and the result follows by Remarks 2.3 and 2.4. $\square$

Proposition 3.7. *Let $0\to M'\to M\to M''\to 0$ be a short exact sequence of D-modules. Suppose that R contains the rational numbers. If M' and M'' are Σ-finite, then M is Σ-finite. Moreover, $\mathcal{C}(M)=\mathcal{C}(M')\cup\mathcal{C}(M')$.*

Proof. Let $v\in M$. We have a short exact sequence

$$0\to M'\cap D\cdot v\to D\cdot v\to D\cdot\bar{v}\to 0.$$

$M'\cap D\cdot v$ is finitely generated because D is Noetherian by Remark 2.1. Then, $M'\cap D\cdot v$ has finite length by Lemma 3.5, and so $D\cdot v$ has finite length. Therefore, $M=\bigcup_{N\in\operatorname{Fin}(M)} N$.

Let $N\in\operatorname{Fin}(M)$. Then, $N\cap M'\in\operatorname{Fin}(M')$ and $\rho(N)\in\operatorname{Fin}(M'')$. We have a short exact sequence

$$0\to N\cap M'\to N\to\rho(N)\to 0$$

of finite length D-modules, and then result follows by Remarks 2.3 and 2.4. $\square$

Proposition 3.8. *Let M be a Σ-finite D-module. Then, M_f is Σ-finite for every $f\in S$.*

Proof. Let $N\subset M_f$ be a module of finite length. We have that N is a finitely generated D-module. Then there exists a finitely generated D-submodule N' of M such that $N\subset N'_f$. We have that N'_f has finite length and $\mathcal{C}(N'_f)=\bigcup_{V\in\mathcal{C}(N)}\mathcal{C}(V_f)$ because V_f is in $C(S/mS,R/m)$ [13]. Then,

$$M_f = \bigcup_{N\subset\operatorname{Fin}(M_f)} N \subset \bigcup_{N\subset\operatorname{Fin}(M)} N_f = M_f$$

and the result follows. $\square$

Lemma 3.9. *Let M and M' be Σ-finite D-modules. Then, $M \oplus M'$ is also Σ-finite.*

Proof. It is clear that $M \oplus M'$ is supported on mS. For every $(v, v') \in M \oplus M'$, there exist N and N', D-modules of finite length, such that $v \in N$ and $v' \in N'$. Then, $N \oplus N' \subset M \oplus M'$ has finite length and $(v, v') \in N \oplus N'$. Therefore,

$$\bigcup_{N \subset \mathrm{Fin}(M), N' \subset \mathrm{Fin}(M')} N \oplus N' = M \oplus M',$$

and the $M \oplus M'$ is the union of its D-modules of finite length. The rest follows from Remarks 2.3 and 2.4. $\square$

Corollary 3.10. *Let M be a Σ-finite D-module. Then, $H^i_I(M)$ is Σ-finite for every ideal $I \subset S$ and $i \in \mathbb{N}$.*

Proof. Let $f_1, \dots, f_\ell$ be generators for I. We have that $\check{C}^i(\underline{f}; M)$ is Σ-finite by Proposition 3.8 and Lemma 3.9. In addition, $\mathrm{Ker}(\check{C}^i(\underline{f}; M) \to \check{C}^{i+1}(\underline{f}; M))$ and $\mathrm{Im}(\check{C}^{i-1}(\underline{f}; M) \to \check{C}^i(\underline{f}; M))$ are Σ-finite by Proposition 3.6. Therefore, $H^i_I(M) = H^i(\check{C}^\bullet(\underline{f}; M))$ is also Σ-finite by Proposition 3.6. $\square$

Proposition 3.11. *Let M_t be an inductive direct system of Σ-finite D-modules. If $\bigcup_t \mathcal{C}(M_t)$ is finite, then $\lim_{\to t} M_t$ is Σ-finite and $\mathcal{C}(M) \subset \bigcup_t \mathcal{C}(M_t)$.*

Proof. Let $M = \lim_{\to t} M_t$ and $\varphi_t : M_t \to M$ the morphism induced by the limit. We have that $\phi_t(M_t)$ is a Σ-finite D-module by Proposition 3.6. We may replace M_t by $\phi_t(M_t)$ by Remarks 2.3 and 2.4, and assume that $M = \bigcup M_t$ and $M_t \subset M_{t+1}$. If $N \subset M$ has finite length as a D-module, then it is finitely generated and there exists t such that $N \subset M_t$. Therefore, $M = \bigcup_t M_t = \bigcup_t \bigcup_{N \in \mathrm{Fin}(M_t)} N$ and the result follows. $\square$

4. Associated Primes

Notation 4.1. Throughout this section (R, m, K) denotes a local ring and S denotes either $R[x_1, \dots, x_n]$ or $R[[x_1, \dots, x_n]]$. In addition, D denotes $D(S, R)$.

Lemma 4.2. *Let $J \subset S$ be an ideal and M be an R-module of finite length. Then, $H^i_J(M \otimes_R S)$ is a $D(S, R)$-module of finite length. Moreover, $\mathcal{C}(H^i_{JS}(M \otimes_R S)) \subset \bigcup_j \mathcal{C}(H^j_{JS}(S/mS))$.*

Proof. Our proof will be by induction on $h = \mathrm{length}_R(M)$. If $h = 1$, $M = R/m$. Then $H^i_{JS}(R/m \otimes_R S) = H^i_{JS}(S/mS)$, which has finite length as a $D(S, R)$-module (see [13, Corollary 3] for power series and [15, Theorem 1.1] for polynomials). Clearly,

$$\mathcal{C}\big(H^i_{JS}(M \otimes_R S)\big) = \mathcal{C}\big(H^i_{JS}(R/m \otimes_R S)\big) \subset \bigcup_j \mathcal{C}\big(H^j_{JS}(S/mS)\big)$$

in this case. Suppose that the statement is true for h and $\mathrm{length}_R(M) = h + 1$. We have a short exact sequence of R-modules, $0 \to K \to M \to M' \to 0$, where $h = \mathrm{length}_R(M')$. Since S is flat over R, we have that $0 \to K \otimes_R S \to M \otimes_R S \to M' \otimes_R S \to 0$ is also exact. Then, we have a long exact sequence

$$\cdots \to H^i_J(K \otimes_R S) \to H^i_J(M \otimes_R S) \to H^i_J\big(M' \otimes_R S\big) \to \cdots.$$

Then $H^i_J(M \otimes_R S)$ has finite length by the induction hypothesis. In addition,

$$\mathcal{C}\big(H^i_J(M\otimes_R S)\big)\subset\mathcal{C}\big(H^i_J\big(M'\otimes_R S\big)\big)\cup\mathcal{C}\big(H^i_J(K\otimes_R S)\big)\subset\bigcup_j\mathcal{C}\big(H^j_J(S/mS)\big),$$

and the result follows by the induction hypothesis and Remark 2.4. □

Proposition 4.3. *Let $I\subset S$ be an ideal containing mS. Then $H^i_I(S)$ is Σ-finite for every $i\in\mathbb{N}$.*

Proof. Let $f_1,\dots,f_d$ be a system of parameters for R and $g_1,\dots,g_\ell$ be a set of generators for I. Let $\underline{f}^t$ denote the sequence $f^t_1,\dots,f^t_d$. Let $T_i=\{T^{p,q}_t\}$ be the double complex of $D(S,R)$-modules given by the tensor product $\mathcal{K}^\bullet(\underline{f};R)\otimes_R \check{C}^bullet(\underline{g};S)$. The direct system of Koszul complexes $\mathcal{K}^\bullet(\underline{f}^t;R)$ that gives the Čech complex induces a direct system of double complexes $\operatorname{Tot}(T_t)\to\operatorname{Tot}(T_{t+1})$. Since $\lim_{\to t}\mathcal{K}^\bullet(\underline{f}^t;R)=\check{C}^\bullet(\underline{f};R)$, we have that $\lim_{\to t}\operatorname{Tot}(T_t)=\check{C}^\bullet(\underline{f,g};S)$. Let $E^{p,q}_{r,t}$ be the spectral sequence associated to T_t. We have that

$$E^{p,q}_{2,t}=H^p_I\big(H^q\big(\mathcal{K}^\bullet\big(\underline{f}^t;S\big)\big)\big)\quad\Rightarrow\quad E^{p,q}_{\infty,t}=H^{p+q}\operatorname{Tot}(T_t).\tag{1}$$

We note that $H^q(\mathcal{K}^\bullet(\underline{f}^t;S))=H^q(\mathcal{K}^\bullet(\underline{f}^t;R))\otimes_R S$, because S is R-flat. Since $H^q(\mathcal{K}^\bullet(\underline{f}^t;R))$ has finite length as an R-module, we have that $E^{p,q}_{2,t}$ is a $D(S,R)$-module of finite length for all $p,q\in\mathbb{N}$ and that $\mathcal{C}(E^{p,q}_{2,t})=\bigcup_j\mathcal{C}(H^i_{JS}(S/mS))$ by Lemma 4.2. Moreover, $E^{p,q}_{r,t}$ is a $D(S,R)$-module of finite length, and

$$\mathcal{C}\big(E^{p,q}_{r,t}\big)\subset\bigcup_{p,q}\mathcal{C}\big(E^{p,q}_{2,t}\big)=\bigcup_j\mathcal{C}\big(H^j_I(S/mS)\big)\tag{2}$$

for $r>2$, where the first containment follows from Proposition 3.6. We note that $H^i(\operatorname{Tot}(T_t))$ comes from taking submodules and quotients of $E^{p,q}_{r,t}$ with $r>2$. Then, $\mathcal{C}(H^i(\operatorname{Tot}(T_t)))\subset\bigcup_j\mathcal{C}(H^i_I(S/mS))$ for every $j,t\in\mathbb{N}$ by combining (1), (2) and Remark 2.4, because all the modules in the spectral sequence have finite length. In particular, $\bigcup_t\mathcal{C}(H^i\operatorname{Tot}(T_t))$ is finite and every element in this union belongs to the subcategory $C(S/mS,R/mR)$. Therefore, $E^{p,q}_{r,t}$ is a Σ-finite $D(S,R)$-module. Moreover,

$$H^i_I(S)=H^i\big(\check{C}^\bullet(\underline{f,g};S)\big)=H^i\Big(\lim_{\to t}\operatorname{Tot}(T_t)\Big)=\lim_{\to t}H^i\big(\operatorname{Tot}(T_t)\big)$$

because the direct limit is exact. Hence, $H^i_I(S)$ is Σ-finite by Proposition 3.11. □

Corollary 4.4. *Let $I\subset S$ be an ideal containing mS and $J_1,\dots,J_\ell\subset S$ be ideals. Then $H^{j_1}_{J_1}\cdots H^{j_\ell}_{J_\ell}H^i_I(S)$ is Σ-finite.*

Proof. This is a consequence of Corollary 3.10 and Proposition 4.3. □

Proof of Theorem 1.3. This is a consequence of Remark 3.4 and Corollary 4.4. □

Proposition 4.5. *Let (R,m,K) be any local ring. Let S denote either $R[x_1,\dots,x_n]$ or $R[[x_1,\dots,x_n]]$. Let $I\subset S$ be an ideal such that* $\dim R/I\cap R\leqslant 1$. *Then,*

$$\operatorname{Ass}_S H^0_m H^i_I(S)$$

is finite for every $i\in\mathbb{N}$.

Proof. Since $\dim R/(I \cap R) \leqslant 1$, there exists $f \in R$ such that $mS \subset \sqrt{I + fS}$. We have the exact sequence [3, Proposition 8.1.2]:

$$\cdots \to H^i_{(I,f)S}(S) \xrightarrow{\alpha_i} H^i_I(S) \xrightarrow{\beta_i} H^i_I(S_f) \to \cdots.$$

Then,

$$\operatorname{Ass}_S H^i_I(S) \cap \mathcal{V}(mS) \subset \big(\operatorname{Ass}_S \operatorname{Im}(\alpha_i) \cap \mathcal{V}(mS)\big) \cup \big(\operatorname{Ass}_S \operatorname{Im}(\beta_i) \cap \mathcal{V}(mS)\big).$$

Since $H^i_{(I,f)S}(S)$ is a Σ-finite $D(S,R)$-module by Proposition 4.3, we have that $\operatorname{Im}(\alpha_1)$ is also Σ-finite by Proposition 3.6, and so $\operatorname{Ass}_S \operatorname{Im}(\alpha_i)$ is finite. Since $\operatorname{Im}(\beta_i) \subset H^i_I(S_f)$, $\operatorname{Ass}_S \operatorname{Im}(\beta_i) \cap \mathcal{V}(mS) = \emptyset$. Therefore,

$$\operatorname{Ass}_S H^i_I(S) \cap \mathcal{V}(mS) = \operatorname{Ass}_S H^0_{mS} H^i_I(S)$$

is finite. □

Proposition 4.6. *Suppose that R is a ring of characteristic* 0 *and that* $\dim R/(I \cap R) \leqslant 1$. *Then $H^j_{mS} H^i_I(S)$ is Σ-finite for every $i, j \in \mathbb{N}$.*

Proof. Since $\dim R/(I \cap R) \leqslant 1$, there exists $g \in R$, such that $mS \subset \sqrt{(I,g)S}$. We have the long exact sequence

$$\cdots \to H^i_{(I,g)S}(S) \to H^i_I(S) \to H^i_I(S_g) \to \cdots.$$

Let $M_i = \operatorname{Ker}(H^i_{(I,g)S}(S) \to H^i_I(S))$, $N_i = \operatorname{Im}(H^i_{(I,g)S}(S) \to H^i_I(S))$ and $W_i = \operatorname{Im}(H^i_I(S) \to H^i_I(S_g))$. We have the following short exact sequences:

$$0 \to M_i \to H^i_{(I,g)S}(S) \to N_i \to 0,$$
$$0 \to N_i \to H^i_I(S) \to W_i \to 0$$

and

$$0 \to W_i \to H^i_I(S_g) \to M_{i+1} \to 0.$$

Since $mS \subset \sqrt{(I,g)S}$, we have that $H^i_{(I,g)S}(S)$ is Σ-finite by Proposition 4.3. Then, M_i and N_i is Σ-finite for every $i \in \mathbb{N}$ by Proposition 3.6. By the long exact sequences

$$\cdots \to H^j_{mS}(M_i) \to H^j_{mS} H^i_{(I,g)S}(S) \to H^j_{mS}(N_i) \to \cdots,$$
$$\cdots \to H^j_{mS}(N_i) \to H^j_{mS} H^i_I(S) \to H^j_{mS}(W_i) \to \cdots$$

and

$$\cdots \to H^j_{mS}(W_i) \to H^j_{mS} H^i_I(S_g) \to H^j_{mS}(M_{i+1}) \to \cdots,$$

$H^j_{mS}(M_i)$, $H^j_{mS} H^i_{(I,g)S}(S)$ and $H^j_{mS}(M_i)$ are Σ-finite for every $i, j \in \mathbb{N}$. Since $H^j_{mS} H^i_I(S_g) = 0$, $H^j_{mS}(W_i) = H^j_{mS}(M_{i+1})$. Then, $H^j_{mS}(W_i)$ is Σ-finite, and so $H^j_{mS} H^i_I(S)$ is Σ-finite by Proposition 3.7. □

5. More examples of Σ-finite D-modules

In the previous section we gave a positive answer for specific cases for Question 1.2. Our method consisted in proving that $H^j_{mS}H^i_I(S)$ is Σ-finite and then applying Remark 3.4. This motivates the following question:

Question 5.1. Is $H^j_{mS}H^i_I(S)$ Σ-finite for every ideal $I \subset S$ and $i, j \in \mathbb{N}$?

In this section, we provide examples where Question 5.1 has an affirmative answer.

Proposition 5.2. *Let (R, m, K) be any local ring. Let S denote either $R[x_1, \dots, x_n]$ or $R[[x_1, \dots, x_n]]$. Let $I \subset S$ be an ideal such that* $\operatorname{depth}_S I = \operatorname{cd}_S I$. *Then, $H^i_{mS}H^{\operatorname{cd}_S I}_I(S)$ is Σ-finite for every $i \in \mathbb{N}$.*

Proof. Let $f_1, \dots, f_s \in S$ and $g_1, \dots, g_\ell \in S$ be generators for mS and I respectively. We have that a spectral sequence

$$E_2^{p,q} = H^p_{mS}H^q_I(S) \quad \Rightarrow \quad E_\infty^{p,q} = H^{p+q}_{(I,m)S}(S) \tag{3}$$

given by the tensor product $\check{C}^\bullet(\underline{f}, S) \otimes_S \check{C}^\bullet(\underline{g}, S)$. Since $\operatorname{depth}_S I = \operatorname{cd}_S I$, there is only one row in the double complex given by $\check{C}^\bullet(\underline{f}, S) \otimes \check{C}^\bullet(\underline{g}, S)$, once we take the cohomology corresponding to $\underline{g}$. The only row that appears is $\check{C}^\bullet(\underline{f}, S) \otimes_R H^{\operatorname{cd}_S I}_I(S)$. Therefore the spectral sequence in (3) degenerates and converges at the second spot. Hence,

$$H^p_{mS}H^q_I(S) = H^{p+q}_{(I,m)S}(S)$$

and the result follows by Proposition 4.3. $\square$

Proposition 5.3. *Let (R, m, K) be any local ring. Let S denote either $R[x_1, \dots, x_n]$ or $R[[x_1, \dots, x_n]]$. Suppose that $I \subset S$ is an ideal such that* $\operatorname{Ext}^i_S(S/mS, H^j_I(S))$ *is a D-module in the category $C(R, S)$ for every $i \in \mathbb{N}$. Then, $H^i_{mS}H^j_I(S)$ is a Σ-finite $D(S, R)$-module for every $i, j \in \mathbb{N}$.*

Proof. We claim that $\operatorname{Ext}^i_S(N \otimes_R S, H^j_I(S))$ is a $D(S/mS, K)$-module in $C(S/mS, K)$ for every $i \in \mathbb{N}$ and every finite length R-module N. Moreover, $\mathcal{C}(\operatorname{Ext}^i_S(N \otimes_R S, H^j_I(S))) \subset \bigcup_i \mathcal{C}(\operatorname{Ext}^i_S(K \otimes_R S, H^j_I(S)))$. The proof of our claim is analogous to Lemma 4.2.

The direct system $\operatorname{Ext}^i(S/m^\ell S, H^j_I(S)) \to \operatorname{Ext}^i(S/m^{\ell+1}S, H^j_I(S))$ satisfies the hypotheses of Proposition 3.11. Hence,

$$H^i_{mS}H^j_I(S) = \varinjlim_{\ell} \operatorname{Ext}^i\big(S/m^\ell S, H^j_I(S)\big)$$

is a Σ-finite $D(S, R)$-module. $\square$

Remark 5.4. The condition that $\operatorname{Ext}^i_S(S/mS, H^j_I(S))$ is a $D(S/mS, K)$-module in $C(S/mS, K)$ for every $i \in \mathbb{N}$ is not necessary.

Let $R = K[[s, t, u, w]]/(us + vt)$, where K is a field. This is the ring given by Hartshorne's example [6]. Let $I = (s, t)A$. Hartshorne showed that $\dim_K \operatorname{Hom}_A(K, H^2_I(A))$ is not finite. Let S be either $R[x_1, \dots, x_n]$ or $R[[x_1, \dots, x_n]]$. Therefore,

$$\operatorname{Ext}^0_S\big(S/mS, H^2_I(S)\big) = \operatorname{Hom}_S\big(S/mS, H^2_I(S)\big) = \operatorname{Hom}_R\big(K, H^2_I(R)\big) \otimes_R S = \bigoplus S/mS,$$

where the direct sum is infinite. Then, $\operatorname{Ext}^0_S(S/mS, H^2_I(S))$ does not belong to $C(S, R)$.

On the other hand, $H^0_m H^2_I(S)$ is a direct limit of finite direct sums of S/mS. This direct limit satisfies the hypotheses of Proposition 3.11. Therefore, $H^0_m H^2_I(S)$ is a Σ-finite $D(S,R)$-module.

Proposition 5.5. *Let (R,m,K) be any local ring and let $S = R[x_1,\dots,x_n]$. Let $I \subset S$ be an ideal. Then, $H^i_{mS} H^0_I(S)$ is Σ-finite for every $i \in \mathbb{N}$. In addition, if $\operatorname{cd}_S I \leqslant 1$, then $H^i_{mS} H^j_I(S)$ is Σ-finite for every $i, j \in \mathbb{N}$.*

Proof. We claim that there exists an ideal $J \subset R$ such that $H^0_I(S) = JS$. We have that $H^0_I(S)$ is a $D(S,R)$-module. For every $f = \sum_\alpha c_\alpha x^\alpha \in H^0_I(S)$ and $\partial \in D(S,R)$, $\partial f \in H^0_I(S)$. Therefore, $c_\alpha \in H^0_I(S)$ and $H^0_I(S) = JS$, where $J = \{c_\alpha \mid \sum_\alpha c_\alpha x^\alpha \in H^0_I(S)\}$. We have that

$$\begin{aligned}
\operatorname{Ext}^i_S\bigl(S/mS, H^0_I(S)\bigr) &= \operatorname{Ext}^i_S(R/mR \otimes_R S, J \otimes_R S) \\
&= \operatorname{Ext}^i_R(K, J) \otimes_S S \\
&= \bigoplus{}^{\mu} S/mS, \quad \text{where } \mu = \dim_K \operatorname{Ext}^i_R(K, J).
\end{aligned}$$

This $D(S,R)$-module belongs to $C(S,R)$ for every $i \in \mathbb{N}$. The first claim follows from Proposition 5.3.

We have that $H^1_I(S) = H^1_{I(S/J)}(S/J)$ [3, Corollary 2.1.7]. In addition, $S/JS = (R/J)[x_1,\dots,x_n]$ and $\operatorname{depth}_{I(S/JS)} = \operatorname{cd}_{I(S/JS)}(S/JS) = 1$. The second claim follows from Proposition 5.2. □

6. Reduction to power series rings

Discussion 6.1. Suppose that (R,m,K) and (S,η,L) are complete local rings and that $\varphi : R \to S$ is a flat extension of local rings with regular closed fiber. Assume that φ maps a coefficient field of R to a coefficient field of S. We pick such coefficient fields, and then $\varphi(K) \subset L$. Thus, $R = K[[x_1,\dots,x_n]]/I$ for some ideal $I \subset K[[x_1,\dots,x_n]]$. Let $A = L \widehat{\otimes}_K R = L[[x_1,\dots,x_n]]/IL[[x_1,\dots,x_n]]$. We note that A is a flat local extension of R, such that mA is the maximal ideal of A. Let $\theta : A \to S$ be the morphism induced by φ and our choice of coefficient fields.

We claim that S is a flat A-algebra. Let F_* be a free resolution of R/mR. Then, $A \otimes_R F_*$ is a free resolution for A/mA. We have that

$$\operatorname{Tor}^A_1(S, A/mA) = H_1(S \otimes_A A \otimes_R F_*) = H_1(S \otimes_R F_*) = \operatorname{Tor}^R_1(S, R/mR) = 0$$

because S is a flat extension. Since mA is the maximal ideal of A, we have that S is a flat A-algebra by the local criterion of flatness [4, Theorem 6.8].

Let $d = \dim(S/mS)$ and $z_1,\dots,z_d \in S$ be preimages of a regular system of parameters for S/mS. Let $\phi : A[[y_1,\dots,y_d]] \to S$ be the morphism given by sending A to S via θ and y_i to z_i. Since $(mA + (z_1,\dots,z_d)A)S = \eta$ and the morphism induced by ϕ in the quotient fields of A and S is an isomorphism, we have that ϕ is an isomorphism.

Proposition 6.2. *Let $\varphi : (R,m,K) \to (S,\eta,L)$ be a flat extension of local rings with regular closed fiber. Suppose that $\widehat{\varphi} : \widehat{R} \to \widehat{S}$, the induced morphism in the completions, maps a coefficient field of the $\widehat{R}$ to a coefficient field of $\widehat{S}$. Then, Questions 1.1 and 1.2 are equivalent.*

Proof. We have that $\operatorname{Ass}_S H^0_{mS} H^i_I(S)$ is finite if and only if $\operatorname{Ass}_S H^0_{m\widehat{S}} H^i_I(\widehat{S})$ is finite. Let A be as in the previous discussion and $d = \dim(S/mS)$. The result follows, because $\widehat{S} = A[[y_1,\dots,y_d]]$ and $mS = (mA)S$. □

Remark 6.3. In the previous proposition, the hypothesis that $\widehat{\varphi}$ maps a coefficient field of $\widehat{R}$ to a coefficient field of $\widehat{S}$ is satisfied when the map induced in the residue fields, $K \to L$, is a separable extension. Now, we prove this statement. Let K' be a coefficient field for $\widehat{R}$. Then $\widehat{\varphi}(K') \cong K$ is a field contained in $\widehat{S}$. We have that the field extension given by $\widehat{\varphi}(K') \to \widehat{S} \twoheadrightarrow L$ is separable. By [17, Theorem 28.3(iii)], there exists a quasi-coefficient field L' in $\widehat{S}$ such that $\widehat{\varphi}(K') \subset L'$. Since $\widehat{S}$ is complete, there is a unique coefficient field of $\widehat{S}$, L'', such that $L' \subset L''$ by [17, Theorem 28.3(iv)]. Hence, $\widehat{\varphi}(K') \subset L''$.

Proof of Theorem 1.4. By Discussion 6.1, we may assume that R is complete and S is a power series ring over R. The rest is a consequence of Proposition 4.5. □

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