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# Degree four cohomological invariants for certain central simple algebras

A.S. Sivatski

*Departamento de Matemática, Universidade Federal do Rio Grande do Norte, Natal, Brazil*

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ABSTRACT

Let $F$ be a field, $\operatorname{char} F \neq 2$. In the first section of the paper we prove that if $A = (a,b) + (c,d)$ is a biquaternion algebra divisible by 2 in the Brauer group $\operatorname{Br}(F)$, and $\langle\langle -1,-1\rangle\rangle_F = 0$, then the symbol $(a,b,c,d) \in H^4(F, \mathbb{Z}/2\mathbb{Z})$ is an invariant, i.e. it does not depend on the decomposition of $A$ into a sum of two quaternions. In the second section we construct an invariant $p$ in $H^4(F, \mathbb{Z}/2\mathbb{Z})$ for elements $C+\alpha \in {}_4\operatorname{Br}(F)$, where $C$ is cyclic of degree at most 4, and $\alpha \in {}_2\operatorname{Br}(F)$. In the case $\sqrt{-1} \in F^*$ we extend the invariant $p$ to elements $D+\alpha \in {}_4\operatorname{Br}(F)$, where $\operatorname{ind} D \leq 4$ and $\alpha \in {}_2\operatorname{Br}(F)$.



Let $F$ be a field of characteristic different from 2. Let $H^n(F)$ be the degree $n$ cohomological group of $F$ with coefficients in $\mathbb{Z}/2\mathbb{Z}$, i.e.

$$H^n(F) = H^n(F, \mathbb{Z}/2\mathbb{Z}) = H^n(Gal(F_{sep}/F), \mathbb{Z}/2\mathbb{Z}).$$

In the first section of the paper we prove that if $A = (a,b) + (c,d)$ is a biquaternion algebra divisible by 2 in the Brauer group $\operatorname{Br}(F)$, and the Pfister form $\langle\langle -1,-1\rangle\rangle_F$ is hyperbolic, then the symbol $(a,b,c,d) \in H^4(F)$ is an invariant, i.e. it does not depend

*E-mail address:* alexander.sivatski@gmail.com.

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on the decomposition of $A$ into a sum of two quaternion algebras. This invariant is well known if $\sqrt{-1} \in F^*$ and in this case it is equal to $\gamma_2(A)$, where $\gamma_2$ is the second divided power operation ${}_2\mathrm{Br}(F) \to H^4(F)$ (note that if $\sqrt{-1} \in F^*$, then the whole group $\mathrm{Br}(F)$ is 2-divisible). However, our invariant is defined only for biquaternion algebras, and under the same hypothesis can hardly be extended to the subgroup of 2-divisible elements of ${}_2\mathrm{Br}(F)$.

In the second section we construct another invariant in $H^4(F)$ for elements $D + \alpha \in {}_4\mathrm{Br}(F)$ such that $\operatorname{ind} D \leq 4$, and $\alpha \in {}_2\mathrm{Br}(F)$.

In the proofs of the statements below we use cohomology groups, Milnor's $K$-groups and some rudimentary facts from the theory of quadratic forms over fields. All the needed information on these objects is contained in the books [3], [6], and [1].

A few words about notation. Let $F$ be a field, $m, n$ positive integers. Assume that $n$ is not divisible by $\operatorname{char} F$, and $F$ contains the group of roots of unity of degree $n$, which is denoted by $\mu_n$. The choice of a primitive root of unity $\xi_n$ determines the isomorphism $\mathbb{Z}/n\mathbb{Z} \simeq \mu_n$, under which 1 takes to $\xi_n$. This isomorphism allows to identify the groups $H^1(F, \mathbb{Z}/n\mathbb{Z})$ and $H^1(F, \mu_n) = F^*/F^{*n}$. Let $a_1, \dots, a_m \in F^*$. Consider the cup-product $H^1(F, \mathbb{Z}/n\mathbb{Z}) \otimes \cdots \otimes H^1(F, \mathbb{Z}/n\mathbb{Z}) \to H^m(F, \mathbb{Z}/n\mathbb{Z})$. The elements $a_i \in F^*/F^{*n}$ can be considered as elements of the group $H^1(F, \mathbb{Z}/n\mathbb{Z})$, provided a primitive root of unity $\xi_n$ is chosen. The element $(a_1) \cup (a_2) \cdots \cup (a_m) \in H^m(F, \mathbb{Z}/n\mathbb{Z})$ is denoted by $(a_1, \dots, a_m)_n$. If $n = 2$ we omit the index $n$ in the last symbol and write it merely as $(a_1, \dots, a_m)$.

By ${}_\mathrm{n}\mathrm{Br}(F)$ we denote the $n$-torsion of the Brauer group $\mathrm{Br}(F)$. If $F$ contains the group of roots of unity of degree $n$ and $\xi_n$ is a fixed primitive root of unity, then by $(a, b)_{\xi_n} \in {}_\mathrm{n}\mathrm{Br}(F)$ we denote the class of the symbol algebra of degree $n$ with the generators $i$ and $j$, and the relations $i^n = a$, $j^n = b$, and $ij = \xi_n ji$. There is an isomorphism $\varphi : {}_\mathrm{n}\mathrm{Br}(F) \simeq H^2(F, \mathbb{Z}/n\mathbb{Z})$ taking $(a, b)_{\xi_n}$ to $(a, b)_n$, which permits us to identify the elements $(a, b)_n \in H^2(F, \mathbb{Z}/n\mathbb{Z})$ and $(a, b)_{\xi_n} \in {}_\mathrm{n}\mathrm{Br}(F)$. If $n = km$ and $\xi_n \in F$, then $k(a, b)_n = (a, b)_m \in \mathrm{Br}(F)$, where the roots of unity $\xi_n$ and $\xi_m$ are related by the equality $\xi_n^k = \xi_m$.

For $m \geq 0$, let $K_m(F)$ denote the $m$th Milnor $K$-group of the field $F$. Assume that $\xi_n \in F^*$. The symbols $(a_1, \dots, a_m)_n$ satisfies the same basic relations as the symbols $\{a_1, \dots, a_m\}$ in $K_m(F)/n$, hence there exists a natural homomorphism $K_m(F)/n \to H^m(F, \mathbb{Z}/n\mathbb{Z})$ called the norm residue homomorphism, which takes the symbol $\{a_1, \dots, a_m\}$ to the symbol $(a_1, \dots, a_m)_n$. Suppose that $n = 2^m$, and $-1 \in F^{*n}$, i.e. $\xi_{2n} \in F^*$. Let $\gamma_2 : K_2(F)/n \to K_4(F)/n$ be the second divided power operation on $K_2(F)/n$, which is well defined by the rule $\gamma_2(\sum\limits_i \alpha_i) = \sum\limits_{i<j} \alpha_i \alpha_j$, where all $\alpha_i$ are symbols in $K_2(F)/n$ ([4], [15]). We call the composition of the second divided power operation and the norm residue homomorphism

$$K_2(F)/n \to K_4(F)/n \to H^4(F, \mathbb{Z}/n\mathbb{Z})$$

the second divided power operation as well, and also denote it by $\gamma_k$. Moreover, the norm residue homomorphism $K_2(F)/n \to H^2(F, \mathbb{Z}/n\mathbb{Z})$ is bijective ([9], Th. 11.5), which permits us to define in an obvious way the second divided power operations $H^2(F, \mathbb{Z}/n\mathbb{Z}) \to K_4(F)/n$ and $H^2(F, \mathbb{Z}/n\mathbb{Z}) \to H^4(F, \mathbb{Z}/n\mathbb{Z})$, which also will be denoted by $\gamma_2$. Obviously, the divided power operations commute with field extension homomorphisms.

For every $m \geq 0$ there is an operation $w_m : \widehat{W}(F) \to H^m(F, \mathbb{Z}/2\mathbb{Z})$ on the Witt-Grothendieck ring $\widehat{W}(F)$, which is called the $m$th Stiefel-Whitney map. These maps satisfy equalities $w_0 = 1$ and

$$w_m(\langle a_1, \ldots, a_n \rangle) = \sum_{i_1 < i_2 < \cdots < i_m} (a_{i_1}, a_{i_2}, \ldots a_{i_m})$$

for $m \geq 1$ and $a_i \in F^*$. It readily follows that $w_m(\alpha + \beta) = \sum_{i+j=m} w_i(\alpha) w_j(\beta)$ for any $\alpha, \beta \in \widehat{W}(F)$ ([1], §5).

If $L/F$ is a finite field extension, then by $\mathrm{res}_{L/F}$ and $N_{L/F}$ we denote the restriction and the norm maps for the Brauer group, Milnor's K-group, etc. The composition $N_{L/F} \circ \mathrm{res}_{L/F}$ is multiplication by the degree of the extension $L/F$.

Finally, we assume that the characteristic of any field below is different from 2.

## 1. A degree four invariant for biquaternion algebras

We begin with the following theorem, which is an application of the chain lemma for biquaternion algebras from [13].

**Theorem 1.1.** *Let $F$ be a field, let $A$ be a biquaternion. Consider two decompositions of $A$ into a sum of two quaternions:*

$$A = (a_1, b_1) + (c_1, d_1) = (a_2, b_2) + (c_2, d_2).$$

*Then*

$$(a_1, b_1, c_1, d_1) - (a_2, b_2, c_2, d_2) \in (-1, -1) \cup H^2(F) + (-1) \cup [A] \cup H^1(F).$$

**Proof.** Recall the definition of equivalent decompositions of $A$ into sum of two quaternion algebras ([13]). Let $A = D_1 + D_1' = D_2 + D_2'$ be two decompositions of $A$ into a sum of two quaternion algebras (the signs = and + will always mean equality and addition in the Brauer group of $F$). We call these decompositions equal if $D_1 = D_2$ and $D_1' = D_2'$, and simply-equivalent if there exist elements $x, y, a, c \in F$ such that $D_{1F(\sqrt{a})} = D'_{1F(\sqrt{c})} = 0$ and

$$D_2 = D_1 + (a, x^2 - acy^2),\ D_2' = D_1' + (c, x^2 - acy^2). \tag{$*$}$$

Notice that, since $(ac, x^2 - acy^2) = 0$, we have $D_1 + D_1' = D_2 + D_2'$ as soon as the equalities ($*$) hold. We say that two decompositions of $A$ are equivalent if they can be connected by a chain of decompositions in such a way that every two neighboring decompositions in this chain are simply-equivalent. It has been proved in ([13], Prop. 1) that any two decompositions of $A$ are equivalent to one another. Thus, it suffices to verify that if

$$A = (a, b) + (c, d) = (a, b(x^2 - acy^2)) + (c, d(x^2 - acy^2)),$$

then

$$(a, b(x^2 - acy^2), c, d(x^2 - acy^2)) - (a, b, c, d) \in (-1, -1)H^2(F) + (-1) \cup [A] \cup H^1(F).$$

Obviously, we may assume that $y \neq 0$. Put $t = xy^{-1}$. We have

$$\begin{aligned}
(a, b(x^2 - acy^2), c, d(x^2 - acy^2)) - (a, b, c, d) = (a, -bd, c, d(t^2 - ac)) - (a, b, c, d) = \\
(a, -bd, c, d) + (a, -bd, c, t^2 - ac) - (a, b, c, d) = (a, -bd, c, t^2 - ac) = \\
(a, -bd, a, t^2 - ac) = (-1, -bd, a, t^2 - ac) = (-1, -1, a, t^2 - ac) + (-1, bd, a, t^2 - ac) = \\
(-1, -1, a, t^2 - ac) + (-1, bd, a, t^2 - ac) + (-1, ac, d, t^2 - ac) = \\
(-1, -1, a, t^2 - ac) + (-1, t^2 - ac) \cup [A],
\end{aligned}$$

which proves what we need. (In this computation we used the equalities $(u, v_1v_2) = (u, v_1) + (u, v_2)$, $(u, -u) = 0$, for $u, v_1, v_2 \in F^*$, and the equalities $(a, bd) + (ac, d) = (a, b) + (c, d)$ and $(ac, t^2 - ac) = 0$.)

*Second proof.* We will give another proof of the above theorem, which was proposed by the referee. By Jacobson's theorem ([2]) there exists $\lambda \in F^*$ such that

$$\langle a_1, b_1, -a_1b_1, -c_1, -d_1, c_1d_1\rangle \simeq \lambda\langle a_2, b_2, -a_2b_2, -c_2, -d_2, c_2d_2\rangle.$$

Computing the 4th Stiefel-Whitney invariant of both sides, we easily get the equality

$$(-1, -1, a_1, b_1) + (a_1, b_1, c_1, d_1) = (-1, -\lambda, a_2, b_2) + (-1, \lambda, c_2, d_2) + (a_2, b_2, c_2, d_2),$$

which is equivalent to

$$(a_1, b_1, c_1, d_1) - (a_2, b_2, c_2, d_2) = (-1, -1) \cup [(a_1, b_1) + (a_2, b_2)] + (-1) \cup A \cup (\lambda). \quad \square$$

**Corollary 1.2.** *Let $F$ be a field such that $\langle\langle -1, -1\rangle\rangle_F = 0$ (for instance, any field of positive characteristic). Let $A$ be a biquaternion algebra over $F$ such that $(-1) \cup A = 0 \in H^3(F)$. Then the element $(a, b, c, d) \in H^4(F)$ does not depend on the decomposition $A = (a, b) + (c, d)$ of $A$ into a sum of two quaternion algebras. In particular, $(a, b, c, d) = 0$ if $\mathrm{ind}(A) \leq 2$.*

**Proof.** Obvious, in view of Theorem 1.1. $\square$

**Remark.** Under the hypothesis of Corollary 1.2 we will call the invariant $(a, b, c, d) \in H^4(F)$ for the biquaternion $A = (a, b) + (c, d)$ the second divided power operation $\gamma_2(A)$ on $A$, since when $\sqrt{-1} \in F^*$ this is really the case. However, the hypothesis of Corollary 1.2 does not permit to define $\gamma_2$ on the 2-divisible part of the group ${}_2\mathrm{Br}(F)$ by the same formula as in the case $\sqrt{-1} \in F^*$. For instance, consider the decomposition of the trivial algebra as $(a, b) + (a, c) + (a, bc) = 0$. Then, mimicking the computation of $\gamma_2$ of the left-hand part of the last equality in the case $\sqrt{-1} \in F^*$, we get

$$(a, b, a, c) + (a, b, a, bc) + (a, c, a, bc) = (a, b, a, c) + (a, bc, a, bc) =$$
$$(-1, a, b, c) + (-1, -1, a, bc) = (-1, a, b, c) \neq 0,$$

if $\sqrt{-1} \notin F^*$, and $a, b, c$ are indeterminates.

Let $F$ be a field such that $\langle\langle -1, -1 \rangle\rangle_F = 0$. Let $D_1, D_2$ be biquaternion algebras over $F$ such that $D_1 + D_2$ is a biquaternion algebra as well, and $(-1) \cup D_1 = (-1) \cup D_2 = 0$. It would be interesting to determine if $\gamma_2(D_1 + D_2) = \gamma_2(D_1) + \gamma_2(D_2) + D_1 \cup D_2$ just as in the case when $\sqrt{-1} \in F^*$. We do not know the answer in the general case, but at least it is positive in the following two cases, as we show in the following

**Corollary 1.3.** *Let $F$, $D_1, D_2$ be as above. Assume additionally that either*

*a) there exists a quaternion $Q$ such that $D_1 + Q, D_2 + Q$ are quaternions, or*
*b) $D_1$ and $D_2$ have a common biquadratic splitting field extension.*

*Then $\gamma_2(D_1 + D_2) = \gamma_2(D_1) + \gamma_2(D_2) + D_1 \cup D_2$.*

**Proof.** In both cases the pair $(D_1, D_2)$ can be parametrized. In case $a$) we have $D_1 = (a, b) + (c_1, d_1)$, $D_2 = (a, b) + (c_2, d_2)$ for some $a, b, c_i, d_i \in F^*$. Hence $D_1 + D_2 = (c_1, d_1) + (c_2, d_2)$, so $\gamma_2(D_1 + D_2) = (c_1, d_1, c_2, d_2)$. On the other hand,

$$\gamma_2(D_1) + \gamma_2(D_2) + D_1 \cup D_2 = (a, b, c_1, d_1) + (a, b, c_2, d_2)+$$
$$(a, b, a, b) + (a, b, c_1, d_1) + (a, b, c_2, d_2) + (c_1, d_1, c_2, d_2) = (c_1, d_1, c_2, d_2),$$

since $(a, b, a, b) = (-1, -1, a, b) = 0$.

In case $b$) there are some $a, b, c_1, d_1, c_2, d_2 \in F^*$ such that $D_1 = (a, c_1) + (b, d_1)$, $D_2 = (a, c_2) + (b, d_2)$ ([7], Prop. 5.2). Therefore,

$D_1 + D_2 = (a, c_1c_2) + (b, d_1d_2)$, and

$$\gamma_2(D_1 + D_2) = (a, b, c_1c_2, d_1d_2) = (a, b, c_1, d_1) + (a, b, c_1, d_2) + (a, b, c_2, d_1) + (a, b, c_2, d_2).$$

On the other hand,

$$\gamma_2(D_1)+\gamma_2(D_2)+D_1\cup D_2=(a,b,c_1,d_1)+(a,b,c_2,d_2)+$$
$$(a,c_1,a,c_2)+(a,c_1,b,d_2)+(b,d_1,a,c_2)+(b,d_1,b,d_2).$$

Therefore, the desired equality is equivalent to the equality $(a,c_1,a,c_2)=(b,d_1,b,d_2)$, i.e. to $(-1,a,c_1,c_2)=(-1,b,d_1,d_2)$. To prove the last equality note that by hypothesis we have $(-1,a,c_1)=(-1,b,d_1)$, which implies

$$\langle\langle -1,a,c_1\rangle\rangle=\langle\langle -1,b,d_1\rangle\rangle\in W(F). \tag{$*$}$$

Similarly,

$$\langle\langle -1,a,c_2\rangle\rangle=\langle\langle -1,b,d_2\rangle\rangle. \tag{$**$}$$

It follows from ($*$) and ($**$) that

$$\langle\langle -1,a,c_1c_2\rangle\rangle=\langle\langle -1,b,d_1d_2\rangle\rangle. \tag{$***$}$$

Moreover, $\langle\langle c_1\rangle\rangle+\langle\langle c_2\rangle\rangle-\langle\langle c_1c_2\rangle\rangle=\langle\langle c_1,c_2\rangle\rangle$, and similarly, $\langle\langle d_1\rangle\rangle+\langle\langle d_2\rangle\rangle-\langle\langle d_1d_2\rangle\rangle=\langle\langle d_1,d_2\rangle\rangle$. The equality ($*$) + ($**$) − ($***$) is just $\langle\langle -1,a,c_1,c_2\rangle\rangle=\langle\langle -1,b,d_1,d_2\rangle\rangle$, which implies $\{-1,a,c_1,c_2\}=\{-1,b,d_1,d_2\}\in K_4(F)/2$, and finally, $(-1,a,c_1,c_2)=(-1,b,d_1,d_2)$. □

**Remark.** Consider the case $\mathrm{ind}(D_1+D_2)=2$ in Corollary 1.3. Then condition $a$) implies condition $b$). Indeed, assume that

$$D_1=(a,b)+(c_1,d_1),\ D_2=(a,b)+(c_2,d_2).$$

Then, since $\mathrm{ind}(D_1+D_2)=2$, the quaternion algebras $(c_1,d_1)$ and $(c_2,d_2)$ have a common slot, say, $s$. Then $F(\sqrt{a},\sqrt{s})$ is a common splitting field for $D_1$ and $D_2$.

However, condition $b$) does not necessarily hold if $\mathrm{ind}(D_1+D_2)=2$ ([12], Prop. 3), which, in its turn, implies the same for condition $a$).

Finally in this section we give a necessary condition for the biquaternion algebra $A$ to be cyclic.

**Proposition 1.4.** *Under the hypothesis of Corollary 1.2 assume that $A$ is cyclic. Then $\gamma_2(A)=0$.*

**Proof.** Assume $A=a\cup\chi$ for some character $\chi$ of order 4, and $2\chi$ corresponds to an element $l\in F^*/F^{*2}$. Then $(a,l)=a\cup 2\chi=0$. On the other hand,

$$A_{F(\sqrt{a})}=(\sqrt{a})^2\cup\chi=\sqrt{a}\cup 2\chi=(\sqrt{a},l).$$

Hence $A_{F(\sqrt{a})} = (b, l)$ for some $b \in F^*$ ([14], Prop. 2.6). It follows that $A = (a, c) + (b, l)$ for some $c \in F^*$. Therefore, $\gamma_2(A) = (a, c, b, l) = 0$, since $(a, l) = 0$. □

**Question.** Assume that $\gamma_2(A) = 0$. Is $A$ cyclic? In [11] it was proved that this is the case if $\sqrt{-1} \in F^*$.

## 2. A degree four invariant for certain elements of ${}_4\mathrm{Br}(F)$

Let $F$ be a field, let $A \in {}_4\mathrm{Br}(F)$ be an algebra such that $\mathrm{ind}(2A) \leq 2$. In this section we introduce an invariant $p(A)$ with values in $H^4(F)$. To this end note that by ([7], Cor. 5.14) $2A = 2C$ for some cyclic element $C$ of degree at most 4, hence $A = C + \alpha$ for some $\alpha \in {}_2\mathrm{Br}(F)$.

**Theorem 2.1.** *The element $p(A) := (2A) \cup \alpha$ does not depend on the presentation $A = C + \alpha$, hence it is an invariant of $A$.*

**Proof.** We give the proof proposed by the referee. The original proof is longer and more complicated.

Assume that $A = C_1 + \alpha_1 = C_2 + \alpha_2$, where $C_i = \chi_i \cup z_i$ ($\chi_i \in H^1(G, \mathbb{Z}/4\mathbb{Z})$, $z_i \in F^*$), and $2\alpha_i = 0$ for $i = 1, 2$. By hypothesis, $2A = 2C_1 = 2C_2$ is a quaternion algebra. The common slot lemma yields $z \in F^*$ such that

$$2\chi_1 \cup z_1 = 2\chi_1 \cup z = 2\chi_2 \cup z = 2\chi_2 \cup z_2, \tag{1}$$

hence

$$C_1 - C_2 = \chi_1 \cup (zz_1) - \chi_2 \cup (zz_2) + (\chi_2 - \chi_1) \cup z.$$

From (1) it follows that $2\chi_1 \cup (zz_1) = 0$, hence $\chi_1 \cup (zz_1)$ is a biquaternion algebra. The character $2\chi_1$ corresponds to an element $\overline{a_1} \in F^*/F^{*2}$. Obviously, the element $\chi_1 \cup (zz_1)$ is split by $F(\sqrt{a_1}, \sqrt{zz_1})$, hence

$$\chi_1 \cup (zz_1) = (a_1, x_1) + (zz_1, y_1)$$

for some $x_1, y_1 \in F^*$.

Likewise, if the character $2\chi_2$ corresponds to an element $\overline{a_2} \in F^*/F^{*2}$, then there are $x_2, y_2, x_3, y_3 \in F^*$ such that

$$\chi_2 \cup (zz_2) = (a_2, x_2) + (zz_2, y_2), \qquad (\chi_2 - \chi_1) \cup z = (a_1 a_2, x_3) + (z, y_3).$$

This implies

$$\alpha_2 - \alpha_1 = C_1 - C_2 = (a_1, x_1) + (zz_1, y_1) + (a_2, x_2) + (zz_2, y_2) + (a_1 a_2, x_3) + (z, y_3). \tag{2}$$

Since $2A = 2C_1 = (a_1, z_1)$, and $a_1$ is a sum of two squares (because $a_1 = 2\chi_1$), it follows that $(2A) \cup (a_1) = (-1, a_1, z_1) = 0$. Similarly, $(2A) \cup (a_2) = 0$. Moreover, (1) yields $(a_1, zz_1) = (a_2, zz_2) = 0$, hence $(2A) \cup (zz_1) = (2A) \cup (zz_2) = 0$. Finally, since $2A = (a_1, z)$, we have

$$(2A) \cup (z) = (a_1, z, z) = (a_1, -1, z) = 0.$$

Therefore, $(2A) \cup (\alpha_2 - \alpha_1) = 0$ in view of (2), which completes the proof. □

If $\sqrt{-1} \in F^*$ Theorem 2.1 can be a bit strengthened.

**Proposition 2.2.** *Under hypothesis of Theorem 2.1 assume additionally that $\sqrt{-1} \in F^*$, and $A = D + \beta$, where $\operatorname{ind} D \leq 4$, $\beta \in {}_2\mathrm{Br}(F)$. Then $p(A) = (2A) \cup \beta$. In general $p(A)$ is not zero.*

**Proof.** First consider the following particular case.

**Lemma 2.3.** *Let $F$ be a field, $\sqrt{-1} \in F^*$, $Q$ a quaternion algebra over $F$. Let the element $D \in {}_4\mathrm{Br}(F)$ be such that $\operatorname{ind} D = \operatorname{ind}(D + Q) = 4$. Then $(2D) \cup Q = 0 \in H^4(F)$.*

**Proof.** By Risman's theorem ([10]) there exists a quartic extension $L/F$ such that $D_L = Q_L = 0$. We may assume that $F$ has no proper odd degree extension. Then $L = F(\sqrt{x + y\sqrt{a}})$ for some $x, y, a \in F$, hence we have $D_{F(\sqrt{a})} = (x + y\sqrt{a}, p + q\sqrt{a})$ and $Q_{F(\sqrt{a})} = (x + y\sqrt{a}, p_1 + q_1\sqrt{a})$ for some $p, q, p_1, q_1 \in F$. By the projection formula we get

$$\begin{aligned}(2D) \cup Q = N_{F(\sqrt{a})/F}(D_{F(\sqrt{a})}) \cup Q = N_{F(\sqrt{a})/F}(D_{F(\sqrt{a})} \cup Q_{F(\sqrt{a})}) = \\ N_{F(\sqrt{a})/F}(x + y\sqrt{a}, x + y\sqrt{a}, p + q\sqrt{a}, p_1 + q_1\sqrt{a}) = 0,\end{aligned}$$

since $(x + y\sqrt{a}, x + y\sqrt{a}) = (-1, x + y\sqrt{a}) = 0$. □

We return to the proof of Proposition 2.2. By ([8], Th. 1.3) we have $D = C + Q$, where $C$ is cyclic of degree at most 4, and $Q$ is a quaternion algebra. In particular, $A = C + \beta + Q$.

By Lemma 2.3 $(2A) \cup Q = (2D) \cup Q = 0$, hence

$$p(A) = (2A) \cup (\beta + Q) = (2A) \cup \beta.$$

To give an example of an algebra $A$ with $p(A) \neq 0$, consider any field $k$ with $\sqrt{-1} \in k^*$, and put $F = k(u, v, x, y)$, where $u, v, x, y$ are indeterminates. Then $p(A) = (u, v, x, y) \neq 0$, where $A = (u, v)_4 + (x, y)$. This completes the proof of Proposition 2.2. □

**Corollary 2.4.** *Under hypothesis of Lemma 2.3 the division algebra associated with the element $2D + Q$ is cyclic.*

**Proof.** Let $\pi_D$ and $\pi_Q$ be the Pfister forms associated with the quaternion algebras $2D$ and $Q$. Following the proof of Lemma 2.3, it is easy to see that $\pi_D \otimes \pi_Q = 0$. Now the result follows from ([11], Th. 3). $\square$

**Remark.** One cannot expect an invariant similar to $p$ for elements $A = \widetilde{D} + \alpha$, where $\widetilde{D} \in {}_4\mathrm{Br}(F)$, $\operatorname{ind} \widetilde{D} \leq 8$, and $\alpha \in {}_2\mathrm{Br}(F)$. Indeed, let, for instance, $F = k(x, y)$, where $x, y$ are indeterminates, $\alpha = (x, y)$, $D \in {}_4\mathrm{Br}(k)$, $\operatorname{ind} D = 4$, $2D \neq 0$, and $\widetilde{D} = D + \alpha$. Then, clearly, $\operatorname{ind} \widetilde{D} = 8$, and $(2D) \cup \alpha \neq 0 = (2\widetilde{D}) \cup 0$.

Lemma 2.3 remains valid even if $\sqrt{-1} \notin F^*$. However, we omit the proof of this result, since it is a bit technical, and we do not need it in the sequel.

Next we compare the invariants $p(A) = (2D) \cup \alpha \in H^4(F)$ and the second divided power operation $\gamma_2(A) \in K_4(F)/4$, provided $\xi_8 \in F^*$. To do this we need the following

**Lemma 2.5.** *Let $F$ be a field, $\xi_8 \in F^*$, $A \in \mathrm{Br}(F)$, $\operatorname{ind} A = 4$.*

*Let further $\gamma_2 : K_2(F)/4 \to K_4(F)/4$ be the second divided power operation. Then $\gamma_2(A) = 0$.*

**Proof.** Let $\varphi : H^4(F, \mathbb{Z}/2\mathbb{Z}) \to H^4(F, \mathbb{Z}/4\mathbb{Z})$ be the homomorphism induced by the natural embedding $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$. By ([8], Th. 1.3)

$$A = (a, b)_4 + (c, d) = (a, b)_4 + (c, d^2)_4$$

for some $a, b, c, d \in F^*$, hence $\gamma_2(A) = (a, b, c, d^2)_4 = \varphi(a, b, c, d)$. On the other hand, $(a, b, c, d) = 0$ by Lemma 2.3. $\square$

In fact Lemma 2.5 is a particular case of the following

**Proposition 2.6.** *Let $F$ be a field, $\xi_8 \in F^*$, and $A = D + \alpha$, where $D \in {}_4\mathrm{Br}(F)$, $\operatorname{ind} D \leq 4$, $\alpha \in {}_2\mathrm{Br}(F)$. Then $\varphi(p(A)) = \gamma_2(A)$.*

**Proof.** Let $D = \sum\limits_i (u_i, v_i)_4$, $\alpha = \sum\limits_i (a_i, b_i)$. We have

$$\gamma_2(A) = \gamma_2(D) + \gamma_2(\alpha) + (\sum_i (u_i, v_i)_4) \cup (\sum_j (a_j, b_j^2)_4) = \sum_{i,j} (u_i, v_i, a_j, b_j^2)_4,$$

since $\gamma_2(D) = 0$ by Lemma 2.5, and

$$\gamma_2(\alpha) = \sum_{i,j} (a_i, b_i^2, a_j, b_j^2)_4 = 4 \sum_{i,j} (a_i, b_i, a_j, b_j)_4 = 0.$$

On the other hand, $p(A) = (2D) \cup \alpha = \sum_{i,j} (u_i, v_i, a_j, b_j)$. The equality $\varphi(u_i, v_i, a_j, b_j) = (u_i, v_i, a_j, b_j^2)_4$ implies the claim. $\square$

**Remark.** Assume that $A = D + \alpha \in {}_4\mathrm{Br}(F)$, where $\operatorname{ind} D \leq 4$, $\alpha \in {}_2\mathrm{Br}(F)$. We do not know if necessarily $p(A) = (2D) \cup \alpha$ if $\sqrt{-1} \notin F^*$. This would be true if there were a chain $C = D_0, D_1, \ldots, D_n = D$ with a cyclic $C$, and $D_i \in {}_4\mathrm{Br}(F)$, $\operatorname{ind} D_i \leq 4$, $\operatorname{ind}(D_i - D_{i-1}) = 2$. Indeed, in this case we have for $\beta = D - C$

$$2D \cup \beta = (2D) \cup (D - C) = \sum_{i=1}^{n} (2D_{i-1}) \cup (D_i - D_{i-1}) = 0$$

in view of the remark after Corollary 2.4. Hence

$$(2D) \cup \alpha = (2C) \cup (\alpha + \beta) = p(A).$$

However, it seems to be unknown whether such a chain always exists.

Note also that the invariant $\gamma_2$ for biquaternions from Corollary 1.2 can be expressed via $p$ in the following way. Let $A = (a, b) + (c, d)$. Assume that $\langle\langle -1, -1 \rangle\rangle_F = 0$ and $(-1) \cup A = 0$. There exists a decomposition $A = (a_1, b_1) + (c_1, d_1)$, where $(-1, a_1, b_1) = (-1, c_1, d_1) = 0$ ([5], Lemma 1.3). Hence $(a_1, b_1) = 2C$ for some cyclic $C$, $\operatorname{ind} C \leq 4$, and

$$\gamma_2(A) = (a_1, b_1, c_1, d_1) = p(C + (c_1, d_1)).$$

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