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# Ad-nilpotent ideals of a Borel subalgebra II

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## Abstract

We provide an explicit bijection between the ad-nilpotent ideals of a Borel subalgebra of a simple Lie algebra  $\mathfrak{g}$  and the orbits of  $\check{Q}/(h+1)\check{Q}$  under the Weyl group ( $\check{Q}$  being the coroot lattice and  $h$  the Coxeter number of  $\mathfrak{g}$ ). From this result we deduce in a uniform way a counting formula for the ad-nilpotent ideals.

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## 1. Introduction

Let  $\mathfrak{g}$  be a complex simple Lie algebra of rank  $n$ . Let  $\mathfrak{b} \subset \mathfrak{g}$  be a fixed Borel subalgebra, with Cartan component  $\mathfrak{h}$ , and let  $\Delta^+$  be the positive system of the root system  $\Delta$  of  $\mathfrak{g}$  corresponding to the previous choice. For each  $\alpha \in \Delta^+$  let  $\mathfrak{g}_\alpha$  denote the root space of  $\mathfrak{g}$  relative to  $\alpha$ , and  $\mathfrak{n} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$ , so that  $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ .

In this paper we continue the analysis, started in [CP1], of the set  $\mathcal{I}$  of ad-nilpotent ideals of  $\mathfrak{b}$ , i.e. the ideals of  $\mathfrak{b}$  consisting of ad-nilpotent elements. These

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ideals are precisely the ideals of  $\mathfrak{b}$  which are contained in  $\mathfrak{n}$ ; in particular the abelian ideals of  $\mathfrak{b}$  are ad-nilpotent.

The abelian ideals of Borel subalgebras were studied by Kostant in [Ko1, Ko2] in connection with representation theory of semisimple Lie groups, and very recently by Panyushev and Röhrle [PR] in connection with the theory of spherical orbits. In particular, in [Ko2] Kostant detailed the following unpublished theorem of D. Peterson: the abelian ideals of  $\mathfrak{b}$  are  $2^n$  in number, independently of the type of  $\mathfrak{g}$ . In fact, Peterson gives a bijection between the abelian ideals of  $\mathfrak{b}$  and a certain set of elements of the affine Weyl group  $\widehat{W}$  of  $\mathfrak{g}$ . This leads to look for a similar result for all ad-nilpotent ideals. In [CP1] we showed how to associate to any ideal  $\mathfrak{i}$  in  $\mathcal{I}$  a uniquely determined element  $w_{\mathfrak{i}}$  in  $\widehat{W}$ , and we characterized the set  $\{w_{\mathfrak{i}} \mid \mathfrak{i} \in \mathcal{I}\}$  inside  $\widehat{W}$ . In this paper we develop our previous results and prove the following theorem.

Let  $W$  denote the Weyl group of  $\mathfrak{g}$ , and  $Q, \check{Q}$  be the root and coroot lattices, respectively. Moreover let  $h$  be the Coxeter number of  $W$  and  $e_1, \dots, e_n$  be the exponents of  $W$  [Hu, Section 3.19]. We consider the natural action of  $W$  on  $\check{Q}/(h+1)\check{Q}$ .

**Theorem 1.** *There exists an explicit bijection between the set of ad-nilpotent ideals of  $\mathfrak{b}$  and the set of  $W$ -orbits in  $\check{Q}/(h+1)\check{Q}$ . In particular, the number of the ad-nilpotent ideals of  $\mathfrak{b}$  is*

$$\frac{1}{|W|} \prod_{i=1}^n (h + e_i + 1). \tag{1}$$

The fact that formula (1) counts  $W$ -orbits in  $\check{Q}/(h+1)\check{Q}$  follows from a theorem of M. Haiman [Ha, Theorem 7.4.4].

As we shall recall in Section 4, the ad-nilpotent ideals of  $\mathfrak{b}$  are naturally in bijection with the antichains of the root poset  $(\Delta^+, \leq)$ , hence with the  $\oplus$ -sign types of  $\check{\Delta}$ , and with the regions of the Catalan hyperplane arrangement which are contained in the fundamental chamber of  $W$ . So our result affords a uniform enumeration for all these objects. In particular, it answers the question raised in [S, Remark 3.7] regarding the (uniform) enumeration of sign types.

Formula (1) for the ad-nilpotent ideals already appears at the end of the introduction of [KOP], where the authors asked for a proof of it avoiding case by case inspection. Our main theorem also improves the known results on the Catalan arrangement extending to any root system the enumeration formula (1), which was proved by Athanasiadis for the classical systems [At1, At2].

We remark that Sommers [So] gives a generalization of formula (1), expressing the Euler characteristic of the space of partial flags containing a certain regular semisimple nil-elliptic element  $n_t$  in an affine Lie algebra.

Our paper is organized as follows. In the next section we fix notation and recall some basic facts about affine Weyl groups. In Section 3 we prove Theorem 1 after

recalling the results of [CP1] which are needed for the proof. In Section 4 we briefly recall the known bijections between ad-nilpotent ideals, antichains of the root poset  $(\Delta^+, \leq)$ ,  $\oplus$ -sign types of  $\check{\Delta}$ , and regions of the Catalan hyperplane arrangement contained in the fundamental chamber. In Section 5 we illustrate the bijection of Theorem 1 for the root types  $A_2$  and  $B_2$ .

## 2. Notation and preliminaries

Let  $\Pi = \{\alpha_1, \dots, \alpha_n\}$  be the simple roots of  $\Delta^+$ . We set  $V \equiv \mathfrak{h}_{\mathbb{R}}^* = \bigoplus_{i=1}^n \mathbb{R}\alpha_i$  and denote by  $(, )$  the positive symmetric bilinear form induced on  $V$  by the Killing form. We describe the affine root system associated to  $\Delta$  as follows [Kac, Chapter 6]. We extend  $V$  and its inner product setting  $\widehat{V} = V \oplus \mathbb{R}\delta \oplus \mathbb{R}\lambda$ ,  $(\delta, \delta) = (\delta, V) = (\lambda, \lambda) = (\lambda, V) = 0$ , and  $(\delta, \lambda) = 1$ . We still denote by  $(, )$  the resulting (non-degenerate) bilinear form. The affine root system associated to  $\Delta$  is  $\widehat{\Delta} = \Delta + \mathbb{Z}\delta = \{\alpha + k\delta \mid \alpha \in \Delta, k \in \mathbb{Z}\}$ ; remark that the affine roots are non-isotropic vectors. The set of positive affine roots is  $\widehat{\Delta}^+ = (\Delta^+ + \mathbb{N}\delta) \cup (\Delta^- + \mathbb{N}^+\delta)$ , where  $\Delta^- = -\Delta^+$ . We denote by  $\theta$  the highest root of  $\Delta$  and set  $\alpha_0 = -\theta + \delta$ ,  $\widehat{\Pi} = \{\alpha_0, \dots, \alpha_n\}$ . For each  $\alpha \in \widehat{\Delta}^+$  we denote by  $s_\alpha$  the corresponding reflection of  $\widehat{V}$ ; the affine Weyl group associated to  $\Delta$  is the group  $\widehat{W}$  generated by  $\{s_\alpha \mid \alpha \in \widehat{\Delta}^+\}$ . Note that  $w(\delta) = \delta$  for each  $w \in \widehat{W}$ .

$\widehat{W}$  is a semidirect product  $T \rtimes W$ , where  $T = \{t_\tau \mid \tau \in \check{Q}\} \cong \check{Q}$  is the subgroup of translations, and the action of  $W$  on  $T$  is  $vt_\tau v^{-1} = t_{v(\tau)}$ . The action of  $t_\tau$  on  $V \oplus \mathbb{R}\delta$ , in particular on the roots, is given by  $t_\tau(x) = x - (x, \tau)\delta$ , for each  $x \in V \oplus \mathbb{R}\delta$ . (See [Kac] for the general definition of  $t_\tau$  on  $\widehat{V}$ .)

Consider the  $\widehat{W}$ -invariant affine subspace  $E = \{x \in V \mid (x, \delta) = 1\} = V \oplus \mathbb{R}\delta + \lambda$ . Let  $\pi: E \rightarrow V + \lambda$  be the natural projection. For  $w \in \widehat{W}$  we set  $\bar{w} = \pi \circ w|_{V+\lambda}$ . Then the map  $w \mapsto \bar{w}$  is injective. We identify  $V + \lambda$  with  $V$  through the natural projection. In this way  $w \mapsto \bar{w}$  induces an isomorphism of  $\widehat{W}$  onto a group  $W_{af}$  of affine transformations of  $V$ , which is in fact the usual affine representation of the affine Weyl group [B, VI, §2]. For  $v \in W$ ,  $\bar{v}$  is simply the restriction of  $v$  to  $V$ , so we omit the bar. For  $\tau \in \check{Q}$ ,  $\bar{t}_\tau$  is the true translation by  $\tau$ .

For  $k \in \mathbb{N}^+$ , we set

$$C_\infty = \{x \in V \mid (x, \alpha_i) > 0 \text{ for each } i \in \{1, \dots, n\}\},$$

$$C_k = \{x \in C_\infty \mid (x, \theta) < k\}.$$

$C_\infty$  is the fundamental chamber of  $W$  in  $V$ , and  $C_1$  is the fundamental alcove of  $W_{af}$  in  $V$ . The closures  $\bar{C}_\infty, \bar{C}_1$  are fundamental domains for the actions on  $V$  of  $W, W_{af}$ , respectively.

As usual we denote by  $\{\check{\omega}_1, \dots, \check{\omega}_n\}$  the dual basis of  $\{\alpha_1, \dots, \alpha_n\}$  and set  $\alpha_i = \check{\omega}_i/m_i$ , where  $\theta = m_1\alpha_1 + \dots + m_n\alpha_n$ . For each  $k \in \mathbb{N}^+$ ,  $\bar{C}_k (= k\bar{C}_1)$  is the simplex whose vertices are  $0, k\alpha_1, \dots, k\alpha_n$ .

Let  $\bar{T}^k = \{\bar{\tau}_\tau \mid \tau \in k\check{Q}\}$  and set  $W_k = \bar{T}^k \rtimes W$ . Note that  $W_k$  is the affine Weyl group of  $\frac{1}{k}\Delta$  and  $C_k$  is its fundamental chamber with respect to  $\frac{1}{k}\Pi$ . In particular,  $\check{Q} \cap \bar{C}_k$  is a set of representatives of the orbits of  $\check{Q}$  under the natural action of  $W_k$ . For  $\tau \in \check{Q}$ , consider its orbit  $W_k(\tau)$ . We have  $W_k(\tau) = W(\tau) + k\check{Q} = W(\tau + k\check{Q})$ , hence the orbits of  $\check{Q}$  under  $W_k$  naturally correspond to the orbits of  $\check{Q}/k\check{Q}$  under the action of  $W$  [Ha]. In fact, in order to prove Theorem 1, we shall prove that  $\mathcal{I}$  is in bijection with  $\check{Q} \cap \bar{C}_{h+1}$ .

### 3. Proof of Theorem 1

In [CP1] we found an explicit encoding of the elements of  $\mathcal{I}$  by means of certain elements of  $\widehat{W}$ . We briefly recall this result. By definition any ad-nilpotent ideal  $\mathfrak{i}$  of  $\mathfrak{b}$  is included in  $\mathfrak{n}$ . Such an ideal  $\mathfrak{i}$  is, in particular,  $\text{ad}(\mathfrak{h})$ -stable, so there exists  $\Phi_i \subseteq \Delta^+$  such that  $\mathfrak{i} = \bigoplus_{\alpha \in \Phi_i} \mathfrak{g}_\alpha$ . We set  $\Phi_i^1 = \Phi_i$  and  $\Phi_i^{k+1} = (\Phi_i^k + \Phi_i) \cap \Delta^+$ , for each  $k \in \mathbb{N}^+$ , so that  $\bigoplus_{\alpha \in \Phi_i^k} \mathfrak{g}_\alpha$  equals  $\mathfrak{i}^{(k)}$ , the  $k$ th element of the descending central series of  $\mathfrak{i}$ . Then we associate to  $\mathfrak{i}$  the following set of positive affine roots:  $L_i = \bigcup_{k \geq 1} (-\Phi_i^k + k\delta)$ . In [CP1, Proposition 2.4] we proved that there exists a (unique)  $w_i \in \widehat{W}$  such that  $L_i = N(w_i) = \{\alpha \in \widehat{\Delta}^+ \mid w_i^{-1}(\alpha) < 0\}$ . Then  $\mathfrak{i} \mapsto w_i$  is the required encoding. We remark that  $w_i$  is explicitly determined by  $L_i$ . We also gave the following characterization, which will play a crucial role in the sequel.

**Proposition 1** [CP1, Proposition 2.12]. *Let  $w \in \widehat{W}$ . Then  $w = w_i$  for some  $\mathfrak{i} \in \mathcal{I}$  if and only if the following conditions hold:*

- (a)  $w^{-1}(\alpha) > 0$  for each  $\alpha \in \Pi$ ;
- (b) if  $w(\alpha) < 0$  for some  $\alpha \in \widehat{\Pi}$ , then  $w(\alpha) = \beta - \delta$  for some  $\beta \in \Delta^+$ .

For  $\alpha \in \Delta^+$  and  $l \in \mathbb{Z}$  set  $H_{\alpha,l} = \{x \in V \mid (x, \alpha) = l\}$ . We recall that, for  $\alpha \in \Delta^+$ ,  $l \in \mathbb{N}^+$ ,  $m \in \mathbb{N}$ ,  $w \in \widehat{W}$ , we have  $w^{-1}(-\alpha + l\delta) < 0$  if and only if  $H_{\alpha,l}$  separates  $C_1$  and  $\bar{w}(C_1)$ , and  $w^{-1}(\alpha + m\delta) < 0$  if and only if  $H_{\alpha,-m}$  separates  $C_1$  and  $\bar{w}(C_1)$ . From Proposition 1 we obtain the following characterization.

**Proposition 2.** *Let  $w \in \widehat{W}$ ,  $w = t_\tau v$ ,  $\tau \in \check{Q}$ ,  $v \in W$ . Set  $\beta_i = v(\alpha_i)$  for  $i \in \{1, \dots, n\}$ . Then  $w = w_i$  for some  $\mathfrak{i} \in \mathcal{I}$  if and only if the following conditions hold:*

- (i)  $\bar{w}(C_1) \subset C_\infty$ ;
- (ii)  $(\tau, \beta_i) \leq 1$  for each  $i \in \{1, \dots, n\}$  and  $(\tau, v(\theta)) \geq -2$ .

**Proof.** It is clear that condition (a) of Proposition 1 is equivalent to  $\bar{w}(C_1) \subset C_\infty$ .

Assume  $w = w_i$  for some  $i \in \mathcal{I}$ . Then (i) holds and this implies, in particular, that  $\tau \in \overline{C}_\infty$ . Since  $\beta_i \in \Delta$ , if  $\beta_i < 0$  then  $(\tau, \beta_i) \leq 0$ . So we may assume  $\beta_i > 0$ . We have  $w(\alpha_i) = \beta_i - (\tau, \beta_i)\delta$ , hence if  $(\tau, \beta_i) > 0$  we obtain  $w(\alpha_i) < 0$ . Then by Proposition 1,  $w(\alpha_i) = \beta_i - \delta$  and thus  $(\tau, \beta_i) = 1$ . Then we consider  $v(\theta)$ . We have  $w(\alpha_0) = -v(\theta) + ((\tau, v(\theta)) + 1)\delta$ . If  $(\tau, v(\theta)) < -1$  then  $w(\alpha_0) < 0$ , hence by Proposition 1,  $(\tau, v(\theta)) + 1 = -1$ , hence  $(\tau, v(\theta)) = -2$ .

Conversely, assume that (i) and (ii) hold. Then condition (a) of Proposition 1 holds. Let  $1 \leq i \leq n$  and  $w(\alpha_i) < 0$ . Then either  $(\tau, \beta_i) > 0$ , or  $(\tau, \beta_i) = 0$  and  $\beta_i < 0$ . The latter case cannot occur, otherwise, for  $x \in C_1$  we would have  $(\overline{w}(x), \beta_i) = (v(x), \beta_i) = (x, \alpha_i) > 0$ , which is impossible, since  $\overline{w}(x)$  belongs to  $\overline{C}_\infty$  and  $\beta_i$  is negative. So we have  $(\tau, \beta_i) > 0$ , hence, by assumption  $(\tau, \beta_i) = 1$ , so that  $w(\alpha_i) = \beta_i - \delta$ . Finally assume  $w(\alpha_0) < 0$ . Then either  $(\tau, v(\theta)) + 1 < 0$ , or  $(\tau, v(\theta)) + 1 = 0$  and  $v(\theta) > 0$ . As above we see that the latter case cannot occur, so, by assumption,  $(\tau, v(\theta)) = -2$ . This implies  $v(\theta) < 0$  and  $w(\alpha_0) = -v(\theta) - \delta$ , hence the claim.  $\square$

Set

$$D = \{ \tau \in \check{Q} \mid (\tau, \alpha_i) \leq 1 \text{ for each } i \in \{1, \dots, n\} \text{ and } (\tau, \theta) \geq -2 \}.$$

Assume  $w_i = t_{\tau_i} v_i$  for some  $i \in \mathcal{I}$ ,  $\tau_i \in \check{Q}$ ,  $v_i \in W$ . Then by Proposition 2 we have  $(\tau_i, \beta_j) \leq 1$  for each  $j \in \{1, \dots, n\}$  and  $(\tau_i, v_i(\theta)) \geq -2$ , hence  $(v_i^{-1}(\tau_i), \alpha_j) \leq 1$  for each  $j \in \{1, \dots, n\}$  and  $(v_i^{-1}(\tau_i), \theta) \geq -2$ . It follows that  $t_{\tau_i} v_i \mapsto v_i^{-1}(\tau_i)$  is a map from  $\{w_i \mid i \in \mathcal{I}\}$  to  $D$ .

**Proposition 3.** *The map  $F : w_i = t_{\tau_i} v_i \mapsto v_i^{-1}(\tau_i)$  is a bijection between  $\{w_i \mid i \in \mathcal{I}\}$  and  $D$ .*

**Proof.** Set, for notational simplicity,  $w_i = t_\tau v$ ,  $w_j = t_\sigma u$  for some  $i$  and  $j$  in  $\mathcal{I}$ ,  $\tau, \sigma \in \check{Q}$  and  $v, u \in W$ . Assume  $v^{-1}(\tau) = u^{-1}(\sigma)$ . Since  $\tau, \sigma \in \overline{C}_\infty$ , which is a fundamental domain for  $W$ , we have  $\tau = \sigma$  and  $vu^{-1}(\tau) = \tau$ . Hence  $\bar{t}_\tau v(C_1) = \bar{t}_\tau vu^{-1}u(C_1) = vu^{-1}(\bar{t}_\tau u(C_1)) = vu^{-1}(\bar{t}_\sigma u(C_1)) \subset vu^{-1}(C_\infty)$ . But  $\bar{t}_\tau v(C_1) \subset C_\infty$ , hence  $vu^{-1} = 1$ . Thus  $F$  is injective. Next let  $\sigma \in D$ . We first see that there exists  $v \in W$  such that  $t_{v(\sigma)}v(C_1) \subset C_\infty$ : simply take the unique  $v \in W$  such that  $v(\sigma + C_1) \subset C_\infty$ . Now it is immediate that, since  $\sigma \in D$ ,  $t_{v(\sigma)}v$  also satisfies condition (ii) of Proposition 2, hence  $t_{v(\sigma)}v = w_i$  for some  $i$  in  $\mathcal{I}$ . It is obvious that  $F$  maps  $t_{v(\sigma)}v$  to  $\sigma$ , thus  $F$  is surjective.  $\square$

**Remark.** In a forthcoming paper [CP2] we provide characterizations for the elements of  $D$  corresponding through  $F$  to abelian ideals and, among them, for those encoding maximal abelian ideals.

Let  $\check{P} = \mathbb{Z}\check{\omega}_1 + \dots + \mathbb{Z}\check{\omega}_n$  be the coweight lattice of  $W$ . We denote by  $W'_{af}$  the extended affine Weyl group,  $W'_{af} = \bar{T}' \rtimes W$ , with  $\bar{T}' = \{\bar{t}_\tau \mid \tau \in \check{P}\}$ ,  $\bar{t}_\tau$  the translation by  $\tau$ . As usual, we set  $f = [W'_{af} : W_{af}] = [\check{P} : \check{Q}]$ .

**Lemma 1.** *Assume that  $k$  and  $f$  are relatively prime. Then for each  $w' \in W'_{af}$  there exists  $w \in W_{af}$  such that  $w'(C_k) = w(C_k)$ .*

**Proof.** Let  $\theta = \sum_{i=1}^n m_i \alpha_i$  and  $J = \{i \mid m_i = 1\}$ . By [IM, Sections 1.7 and 1.8],  $\{0\} \cup \{\check{\omega}_j \mid j \in J\}$  is a set of representatives of  $\check{P}/\check{Q}$ . Moreover, for each  $j \in J$ ,  $C_k = t_{k\check{\omega}_j} w_0^j w_0(C_k)$ , where  $w_0$  is the longest element of  $W$  and  $w_0^j$  is the longest element in the maximal parabolic subgroup of  $W$  generated by the reflections with respect to the  $\alpha_i$  with  $i \neq j$ . It suffices to prove the lemma for  $w' \in \bar{T}'$ ; let  $w' = \bar{t}_\sigma$  with  $\sigma \in \check{P}$ . Then we have  $w'(C_k) = \bar{t}_\sigma(C_k) = \bar{t}_{\sigma+k\check{\omega}_j} w_0^j w_0(C_k)$ , for each  $j \in J$ . If  $k$  and  $f$  are relatively prime, then  $\{0\} \cup \{k\check{\omega}_j \mid j \in J\}$  and hence  $\{\sigma\} \cup \{\sigma + k\check{\omega}_j \mid j \in J\}$  still are sets of representatives of  $\check{P}/\check{Q}$ . It follows that exactly one element in  $\{\sigma\} \cup \{\sigma + k\check{\omega}_j \mid j \in J\}$  belongs to  $\check{Q}$ , hence one among  $\bar{t}_\sigma, \bar{t}_{\sigma+k\check{\omega}_j} w_0^j w_0, j \in J$ , belongs to  $W_{af}$ .  $\square$

**Remark.** A direct check shows that the prime divisors of  $f$  also divide the Coxeter number of  $W$ . Hence the assumption of Lemma 1 is satisfied by any integer  $k$  relatively prime to  $h$ .

**Theorem 1.**  *$\mathcal{I}$  is in bijection with the orbits of  $\check{Q}/(h+1)\check{Q}$  under  $W$ .*

**Proof.** Let  $X = \{x \in V \mid (x, \alpha_i) \leq 1 \text{ for each } i \in \{1, \dots, n\} \text{ and } (x, \theta) \geq -2\}$  and  $\check{\rho} = \check{\omega}_1 + \dots + \check{\omega}_n$  be the half sum of positive coroots. We have that  $(\check{\rho}, \theta) = h - 1$ , thus  $X$  is the simplex whose vertices are  $\check{\rho}$  and  $\check{\rho} - (h + 1)\alpha_i$ , for  $i = 1, \dots, n$ . Hence  $X = t_{\check{\rho}} w_0(\bar{C}_{h+1})$ . By Lemma 1 and the above remark there exists  $w \in W_{af}$  such that  $X = w(\bar{C}_{h+1})$ . Such a  $w$  gives a bijection from  $\bar{C}_{h+1} \cap \check{Q}$  to  $D = X \cap \check{Q}$ . If  $i \in \mathcal{I}$  and  $w_i = t_{\tau_i} v_i$ , with  $\tau_i \in \check{Q}$  and  $v_i \in W$ , then, using Proposition 3, we obtain that  $w^{-1} v_i^{-1}(\tau_i)$  belongs to  $\bar{C}_{h+1} \cap \check{Q}$  and  $i \mapsto w^{-1} v_i^{-1}(\tau_i)$  is a bijection between  $\mathcal{I}$  and  $\bar{C}_{h+1} \cap \check{Q}$ . This concludes the proof, since, as we observed in Section 2, the cosets of the elements in  $\bar{C}_{h+1} \cap \check{Q}$  are a natural set of representatives of the orbits of  $\check{Q}/(h+1)\check{Q}$  under the action of  $W$ .  $\square$

We can explicitly determine the element  $w$  which appears in the above proof. Indeed we shall compute  $w^{-1}$ . If  $\check{\rho} \in \check{Q}$ , then trivially  $w^{-1} = w_0 t_{-\check{\rho}}$ . Otherwise, according to the proofs of Lemma 1 and of Theorem 1, there exists a unique  $j \in J$  such that the vertex  $\check{\rho} - (h + 1)\alpha_j = \check{\rho} - (h + 1)\check{\omega}_j$  of  $X$  belongs to  $\check{Q}$ . Now observe that  $w_0^j$  maps  $\{\alpha_i \mid i \neq j\}$  to  $\{-\alpha_i \mid i \neq j\}$  and maps  $\alpha_j$  and  $\theta$  into positive

roots. For any root  $\alpha$  let  $\text{ht}(\alpha) = (\alpha, \check{\rho})$  be the height of  $\alpha$ . Then  $\text{ht}(\theta) = h - 1$ , and, since  $j \in J$ ,  $\text{ht}(w_0^j(\theta)) = \text{ht}(w_0^j(\alpha_j)) - (h - 2)$ . Since  $w_0^j(\theta)$  is positive this implies that  $\text{ht}(w_0^j(\alpha_j)) = h - 1$ , hence  $w_0^j(\alpha_j) = \theta$  and  $w_0^j(\theta) = \alpha_j$ . It is easily seen that this implies that  $w_0^j t_{-\check{\rho} + (h+1)\check{\omega}_j}(X) = \bar{C}_{h+1}$ . Hence in order to determine  $w$  it suffices to determine the above  $j$ .

Numbering the fundamental weights as in [B], by a direct computation we obtain:

$$\begin{aligned}
 A_n: \quad & \check{\rho} \in \check{Q} \text{ for } n \text{ even}; \quad j = \frac{n+1}{2} \text{ for } n \text{ odd}; \\
 B_n: \quad & \check{\rho} \in \check{Q} \text{ for } n \equiv 0, 3 \pmod{4}; \quad j = 1 \text{ for } n \equiv 1, 2 \pmod{4}; \\
 C_n: \quad & j = n; \\
 D_n: \quad & \check{\rho} \in \check{Q} \text{ for } n \equiv 0, 1 \pmod{4}; \quad j = 1 \text{ for } n \equiv 2, 3 \pmod{4}; \\
 E_7: \quad & j = 7; \\
 E_6, E_8, F_4, G_2: \quad & \check{\rho} \in \check{Q}.
 \end{aligned}$$

#### 4. The other bijections

##### 4.1. A bijection between ad-nilpotent ideals of $\mathfrak{b}$ and antichains of the root poset

In Section 3 we observed that any ad-nilpotent ideal of  $\mathfrak{b}$  is a sum of (positive) root spaces. For  $\Phi \subseteq \Delta^+$ , set  $\mathfrak{i}_\Phi = \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$ . If  $\mathfrak{i}_\Phi$  is an ideal of  $\mathfrak{b}$ , then  $\alpha \in \Phi$ ,  $\beta \in \Delta^+$ ,  $\alpha + \beta \in \Delta$  implies  $\alpha + \beta \in \Phi$ . If we endow  $\Delta^+$  with the usual partial order (i.e.  $\alpha \leq \beta$  if  $\beta - \alpha = \sum_{\gamma \in \Delta^+} c_\gamma \gamma$ ,  $c_\gamma \in \mathbb{N}$ ), then, by definition,  $\Phi$  is a dual order ideal of  $(\Delta^+, \leq)$ .

It is a general fact that, in a finite poset  $P$ , dual order ideals and antichains (i.e. sets consisting of pairwise non-comparable elements) are in canonical bijection: map the antichain  $\{a_1, \dots, a_k\}$  to the dual order ideal which is the union of the principal dual order ideals  $V_{a_1}, \dots, V_{a_k}$ , where  $V_a = \{b \in P \mid b \geq a\}$ ; the inverse map sends a dual order ideal into the set of its minimal elements. It is clear that  $\mathfrak{i}_\Phi \mapsto \Phi$  is the required bijection.

**Remark.** In combinatorial literature the antichains of the root poset  $(\Delta^+, \leq)$  are called *non-nesting partitions* [R, Remark 2]. This name derives from the analysis of the definition in type  $A_n$ . In that case, write the positive roots with respect to the standard basis  $\{\varepsilon_i\}_{i=1}^{n+1}$  of  $\mathbb{R}^{n+1}$ , so that  $\Delta^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq n + 1\}$ . Then to an antichain  $A$  we can associate a partition of  $\{1, \dots, n + 1\}$  by putting in the same block  $i, j$  whenever  $\varepsilon_i - \varepsilon_j \in A$ . It turns out that partitions arising in this way are the ones characterized by the following property: if  $a, e$  appear in a block  $B$  and  $b, d$  appear in a different block  $B'$  where  $a < b < d < e$ , then there exists  $c \in B$  satisfying  $b < c < d$ .

4.2. A bijection between antichains of the root poset and  $\oplus$ -sign types of  $\check{\Delta}$  or regions of the Catalan arrangement which are contained in the fundamental chamber

First we recall the definition of  $\oplus$ -sign type for the root system  $\Delta$ . For  $\alpha \in \Delta^+$  set  $H_{\alpha,+} = \{v \in V \mid (v, \check{\alpha}) > 1\}$ ,  $H_{\alpha,0} = \{v \in V \mid 0 < (v, \check{\alpha}) < 1\}$ ,  $H_{\alpha,-} = \{v \in V \mid (v, \check{\alpha}) < 0\}$ . Then a subset  $S \subset V$  is a *sign type* (respectively  $\oplus$ -*sign type*) if it is of the form  $S = \bigcap_{\alpha \in \Delta^+} H_{\alpha, X_\alpha}$  for some collection  $(X_\alpha)_{\alpha \in \Delta^+}$  with  $X_\alpha \in \{+, 0, -\}$  (respectively  $X_\alpha \in \{+, 0\}$ ).

We describe a bijection between dual order ideals and  $\oplus$ -sign types, according to Shi [S, Theorem 1.4]. Given a dual order ideal  $\Phi \subseteq \Delta^+$ , map it to the  $\oplus$ -sign type  $(X_{\check{\alpha}})_{\check{\alpha} \in \check{\Delta}^+}$  defined by

$$X_{\check{\alpha}} = \begin{cases} 0, & \text{if } \alpha \notin \Phi, \\ +, & \text{if } \alpha \in \Phi. \end{cases}$$

This bijection appears also in a different context. Recall the two following remarkable arrangements of real hyperplanes (cf. [At2, Section 3]). The *Shi arrangement*  $\mathcal{S}$ , relative to  $\Delta$ , is the set of hyperplanes of  $V$  having equations

$$(x, \alpha) = 0, \quad (x, \alpha) = 1, \quad \alpha \in \Delta^+;$$

the *Catalan arrangement*  $\mathcal{C}$  is the set of hyperplanes of  $V$  having equations

$$(x, \alpha) = 0, \quad (x, \alpha) = 1, \quad (x, \alpha) = -1, \quad \alpha \in \Delta^+.$$

We call *regions* of the hyperplane arrangement the connected components of the complement in  $V$  of the union of all hyperplanes in the arrangement. By the definition of  $\mathcal{S}$  and  $\mathcal{C}$  it is clear that both arrangements have the same number of regions inside the fundamental chamber of  $W$ . A bijection between antichains in  $\Delta^+$  and regions of  $\mathcal{S}$  or  $\mathcal{C}$  lying in the fundamental chamber (which in [At1, 6.1] is attributed to Postnikov) can be made explicit mapping an antichain  $A$  to the region

$$X_A = \left\{ x \in C_\infty \mid \begin{cases} (\beta, x) > 1 & \text{if } \beta \geq \alpha \text{ for some } \alpha \in A, \\ (\beta, x) < 1 & \text{otherwise} \end{cases} \right\}.$$

We illustrate in Table 1 the above bijections in the easy case of a root system of type  $A_2$ .

Table 1

Ideals in $\mathcal{I}$	Antichains	Regions of $\mathcal{C}$ within $C_\infty$
$i_1 = 0$	$\emptyset$	$X_1 = \{x \mid (x, \alpha_1) < 1, (x, \alpha_2) < 1, (x, \theta) < 1\}$
$i_2 = \mathfrak{g}\theta$	$\{\theta\}$	$X_2 = \{x \mid (x, \alpha_1) < 1, (x, \alpha_2) < 1, (x, \theta) > 1\}$
$i_3 = \mathfrak{g}\alpha_1 \oplus \mathfrak{g}\theta$	$\{\alpha_1\}$	$X_3 = \{x \mid (x, \alpha_1) > 1, (x, \alpha_2) < 1, (x, \theta) > 1\}$
$i_4 = \mathfrak{g}\alpha_2 \oplus \mathfrak{g}\theta$	$\{\alpha_2\}$	$X_4 = \{x \mid (x, \alpha_1) < 1, (x, \alpha_2) > 1, (x, \theta) > 1\}$
$i_5 = \mathfrak{g}\alpha_1 \oplus \mathfrak{g}\alpha_2 \oplus \mathfrak{g}\theta$	$\{\alpha_1, \alpha_2\}$	$X_5 = \{x \mid (x, \alpha_1) > 1, (x, \alpha_2) > 1, (x, \theta) > 1\}$

**Remark.** It is worthwhile to recall that formula (1) also counts the number of conjugacy classes of elements of order dividing  $h + 1$  in a maximal torus  $T$  of the connected simply connected simple algebraic group  $G$  corresponding to  $\mathfrak{g}$ . Indeed these classes are in bijection with  $W$ -orbits on  $\check{Q}/(q + 1)\check{Q}$ . In fact, regard coroots as cocharacters of  $T$ , i.e. as morphism of algebraic groups  $\mathbb{C}^* \rightarrow T$ . Fix a primitive  $r$ th root of unity  $z$ ; then, given  $\tau \in \check{Q}$ , the map  $\tau \mapsto \tau(z)$  is bijection from  $\check{Q}/r\check{Q}$  to  $T_r = \{t \in T \mid t^r = 1\}$  and induces a bijection between the  $W$ -orbits in  $\check{Q}/r\check{Q}$  and the conjugacy classes of elements in  $T_r$ .

### 5. Examples

We illustrate the bijection of Theorem 1 when  $\Delta$  is of type  $A_2$  or  $B_2$ . For this purpose we first need to give explicitly the elements  $w_i \in \widehat{W}$  corresponding to the ideals  $i \in \mathcal{I}$ .

In the case of  $A_2$  the map from  $\mathcal{I}$  into  $\widehat{W}$  is given in Table 2.

We have  $h = 3$  and  $\check{Q} = Q = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2$ ; we have also  $\check{\rho} = \rho = \theta = \alpha_1 + \alpha_2$ . The bijections of Proposition 3 and Theorem 1 are given in Table 3 (regarding the bijection of Theorem 1, we write down the element of  $\overline{C}_{h+1} \cap \check{Q}$  corresponding to each ideal).

Now we consider the root type  $B_2$ . Here  $h = 4$ , and  $\check{Q} = \mathbb{Z}\alpha_1 + 2\mathbb{Z}\alpha_2$ . Then  $\check{\rho} = 2\alpha_1 + 3\alpha_2 \notin \check{Q}$ , but  $\check{\omega}_1 = \alpha_1 + \alpha_2$  so that  $\check{\rho} - 5\check{\omega}_1 = -3\alpha_1 - 2\alpha_2 \in \check{Q}$ . Moreover, in the notation of the proof of Theorem 1,  $w_0^j = w_0^1 = s_2$ . The injection of  $\mathcal{I}$  in  $\widehat{W}$  is given in Table 4 (for shortness we do not write  $N(w_i)$ ).

The bijection with  $\overline{C}_{h+1} \cap \check{Q}$  is made explicit in Table 5.

Table 2

Ideals in $\mathcal{I}$	$N(w_i)$	$w_i$
$i_1 = 0$	$\emptyset$	1
$i_2 = \mathfrak{g}_\theta$	$\{-\theta + \delta\}$	$s_0$
$i_3 = \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_\theta$	$\{-\theta + \delta, -\alpha_1 + \delta\}$	$s_0 s_2$
$i_4 = \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_\theta$	$\{-\theta + \delta, -\alpha_2 + \delta\}$	$s_0 s_1$
$i_5 = \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_\theta$	$\{-\theta + \delta, -\alpha_2 + \delta, -\theta + 2\delta, -\alpha_1 + \delta\}$	$s_0 s_1 s_2 s_1$

Table 3

$w_i = t_{\tau_i} v_i$	$v_i^{-1}(\tau_i)$	$w_0 t_{-\check{\rho}}(v_i^{-1}(\tau_i))$
1	0	$\theta$
$s_0 = t_\theta s_1 s_2 s_1$	$-\theta$	$2\theta$
$s_0 s_2 = t_\theta s_2 s_1$	$-\alpha_1$	$\alpha_1 + 2\alpha_2$
$s_0 s_1 = t_\theta s_1 s_2$	$-\alpha_2$	$2\alpha_1 + \alpha_2$
$s_0 s_1 s_2 s_1 = t_\theta$	$\theta$	0

Table 4

$i_1 = 0$	$w_{i_1} = 1$
$i_2 = \mathfrak{g}_\theta$	$w_{i_2} = s_0$
$i_3 = \mathfrak{g}_{\alpha_1 + \alpha_2} \oplus \mathfrak{g}_\theta$	$w_{i_3} = s_0 s_2$
$i_4 = \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_1 + \alpha_2} \oplus \mathfrak{g}_\theta$	$w_{i_4} = s_0 s_2 s_0$
$i_5 = \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1 + \alpha_2} \oplus \mathfrak{g}_\theta$	$w_{i_5} = s_0 s_2 s_1 s_2$
$i_6 = \mathfrak{n}$	$w_{i_6} = s_0 s_2 s_1 s_2 s_0 s_2 s_0$

Table 5

$w_i = t_{\tau_i} v_i$	$v_i^{-1}(\tau_i)$	$w_0^j t_{-\check{\rho} + (h+1)\check{\omega}_j}(v_i^{-1}(\tau_i))$
$w_{i_1} = 1$	0	$3\check{\alpha}_1 + 2\check{\alpha}_2$
$w_{i_2} = t_{\check{\theta}} s_2 s_1 s_2$	$-\check{\alpha}_1 - \check{\alpha}_2$	$2\check{\alpha}_1 + 2\check{\alpha}_2$
$w_{i_3} = t_{\check{\theta}} s_2 s_1$	$-\check{\alpha}_1$	$2\check{\alpha}_1 + \check{\alpha}_2$
$w_{i_4} = t_{2\check{\alpha}_1 + \check{\alpha}_2} s_1 s_2 s_1$	$-2\check{\alpha}_1 - \check{\alpha}_2$	$\check{\alpha}_1 + \check{\alpha}_2$
$w_{i_5} = t_{\check{\theta}}$	$\check{\alpha}_1 + \check{\alpha}_2$	$4\check{\alpha}_1 + 2\check{\alpha}_2$
$w_{i_6} = t_{3\check{\alpha}_1 + 2\check{\alpha}_2} s_1 s_2 s_1$	$-3\check{\alpha}_1 - \check{\alpha}_2$	0

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