



ELSEVIER

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra



# On the weak Lefschetz property for almost complete intersections generated by uniform powers of general linear forms

Rosa M. Miró-Roig<sup>a</sup>, Quang Hoa Tran<sup>b,\*</sup>

<sup>a</sup> *Universitat de Barcelona, Departament de Matemàtiques i Informàtica, Gran Via de les Corts Catalanes 585, 08007 Barcelona, Spain*

<sup>b</sup> *University of Education, Hue University, 34 Le Loi St., Hue City, Viet Nam*

## ARTICLE INFO

*Article history:*

Received 22 July 2019

Available online 28 January 2020

Communicated by Kazuhiko Kurano

*MSC:*

primary 14C20, 13E10

secondary 13C40, 13C13, 13D02,

13D40

*Keywords:*

Almost complete intersections

Artinian algebras

General linear forms

Linear systems of general points

Powers of linear forms

Weak Lefschetz property

## ABSTRACT

In [10], Conjecture 6.6, Migliore, the first author, and Nagel conjectured that, for all  $n \geq 4$ , the artinian ideal  $I = (L_0^d, \dots, L_{2n+1}^d) \subset R = k[x_0, \dots, x_{2n}]$  generated by the  $d$ -th powers of  $2n + 2$  general linear forms fails to have the weak Lefschetz property if and only if  $d > 1$ . This paper is entirely devoted to prove partially this conjecture. More precisely, we prove that  $R/I$  fails to have the weak Lefschetz property, provided  $4 \leq n \leq 8$ ,  $d \geq 4$  or  $d = 2r$ ,  $1 \leq r \leq 8$ ,  $4 \leq n \leq 2r(r + 2) - 1$ .

© 2020 Elsevier Inc. All rights reserved.

## 1. Introduction

Ideals generated by powers of linear forms have attracted great deal of attention recently. For instance, their Hilbert function has been the focus of the papers [2,8,16];

\* Corresponding author.

E-mail addresses: [miro@ub.edu](mailto:miro@ub.edu) (R.M. Miró-Roig), [tranquanghoa@hueuni.edu.vn](mailto:tranquanghoa@hueuni.edu.vn) (Q.H. Tran).

and the presence or failure of the weak Lefschetz property has been deeply studied in [8,10,12–14], among others.

Let  $k$  be a field of characteristic zero and  $R = k[x_0, \dots, x_n]$  be the standard graded polynomial ring over  $k$  in  $n + 1$  variables. A graded artinian  $k$ -algebra  $A := R/I$  is said to have the *weak Lefschetz property* (WLP for short) if there is a linear form  $\ell \in [A]_1$  such that the multiplication

$$\times \ell : [A]_i \longrightarrow [A]_{i+1}$$

has maximal rank for all  $i$ , i.e.,  $\times \ell$  is either injective or surjective, for all  $i$ . On the contrary, we say that  $A$  *fails to have the WLP* if there is an integer  $i$  such that the above multiplication does not have maximal rank for any linear form  $\ell$ . There has been a long series of papers determining classes of algebras holding/failing the WLP but much more work remains to be done.

The first result in this direction is due to Stanley [15] and Watanabe [17] and it asserts that the WLP holds for *any* artinian complete intersection ideal  $I$  generated by powers of linear forms. In fact, they showed that there is a linear form  $\ell \in [A]_1$  such that the multiplication

$$\times \ell^s : [A]_i \longrightarrow [A]_{i+s}$$

has maximal rank for all  $i, s$ . When this property holds, the algebra is said to have the *strong Lefschetz property* (briefly SLP). In [14], Schenck and Seceleanu gave the nice result that *any* artinian ideal  $I \subset R = k[x, y, z]$  generated by powers of linear forms has the WLP. Moreover, when these linear forms are general, the SLP of  $R/I$  has also been studied, in particular, the multiplication by the square  $\ell^2$  of a general linear form  $\ell$  induces a homomorphism of maximal rank in any graded component of  $R/I$ , see [1,11]. However, Migliore, the first author, and Nagel showed by examples that in 4 variables, an ideal generated by the  $d$ -th powers of five general linear forms fails to have the WLP for  $d = 3, \dots, 12$  [10]. Therefore, it is natural to ask when the WLP holds for artinian ideals  $I \subset k[x_0, \dots, x_n]$  generated by powers of  $\geq n + 2$  general linear forms. In [10], Migliore, the first author, and Nagel studied this question where the ideal is an almost complete intersection and they also proposed the following conjecture in order to complete this investigation.

**Conjecture 1.1.** [10, Conjecture 6.6] *Let  $R = k[x_0, \dots, x_{2n}]$  be the polynomial ring over a field of characteristic zero. Consider an artinian ideal  $I = (L_0^d, \dots, L_{2n+1}^d) \subset R$  generated by the  $d$ -th powers of general linear forms. If  $n \geq 4$ , then the ring  $R/I$  fails to have the WLP if and only if  $d > 1$ . Furthermore, if  $n = 3$ , then  $R/I$  fails to have the WLP when  $d = 3$ .*

The first author has shown that  $R/I$  fails to have the WLP when  $d = 2$  [12] and in the recent paper [13], Nagel and Trok have established Conjecture 1.1 for  $n \gg 0$  and

$d \gg 0$ . The last part of the conjecture was proved by Di Gennaro, Iardi, and Vallès in [3, Proposition 5.5]. Unfortunately, there was a gap in their proof. However, it was corrected in [4] and then in [9], the last part of Conjecture 1.1 was proved by Iardi and Vallès. The goal of this note is to solve partially the conjecture. More precisely, we prove the following (see Corollaries 3.3–3.10, Theorem 4.1 and Remark 4.2).

**Theorem.** *Let  $R = k[x_0, \dots, x_{2n}]$  be the polynomial ring over a field of characteristic zero and consider an artinian ideal  $I = (L_0^d, \dots, L_{2n+1}^d) \subset R$  generated by the  $d$ -th powers of general linear forms.*

- (1) *If  $d = 2r$ ,  $2 \leq r \leq 8$  and  $4 \leq n \leq 2r(r + 2) - 1$ , then  $R/I$  fails to have the WLP.*
- (2) *If  $4 \leq n \leq 8$  and  $d \geq 4$ , then  $R/I$  fails to have the WLP.*

Therefore, Theorem answers partially Conjecture 1.1 for  $4 \leq n \leq 8$ , missing only the case  $d = 3$ , since the case  $d = 2$  is shown by the first author [12]. Our approach is based on the connection between computing the dimension of  $R/(I, \ell)$ , where  $\ell$  is a general linear form and the dimension of linear system of fat points. More precisely, we prove the following result (see Theorem 3.1).

**Theorem.** *If  $\ell$  is a general linear form and  $j = \lfloor \frac{(2n^2-1)(d-1)}{2n-1} \rfloor$ , then*

$$\dim_k[R/(I, \ell)]_j = \dim_k \mathfrak{L}_{2n-1}(j; (j + 1 - d)^{2n+2})$$

$$= \begin{cases} \dim_k \mathfrak{L}_{2n-1}(e; 0^{2n+2}) & \text{if } d = (2n - 1)e + 1 \\ \dim_k \mathfrak{L}_{2n-1}(e + n - r + 1; (n - r)^{2n+2}) & \text{if } d = (2n - 1)e + 2r \\ \dim_k \mathfrak{L}_{2n-1}(e + 2n - r + 1; (2n - r - 1)^{2n+2}) & \text{if } d = (2n - 1)e + 2r + 1 \end{cases}$$

where  $e, r$  are non-negative integers such that  $1 \leq r \leq n - 1$ .

## 2. Preparatory results

Throughout this paper  $R$  denotes a polynomial ring  $k[x_0, \dots, x_n]$  over a field  $k$  of characteristic zero, with its standard grading where  $\deg(x_i) = 1$ . If  $I \subset R$  is a homogeneous ideal, then the  $k$ -algebra  $A = \bigoplus_{j \geq 0} [A]_j$  is standard graded. Its Hilbert function is a map  $h_A : \mathbb{N} \rightarrow \mathbb{N}$ ,  $h_A(j) = \dim_k[A]_j$ .

For any artinian ideal  $I \subset R$  and a general linear form  $\ell \in R$ , the exact sequence

$$[R/I]_{j-1} \rightarrow [R/I]_j \rightarrow [R/(I, \ell)]_j \rightarrow 0$$

gives, in particular, that the multiplication by  $\ell$  will fail to have maximal rank exactly when

$$\dim_k[R/(I, \ell)]_j \neq \max\{\dim_k[R/I]_j - \dim_k[R/I]_{j-1}, 0\}. \tag{2.1}$$

In this case, we will say that  $R/I$  fails to have the WLP in degree  $j - 1$ .

We first recall a result of Emsalem and Iarrobino giving a duality between powers of linear forms and ideals of fat points in  $\mathbb{P}^n$ . We quote it in the form that we need.

**Lemma 2.1.** *[6, Theorem I] Let  $\mathfrak{p}_1, \dots, \mathfrak{p}_s$  be the ideals of  $s$  distinct points in  $\mathbb{P}^n$  that are dual to the linear forms  $\ell_1, \dots, \ell_s \in R$  and choose the positive integers  $a_1, \dots, a_s$ . Then, for each integer  $j \geq -1 + \max\{a_1, \dots, a_s\}$ ,*

$$\dim_k [R/(\ell_1^{a_1}, \dots, \ell_s^{a_s})]_j = \dim_k \left[ \bigcap_{a_i \leq j} \mathfrak{p}_i^{j+1-a_i} \right]_j.$$

If the points defined by the ideals  $\mathfrak{p}_i$  are general points, then the dimension of the linear system  $[\mathfrak{p}_1^{b_1} \cap \dots \cap \mathfrak{p}_s^{b_s}]_j \subset R_j$  depends only on the numbers  $n, j, b_1, \dots, b_s$ . In order to simplify notation, in this case we denote by

$$\mathfrak{L}_n(j; b_1, \dots, b_s)$$

the linear system  $[\mathfrak{p}_1^{b_1} \cap \dots \cap \mathfrak{p}_s^{b_s}]_j \subset R_j$ . We use superscripts to indicate repeated entries. For example,  $\mathfrak{L}_4(j; 2^4, 4^2) = \mathfrak{L}_4(j; 2, 2, 2, 2, 4, 4)$ . Notice that, for every linear system  $\mathfrak{L}_n(j; b_1, \dots, b_s)$ , one has

$$\dim_k \mathfrak{L}_n(j; b_1, \dots, b_s) \geq \max \left\{ 0, \binom{n+j}{n} - \sum_{i=1}^s \binom{n+b_i-1}{n} \right\}.$$

To study the WLP, the following is useful to compute the left-hand side of (2.1).

**Lemma 2.2.** *[10, Proposition 3.4] Let  $(\ell_1^{a_1}, \dots, \ell_s^{a_s})$  be an ideal of  $R$  generated by powers of  $s$  general linear forms, and let  $\ell$  be a general linear form. Then, for each integer  $j \geq -1 + \max\{a_1, \dots, a_s\}$ ,*

$$\dim_k [R/(\ell_1^{a_1}, \dots, \ell_s^{a_s}, \ell)]_j = \dim_k \mathfrak{L}_{n-1}(j; j+1-a_1, \dots, j+1-a_s).$$

Using Cremona transformations, one can relate two different linear systems. This is often stated only for general points.

**Lemma 2.3.** *[5, Theorem 3] Let  $s > n \geq 2$  and  $j, b_1, \dots, b_s$  be non-negative integers, with  $b_1 \geq \dots \geq b_s$ . Set  $t = (n-1)j - (b_1 + \dots + b_{n+1})$ . If  $b_i + t \geq 0$  for all  $i = 1, \dots, n+1$ , then*

$$\dim_k \mathfrak{L}_n(j; b_1, \dots, b_s) = \dim_k \mathfrak{L}_n(j+t; b_1+t, \dots, b_{n+1}+t, b_{n+2}, \dots, b_s).$$

In this note, we are interested in certain almost complete intersections. Then one can compute the right-hand side of (2.1). For any integer  $m$ , we denote

$$[m]_+ = \max\{m, 0\}.$$

**Lemma 2.4.** [10, Lemma 3.7] *Let  $I = (L_0^{a_0}, \dots, L_{n+1}^{a_{n+1}}) \subset R$  be an almost complete intersection generated by powers of  $n + 2$  general linear forms. Set  $A = R/(L_0^{a_0}, \dots, L_n^{a_n})$ . Then, for each integer  $j$ ,*

$$\dim_k[R/I]_j - \dim_k[R/I]_{j-1} = [h_A(j) - h_A(j - a_{n+1})]_+ - [h_A(j - 1) - h_A(j - a_{n+1} - 1)]_+.$$

Furthermore, if  $j \leq \frac{1}{2}a_{n+1} + \frac{1}{2} \sum_{i=0}^n (a_i - 1)$ , then the formula simplifies to

$$\dim_k[R/I]_j - \dim_k[R/I]_{j-1} = [h_A(j) - h_A(j - 1)] - [h_A(j - a_{n+1}) - h_A(j - a_{n+1} - 1)].$$

### 3. Almost uniform powers of general linear forms

Throughout this section, we always denote  $R = k[x_0, \dots, x_{2n}]$  and consider an artinian ideal  $I = (L_0^d, \dots, L_{2n+1}^d)$  of  $R$  generated by the  $d$ -th powers of general linear forms and fix  $j = \lfloor \frac{(2n^2-1)(d-1)}{2n-1} \rfloor$ .

**Theorem 3.1.** *If  $\ell$  is a general linear form, then*

$$\begin{aligned} \dim_k[R/(I, \ell)]_j &= \dim_k \mathfrak{L}_{2n-1}(j; (j + 1 - d)^{2n+2}) \\ &= \begin{cases} \dim_k \mathfrak{L}_{2n-1}(e; 0^{2n+2}) & \text{if } d = (2n - 1)e + 1 \\ \dim_k \mathfrak{L}_{2n-1}(e + n - r + 1; (n - r)^{2n+2}) & \text{if } d = (2n - 1)e + 2r \\ \dim_k \mathfrak{L}_{2n-1}(e + 2n - r + 1; (2n - r - 1)^{2n+2}) & \text{if } d = (2n - 1)e + 2r + 1 \end{cases} \end{aligned}$$

where  $e, r$  are non-negative integers such that  $1 \leq r \leq n - 1$  and

$$\dim_k[R/I]_j - \dim_k[R/I]_{j-1} = \sum_{k=0}^n (-1)^k \binom{2n+2}{k} \binom{2n-1+j-kd}{2n-1}.$$

**Proof.** It follows from Lemma 2.2 that

$$D := \dim_k[R/(I, \ell)]_j = \dim_k \mathfrak{L}_{2n-1}(j; (j + 1 - d)^{2n+2}).$$

Set

$$t = (2n - 2)j - 2n(j + 1 - d) = -2j + 2n(d - 1).$$

As  $j = \lfloor \frac{(2n^2-1)(d-1)}{2n-1} \rfloor$ , we get

$$j + 1 - d + t = -j + (2n - 1)(d - 1) \geq \frac{2(n - 1)^2(d - 1)}{2n - 1} \geq 0.$$

Using Lemma 2.3 ( $n + 1$ ) times, in each step the Cremona transformation changes the multiplicities of linear system  $\mathfrak{L}_{2n-1}(j; (j + 1 - d)^{2n+2})$  by  $t$ , we obtain

$$\begin{aligned}
 D &= \dim_k \mathfrak{L}_{2n-1}(j; (j+1-d)^{2n+2}) \\
 &= \dim_k \mathfrak{L}_{2n-1}(-j+2n(d-1); (j+1-d)^2, (-j+(2n-1)(d-1))^{2n}) \\
 &= \dim_k \mathfrak{L}_{2n-1}(-3j+4n(d-1); (-j+(2n-1)(d-1))^4, \\
 &\quad (-3j+(4n-1)(d-1))^{2n-2}) \\
 &\quad \vdots \\
 &= \dim_k \mathfrak{L}_{2n-1}(-(2n+1)j+2n(n+1)(d-1); (-(2n-1)j+(2n^2-1)(d-1))^{2n+2}).
 \end{aligned}$$

These computations are correct and has a chance of resulting in a non-empty linear system if

$$-(2n+1)j+2n(n+1)(d-1) > -(2n-1)j+(2n^2-1)(d-1) \geq 0,$$

which is true since  $j = \lfloor \frac{(2n^2-1)(d-1)}{2n-1} \rfloor$ . Thus

$$D = \dim_k \mathfrak{L}_{2n-1}(2n(n+1)(d-1) - (2n+1)j; ((2n^2-1)(d-1) - (2n-1)j)^{2n+2}). \tag{3.1}$$

Now we consider three cases:

**Case 1:**  $d = (2n-1)e + 1$ , hence  $j = (2n^2-1)e$ . By (3.1) and a simple computation shows that

$$D = \dim_k \mathfrak{L}_{2n-1}(e; 0^{2n+2}).$$

**Case 2:**  $d = (2n-1)e + 2r$ ,  $1 \leq r \leq n-1$ . A straightforward computation shows that

$$j = (2n^2-1)e + 2nr + r - n - 1.$$

Therefore, by (3.1), we obtain

$$D = \dim_k \mathfrak{L}_{2n-1}(e+n-r+1; (n-r)^{2n+2}).$$

**Case 3:**  $d = (2n-1)e + 2r + 1$ ,  $1 \leq r \leq n-1$ . It is easy to show that

$$j = (2n^2-1)e + 2nr + r - 1.$$

By (3.1) we get that

$$D = \dim_k \mathfrak{L}_{2n-1}(e+2n-r+1; (2n-r-1)^{2n+2}).$$

Finally, let  $A = R/(L_0^d, \dots, L_{2n}^d)$ , hence  $A$  is a complete intersection and it has the SLP (see, e.g., [15] or [17]), one has

$$\dim_k[R/I]_j - \dim_k[R/I]_{j-1} = [h_A(j) - h_A(j - 1)] - [h_A(j - d) - h_A(j - d - 1)],$$

by Lemma 2.4 since  $j \leq \frac{d}{2} + \frac{(2n+1)(d-1)}{2}$ . Resolving  $A$  over  $R$  using the Koszul resolution, we get for the Hilbert function of  $A$

$$h_A(j) = \sum_{k=0}^{2n+1} (-1)^k \binom{2n+1}{k} \binom{2n+j-kd}{2n}.$$

As  $j = \lfloor \frac{(2n^2-1)(d-1)}{2n-1} \rfloor$ , hence  $j - kd < 0$  if  $k \geq n + 1$ . It follows that

$$h_A(j) = \sum_{k=0}^n (-1)^k \binom{2n+1}{k} \binom{2n+j-kd}{2n}.$$

A straightforward computation gives

$$\dim_k[R/I]_j - \dim_k[R/I]_{j-1} = \sum_{k=0}^n (-1)^k \binom{2n+2}{k} \binom{2n-1+j-kd}{2n-1}. \quad \square$$

**Proposition 3.2.** *Assume that  $d = (2n - 1)e + 2r$ ,  $e$  and  $r$  are non-negative integers such that  $1 \leq r \leq n$ . If  $n \leq 2r(r + 2) - 1$  then*

$$\dim_k[R/(I, \ell)]_j > 0,$$

where  $\ell$  is a general linear form in  $R$ .

**Proof.** As  $d = (2n - 1)e + 2r$ ,  $1 \leq r \leq n$ , by Theorem 3.1 we get that

$$\dim_k[R/(I, \ell)]_j = \dim_k \mathfrak{L}_{2n-1}(e + n - r + 1; (n - r)^{2n+2}),$$

where  $e$  is a non-negative integer. It is enough to show that

$$\dim_k \mathfrak{L}_{2n-1}(n - r + 1; (n - r)^{2n+2}) > 0.$$

Lemma 2.1 shows that

$$\dim_k \mathfrak{L}_{2n-1}(n - r + 1; (n - r)^{2n+2}) = \dim_k \left[ k[x_0, \dots, x_{2n-1}] / (\ell_0^2, \dots, \ell_{2n+1}^2) \right]_{n-r+1},$$

where  $\ell_0, \dots, \ell_{2n+1}$  are general linear forms in  $k[x_0, \dots, x_{2n-1}]$ . Setting

$$P = k[x_0, \dots, x_{2n-1}] / (\ell_0^2, \dots, \ell_{2n}^2)$$

then, by [12, Proposition 3.4], for every  $0 \leq t \leq n$ ,

$$\dim_k [P]_t = \binom{2n}{t} - \binom{2n}{t-2}.$$

It follows that

$$\begin{aligned} \dim_k \mathfrak{L}_{2n-1}(n-r+1; (n-r)^{2n+2}) &\geq \dim_k [P]_{n-r+1} - \dim_k [P]_{n-r-1} \\ &= \binom{2n}{n-r+1} - 2\binom{2n}{n-r-1} + \binom{2n}{n-r-3}. \end{aligned}$$

We have

$$\begin{aligned} &\binom{2n}{n-r+1} - 2\binom{2n}{n-r-1} + \binom{2n}{n-r-3} > 0 \\ &\Leftrightarrow \frac{(2n)!}{(n-r+1)!(n+r-1)!} - \frac{(2n)!}{(n-r-1)!(n+r+1)!} \\ &> \frac{(2n)!}{(n-r-1)!(n+r+1)!} - \frac{(2n)!}{(n-r-3)!(n+r+3)!} \\ &\Leftrightarrow \frac{2r(2n+1)}{(n-r+1)!(n+r+1)!} > \frac{2(r+2)(2n+1)}{(n-r-1)!(n+r+3)!} \\ &\Leftrightarrow \frac{r}{(n-r)(n-r+1)} > \frac{r+2}{(n+r+2)(n+r+3)} \\ &\Leftrightarrow n^2 - (2r^2 + 4r - 1)n - (2r^2 + 4r) < 0 \\ &\Leftrightarrow -1 < n < 2r(r+2). \end{aligned}$$

Thus  $\dim_k [R/(I, \ell)]_j$  for any  $r \leq n \leq 2r(r+2) - 1$ .  $\square$

**Corollary 3.3.** *If  $2 \leq n \leq 15$  and  $d = 4$ , then  $R/I$  fails to have the WLP.*

**Proof.** In this case, we have  $j = 3n + 1$ . By Proposition 3.2, for any  $2 \leq n \leq 15$ ,

$$\dim_k [R/(I, \ell)]_{3n+1} > 0,$$

where  $\ell$  is a general linear form in  $R$ .

On other hand, by Theorem 3.1, we have

$$\dim_k [R/I]_{3n+1} - \dim_k [R/I]_{3n} = \sum_{k=0}^n (-1)^k \binom{2n+2}{k} \binom{5n-4k}{2n-1}.$$

Using Macaulay2 [7], we see that for any  $2 \leq n \leq 15$ ,

$$\dim_k [R/I]_{3n+1} - \dim_k [R/I]_{3n} \leq 0.$$

It follows that  $R/I$  fails to have the WLP since the surjectivity does not hold.  $\square$

**Remark 3.4.** Set

$$S_n = \sum_{k=0}^n (-1)^k \binom{2n+2}{k} \binom{5n-4k}{2n-1}, n \geq 2.$$

Examples suggest that the sequence  $(S_n)_{n \geq 2}$  of integers is strictly decreasing with  $S_2 = 0$ , and so all these are non-positive.

**Corollary 3.5.** *If  $3 \leq n \leq 29$  and  $d = 6$ , then  $R/I$  fails to have the WLP.*

**Proof.** In this case, we have  $j = 5n + 2$ . Let  $\ell$  be a general linear form in  $R$ . By Proposition 3.2 we get that

$$\dim_k [R/(I, \ell)]_{5n+2} > 0,$$

for any  $3 \leq n \leq 29$ .

On other hand, by Theorem 3.1, we have

$$\dim_k [R/I]_{5n+2} - \dim_k [R/I]_{5n+1} = \sum_{k=0}^n (-1)^k \binom{2n+2}{k} \binom{7n+1-6k}{2n-1}.$$

Using Macaulay2 [7], we see that for any  $3 \leq n \leq 29$ ,

$$\dim_k [R/I]_{5n+2} - \dim_k [R/I]_{5n+1} < 0,$$

which shows that  $R/I$  fails to have the WLP since the surjectivity does not hold.  $\square$

**Corollary 3.6.** *If  $4 \leq n \leq 47$  and  $d = 8$ , then  $R/I$  fails to have the WLP.*

**Proof.** Let  $\ell$  be a general linear form. In this case, we have  $j = 7n + 3$ . By Proposition 3.2, for any  $4 \leq n \leq 47$ ,

$$\dim_k [R/(I, \ell)]_{7n+3} > 0.$$

On other hand, by Theorem 3.1, we have

$$\dim_k [R/I]_{7n+3} - \dim_k [R/I]_{7n+2} = \sum_{k=0}^n (-1)^k \binom{2n+2}{k} \binom{9n+2-8k}{2n-1}.$$

Using Macaulay2 [7], we see that for any  $4 \leq n \leq 47$ ,

$$\dim_k [R/I]_{7n+3} - \dim_k [R/I]_{7n+2} < 0,$$

which shows that  $R/I$  fails to have the WLP since the surjectivity does not hold.  $\square$

**Corollary 3.7.** *If  $5 \leq n \leq 69$  and  $d = 10$ , then  $R/I$  fails to have the WLP.*

**Proof.** Let  $\ell$  be a general linear form in  $R$ . In this case, one has  $j = 9n + 4$ . By Proposition 3.2, for any  $5 \leq n \leq 69$ ,

$$\dim_k[R/(I, \ell)]_{9n+4} > 0.$$

On other hand, by Theorem 3.1, we have

$$\dim_k[R/I]_{9n+4} - \dim_k[R/I]_{9n+3} = \sum_{k=0}^n (-1)^k \binom{2n+2}{k} \binom{11n+3-10k}{2n-1}.$$

Using Macaulay2 [7], we see that for any  $5 \leq n \leq 69$ ,

$$\dim_k[R/I]_{9n+4} - \dim_k[R/I]_{9n+3} < 0,$$

which shows that  $R/I$  fails to have the WLP since the surjectivity does not hold.  $\square$

**Corollary 3.8.** *If  $6 \leq n \leq 95$  and  $d = 12$ , then  $R/I$  fails to have the WLP.*

**Proof.** In this case, one has  $j = 11n + 5$ . For any  $6 \leq n \leq 95$ , one has

$$\dim_k[R/(I, \ell)]_{11n+5} > 0,$$

by Proposition 3.2, where  $\ell$  is a general linear form in  $R$ .

On other hand, by Theorem 3.1, we have

$$\dim_k[R/I]_{11n+5} - \dim_k[R/I]_{11n+4} = \sum_{k=0}^n (-1)^k \binom{2n+2}{k} \binom{13n+4-12k}{2n-1}.$$

Using Macaulay2 [7], we see that for any  $6 \leq n \leq 95$ ,

$$\dim_k[R/I]_{11n+5} - \dim_k[R/I]_{11n+4} < 0,$$

which shows that  $R/I$  fails to have the WLP since the surjectivity does not hold.  $\square$

**Corollary 3.9.** *If  $7 \leq n \leq 125$  and  $d = 14$ , then  $R/I$  fails to have the WLP.*

**Proof.** Let  $\ell$  be a general linear form in  $R$ . In this case, one has  $j = 13n + 6$ . Proposition 3.2 follows that

$$\dim_k[R/(I, \ell)]_{13n+6} > 0,$$

for any  $7 \leq n \leq 125$ .

On other hand, by Theorem 3.1, we have

$$\dim_k[R/I]_{13n+6} - \dim_k[R/I]_{13n+5} = \sum_{k=0}^n (-1)^k \binom{2n+2}{k} \binom{15n+5-14k}{2n-1}.$$

Using Macaulay2 [7], we see that for any  $7 \leq n \leq 125$ ,

$$\dim_k[R/I]_{13n+6} - \dim_k[R/I]_{13n+5} < 0,$$

which shows that  $R/I$  fails to have the WLP since the surjectivity does not hold.  $\square$

**Corollary 3.10.** *If  $8 \leq n \leq 159$  and  $d = 16$ , then  $R/I$  fails to have the WLP.*

**Proof.** Let  $\ell$  be a general linear form in  $R$ . Computation shows that  $j = 15n + 7$ . Proposition 3.2 follows that

$$\dim_k[R/(I, \ell)]_{15n+7} > 0,$$

for any  $8 \leq n \leq 159$ .

On other hand, by Theorem 3.1, we have

$$\dim_k[R/I]_{15n+7} - \dim_k[R/I]_{15n+6} = \sum_{k=0}^n (-1)^k \binom{2n+2}{k} \binom{17n+6-16k}{2n-1}.$$

Using Macaulay2 [7], we see that for any  $7 \leq n \leq 159$ ,

$$\dim_k[R/I]_{15n+7} - \dim_k[R/I]_{15n+6} < 0,$$

which shows that  $R/I$  fails to have the WLP since the surjectivity does not hold.  $\square$

**Proposition 3.11.** *Assume that  $n, d \geq 2$  and  $\ell$  is a general linear form in  $R$ . Then*

$$\dim_k[R/(I, \ell)]_j > 0$$

*if one of the following conditions is satisfied*

- (i)  $2n - 1$  or  $2n + 1$  divides  $d - 1$ .
- (ii)  $2n - 1$  divides  $d + 1$ .
- (iii)  $2n - 1$  divides  $d + 3$ .
- (iv)  $2n - 1$  divides  $d + 5$ .
- (v)  $d \geq 4n^2 - 2n + 2$ .

**Proof.** Set  $t = \lfloor \frac{2n(n+1)(d-1)}{2n+1} \rfloor$ . It is easy to show that  $j \leq t$ . It follows from [13, Proposition 4.1] that

$$\dim_k[R/(I, \ell)]_t > 0$$

if  $2n + 1$  divides  $d - 1$  or  $d \geq 4n^2 - 2n + 2$ . Hence

$$\dim_k[R/(I, \ell)]_j > 0$$

if  $2n + 1$  divides  $d - 1$  or  $d \geq 4n^2 - 2n + 2$  as claimed in the item (v) and the last part of the item (i). Now if  $2n - 1$  divides  $d - 1$ , then, by Theorem 3.1,

$$\dim_k[R/(I, \ell)]_j = \dim_k \mathfrak{L}_{2n-1}(e; 0^{2n+2}) > 0, \forall e \geq 1,$$

which complete the proof of the item (i).

If  $d + 1 = (2n - 1)e, e \geq 1$ , then  $d = (2n - 1)(e - 1) + 2(n - 1)$ . By Theorem 3.1,

$$\dim_k[R/(I, \ell)]_j = \dim_k \mathfrak{L}_{2n-1}(e + 1; 1^{2n+2}) > 0, \forall e \geq 1, n \geq 2$$

as claimed in the item (ii).

If  $d + 3 = (2n - 1)e, e \geq 1$ , then  $d = (2n - 1)(e - 1) + 2(n - 2)$ . If  $n \geq 3$ , by Theorem 3.1,

$$\dim_k[R/(I, \ell)]_j = \dim_k \mathfrak{L}_{2n-1}(e + 2; 2^{2n+2}) \geq \dim_k \mathfrak{L}_{2n-1}(3; 2^{2n+2}).$$

As

$$\begin{aligned} \dim_k \mathfrak{L}_{2n-1}(3; 2^{2n+2}) &\geq \binom{2n+2}{2n-1} - (2n+2) \binom{2n}{2n-1} \\ &= \frac{2n(n+1)(2n-5)}{3} > 0. \end{aligned}$$

If  $n = 2$ , then  $d + 3 = 3e, e \geq 2$ . It follows that

$$\dim_k[R/(I, \ell)]_j = \dim_k \mathfrak{L}_3(e + 2; 2^6) > 0.$$

Thus (iii) is proved.

It remains to show (iv). Since  $d + 5 = (2n - 1)e$  with  $e \geq 1$  if  $n \geq 4$ , one has

$$d = (2n - 1)(e - 1) + 2(n - 3).$$

If  $n \geq 5$ , by Theorem 3.1,

$$\begin{aligned} \dim_k[R/(I, \ell)]_j &= \dim_k \mathfrak{L}_{2n-1}(e + 3; 3^{2n+2}) \\ &\geq \dim_k \mathfrak{L}_{2n-1}(4; 3^{2n+2}) \\ &\geq \binom{2n+3}{2n-1} - (2n+2) \binom{2n+1}{2n-1} \\ &= \frac{n(n+1)(2n+1)(2n-9)}{6} > 0. \end{aligned}$$

Note that  $e \geq \begin{cases} 3 & \text{if } n = 2 \\ 2 & \text{if } n = 3 \end{cases}$ . Thus

$$\dim_k[R/(I, \ell)]_j = \begin{cases} \dim_k \mathfrak{L}_3(e - 2; 0^6) & \text{if } n = 2, e \geq 3 \\ \dim_k \mathfrak{L}_5(e + 3; 3^8) & \text{if } n = 3, e \geq 2 \\ \dim_k \mathfrak{L}_7(e + 3; 3^{10}) & \text{if } n = 4, e \geq 1. \end{cases}$$

Therefore,  $\dim_k[R/(I, \ell)]_j > 0$  if  $n \in \{2, 3\}$ . If  $n = 4$ , then

$$\dim_k[R/(I, \ell)]_j = \dim_k \mathfrak{L}_7(e + 3; 3^{10}) \geq \dim_k \mathfrak{L}_7(4; 3^{10}).$$

Set  $P = k[x_0, \dots, x_7]/(x_0^2, \dots, x_7^2)$ . We have

$$\dim_k \mathfrak{L}_7(4; 3^{10}) \geq h_P(4) - 2h_P(2) + h_P(0) = 15 > 0.$$

This completes the argument.  $\square$

We close this section by giving the following result. It is similar to a result of Nagel and Trok in [13, Proposition 6.3].

**Proposition 3.12.** *Given integers  $n \geq 2$  and  $0 \leq q \leq 2(n - 1)$ , define a polynomial function  $P_{n,q} : \mathbb{R} \rightarrow \mathbb{R}$  by*

$$P_{n,q}(t) = \sum_{k=0}^n (-1)^k \binom{2n+2}{k} \binom{n-1 + \lfloor \frac{(n-1)q}{2n-1} \rfloor + (q+1)(n-k) + t[2n^2 - 1 - (2n-1)k]}{2n-1}.$$

Then one has:

- (a) *If for some  $q$  with  $1 \leq q \leq 2(n - 1)$ ,  $P_{n,q}(t) \leq 0$  for every  $t \geq 0$ , then Conjecture 1.1 is true for every  $d \geq 4n^2 - 2n + 2$  such that  $d - 1 - q$  is divisible by  $2n - 1$ .*
- (b) *If  $P_{n,0}(t) \leq 0$  for every  $t \geq 1$ , then Conjecture 1.1 is true for every  $d$  such that  $d - 1$  is divisible by  $2n - 1$ .*
- (c) *If  $\sum_{k=0}^n (-1)^k \binom{2n+2}{k} [2n^2 - 1 - (2n - 1)k]^{2n-1} < 0$  for each integer  $n \geq 4$ , then Conjecture 1.1 is true for every  $d \gg 0$ .*

**Proof.** Let  $\ell$  be a general linear form in  $R$ . It follows from Proposition 3.11 that

$$\dim_k[R/(I, \ell)]_j > 0,$$

provided  $d \geq 4n^2 - 2n + 2$  or  $d + i$  is divisible by  $2n - 1$  with  $i \in \{-1, 1, 3, 5\}$ . It follows that under these assumptions the multiplication

$$\times \ell : [R/I]_{j-1} \longrightarrow [R/I]_j$$

fails to have maximal rank if and only if it fails surjectivity. It is enough to show that

$$\dim_k[R/I]_j - \dim_k[R/I]_{j-1} \leq 0.$$

Now write  $d - 1 = (2n - 1)t + q$  with integers  $t$  and  $q$  where  $0 \leq q \leq 2(n - 1)$ . Then a straightforward computation gives

$$j = (2n^2 - 1)t + nq + \lfloor \frac{(n - 1)q}{2n - 1} \rfloor.$$

By Theorem 3.1,

$$\begin{aligned} \dim_k[R/I]_j - \dim_k[R/I]_{j-1} &= \sum_{k=0}^n (-1)^k \binom{2n+2}{k} \binom{2n-1+j-kd}{2n-1} \\ &= \sum_{k=0}^n (-1)^k \binom{2n+2}{k} \binom{n-1 + \lfloor \frac{(n-1)q}{2n-1} \rfloor + (q+1)(n-k) + t[2n^2-1 - (2n-1)k]}{2n-1} \\ &= P_{n,q}(t). \end{aligned}$$

Now, if for some integer  $t \geq 0$  we have  $P_{n,q}(t) \leq 0$ , then

$$\dim_k[R/(I, \ell)]_j \neq \max\{\dim_k[R/I]_j - \dim_k[R/I]_{j-1}, 0\}.$$

This proves assertions (a) and (b).

Note that  $P_{n,q}(t)$  is a polynomial in  $t$  of degree  $2n - 1$  and

$$c_n := \sum_{k=0}^n (-1)^k \binom{2n+2}{k} [2n^2 - 1 - (2n - 1)k]^{2n-1}$$

is the coefficient of  $t^{2n-1}$  in  $P_{n,q}(t)$ . Since  $c_n < 0$  by assumption, it follows that  $P_{n,q}(t) < 0$  for all  $t \gg 0$  independent of  $q$ , and thus the claim (c) is proved.  $\square$

Based on computations, we conjecture that

$$c_n := \sum_{k=0}^n (-1)^k \binom{2n+2}{k} [2n^2 - 1 - (2n - 1)k]^{2n-1} < 0, \text{ for any } n \geq 2.$$

In facts that computations suggest that the sequence  $(c_n)_{n \geq 2}$  of integers is strictly decreasing with  $c_2 = -26$ , and so all these are negatives. Thank to Macaulay2 [7], we can check it  $c_n < 0$  for any  $2 \leq n \leq 400$ . This conjecture implies that Conjecture 1.1 is true for  $d \gg 0$ .

**4. Almost uniform powers of general linear forms in a few variables**

Our main result of this section is the following.

**Theorem 4.1.** *Let  $R = k[x_0, \dots, x_{2n}]$  and  $I = (L_0^d, \dots, L_{2n+1}^d)$ , where  $L_0, \dots, L_{2n+1}$  are general linear forms in  $R$ . If  $4 \leq n \leq 8$  and  $d \geq 4$ , then  $R/I$  fails to have the WLP.*

**Proof.** Let  $\ell \in R$  be a general linear form and we will show that the multiplication

$$\times \ell : [R/I]_{j-1} \longrightarrow [R/I]_j$$

fails to have maximal rank with  $j = \lfloor \frac{(2n^2-1)(d-1)}{2n-1} \rfloor$ , provided  $4 \leq n \leq 8$  and  $d \geq 4$ . To do this, we will show

$$\dim_k [R/(I, \ell)]_j \neq \max\{\dim_k [R/I]_j - \dim_k [R/I]_{j-1}, 0\}.$$

First, we prove the following assertion.

**Claim 1.**  $D := \dim_k [R/(I, \ell)]_j > 0$  for any  $4 \leq n \leq 8$  and  $d \geq 4$ .

Indeed, Theorem 3.1 shows that

$$D = \begin{cases} \dim_k \mathfrak{L}_{2n-1}(e; 0^{2n+2}) & \text{if } d = (2n-1)e + 1 \\ \dim_k \mathfrak{L}_{2n-1}(e + n - r + 1; (n-r)^{2n+2}) & \text{if } d = (2n-1)e + 2r \\ \dim_k \mathfrak{L}_{2n-1}(e + 2n - r + 1; (2n-r-1)^{2n+2}) & \text{if } d = (2n-1)e + 2r + 1 \end{cases}$$

where  $e, r$  are non-negative integers and  $1 \leq r \leq n-1$ . Note that the dimension of linear systems satisfies

$$\dim_k \mathfrak{L}_{2n-1}(i; a^{2n+2}) \geq \binom{2n-1+i}{2n-1} - (2n+2) \binom{2n-2+a}{2n-1}. \tag{4.1}$$

We now consider the following cases:

**Case 1: n=4.** Using (4.1), computations show that these linear systems are not empty for every

$$e \geq \begin{cases} 1 & \text{if } d-1 \equiv 0, 1, 2 \pmod{7} \\ 0 & \text{if } d-1 \equiv 3, 4, 5, 6 \pmod{7}. \end{cases}$$

In other words,  $D \neq 0$  for any  $d \geq 4$ .

**Case 2:  $n=5$ .** Using (4.1), computations show that these linear systems are not empty, provided

$$e \geq \begin{cases} 1 & \text{if } d - 1 \equiv 0, 1, 2, 4 \pmod{9} \\ 0 & \text{if } d - 1 \equiv 3, 5, 6, 7, 8 \pmod{9}. \end{cases}$$

In other words,  $D \neq 0$  for all  $d \geq 4$  and  $d \neq 5$ .

Let  $\ell_0, \dots, \ell_{2n+1}$  be general linear forms in  $k[x_0, \dots, x_{2n-1}]$  and set

$$P_{n,s} = k[x_0, \dots, x_{2n-1}]/(\ell_0^s, \dots, \ell_{2n-1}^s), \quad Q_{n,s} = k[x_0, \dots, x_{2n-1}]/(\ell_0^s, \dots, \ell_{2n}^s)$$

and  $R_{n,s} = k[x_0, \dots, x_{2n-1}]/(\ell_0^s, \dots, \ell_{2n+1}^s)$ . The exact sequence

$$[Q_{n,s}]_{i-s} \xrightarrow{\times \ell_{2n+1}^s} [Q_{n,s}]_i \longrightarrow [R_{n,s}]_i \longrightarrow 0$$

deduces that

$$\begin{aligned} h_{R_{n,s}}(i) &\geq h_{Q_{n,s}}(i) - h_{Q_{n,s}}(i - s) \\ &= h_{P_{n,s}}(i) - 2h_{P_{n,s}}(i - s) + h_{P_{n,s}}(i - 2s) \end{aligned}$$

where the last equality deduces from the fact that  $P_{n,s}$  is a complete intersection and has the SLP (see [15] or [17]).

Now we need to show  $D \neq 0$  for  $d = 5$ . Indeed, in this case, one has

$$\begin{aligned} D = \dim_k \mathfrak{L}_9(9; 7^{12}) &= \dim_k [R_{5,3}]_9 \\ &\geq h_{P_{5,3}}(9) - 2h_{P_{5,3}}(6) + h_{P_{5,3}}(3) \\ &= 8350 - 2 \times 2850 + 210 \\ &= 2860 \end{aligned}$$

which shows  $D \neq 0$  for  $d = 5$ .

**Case 3:  $n=6$ .** Using (4.1), computations show that these linear systems are not empty for any  $d \geq 6$  and  $d \neq 7, 9$ . We need to show  $D \neq 0$  for  $d = 4, 5, 7, 9$ . If  $d = 4$ , then  $D \neq 0$ , by Proposition 3.2. With the notations as in the case 2, one has

$$D = \begin{cases} \dim_k \mathfrak{L}_{11}(11; 9^{14}) = \dim [R_{6,3}]_{11} & \text{if } d = 5 \\ \dim_k \mathfrak{L}_{11}(10; 8^{14}) = \dim [R_{6,3}]_{10} & \text{if } d = 7 \\ \dim_k \mathfrak{L}_{11}(9; 7^{14}) = \dim [R_{6,3}]_9 & \text{if } d = 9. \end{cases}$$

The  $h$ -vector of  $P_{6,3}$  is

$$h_{P_{6,3}} = (1, 12, 78, 352, 1221, 3432, 8074, 16236, 28314, 43252, 58278, 69576, 73789, 69576, 58278, 43252, 28314, 16236, 8074, 3432, 1221, 352, 78, 12, 1).$$

It is easy to see

$$\dim[R_{6,3}]_i \geq h_{P_{6,3}}(i) - 2h_{P_{6,3}}(i - 3) + h_{P_{6,3}}(i - 6) > 0,$$

for each  $i \in \{9, 10, 11\}$ .

Thus,  $D > 0$  for every  $d \geq 4$ .

**Case 4:  $n=7$ .** Using (4.1), computations show that these linear systems are not empty for  $d \geq 4$  and  $d \neq 5, 6, 7, 9, 11, 16$ . We need to show  $D \neq 0$  for  $d = 5, 6, 7, 9, 11, 16$ . By Proposition 3.2 we get that  $D \neq 0$  for  $d = 6$ . With the notations as in the case 2, one has

$$D = \begin{cases} \dim_k \mathfrak{L}_{13}(13; 11^{16}) = \dim[R_{7,3}]_{13} & \text{if } d = 5 \\ \dim_k \mathfrak{L}_{13}(12; 10^{16}) = \dim[R_{7,3}]_{12} & \text{if } d = 7 \\ \dim_k \mathfrak{L}_{13}(11; 9^{16}) = \dim[R_{7,3}]_{11} & \text{if } d = 9 \\ \dim_k \mathfrak{L}_{13}(10; 8^{16}) = \dim[R_{7,3}]_{10} & \text{if } d = 11 \\ \dim_k \mathfrak{L}_{13}(15; 12^{16}) = \dim[R_{7,4}]_{15} & \text{if } d = 16. \end{cases}$$

As  $h$ -vector of  $Q_{7,3}$  is

$$h_{Q_{7,3}} = (1, 14, 105, 545, 2170, 6993, 18837, 43290, 85995, 148785, 224796, 295659, 334425, 315420, 227475, 83097)$$

we get  $D > 0$  for  $d = 5, 7, 9, 11$ . Similarly, one can easy show that  $D > 0$  for  $d = 16$ . Thus,  $D > 0$  for every  $d \geq 4$ .

**Case 5:  $n=8$ .** By Proposition 3.2, we have  $D \neq 0$  for  $d = 15e+2r$ ,  $e$  and  $r$  are non-negative integers such that  $2 \leq r \leq 8$ . Using (4.1), we can also show that  $D \neq 0$  for  $d \geq 4$  and  $d \neq 5, 7, 9, 11, 13, 18, 20$ . We now need to prove  $D \neq 0$  for  $d \neq 5, 7, 9, 11, 13, 18, 20$ . With the notations as in the case 2, one has

$$D = \begin{cases} \dim_k \mathfrak{L}_{15}(15; 13^{18}) = \dim[R_{8,3}]_{15} & \text{if } d = 5 \\ \dim_k \mathfrak{L}_{15}(14; 12^{18}) = \dim[R_{8,3}]_{14} & \text{if } d = 7 \\ \dim_k \mathfrak{L}_{15}(13; 11^{18}) = \dim[R_{8,3}]_{13} & \text{if } d = 9 \\ \dim_k \mathfrak{L}_{15}(12; 10^{18}) = \dim[R_{8,3}]_{12} & \text{if } d = 11 \\ \dim_k \mathfrak{L}_{15}(11; 9^{18}) = \dim[R_{8,3}]_{11} & \text{if } d = 13 \\ \dim_k \mathfrak{L}_{15}(17; 14^{18}) = \dim[R_{8,4}]_{17} & \text{if } d = 18 \\ \dim_k \mathfrak{L}_{15}(16; 13^{18}) = \dim[R_{8,4}]_{16} & \text{if } d = 20. \end{cases}$$

As  $h$ -vector of  $Q_{8,3}$  is

$$h_{Q_{8,3}} = (1, 16, 136, 799, 3604, 13192, 40528, 106828, 245242, 495312, 885768, 1406886, 1983696, 2469624, 2677704, 2448816, 1730787, 625992)$$

we get  $D > 0$  for  $d = 5, 7, 9, 11, 13$ . Similarly, the Hilbert functions of  $Q_{8,4}$  up to degree 17 are

$$h_{Q_{8,4}}(t) = (1, 16, 136, 816, 3859, 15232, 51952, 156672, 424558, 1046112, 2364768, 4937888, 9574978, 17312256, 29277264, 46411904, 69063979, 96521904)$$

which show  $D > 0$  for  $d = 18, 20$ . Thus,  $D > 0$  for every  $d \geq 4$ .

Therefore, Claim 1 is completely proved.

Second, to prove failure of the WLP in degree  $j$  it remains to show the following assertion.

**Claim 2.**  $E := \dim_k[R/I]_j - \dim_k[R/I]_{j-1} \leq 0$  for all  $4 \leq n \leq 8$  and  $d \geq 4$ .

Theorem 3.1 gives

$$E := \dim_k[R/I]_j - \dim_k[R/I]_{j-1} = \sum_{k=0}^n (-1)^k \binom{2n+2}{k} \binom{2n-1+j-kd}{2n-1}.$$

We consider the following cases:

**Case 1:  $n=4$ .** We consider seven cases for  $d-1 = 7e+m, 0 \leq m \leq 6$ . Thank to Macaulay2 [7], we can show that  $E < 0$  for any  $d \geq 4$ .

Subcase 1: If  $d = 7e + 1$ , then  $j = 31e$  and hence

$$\begin{aligned} E &= \binom{31e+7}{7} - 10 \binom{24e+6}{7} + 45 \binom{17e+5}{7} - 120 \binom{10e+4}{7} + 210 \binom{3e+3}{7} \\ &= \frac{1}{7!} (-1086400574e^7 - 914853422e^6 - 328170248e^5 - 60270140e^4 - 5015486e^3 \\ &\quad + 102442e^2 + 60228e + 5040) < 0 \quad \text{for any } e \geq 1. \end{aligned}$$

Subcase 2: If  $d = 7e + 2$ , then  $j = 31e + 4$  and we have

$$\begin{aligned} E &= \binom{31e+11}{7} - 10 \binom{24e+9}{7} + 45 \binom{17e+7}{7} - 120 \binom{10e+5}{7} + 210 \binom{3e+3}{7} \\ &= \frac{1}{7!} (-1086400574e^7 - 1829706844e^6 - 1272885740e^5 - 457929640e^4 \\ &\quad - 84318206e^3 - 5316556e^2 + 535080e + 75600) < 0 \quad \text{for any } e \geq 1. \end{aligned}$$

Subcase 3: If  $d = 7e + 3$ , then  $j = 31e + 8$ . It follows that

$$\begin{aligned}
 E &= \binom{31e + 15}{7} - 10 \binom{24e + 12}{7} + 45 \binom{17e + 9}{7} - 120 \binom{10e + 6}{7} + 210 \binom{3e + 3}{7} \\
 &= \frac{1}{7!} (-1086400574e^7 - 2744560266e^6 - 2847411560e^5 - 1530367860e^4 \\
 &\quad - 431507006e^3 - 50737554e^2 + 1747620e + 680400) < 0 \quad \text{for any } e \geq 1.
 \end{aligned}$$

Subcase 4: If  $d = 7e + 4$ , then  $j = 31e + 13$ . One has

$$\begin{aligned}
 E &= \binom{31e + 20}{7} - 10 \binom{24e + 16}{7} + 45 \binom{17e + 12}{7} - 120 \binom{10e + 8}{7} + 210 \binom{3e + 4}{7} \\
 &= \frac{1}{7!} (-1086400574e^7 - 4059690376e^6 - 6472447730e^5 - 5696621560e^4 \\
 &\quad - 2981962256e^3 - 925181824e^2 - 156720480e - 11088000) < 0 \quad \text{for any } e \geq 0.
 \end{aligned}$$

Subcase 5: If  $d = 7e + 5$ , then  $j = 31e + 17$  and hence

$$\begin{aligned}
 E &= \binom{31e + 24}{7} - 10 \binom{24e + 19}{7} + 45 \binom{17e + 14}{7} - 120 \binom{10e + 9}{7} + 210 \binom{3e + 4}{7} \\
 &= \frac{1}{7!} (-1086400574e^7 - 4974543798e^6 - 9666743618e^5 - 10305716610e^4 \\
 &\quad - 6484301936e^3 - 2393744472e^2 - 475568352e - 38586240) < 0 \quad \text{for any } e \geq 0.
 \end{aligned}$$

Subcase 6: If  $d = 7e + 6$ , then  $j = 31e + 22$  and therefore

$$\begin{aligned}
 E &= \binom{31e + 29}{7} - 10 \binom{24e + 23}{7} + 45 \binom{17e + 17}{7} - 120 \binom{10e + 11}{7} + 210 \binom{3e + 5}{7} \\
 &= \frac{1}{7!} (-1086400574e^7 - 6289673908e^6 - 15592053428e^5 - 21447402760e^4 \\
 &\quad - 17672567486e^3 - 8719279492e^2 - 2383703952e - 278359200) < 0 \quad \text{for any } e \geq 0.
 \end{aligned}$$

Subcase 7: If  $d = 7e + 7$ , then  $j = 31e + 26$ . It follows that

$$\begin{aligned}
 E &= \binom{31e + 33}{7} - 10 \binom{24e + 26}{7} + 45 \binom{17e + 19}{7} - 120 \binom{10e + 12}{7} + 210 \binom{3e + 5}{7} \\
 &= \frac{1}{7!} (-1086400574e^7 - 7204527330e^6 - 20406119384e^5 - 31980364500e^4 \\
 &\quad - 29926695806e^3 - 16705543050e^2 - 5144220396e - 673001280) < 0 \quad \text{for any } e \geq 0.
 \end{aligned}$$

Thus  $E < 0$  for any  $d \geq 4$ . Claim 2 is proved for  $n = 4$ .

**Case 2:  $n=5$ .** We write  $d - 1 = 9e + m, 0 \leq m \leq 8$ . We will prove that  $E < 0$  for any  $d \geq 4$ . Thank to Macaulay2 [7], a straightforward computation gives

Subcase 1: If  $d = 9e + 1$ , then  $j = 49e$ . It follows that

$$\begin{aligned}
 E &= \sum_{k=0}^5 (-1)^k \binom{12}{k} \binom{(49-9k)e + 9 - k}{9} \\
 &= \frac{1}{9!} (-32649547827918e^9 - 29495874488598e^8 - 11942585863236e^7 \\
 &\quad - 2793889960092e^6 - 406323342558e^5 - 35868202902e^4 - 1535113104e^3 \\
 &\quad + 29687112e^2 + 7209216e + 362880) < 0 \text{ for any } e \geq 1.
 \end{aligned}$$

Analogously we can check the another cases.

Subcase 2: If  $d = 9e + 2r$ ,  $1 \leq r \leq 4$ , then  $j = 49e + 11r - 6$  and

$$E = \sum_{k=0}^5 (-1)^k \binom{12}{k} \binom{(49-9k)e + (11-2k)r + 3}{9}.$$

We compute with `Macaulay2` to show that if  $r = 1$ , then  $E < 0$  for any  $e \geq 1$  and if  $r \in \{2, 3, 4\}$  then  $E < 0$  for any  $e \geq 0$ .

Subcase 3: If  $d = 9e + 2r + 1$ ,  $1 \leq r \leq 4$ , then  $j = 49e + 11r - 1$  and

$$E = \sum_{k=0}^5 (-1)^k \binom{12}{k} \binom{(49-9k)e + (11-2k)r - k + 8}{9}.$$

Similarly, we can show that if  $r = 1$ , then  $E < 0$  for any  $e \geq 1$  and if  $r \in \{2, 3, 4\}$  then  $E < 0$  for any  $e \geq 0$ .

It follows that  $E < 0$  for any  $d \geq 4$ . Claim 2 is proved for  $n = 5$ .

**Case 3: n=6.** Write  $d - 1 = 11e + m$ ,  $0 \leq m \leq 10$ . Thank to `Macaulay2` [7], we will show that  $E < 0$  for any  $d \geq 2$ .

Subcase 1: If  $d = 11e + 1$ , then  $j = 71e$  and

$$\begin{aligned}
 E &= \sum_{k=0}^6 (-1)^k \binom{14}{k} \binom{(71-11k)e + 11 - k}{11} = \frac{1}{11!} (-2310696921327619572e^{11} \\
 &\quad - 2159206229822458212e^{10} - 925836626096405100e^9 - 238845827273630940e^8 \\
 &\quad - 40895244843536556e^7 - 4822097086873836e^6 - 390251062386900e^5 \\
 &\quad - 20387890763460e^4 - 526999267872e^3 + 8455070448e^2 \\
 &\quad + 1189900800e + 39916800) < 0 \text{ for any } e \geq 1.
 \end{aligned}$$

Analogously we can check the another cases.

Subcase 2: If  $d = 11e + 2r$ ,  $1 \leq r \leq 5$ , then  $j = 71e + 13r - 7$ . For each  $1 \leq r \leq 5$ , computations with `Macaulay2` show that

$$E = \sum_{k=0}^6 (-1)^k \binom{14}{k} \binom{(71-11k)e + (13-2k)r + 4}{11} < 0 \quad \text{for any } e \geq 0.$$

Subcase 3: If  $d = 11e + 2r + 1$ ,  $1 \leq r \leq 5$ , then  $j = 71e + 13r - 1$ . For each  $1 \leq r \leq 5$ , computations with `Macaulay2` show that

$$E = \sum_{k=0}^6 (-1)^k \binom{14}{k} \binom{(71-11k)e + (13-2k)r - k + 10}{11} < 0 \quad \text{for any } e \geq 0.$$

It follows that  $E < 0$  for any  $d \geq 2$ . Claim 2 is proved for  $n = 6$ .

**Case 4: n=7.** We write  $d - 1 = 13e + m$ ,  $0 \leq m \leq 12$ . Thank to `Macaulay2` [7], we will show that  $E < 0$  for any  $d \geq 2$ .

Subcase 1: If  $d = 13e + 1$ , then  $j = 97e$  and hence

$$\begin{aligned} E &= \sum_{k=0}^7 (-1)^k \binom{16}{k} \binom{(97-13k)e + 13 - k}{13} \\ &= \frac{1}{13!} (-334688414610649890510291e^{13} - 318779633066827110608001e^{12} \\ &\quad - 141329943714960759520905e^{11} - 38495945182007845679433e^{10} \\ &\quad - 7165747937184385180203e^9 - 958746457198704734703e^8 \\ &\quad - 94270438988259145755e^7 - 6819523889292264579e^6 \\ &\quad - 354359614333473606e^5 - 12260161531299396e^4 - 210757791455640e^3 \\ &\quad + 2848164688512e^2 + 260089315200e + 6227020800) < 0 \quad \text{for any } e \geq 1. \end{aligned}$$

Analogously we can check the another cases.

Subcase 2: If  $d = 13e + 2r$ ,  $1 \leq r \leq 6$ , then  $j = 97e + 15r - 8$ . For each  $1 \leq r \leq 6$ , computations with `Macaulay2` show that

$$E = \sum_{k=0}^7 (-1)^k \binom{16}{k} \binom{(97-13k)e + (15-2k)r + 5}{13} < 0 \quad \text{for any } e \geq 0.$$

Subcase 3: If  $d = 13e + 2r + 1$ ,  $1 \leq r \leq 6$ , then  $j = 97e + 15r - 1$ . For each  $1 \leq r \leq 6$ , computations with `Macaulay2` show that

$$E = \sum_{k=0}^7 (-1)^k \binom{16}{k} \binom{(97-13k)e + (15-2k)r - k + 12}{13} < 0 \quad \text{for any } e \geq 0.$$

It follows that  $E < 0$  for all  $d \geq 2$  as desired.

**Case 5: n=8.** Write  $d - 1 = 15e + m$ ,  $0 \leq m \leq 14$ . Thank to `Macaulay2` [7], we will show that  $E < 0$  for any  $d \geq 2$ .

Subcase 1: If  $d = 15e + 1$ , then  $j = 127e$  and hence

$$\begin{aligned}
 E &= \sum_{k=0}^8 (-1)^k \binom{18}{k} \binom{(127 - 15k)e + 15 - k}{15} \\
 &= \frac{1}{15!} (-89416180762084130597433031670e^{15} - 86189264600012090365415830692e^{14} \\
 &\quad - 39053028448507299529147674830e^{13} - 11015489694695869227569915190e^{12} \\
 &\quad - 2159771261721698841245859830e^{11} - 311249224672723122942089934e^{10} \\
 &\quad - 3397943759406996655524110e^9 - 2851416027092144483798970e^8 \\
 &\quad - 184330466550469812352780e^7 - 9076381685750429456406e^6 \\
 &\quad - 329488524726287066140e^5 - 8109990225233736840e^4 - 98835121150056720e^3 \\
 &\quad + 1138217439820032e^2 + 71328551374080e + 1307674368000) < 0 \text{ for any } e \geq 1.
 \end{aligned}$$

Analogously we can check the another cases.

Subcase 2: If  $d = 15e + 2r$ ,  $1 \leq r \leq 7$ , then  $j = 127e + 17r - 9$ . For each  $1 \leq r \leq 7$ , computations with `Macaulay2` show that

$$E = \sum_{k=0}^8 (-1)^k \binom{18}{k} \binom{(127 - 15k)e + (17 - 2k)r + 6}{15} < 0 \text{ for any } e \geq 0.$$

Subcase 3: If  $d = 15e + 2r + 1$ ,  $1 \leq r \leq 7$ , then  $j = 127e + 17r - 1$ . For each  $1 \leq r \leq 7$ , computations with `Macaulay2` show that

$$E = \sum_{k=0}^8 (-1)^k \binom{18}{k} \binom{(127 - 15k)e + (17 - 2k)r - k + 14}{15} < 0 \text{ for any } e \geq 0.$$

It follows that  $E < 0$  for all  $d \geq 2$  and  $n = 8$ .

Thus Claim 2 is completely proved.

Finally, Theorem 4.1 follows from the above two claims.  $\square$

**Remark 4.2.**

- (1) The first author has shown that an artinian ideal  $I = (L_0^2, \dots, L_{2n+1}^2) \subset R$  generated by the quadratic powers of general linear forms fails to have the WLP [12]. Therefore, Theorem 4.1 answers partially Conjecture 1.1 for  $4 \leq n \leq 8$ , missing only the case  $d = 3$ .
- (2) Theorem 4.1 together with Corollaries 3.3–3.10 says that  $R/I$  fails to have the WLP for all  $d = 2r$ ,  $2 \leq r \leq 8$  and  $4 \leq n \leq 2r(r + 2) - 1$ .

## Acknowledgments

Computations using the algebra software `Macaulay2` [7] were essential to get the ideas behind some of the proofs. The authors thank the referee for a careful reading and useful comments that improved the presentation of the article. The first author was partially supported by the grant MTM2016-78623-P. The second author was partially supported by the project “Àlgebra i Geometria Algebraica” under grant number 2017SGR00585 and by Vietnam’s National Foundation for Science and Technology Development (NAFOS-TED) under grant number 101.04-2019.07. The second author would like to thank Hue University for the support.

## References

- [1] Charles Almeida, Aline V. Andrade, Lefschetz property and powers of linear forms in  $\mathbb{K}[x, y, z]$ , *Forum Math.* 30 (4) (2018) 857–865.
- [2] Federico Ardila, Alexander Postnikov, Combinatorics and geometry of power ideals, *Trans. Am. Math. Soc.* 362 (8) (2010) 4357–4384.
- [3] Roberta Di Gennaro, Giovanna Ilardi, Jean Vallès, Singular hypersurfaces characterizing the Lefschetz properties, *J. Lond. Math. Soc.* (2) 89 (1) (2014) 194–212.
- [4] Roberta Di Gennaro, Giovanna Ilardi, Jean Vallès, Comments on the WLP in degree 3 for 8 cubes in  $\mathbb{P}^6$ , Working paper or preprint: hal-02321118, October 2019.
- [5] Marcin Dumnicki, An algorithm to bound the regularity and nonemptiness of linear systems in  $\mathbb{P}^n$ , *J. Symb. Comput.* 44 (10) (2009) 1448–1462.
- [6] Jacques Emsalem, Anthony Iarrobino, Inverse system of a symbolic power. I, *J. Algebra* 174 (3) (1995) 1080–1090.
- [7] Daniel R. Grayson, Michael E. Stillman, `Macaulay2`, a software system for research in algebraic geometry, Available at <http://www.math.uiuc.edu/Macaulay2/>.
- [8] Brian Harbourne, Hal Schenck, Alexandra Seceleanu, Inverse systems, Gelfand-Tsetlin patterns and the weak Lefschetz property, *J. Lond. Math. Soc.* (2) 84 (3) (2011) 712–730.
- [9] Giovanna Ilardi, Jean Vallès, Eight cubes of linear forms in  $\mathbb{P}^6$ , arXiv e-prints: arXiv:1910.04035 [math.AG], Oct. 2019.
- [10] Juan C. Migliore, Rosa M. Miró-Roig, Uwe Nagel, On the weak Lefschetz property for powers of linear forms, *Algebra Number Theory* 6 (3) (2012) 487–526.
- [11] Juan C. Migliore, Rosa M. Miró-Roig, On the strong Lefschetz problem for uniform powers of general linear forms in  $k[x, y, z]$ , *Proc. Am. Math. Soc.* 146 (2) (2018) 507–523.
- [12] Rosa M. Miró-Roig Harbourne, Schenck and Seceleanu’s conjecture, *J. Algebra* 462 (2016) 54–66.
- [13] Uwe Nagel, Bill Trok, Interpolation and the weak Lefschetz property, *Trans. Am. Math. Soc.* 372 (12) (2019) 8849–8870.
- [14] Hal Schenck, Alexandra Seceleanu, The weak Lefschetz property and powers of linear forms in  $\mathbb{K}[x, y, z]$ , *Proc. Am. Math. Soc.* 138 (7) (2010) 2335–2339.
- [15] Richard P. Stanley, Weyl groups, the hard Lefschetz theorem, and the Sperner property, *SIAM J. Algebraic Discrete Methods* 1 (2) (1980) 168–184.
- [16] Bernd Sturmfels, Zhiqiang Xu, Sagbi bases of Cox-Nagata rings, *J. Eur. Math. Soc.* 12 (2) (2010) 429–459.
- [17] Junzo Watanabe, The Dilworth number of Artinian rings and finite posets with rank function, in: *Commutative Algebra and Combinatorics*, Kyoto, 1985, in: *Adv. Stud. Pure Math.*, vol. 11, North-Holland, Amsterdam, 1987, pp. 303–312.