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Denominator identities for the periplectic Lie superalgebra [☆]

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We prove denominator identities for the periplectic Lie superalgebra $\mathfrak{p}(n)$, thereby completing the problem of finding denominator identities for all simple classical finite-dimensional Lie superalgebras.

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1. Introduction

The classical Weyl character formula describes the character of a simple finite-dimensional module over a Lie algebra in terms of characters of modules which are easy to compute. This formula admits generalizations to infinite-dimensional Lie algebras as well as to some Lie superalgebras. It turns out that a particularly interesting case is when the formula is specialized to the trivial representation. In this case, one side of the equality is easy to understand, namely it is 1, and since the other side is a fraction, the resulting identity is called the *denominator identity*.

Denominator identities for Lie algebras have numerous applications to algebra, combinatorics and number theory. For example, the denominator identities for affine Lie algebras [16] turned out to be the famous Macdonald identities [20]. The simplest case of this is the Jacobi triple product identity, which is the Macdonald identity for the affine root system of type A_1 and is the denominator identity for the affine Lie algebra $\widehat{\mathfrak{sl}(2)}$.

Denominator identities for Lie superalgebras are also fascinating and useful. For example, V. Kac and M. Wakimoto showed that a specialization of the affine denominator gives formulas for computing the number of ways to decompose an integer as a sum of k-squares or as a sum of triangular numbers for certain values of k in [19, Section 5]. In addition, super denominator identities were used to determine the simplicity conditions of various W-algebras in [6,14] and to recover the Theta correspondence for compact dual pairs of real Lie groups in [7].

The denominator identity for simple Lie algebras and symmetrizable Kac–Moody algebras takes the form

$$e^{\rho}R = \sum_{w\in W} (\operatorname{sgn} w)w\left(e^{\rho}\right),$$

where R is the denominator, W is the Weyl group and ρ is a certain element in the dual of the Cartan subalgebra [16]. For Lie superalgebras, one needs another ingredient, namely the notion of a maximal isotropic set of roots, which was introduced by V. Kac and M. Wakimoto in [19] and now plays a key role in character formulas for Lie superalgebras.

Denominator identities for simple basic classical Lie superalgebras and queer Lie superalgebras, and for their non-twisted affinizations were formulated by V. Kac and M. Wakimoto in [19], where they proved the defect one case. The proofs of these denominator identities were completed by M. Gorelik and D. Zagier in [9–11,26], and generalizations appeared in [12,22,7].

The only classical finite-dimensional Lie superalgebra that remained was the periplectic Lie superalgebra $\mathfrak{p}(n)$. This algebra first appeared in V. Kac's well-known classification of simple finite-dimensional Lie superalgebras [17]. The family $\mathfrak{p}(n)$ is one of the families of "strange" Lie superalgebras, which have no even invariant bilinear form. In fact, $\mathfrak{p}(n)$ does not admit a nondegenerate even or odd invariant bilinear form, and thus has no affinization.

The representation theory of $\mathfrak{p}(n)$ has been studied by V. Serganova in [23] and others in [21,3]; however, one difficulty arises owing to the lack of a quadratic Casimir element (see [8] for a description of the center of the universal enveloping algebra of $\mathfrak{p}(n)$). Recently, a large breakthrough in the representation theory of $\mathfrak{p}(n)$ was accomplished after the introduction of a "fake Casimir", a.k.a. tensor Casimir [1]. This advancement has promoted a resurgence of interest in the Lie superalgebra $\mathfrak{p}(n)$; see for example [2,4,5,15].

In this paper, we state and prove two different denominator identities for the periplectic Lie superalgebra, namely for two nonconjugate Borel subalgebras $\mathfrak{b}^{\text{thick}}$ and $\mathfrak{b}^{\text{thin}}$ (see Section 2.2). These identities take the form

$$e^{\rho}R = \sum_{w\in W} (\operatorname{sgn} w) w \left(\frac{e^{\rho}}{(1-e^{-\beta_1})(1-e^{-\beta_1-\beta_2})\cdots(1-e^{-\beta_1-\dots-\beta_r})} \right),$$

where $\{\beta_1, \dots, \beta_r\}$ is an explicitly defined maximal set of mutually orthogonal odd roots. These denominator identities are similar to the identities given in [18, Thm. 1.1] for basic Lie superalgebras. For the Borel subalgebra $\mathfrak{b}^{\text{thin}}$, we also have an identity in a form similar to [19, Thm 2.1]. We note that other Borel subalgebras of $\mathfrak{p}(n)$ cannot admit a denominator identity in the classical form, as $e^{\rho}R$ is not W-anti-invariant in these other cases (see Remark 2.3).

Our paper is organized as follows. Section 2 contains preliminary definitions and lemmas. In Sections 3 and 4, we state and prove the thin and thick denominator identities, respectively. The proofs are of a combinatorial nature and do not rely on deep theorems. The final section of our paper contains a remark on character formulas and an open problem concerning the homological complex behind these denominator identities.

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2. The periplectic Lie superalgebra

2.1. Lie superalgebras

Let $\mathfrak{gl}(n|n)$, the general linear Lie superalgebra over $\mathbb{C}$, and let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a $\mathbb{Z}_2$-graded vector superspace. The parity of a homogeneous vector $v \in V_{\bar{0}}$ is defined as $\bar{v} = \bar{0} \in \mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$, while the parity of an odd vector $v \in V_{\bar{1}}$ is defined as $\bar{v} = \bar{1}$. If the parity of a vector v is $\bar{0}$ or $\bar{1}$, then v has degree 0 or 1, respectively. If the notation $\bar{v}$ appears in formulas, we will assume that v is homogeneous.

The general linear Lie superalgebra may be identified with the endomorphism algebra $\operatorname{End}(V_{\bar{0}} \oplus V_{\bar{1}})$, where $\dim V_{\bar{0}} = \dim V_{\bar{1}} = n$. Then $\mathfrak{gl}(n|n) = \mathfrak{gl}(n|n)_{\bar{0}} \oplus \mathfrak{gl}(n|n)_{\bar{1}}$, where

$$\mathfrak{gl}(n|n)_{\bar{0}} = \operatorname{End}(V_{\bar{0}}) \oplus \operatorname{End}(V_{\bar{1}}) \qquad \text{and} \qquad \mathfrak{gl}(n|n)_{\bar{1}} = \operatorname{Hom}(V_{\bar{0}}, V_{\bar{1}}) \oplus \operatorname{Hom}(V_{\bar{1}}, V_{\bar{0}}).$$

So $\mathfrak{gl}(n|n)_{\bar{0}}$ consists of parity-preserving linear maps while $\mathfrak{gl}(n|n)_{\bar{1}}$ consists of parity-switching maps. We also have a bilinear operation on $\mathfrak{gl}(n|n)$:

$$[x, y] = xy - (-1)^{\bar{x}\bar{y}} yx$$

on homogeneous elements, which then extends linearly to all of $\mathfrak{gl}(n|n)$. By fixing a basis of $V_{\bar{0}}$ and $V_{\bar{1}}$, the Lie superalgebra $\mathfrak{gl}(n|n)$ can be realized as the set of $2n \times 2n$ matrices, where

$$\mathfrak{gl}(n|n)_{\bar{0}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : A, B \in M_{n,n} \right\} \quad \text{and} \quad \mathfrak{gl}(n|n)_{\bar{1}} = \left\{ \begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix} : C, D \in M_{n,n} \right\},$$

where $M_{n,n}$ are $n \times n$ complex matrices.

Let V be an $(n|n)$-dimensional vector superspace equipped with a non-degenerate odd symmetric form

$$\beta : V \otimes V \to \mathbb{C}, \quad \beta(v, w) = \beta(w, v), \quad \text{and} \quad \beta(v, w) = 0 \quad \text{if} \quad \bar{v} = \bar{w}. \tag{1}$$

Then $\mathrm{End}_{\mathbb{C}}(V)$ inherits the structure of a vector superspace from V.

The periplectic superalgebra $\mathfrak{p}(n)$ is defined to be the Lie superalgebra of all $X \in \mathrm{End}_{\mathbb{C}}(V)$ preserving β, i.e., β satisfies the condition

$$\beta(Xv, w) + (-1)^{\bar{X}\bar{v}} \beta(v, Xw) = 0.$$

Remark 2.1. With respect to fixed bases for V, the matrix of $X \in \mathfrak{p}(n)$ has the form $\left(\begin{smallmatrix} A & B \\ C & -A^t \end{smallmatrix}\right)$, where A, B, C are $n \times n$ matrices such that $B^t = B$ and $C^t = -C$. Note that $\mathfrak{p}(n)$ is not itself simple; however, the subalgebra $\mathfrak{sp}(n)$ obtained by imposing the additional condition $\operatorname{tr} A = 0$ is simple and has codimension 1 in $\mathfrak{p}(n)$.

2.2. Root systems

From this point on, we will let $\mathfrak{g} := \mathfrak{p}(n)$. Fix the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ which consists of diagonal matrices and let $\{\varepsilon_1, \dots, \varepsilon_n\}$ be the standard basis of $\mathfrak{h}^*$. Note that $\mathfrak{h} \subset \mathfrak{g}_{\bar{0}}$. We have a root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha\right)$, where Δ denotes the set of roots of $\mathfrak{g}$. The set Δ decomposes as

$$\Delta = \Delta(\mathfrak{g}_{-1}) \cup \Delta(\mathfrak{g}_0) \cup \Delta(\mathfrak{g}_1),$$

where

$$\Delta(\mathfrak{g}_0) = \{\varepsilon_i - \varepsilon_j : 1 \leq i \neq j \leq n\},$$

$$\Delta(\mathfrak{g}_1) = \{\varepsilon_i + \varepsilon_j : 1 \leq i \leq j \leq n\} \quad \text{and} \quad \Delta(\mathfrak{g}_{-1}) = \{-(\varepsilon_i + \varepsilon_j) : 1 \leq i < j \leq n\},$$

and moreover, $\mathfrak{g}$ has a (short) $\mathbb{Z}$-grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_k := \bigoplus_{\alpha\in\Delta(\mathfrak{g}_k)} \mathfrak{g}_\alpha$. This $\mathbb{Z}$-grading is compatible with the Lie superalgebra structure on $\mathfrak{g}$ as $\mathfrak{g}_{\bar{0}} = \mathfrak{g}_0$ and $\mathfrak{g}_{\bar{1}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$.

Additionally, Δ decomposes into even and odd roots $\Delta = \Delta_{\bar{0}} \cup \Delta_{\bar{1}}$, where $\Delta_{\bar{0}} = \Delta_0$ and $\Delta_{\bar{1}} = \Delta_{-1} \cup \Delta_1$. We can choose a set of positive roots $\Delta^+ \subset \Delta$ and consider the corresponding Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha\right)$. In what follows, we will always assume that

$$\Delta_{\bar{0}}^+ = \{\varepsilon_i - \varepsilon_j : 1 \le i < j \le n\}$$

and that $\Delta_{\bar{1}}^+$ is either $\Delta(\mathfrak{g}_1)$ or $\Delta(\mathfrak{g}_{-1})$. We denote the corresponding Borel subalgebras of $\mathfrak{g}$ by $\mathfrak{b}^{\text{thick}} := \mathfrak{b}_0 \oplus \mathfrak{g}_1$ and $\mathfrak{b}^{\text{thin}} := \mathfrak{b}_0 \oplus \mathfrak{g}_{-1}$, respectively. Let $\rho_{\bar{0}} := \frac{1}{2}\left(\sum_{\alpha\in\Delta_{\bar{0}}^+} \alpha\right)$, $\rho_{\bar{1}} := \frac{1}{2}\left(\sum_{\alpha\in\Delta_{\bar{1}}^+} \alpha\right)$ and $\rho = \rho_n := \rho_{\bar{0}} - \rho_{\bar{1}}$.

We will work over the rational function field generated by e^λ, where $\lambda \in \mathfrak{h}^*$. Let

$$R_{0,n} = \prod_{\alpha\in\Delta^+(\mathfrak{g}_0)} (1-e^{-\alpha}), \qquad R_{1,n} = \prod_{\alpha\in\Delta(\mathfrak{g}_1)} (1-e^{-\alpha}), \qquad R_{-1,n} = \prod_{\alpha\in\Delta(\mathfrak{g}_{-1})} (1-e^{-\alpha}).$$

We will write R_0, R_1 R_{-1}, respectively, when it is clear what the algebra is. We let $R = \frac{R_0}{R_{\bar{1}}}$, where $R_{\bar{1}} = R_1$ or R_{-1}.

Remark 2.2. The inverse of $e^{2\rho}R$ is the supercharacter of $U(\mathfrak{n})$ (up to a sign), where $\mathfrak{n}$ denotes the nilradical of $\mathfrak{b}$ (here $\mathfrak{b} = \mathfrak{b}^{\text{thick}}$ or $\mathfrak{b}^{\text{thick}}$ for $R_{\bar{1}} = R_1$ or R_{-1}, respectively). One can also consider character versions of the denominator identity. See Section 5.1.

A polynomial f is anti-invariant or skew-invariant if $w.f = (\operatorname{sgn} w)f$ for any $w \in W$. Note that R_{-1}, R_1 are W-invariant and $e^\rho R_0$ is W-anti-invariant.

Remark 2.3. The sets $\Delta(\mathfrak{g}_1)$ and $\Delta(\mathfrak{g}_{-1})$ are the only choices of positive odd roots for which $e^\rho R$ is W-anti-invariant with respect to our fixed choice of positive even roots. Thus, there does not exist a denominator identity of this form for other Borel subalgebras, since $e^\rho R$ can not equal an alternating sum over the Weyl group.

Define $\mathcal{F}_W(a) := \sum_{w\in W}(\operatorname{sgn} w)w(a)$. Let $y = \sum_\mu a_\mu e^\mu$, where $a_\mu \in \mathbb{Q}$. The support of y is defined to be

$$\operatorname{supp}(y) = \{\mu : a_\mu \neq 0\}.$$

An element $\lambda = \sum a_i\varepsilon_i$ is regular if and only if it has a distinct coefficient for every ε_i. That is, λ is regular if $w(\lambda) = \lambda$ implies $w = \mathrm{Id}$. Moreover, its orbit is of maximal size.

We cite [11, Lemma 4.1.1 (ii)]:

Lemma 2.4. *For any $\mu \in \mathfrak{h}_{\mathbb{R}}^*$, the stabilizer of μ in W is either trivial or contains a reflection.*

This implies the stabilizer of a non-regular point μ in W contains a reflection. Thus the space of W-anti-invariant elements is spanned by $\mathcal{F}_W(e^\mu)$, where μ is regular.

We call the orbit $W(\mu)$ regular if μ is regular; thus, regular orbits consist of regular points.

Lemma 2.5. *The support of a W-anti-invariant element is a union of regular W-orbits.*

Proof. Since $\mathcal{F}_W\left(e^\lambda\right) = 0$ for non-regular λ, only regular elements appear in the support of $\mathcal{F}_W(a)$. □

3. Thin denominator identity for $\mathfrak{p}(n)$

In this section, we present denominator identities for the Borel subalgebra $\mathfrak{b}^{\text{thin}}$ of $\mathfrak{p}(n)$, namely when $\Delta_{\bar{1}}^+ = \Delta(\mathfrak{g}_{-1})$. In this case, $\rho_{\bar{1}} = \left(\frac{1-n}{2}\right)\sum_{i=1}^n \varepsilon_i$ and $\rho = \sum_{i=1}^n (n-i)\varepsilon_i$.

Let $R = \frac{R_0}{R_{-1}}$, where $R_{-1} = \prod_{1\le i<j\le n}\left(1-e^{\varepsilon_i+\varepsilon_j}\right)$. Set $r = \left\lfloor \frac{n}{2} \right\rfloor$ and let

$$\beta_1 = -(\varepsilon_1+\varepsilon_2), \quad \beta_2 = -(\varepsilon_3+\varepsilon_4), \ldots, \beta_r = -(\varepsilon_{2r-1}+\varepsilon_{2r}).$$

We define

$$\rho^{\Uparrow} := \rho + (n-2)\beta_1 + (n-4)\beta_2 + \ldots + (n-2r)\beta_r = \varepsilon_1 + \varepsilon_3 + \ldots + \varepsilon_{2r-1}.$$

Here is one form of the thin denominator identity.

Theorem 3.1. *Let $\mathfrak{g} = \mathfrak{p}(n)$ and $\Delta_{\bar{1}}^+ = \Delta(\mathfrak{g}_{-1})$. Then*

$$e^\rho R = \frac{1}{r!}\sum_{w\in W}(\operatorname{sgn} w) w\left(\frac{e^{\rho^{\Uparrow}}}{\prod_{\beta\in S}\left(1-e^{-\beta}\right)}\right),$$

where $S = \{\beta_1, \beta_2, \ldots, \beta_r\}$.

Proof. By applying the permutation $\tau_n := (2t-1 \to t; 2t \to n+1-t)$ to the RHS we obtain the equivalent expression

$$e^\rho R = (\operatorname{sgn}\tau_n)\frac{1}{r!}\sum_{w\in W}(\operatorname{sgn} w) w\left(\frac{e^{\varepsilon_1+\varepsilon_2+\ldots+\varepsilon_r}}{\prod_{\beta'\in S'}\left(1-e^{-\beta'}\right)}\right), \tag{2}$$

where $S' = \{-(\varepsilon_1+\varepsilon_n), -(\varepsilon_2+\varepsilon_{n-1}), \ldots, -(\varepsilon_r+\varepsilon_{n+1-r})\}$.

Since R_{-1} is W-invariant, equation (2) is equivalent to

$$(\operatorname{sgn}\tau_n) r! e^{\rho} R_0 = \mathcal{F}_W\left(e^{\varepsilon_1+\varepsilon_2+\ldots+\varepsilon_r} \prod_{\alpha\in\Delta(\mathfrak{g}_{-1})\setminus S'} \left(1-e^{-\alpha}\right)\right). \tag{3}$$

Since $\rho_{\bar{1}}$ is W-invariant, the Weyl denominator identity for $\mathfrak{sl}(n)$ yields the equivalent expression

$$(\operatorname{sgn}\tau_n) r! \mathcal{F}_W\left(e^{\rho}\right) = \mathcal{F}_W\left(e^{\varepsilon_1+\varepsilon_2+\ldots+\varepsilon_r} \prod_{(i,j)\in U} \left(1-e^{\varepsilon_i+\varepsilon_j}\right)\right), \tag{4}$$

where

$$U := \{(i,j) \in \mathbb{Z}_{>0}\times\mathbb{Z}_{>0} \mid 1 \le i < j \le n,\ i+j \ne n+1\}.$$

Note that the Weyl denominator identity for the Lie algebra $\mathfrak{g}_{\bar{0}}$ gives $e^{\rho_{\bar{0}}} R_0 = \mathcal{F}_W\left(e^{\rho_{\bar{0}}}\right)$. So we can multiply both sides by $e^{-\rho_{\bar{1}}}$ and pass it through $\mathcal{F}_W$ since it is invariant. We also note that both sides of (4) are W-skew-invariant. Let

$$\mathcal{A} := e^{\varepsilon_1+\varepsilon_2+\ldots+\varepsilon_r} \prod_{(i,j)\in U} \left(1-e^{\varepsilon_i+\varepsilon_j}\right) = \sum_{\nu} a_\nu e^{\nu}. \tag{5}$$

Then

$$\begin{aligned} \operatorname{supp}(\mathcal{A}) \subset \Bigg\{ \sum_{i=1}^{n} k_i\varepsilon_i : 0 \le k_i \le n-1,\ 1 \le k_1,\ldots,k_r \le n-1, \\ 0 \le k_{n+1-r},\ldots,k_n \le n-2 \Bigg\}, \end{aligned} \tag{6}$$

since $\mathcal{A}$ can be expressed using k_i, i.e., we get the bounds on the coefficients k_i by counting the elements in the set U, and the regular elements in $\operatorname{supp}(\mathcal{A})$ lie in the orbit $W(\rho)$ since elements in the orbits of $W(\rho)$ have coefficients $0, 1, \ldots, n-1$ (in some order). Now since $\mathcal{F}_W\left(e^{\mu}\right) = 0$ for non-regular elements $\mu \in \mathfrak{h}^*$, we have that the RHS of (4) equals

$$\mathcal{F}_W(\mathcal{A}) = \mathcal{F}_W\left(\sum_{y\in W} a_{y\rho} e^{y\rho}\right) = j\ \mathcal{F}_W\left(e^{\rho}\right),$$

where

$$j := \sum_{y\in W} (\operatorname{sgn} y)\ a_{y\rho},$$

by switching the sums and then changing the indexing set. We get a summation of the form $\mathcal{F}_W(e^{y\rho})$, and we reindex since the sum is over all of W. Hence to prove (4) and deduce the theorem, it remains to show that $j = (\operatorname{sgn} \tau_n) r!$.

Consider the following embedding $\iota : S_r \to S_n$: each permutation $\sigma \in S_r$ maps to the corresponding permutation of the set $1, 2, \dots, r$ and the corresponding permutation of the set $n, n-1, \dots, n+1-r$ (for instance, for $n = 5$ we have $\iota((12)) = (12)(54)$); note that $\iota(S_r)$ consists of even permutations. Clearly, $\mathcal{A}$ is $\iota(S_r)$-invariant, as

$$\mathcal{A} = \frac{e^{\varepsilon_1+\varepsilon_2+\ldots+\varepsilon_r} R_{-1}}{\prod_{\beta' \in S'} (1 - e^{-\beta'})}.$$

So by fixing a set of representatives of the left cosets of $\iota(S_r)$ in S_n to be

$$S_n/S_r := \{\sigma \in S_n : \sigma(n-(r-1)) < \ldots < \sigma(n-1) < \sigma(n)\},$$

we have

$$j = r! \sum_{y \in S_n/S_r} (\operatorname{sgn} y) a_{y\rho}.$$

We derive from (5) that

$$\mathcal{A} = \sum_{P \subset U} (-1)^{|P|} e^{\varepsilon_1+\varepsilon_2+\ldots+\varepsilon_r+\sum_{(i,j)\in P}(\varepsilon_i+\varepsilon_j)},$$

where each P is a subset of U, and $|P|$ denotes the number of elements in P. Suppose

$$\varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_r + \sum_{(i,j)\in P'} (\varepsilon_i + \varepsilon_j) = \sum_{i=1}^{n} k_i \varepsilon_i = y'\rho$$

for some $y' \in S_n/S_r$ and $P' \subset U$. We will prove that necessarily $y' = \mathrm{Id}$ and

$$P' = \{(i,j) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} : i < j,\ i + j \le n\}.$$

First, note that

$$\{k_1, \dots, k_n\} = \{0, 1, \dots, n-1\} \quad \text{and} \quad k_{n+1-r} > \ldots > k_{n-1} > k_n$$

since $\rho = \sum_{i=1}^{n} (n-i)\varepsilon_i$ and $y' \in S_n/S_r$. Also, recall the conditions on $\operatorname{supp}(\mathcal{A})$ given in (6).

We will prove that $k_i = n - i$ for all $i = 1, \dots, n$. Our base case is to show that $k_1 = n-1$, $k_n = 0$, and that $(1, i) \in P' \;\Leftrightarrow\; i \neq 1, n$. Now since $k_i \ge 1$ for $i \le r$ and $k_n < k_i$ for $i \in \{n+1-r, \dots, n-1\}$, we can have either $k_n = 0$ or $k_{r+1} = 0$ (in the case that $n = 2r+1$). However, if $k_{r+1} = 0$ we reach a contradiction that $(n-1)$ could not occur as a coefficient. Indeed, suppose $n = 2r + 1$ and $k_{r+1} = 0$ and take j such that $k_j = n - 1$. Then $j \leq r$ and the elements $(j, j), (j, r + 1), (j, n + 1 - j)$ are not in P'. Since $k_j = n - 1$, two of these pairs must coincide, implying $j = r + 1$, which is a contradiction. Hence, $k_n = 0$ and so $(i, n) \notin P'$ for all i. Take j such that $k_j = n - 1$; then $j \leq r$ and $(j, i) \in P'$ for all $i \neq j, n + 1 - j, n$. Thus $n + 1 - j = n$, that is $j = 1$. Therefore, we obtain $k_1 = n - 1$, $k_n = 0$. It follows that $(1, i) \in P'$ if and only if $i \neq 1, n$.

Let $t \leq r$, and suppose for the induction hypothesis that for all $i = 1, \ldots, t - 1$ we have: $k_i = n - i$, $k_{n+1-i} = i - 1$, and

$$(i, j) \in P' \;\Leftrightarrow\; i \leq \min\{j - 1, n - j\}. \tag{7}$$

One can prove that the induction hypothesis implies that $k_i \geq t$ for all $i \leq r$, and that $k_i < n - t$ for $i > r$. Suppose $k_p = n - t$ and $k_q = t - 1$. Then $t \leq p \leq r$ and $r < q \leq n + 1 - t$. It follows (indirectly) from the induction hypothesis that

$$(p, p), (p, n + 1 - p), (p, q), (p, n), (p, n - 1), (p, n - 2), \ldots, (p, n + 2 - t) \notin P'.$$

This implies that $p + q = n + 1$. It follows that $k_t = n - t$ since $p \neq q$ and $k_t > k_i$ for all $r < i < t$. Hence, $k_{n+1-t} = t - 1$. Finally, since the elements $(t, t), (t, n), (t, n - 1), \ldots, (t, n + 1 - t)$ are not in P' and yet $k_t = n - t$, we see that condition (7) also holds for $i = t$. This concludes the induction proof. Hence, $k_i = n - i$ for each i and $y' = \mathrm{Id}$. Therefore, $a_{y\rho} = 0$ for $y \in S_n/S_r$ such that $y \neq \mathrm{Id}$.

Next we will prove that $a_\rho = \operatorname{sgn} \tau_n$. For this, we need to show that $\operatorname{sgn} \tau_n = (-1)^{|P'|}$, where

$$\varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_r + \sum_{(i,j) \in P'} (\varepsilon_i + \varepsilon_j) = \rho.$$

We claim that $\operatorname{sgn} \tau_n = 1$ if n is even, and $\operatorname{sgn} \tau_n = (-1)^k$ for $n = 2k + 1$. Indeed, one can check directly that $\tau_1, \tau_2 = \mathrm{Id}$,

$$\tau_{2k+1} = (k + 1, k + 2, \ldots, 2k + 1)\tau_{2k}, \quad \tau_{2k} = (k + 1, k + 2, \ldots, 2k)\tau_{2k-1}.$$

Thus, $\operatorname{sgn} \tau_{2k+2} = \operatorname{sgn} \tau_{2k} = 1$, while $\operatorname{sgn} \tau_{2k+1} = (-1)(\operatorname{sgn} \tau_{2k-1}) = (-1)^k$, and the claim follows by induction. On the other hand, counting coefficients for $\rho - (\varepsilon_1 + \varepsilon_3 + \ldots + \varepsilon_{2r-1})$ yields

$$|P'| = \frac{1}{2}\left(\frac{n(n-1)}{2} - \left\lfloor \frac{n}{2} \right\rfloor\right).$$

If n is even then $|P'| = \frac{n(n-2)}{4}$, which is always even. If n is odd then $|P'| = \left(\frac{n-1}{2}\right)^2$, which is even precisely when $n = 4k + 1$ for some $k \in \mathbb{N}$. Thus $a_\rho = (-1)^{|P'|} = \operatorname{sgn} \tau$.

Therefore, $j = (\operatorname{sgn} \tau) r!$, and the theorem follows. $\square$

Here is another form of the thin denominator identity.

Theorem 3.2. *Let $\mathfrak{g}=\mathfrak{p}(n)$ and $\Delta_{\bar 1}^{+}=\Delta(\mathfrak{g}_{-1})$. Then*

$$e^{\rho}R=\sum_{w\in W}(\operatorname{sgn} w)w\left(\frac{e^{\rho}}{(1-e^{-\beta_1})(1-e^{-\beta_1-\beta_2})\cdots(1-e^{-\beta_1-\ldots-\beta_r})}\right),$$

where $r=\lfloor n/2\rfloor$ and $\beta_1=-(\varepsilon_1+\varepsilon_2),\beta_2=-(\varepsilon_3+\varepsilon_4),\ldots,\beta_r=-(\varepsilon_{2r-1}+\varepsilon_{2r})$.

Proof. For $\mu\in\mathfrak{h}^*$, write

$$X_\mu:=\mathcal{F}_W\left(\frac{e^{\mu}}{(1-e^{-\beta_1})(1-e^{-\beta_1-\beta_2})\cdots(1-e^{-\beta_1-\ldots-\beta_r})}\right).$$

We show that

$$X_\rho=\frac{1}{r!}\mathcal{F}_W\left(\frac{e^{\rho^{\Uparrow}}}{(1-e^{-\beta_1})(1-e^{-\beta_2})\cdots(1-e^{-\beta_r})}\right), \tag{8}$$

where $\rho^{\Uparrow}=\varepsilon_1+\varepsilon_3+\ldots+\varepsilon_{2r-1}$.

We first claim that $X_\rho=X_{\rho^{\Uparrow}}$. We expand X_ρ and $X_{\rho^{\Uparrow}}$ as a geometric series in the domain $|e^{\alpha}|<1$ for $\alpha>0$. Note that $w\beta_i>0$ for every $w\in W$. Thus,

$$\begin{aligned}X_\rho&=\sum_{m_1\geq m_2\geq\ldots\geq m_r\geq 0}\mathcal{F}_W\left(e^{\rho-m_1\beta_1-\ldots-m_r\beta_r}\right)\\&=\sum_{m_1\geq m_2\geq\ldots\geq m_r\geq 0}\mathcal{F}_W\left(e^{(n-1+m_1)\varepsilon_1+(n-2+m_1)\varepsilon_2+(n-3+m_2)\varepsilon_3+(n-4+m_3)\varepsilon_4+\ldots+(1+m_r)\varepsilon_{r-1}+m_r\varepsilon_r}\right)\end{aligned}$$

and

$$\begin{aligned}X_{\rho^{\Uparrow}}&=\sum_{m_1\geq m_2\geq\ldots\geq m_r\geq 0}\mathcal{F}_W\left(e^{\rho^{\Uparrow}-m_1\beta_1-\ldots-m_r\beta_r}\right)\\&=\sum_{m_1\geq m_2\geq\ldots\geq m_r\geq 0}\mathcal{F}_W\left(e^{(1+m_1)\varepsilon_1+m_1\varepsilon_2+(1+m_2)\varepsilon_3+m_2\varepsilon_4+\ldots+(1+m_r)\varepsilon_{r-1}+m_r\varepsilon_r}\right).\end{aligned}$$

Note that

$$\mathcal{F}_W\left(e^{(1+m_1)\varepsilon_1+m_1\varepsilon_2+(1+m_2)\varepsilon_3+m_2\varepsilon_4+\ldots+(1+m_r)\varepsilon_{r-1}+m_r\varepsilon_r}\right)$$

is nonzero only if $m_1,\ldots,m_r,m_r+1,\ldots,m_r+1$ are distinct. Since $m_1\geq m_2\geq\ldots\geq m_r\geq 0$, we get that $m_r\geq 0$, $m_{r-1}\geq 2$, $m_{r-2}\geq 4,\ldots,m_1\geq 2r-1$. Thus all the nonzero terms in X_ρ and $X_{\rho^{\Uparrow}}$ are the same and we get the equality.

Now

$$\begin{aligned}
X_{\rho^{\Uparrow}} &= \sum_{m_1>m_2>\ldots>m_r>0} \mathcal{F}_W\left(e^{(1+m_1)\varepsilon_1+m_1\varepsilon_2+(1+m_2)\varepsilon_3+m_2\varepsilon_4+\ldots+(1+m_r)\varepsilon_{r-1}+m_r\varepsilon_r}\right)\\
&= \frac{1}{r!}\sum_{m_1\neq m_2\neq\ldots\neq m_r\neq 0} \mathcal{F}_W\left(e^{(1+m_1)\varepsilon_1+m_1\varepsilon_2+(1+m_2)\varepsilon_3+m_2\varepsilon_4+\ldots+(1+m_r)\varepsilon_{r-1}+m_r\varepsilon_r}\right)\\
&= \frac{1}{r!}\sum_{m_1,m_2,\ldots,m_r\geq 0} \mathcal{F}_W\left(e^{(1+m_1)\varepsilon_1+m_1\varepsilon_2+(1+m_2)\varepsilon_3+m_2\varepsilon_4+\ldots+(1+m_l)\varepsilon_{r-1}+m_r\varepsilon_r}\right)\\
&= \frac{1}{r!}\mathcal{F}_W\left(\frac{e^{\rho^{\Uparrow}}}{(1-e^{-\beta_1})(1-e^{-\beta_2})\cdots(1-e^{-\beta_r})}\right),
\end{aligned}$$

and the claim follows from (8) and Theorem 3.1. □

Remark 3.3. After completing this paper, it was pointed out to us by V. Serganova that this theorem can also be proven using the following method: let V be the natural representation of $\mathfrak{gl}(n)$. Then $R_{-1}^{-1} = \operatorname{ch}\operatorname{Sym}(\bigwedge^2 V)$. Since the character of a simple $\mathfrak{gl}$-module of highest weight λ is $e^{-\rho}R_0^{-1}\cdot\mathcal{F}_W\left(e^{w(\lambda+\rho)}\right)$, one can use the decomposition of $\operatorname{Sym}(\bigwedge^2 V)$ into direct sum of simple $\mathfrak{gl}(n)$-modules to prove the formula. See [13, Prop. 2] and [24, Thm. 2D] for the decomposition.

4. Thick denominator identity for $\mathfrak{p}(n)$

In this section, we present denominator identities for the Borel subalgebra $\mathfrak{b}^{\text{thick}}$ of $\mathfrak{p}(n)$, namely when $\Delta_{\bar 1}^+ = \Delta(\mathfrak{g}_1)$. In this case, $\Delta_{\bar 1}^+ = \Delta_1$, so $\rho_{\bar 1} = \frac{n}{2}\sum_{i=1}^n \varepsilon_i$ and $\rho = \rho_n = -\sum_{i=1}^n i\varepsilon_i$. Let $R = \frac{R_0}{R_1}$, where $R_1 = \prod_{1\leq i\leq j\leq n}\left(1-e^{-(\varepsilon_i+\varepsilon_j)}\right)$.

We have the following theorem.

Theorem 4.1. *Let $\mathfrak{g} = \mathfrak{p}(n)$ and $\Delta_{\bar 1}^+ = \Delta(\mathfrak{g}_1)$. Then*

$$e^\rho R = \sum_{w\in W}(\operatorname{sgn} w)w\left(\frac{e^\rho}{(1-e^{-\beta_1})(1-e^{-\beta_1-\beta_2})\cdots(1-e^{-\beta_1-\ldots-\beta_n})}\right),$$

where $\beta_1 = 2\varepsilon_n,\ \beta_2 = 2\varepsilon_{n-1},\ldots,\beta_n = 2\varepsilon_1$.

Proof. We prove the identity by induction on n. For $n = 1$, the Weyl group W consists of the identity element and the only root is $\beta_1 = 2\varepsilon_1$. Thus the identity is evident. Suppose that the identity holds for $\mathfrak{p}(n-1)$.

Fix the obvious root embedding of $\mathfrak{p}(n-1)$ in $\mathfrak{p}(n)$ for which $\mathfrak{h}^*_{\mathfrak{p}(n-1)} = \operatorname{span}\{\varepsilon_2,\ldots,\varepsilon_n\}$. For this embedding, $\rho_{n-1} = -\sum_{i=1}^{n-1} i\varepsilon_{i+1}$ and $\rho_n = \rho_{n-1} - \varepsilon_1 - \ldots - \varepsilon_n$. Then

$$\text{RHS} = \mathcal{F}_W\left(\frac{e^{\rho_n}}{(1-e^{-2\varepsilon_n})(1-e^{-2\varepsilon_{n-1}-2\varepsilon_n})\cdots(1-e^{-2\varepsilon_1-\ldots-2\varepsilon_n})}\right)$$

$$= \frac{e^{-\varepsilon_1-\ldots-\varepsilon_n}}{1-e^{-2\varepsilon_1-\ldots-2\varepsilon_n}}\mathcal{F}_{S_n}\left(\frac{e^{\rho_{n-1}}}{(1-e^{-2\varepsilon_n})(1-e^{-2\varepsilon_{n-1}-2\varepsilon_n})\cdots(1-e^{-2\varepsilon_2-\ldots-2\varepsilon_n})}\right)$$
$$= \frac{e^{-\varepsilon_1-\ldots-\varepsilon_n}}{1-e^{-2\varepsilon_1-\ldots-2\varepsilon_n}}\mathcal{F}_{S_n/S_{n-1}}$$
$$\times \mathcal{F}_{S_{n-1}}\left(\frac{e^{\rho_{n-1}}}{(1-e^{-2\varepsilon_n})(1-e^{-2\varepsilon_{n-1}-2\varepsilon_n})\cdots(1-e^{-2\varepsilon_2-\ldots-2\varepsilon_n})}\right)$$
$$\overset{\text{induction}}{=} \frac{e^{-\varepsilon_1-\ldots-\varepsilon_n}}{1-e^{-2\varepsilon_1-\ldots-2\varepsilon_n}}\mathcal{F}_{S_n/S_{n-1}}\left(e^{\rho_{n-1}}R_{n-1}\right)$$
$$= \frac{1}{e^{\varepsilon_1+\ldots+\varepsilon_n}-e^{-\varepsilon_1-\ldots-\varepsilon_n}}\mathcal{F}_{S_n/S_{n-1}}\left(e^{\rho_{n-1}}R_{n-1}\right),$$

where S_n/S_{n-1} denotes a set of left coset representatives. Thus the theorem is equivalent to

$$\left(e^{\varepsilon_1+\ldots+\varepsilon_n}-e^{-\varepsilon_1-\ldots-\varepsilon_n}\right)e^{\rho_n}R_n = \mathcal{F}_{S_n/S_{n-1}}\left(e^{\rho_{n-1}}R_{n-1}\right). \tag{9}$$

To translate this identity to be an identity of finite expressions (and not rational functions), we multiply both sides of (9) by $R_{1,n} = R_{1,n-1}\prod_{i=1}^{n}\left(1-e^{-\varepsilon_1-\varepsilon_i}\right)$, which is W-invariant, and we get

$$\left(e^{\varepsilon_1+\ldots+\varepsilon_n}-e^{-\varepsilon_1-\ldots-\varepsilon_n}\right)e^{\rho_n}R_{0,n} = \mathcal{F}_{S_n/S_{n-1}}\left(e^{\rho_{n-1}}R_{0,n-1}\prod_{i=1}^{n}\left(1-e^{-\varepsilon_1-\varepsilon_i}\right)\right).$$

By the denominator identity of $\mathfrak{sl}(n)$ and the fact that $\rho_{n,\bar{0}}-\rho_n$ is S_n-invariant, we have

$$e^{\rho_n}R_{0,n} = \mathcal{F}_{S_n}\left(e^{\rho_n}\right) \quad \text{and} \quad e^{\rho_{n-1}}R_{0,n} = \mathcal{F}_{S_{n-1}}\left(e^{\rho_{n-1}}\right).$$

So the identity becomes

$$\left(e^{\varepsilon_1+\ldots+\varepsilon_n}-e^{-\varepsilon_1-\ldots-\varepsilon_n}\right)\mathcal{F}_{S_n}\left(e^{\rho_n}\right) = \mathcal{F}_{S_n/S_{n-1}}\left(\mathcal{F}_{S_{n-1}}\left(e^{\rho_{n-1}}\right)\prod_{i=1}^{n}\left(1-e^{-\varepsilon_1-\varepsilon_i}\right)\right). \tag{10}$$

Since the term $\prod_{i=1}^{n}\left(1-e^{-\varepsilon_1-\varepsilon_i}\right)$ is S_{n-1}-invariant, the RHS of (10) equals

$$\mathcal{F}_{S_n/S_{n-1}}\left(\mathcal{F}_{S_{n-1}}\left(e^{\rho_{n-1}}\right)\prod_{i=1}^{n}\left(1-e^{-\varepsilon_1-\varepsilon_i}\right)\right)$$
$$= \mathcal{F}_{S_n/S_{n-1}}\left(\mathcal{F}_{S_{n-1}}\left(e^{\rho_{n-1}}\prod_{i=1}^{n}\left(1-e^{-\varepsilon_1-\varepsilon_i}\right)\right)\right)$$
$$= \mathcal{F}_{S_n}\left(e^{\rho_{n-1}}\prod_{i=1}^{n}\left(1-e^{-\varepsilon_1-\varepsilon_i}\right)\right).$$

Hence, as $(e^{\varepsilon_1+\ldots+\varepsilon_n} - e^{-\varepsilon_1-\ldots-\varepsilon_n})$ is S_n-invariant, (10) becomes

$$\mathcal{F}_{S_n}\left(e^{\rho_n}\left(e^{\varepsilon_1+\ldots+\varepsilon_n} - e^{-\varepsilon_1-\ldots-\varepsilon_n}\right)\right) = \mathcal{F}_{S_n}\left(e^{\rho_{n-1}}\prod_{i=1}^{n}\left(1-e^{-\varepsilon_1-\varepsilon_i}\right)\right). \tag{11}$$

Finally, we are left to prove an equality between two S_n-anti-invariant finite expressions, and by Lemma 2.5, we are reduced to studying regular elements. The LHS of (11) has two S_n-orbits, which correspond to $\mathcal{F}_{S_n}\left(e^{\rho_n+\varepsilon_1+\ldots+\varepsilon_n}\right)$ and $\mathcal{F}_{S_n}\left(-e^{\rho_n-\varepsilon_1-\ldots-\varepsilon_n}\right)$.

By expanding the inside of the RHS of (11) we obtain

$$e^{\rho_{n-1}}\prod_{i=1}^{n}\left(1-e^{-\varepsilon_1-\varepsilon_i}\right) = \sum_{A\subset\{\varepsilon_1+\varepsilon_i|i=1,\ldots,n\}} a_{\lambda_A}e^{\lambda_A}, \tag{12}$$

where $\lambda_A = \rho_{n-1} - \sum_{\alpha\in A}\alpha$ and $a_{\lambda_A} = (-1)^{|A|}$.

If A is empty, then $\lambda_A = \rho_{n-1} = \rho_n + \varepsilon_1 + \ldots + \varepsilon_n$ is regular and $a_{\lambda_A} = 1$. If A is the entire set, then $\lambda_A = -2\varepsilon_2 - 3\varepsilon_3 - \ldots - n\varepsilon_n - (n+1)\varepsilon_1$ is regular and $a_{\lambda_A} = (-1)^n$. In the latter case, $\lambda_A = y(\rho_n - \varepsilon_1 - \ldots - \varepsilon_n)$ where y is the permutation $(12\ldots n)$ and $\operatorname{sgn} y = (-1)^{n-1}$. We claim that if n is odd then these are the only two regular elements in the RHS of (11), while if n is even then we have two more regular elements that cancel each other in the sum. This will imply that (12) holds as required.

Now suppose $A \subsetneq \{\varepsilon_1 + \varepsilon_i \mid i = 1, \ldots, n\}$ and A is nonempty. Write

$$\lambda_A = \sum_{i=1}^{n} b_i\varepsilon_i.$$

Then the coefficients $b_1, \ldots, b_n$ are contained in $\{1, 2, \ldots, n\}$, and they are distinct since λ_A is assumed to be regular. Moreover, for $k \geq 2$, either $b_k = k - 1$ or $b_k = k$. Let $k \geq 2$ be the smallest integer for which $b_k = k$. Then $b_i = i$ for all $i \geq k$, since $b_1, \ldots, b_n$ are distinct. It follows that $k > 2$ since we assume that A is not the entire set and that λ_A is regular. We have two possibilities: $A = \{\varepsilon_1 + \varepsilon_k, \ldots, \varepsilon_1 + \varepsilon_n\}$ and $A' = \{\varepsilon_1 + \varepsilon_k, \ldots, \varepsilon_1 + \varepsilon_n, 2\varepsilon_1\}$. Regularity implies that $k = \frac{n}{2} + 1$ in the former case, while $k = \frac{n}{2} + 2$ in the latter case. Clearly, this implies that n is even. Finally, since λ_A differs from $\lambda_{A'}$ only by the transposition $(1\ \frac{n}{2})$, we conclude that these two additional regular elements cancel each other in the sum. □

Remark 4.2. Similarly to Remark 3.3, in the thick case, $R_1^{-1} = \operatorname{ch}\left(\operatorname{Sym}(\operatorname{Sym}^2 V)\right)$ where V is the natural $\mathfrak{gl}(n)$-module. Then Theorem 4.1 can also be proven using the decomposition of $\operatorname{Sym}(\operatorname{Sym}^2 V)$ into simple $\mathfrak{gl}(n)$-modules. See [13, Prop. 1], [24, Thm. 2C], and [25, Thm. III] for the decomposition.

5. Some remarks

5.1. The character version of the denominator identity

The denominator identities written in this paper are given in terms of supercharacters. One can translate them into characters. In this case R_0 stays the same, $R_1 = \prod_{\alpha\in\Delta(\mathfrak{g}_1)}(1+e^{-\alpha})$, the identity in Theorem 3.1 takes the form

$$e^{\rho}R = \frac{1}{r!}\sum_{w\in W}(\operatorname{sgn} w)w\left(\frac{e^{\rho^{\Uparrow}}}{\prod_{\beta\in S}(1+e^{-\beta})}\right),$$

and the identities in Theorem 3.2 and Theorem 4.1 take the form

$$e^{\rho}R = \mathcal{F}_W\left(\frac{e^{\rho}}{(1+e^{-\beta_1})(1-e^{-\beta_1-\beta_2})\cdots(1+(-1)^r e^{-\beta_1-\ldots-\beta_r})}\right)$$

for the appropriate choices of $\beta_1,\ldots,\beta_r$.

5.2. Representation-theoretical meaning of the denominator identity

It would be interesting to find a complex of thin Kac modules (or thick Kac modules) whose Euler characteristic yields the denominator identity.

For a given dominant integral weight λ, we let $V(\lambda)$ denote the simple $\mathfrak{g}_0$-module with highest weight λ with respect to the fixed Borel $\mathfrak{b}_0$ of $\mathfrak{g}_0$. The thin Kac module corresponding to λ is defined to be $\nabla(\lambda) := \operatorname{Coind}^{\mathfrak{g}}_{\mathfrak{g}_0\oplus\mathfrak{g}_1} V(\lambda)$, where we take the parity of the superspace $V(\lambda)$ to be purely even or odd according the sign convention used in [15, Section 2.3], and denote this parity by $\operatorname{sgn}\lambda$. Then the supercharacter of $\nabla(\lambda)$ is

$$\operatorname{sch}\nabla(\lambda) = (\operatorname{sgn}\lambda)\frac{R_{-1}}{e^{\rho}R_0}\cdot\mathcal{F}_W\left(e^{\lambda+\rho}\right)$$

(see [15, Lemma 2.4.1]). After substitution, the formula in Theorem 3.2 takes the form

$$\operatorname{sch} L(0) = \sum_{i_1\geq\ldots\geq i_r\geq 0}(-1)^{i_1+\ldots+i_r}\operatorname{sch}\nabla\left(-i_1\beta_1-\ldots-i_r\beta_r\right).$$

For $r = 2$, we conjecture that we have the following bi-complex:

$$\begin{tikzcd}
\ddots & & & & & & & \\
\dots & \nabla(-n\beta_1 - n\beta_2) & & & & & & \\
\dots & \vdots \arrow[u] & \ddots & & & & & \\
\dots & \nabla(-n\beta_1 - 2\beta_2) \arrow[u] & \dots \arrow[l] & \nabla(-2\beta_1 - 2\beta_2) \arrow[l] & & & & \\
\dots & \nabla(-n\beta_1 - \beta_2) \arrow[u] & \dots \arrow[l] & \nabla(-2\beta_1 - \beta_2) \arrow[l] \arrow[u] & \nabla(-\beta_1 - \beta_2) \arrow[l] & & & \\
\dots & \nabla(-n\beta_1) \arrow[u] & \dots \arrow[l] & \nabla(-2\beta_1) \arrow[l] \arrow[u] & \nabla(-\beta_1) \arrow[l] \arrow[u] & \nabla(0) \arrow[l] & L(0) \arrow[l] & 0. \arrow[l]
\end{tikzcd}$$

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