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Journal of Algebra

www.elsevier.com/locate/jalgebra



Bailey type summation formulas associated with the root system $F_4^{\vee \star}$

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ARTICLE INFO

Article history:

Received 28 February 2008

Available online 2 October 2008

Communicated by Jean-Yves Thibon

Keywords:

Basic hypergeometric function

Bailey type summation formula

Root systems F_4 and F_4^{\vee}

ABSTRACT

We state three types of hypergeometric summation formulas associated with the exceptional root system F_4^{\vee} , which are motivated from Bailey's summation formula for a very-well-poised-balanced ${}_6\psi_6$ basic hypergeometric series.

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1. Introduction

Throughout this paper, we assume $0 < q < 1$ and denote the q -shifted factorial for all integers N by $(x; q)_{\infty} := \prod_{i=0}^{\infty} (1 - q^i x)$ and $(x; q)_N := (x; q)_{\infty} / (q^N x; q)_{\infty}$.

In this paper, we present three types of basic hypergeometric summation formulas associated with the root system F_4^{\vee} . One of them, which we call the $(1, 1)$ case, is equivalent to the q -Macdonald–Morris constant term identity associated with the root system F_4^{\vee} , and was essentially presented by Garvan and Gonnet [8]. It is stated in a more general form for any root system by Macdonald [19]. Garvan [6] and Garvan and Gonnet [7,8] proved the $(q-)$ Macdonald–Morris identity [18] associated with the root system F_4 and F_4^{\vee} around 1990. Later Cherednik [4] proved it for all irreducible reduced root systems. The aim of this paper is to establish two further summations. The proof is based on an idea of Garvan [6]. Some terminology is defined before reporting the main results.

Let R be an irreducible reduced root system, spanning a real vector space E of dimension n , and let $\langle \cdot, \cdot \rangle$ be a positive definite scalar product on E invariant under the Weyl group W of R . We

[☆] This work was supported in part by Grant-in-Aid for Scientific Research (C) No. 17540034 of the Ministry of Education, Culture, Sports, Science and Technology, Japan.

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denote by R^+ the set of positive roots relative to a fixed basis $\{\alpha_1, \dots, \alpha_n\}$ of R . For each $\alpha \in R$, let $\alpha^\vee = 2\alpha/\langle\alpha, \alpha\rangle$. Let P be the coweight lattice $\{\chi \in E; \langle\alpha, \chi\rangle \in \mathbb{Z} \text{ for any } \alpha \in R\}$ and let Q be the coroot lattice of R defined by $Q = \mathbb{Z}\alpha_1^\vee + \dots + \mathbb{Z}\alpha_n^\vee \subset P$. Let L be any sublattice of P of rank n . We assume L is W -stable, i.e., $L = wL$ for $w \in W$. The scalar product $\langle \cdot, \cdot \rangle$ is uniquely extended linearly to $E_{\mathbb{C}} = E \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C}^n$. For $x \in E_{\mathbb{C}}$, we define

$$\Phi_R(x) = \prod_{i=1}^s \prod_{\substack{\alpha > 0 \\ \alpha: \text{short}}} q_\alpha^{(1/2 - \beta_i)\langle\alpha, x\rangle} \frac{(q_\alpha^{1 - \beta_i + \langle\alpha, x\rangle}; q_\alpha)_\infty}{(q_\alpha^{\beta_i + \langle\alpha, x\rangle}; q_\alpha)_\infty} \prod_{j=1}^\ell \prod_{\substack{\alpha > 0 \\ \alpha: \text{long}}} q_\alpha^{(1/2 - \gamma_j)\langle\alpha, x\rangle} \frac{(q_\alpha^{1 - \gamma_j + \langle\alpha, x\rangle}; q_\alpha)_\infty}{(q_\alpha^{\gamma_j + \langle\alpha, x\rangle}; q_\alpha)_\infty},$$

$$\Delta_R(x) = \prod_{\alpha > 0} (q_\alpha^{-\langle\alpha, x\rangle/2} - q_\alpha^{\langle\alpha, x\rangle/2})$$

where $\beta_i, \gamma_j \in \mathbb{C}$ and $\alpha > 0$ means $\alpha \in R^+$. (If all the roots in R have the same length, we regard all the roots as being short.) For $z \in E_{\mathbb{C}}$, the sum

$$J_R(z; L) := \sum_{\omega \in L} \Phi_R(z + \omega) \Delta_R(z + \omega)$$

over the lattices $L \subset P$ is an extension of the very-well-poised-balanced ${}_6\psi_6$ hypergeometric series. (See the previous paper [17] for an explanation of this extension. See also [1,10,11].) We call $J_R(z; L)$ the *Jackson integral associated with the root system R* , which is the main subject of interest in this paper. Set

$$\Theta_R(x) := \prod_{\substack{\alpha > 0 \\ \alpha: \text{short}}} \frac{q_\alpha^{(s-1)\langle\alpha, x\rangle/2} \theta(q_\alpha^{\langle\alpha, x\rangle}; q_\alpha)}{\prod_{i=1}^s q_\alpha^{\beta_i \langle\alpha, x\rangle} \theta(q_\alpha^{\beta_i + \langle\alpha, x\rangle}; q_\alpha)} \prod_{\substack{\alpha > 0 \\ \alpha: \text{long}}} \frac{q_\alpha^{(\ell-1)\langle\alpha, x\rangle/2} \theta(q_\alpha^{\langle\alpha, x\rangle}; q_\alpha)}{\prod_{j=1}^\ell q_\alpha^{\gamma_j \langle\alpha, x\rangle} \theta(q_\alpha^{\gamma_j + \langle\alpha, x\rangle}; q_\alpha)},$$

where $\theta(x; q)$ denotes the function $(x; q)_\infty (q/x; q)_\infty$. Then the following holds when $L = P$ or Q under the condition $q_\alpha = q$ for all $\alpha \in R$ [12]:

Proposition 1.1. *For $L = P$ or Q , the ratio $J_R(z; L)/\Theta_R(z)$ is a holomorphic function of $z \in E_{\mathbb{C}}$.*

When $L = P$ or Q under the condition $q_\alpha = q$ for all $\alpha \in R$, one of the authors classified the cases where $J_R(z; L)/\Theta_R(z)$ becomes a constant as a function of z by numbering the parameters with (s, ℓ) . See [12] for a list of (s, ℓ) for R and the explicit evaluation of the constants. (See also [14] for the nonreduced case.) Under the condition $q_\alpha = q$ for all $\alpha \in R$, only two entries appear in the (s, ℓ) list for the system R which has two root lengths. In particular, if $R = F_4$, then $(s, \ell) = (1, 1)$ and $(3, 0)$. The explicit form of the constant of the $(1, 1)$ case was essentially evaluated in [8], as mentioned before. (See also [13,16] for the evaluation of the constant of the $(3, 0)$ case.)

In this paper, except for the case $q_\alpha = q$ for all roots α in F_4 , we consider the following condition for the root system F_4 ,

$$q_\alpha = \begin{cases} q^2 & \text{if } \alpha \text{ is short in } F_4, \\ q & \text{if } \alpha \text{ is long in } F_4. \end{cases} \quad (1)$$

In this case, we can find new formulas like Bailey's ${}_6\psi_6$ summation [2], [9, p. 140, Eq. (5.3.1)]. The aim of this paper is to present the formulas for the Jackson integral associated with F_4 under condition (1). In this case, there exist three summation formulas.

Theorem 4.1. *For F_4 under condition (1), $J_{F_4}(z; Q)/\Theta_{F_4}(z)$ does not depend on $z \in E_{\mathbb{C}}$ if and only if $(s, \ell) = (0, 2), (1, 1)$ or $(2, 0)$.*

For each (s, ℓ) in the above theorem, the constant $C_{F_4} := J_{F_4}(z; Q)/\Theta_{F_4}(z)$ is computed explicitly in Theorem 4.3. This is the main result of the paper.

Note that Theorem 4.1 also holds for G_2^\vee under condition

$$q_\alpha = \begin{cases} q^3 & \text{if } \alpha \text{ is short in } G_2^\vee, \\ q & \text{if } \alpha \text{ is long in } G_2^\vee, \end{cases} \quad (2)$$

and hence also has three summation formulas, whose explicit expressions of the constants for G_2^\vee under condition (2) are presented in our previous paper [17].

2. Notation for F_4^\vee and F_4

Let $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$ be the standard basis of \mathbb{R}^4 . We set the notation for F_4^\vee and F_4 following [3]:

- $\left\{ \begin{array}{l} \text{Basis of } F_4^\vee: \alpha_1 = \varepsilon_2 - \varepsilon_3, \alpha_2 = \varepsilon_3 - \varepsilon_4, \alpha_3 = 2\varepsilon_4, \alpha_4 = \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4. \\ \text{Short positive roots of } F_4^\vee: \varepsilon_i \pm \varepsilon_j \ (1 \leq i < j \leq 4). \\ \text{Long positive roots of } F_4^\vee: 2\varepsilon_i \ (1 \leq i \leq 4), \ \varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4. \end{array} \right.$
- $\left\{ \begin{array}{l} \text{Basis of } F_4: \varepsilon_2 - \varepsilon_3, \ \varepsilon_3 - \varepsilon_4, \ \varepsilon_4, \ (\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)/2. \\ \text{Short positive roots of } F_4: \varepsilon_i \ (1 \leq i \leq 4), \ (\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)/2. \\ \text{Long positive roots of } F_4: \varepsilon_i \pm \varepsilon_j \ (1 \leq i < j \leq 4). \end{array} \right.$
- Fundamental coweights of F_4 (fundamental weights of F_4^\vee):

$$\omega_1 = \varepsilon_1 + \varepsilon_2, \quad \omega_2 = 2\varepsilon_1 + \varepsilon_2 + \varepsilon_3, \quad \omega_3 = 3\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4, \quad \omega_4 = 2\varepsilon_1,$$

which satisfy $\langle \alpha_i^\vee, \omega_j \rangle = \delta_{ij}$, where δ_{ij} is the Kronecker delta.

The weight lattice P of F_4^\vee is given by $P := \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 + \mathbb{Z}\omega_3 + \mathbb{Z}\omega_4$ and the root lattice $Q := \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 + \mathbb{Z}\alpha_3 + \mathbb{Z}\alpha_4$ of F_4^\vee coincides with P in the F_4^\vee case. Set

$$P^+ := \mathbb{N}\omega_1 + \mathbb{N}\omega_2 + \mathbb{N}\omega_3 + \mathbb{N}\omega_4,$$

$$Q^+ := \mathbb{N}\alpha_1 + \mathbb{N}\alpha_2 + \mathbb{N}\alpha_3 + \mathbb{N}\alpha_4,$$

where \mathbb{N} denotes the set of nonnegative integers. The Weyl group W of F_4^\vee is generated by the reflections $w_{\alpha_1}, w_{\alpha_2}, w_{\alpha_3}$ and w_{α_4} , where w_α for $\alpha \in R$ is defined by the linear transform $w_\alpha(x) := x - \langle x, \alpha^\vee \rangle \alpha$, $x \in E_{\mathbb{C}} \simeq \mathbb{C}^4$. By definition, $w_{\alpha_1}, w_{\alpha_2}, w_{\alpha_3}$ and w_{α_4} are given explicitly as

$$w_{\alpha_1} : (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \rightarrow (\varepsilon_1, \varepsilon_3, \varepsilon_2, \varepsilon_4),$$

$$w_{\alpha_2} : (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \rightarrow (\varepsilon_1, \varepsilon_2, \varepsilon_4, \varepsilon_3),$$

$$w_{\alpha_3} : (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \rightarrow (\varepsilon_1, \varepsilon_2, \varepsilon_3, -\varepsilon_4),$$

$$w_{\alpha_4} : (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \rightarrow (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)A,$$

where

$$A = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

The order of the Weyl group W of F_4^\vee is 1152. See [6,13] for the explicit expression of the Weyl group. We define the Weyl group action for $f(x)$ of $x \in E_\mathbb{C}$ by

$$wf(x) := f(w^{-1}(x)) \quad \text{for } w \in W,$$

and we denote by $\mathcal{A}f(x)$ the alternating sum over W defined by

$$\mathcal{A}f(x) := \sum_{w \in W} (\text{sgn } w) wf(x).$$

3. Jackson integrals of type (F_4^\vee, F_4)

Under condition (1), for $x \in E_\mathbb{C}$ writing $\Phi_{F_4}(x)$ and $\Delta_{F_4}(x)$ using the terminology of F_4^\vee gives

$$\begin{aligned} \Phi_{F_4}^{(s,\ell)}(\{b_i\}, \{c_j\}; x) &= \prod_{i=1}^s \prod_{\substack{\alpha > 0 \\ \alpha: \text{long} \\ \text{in } F_4^\vee}} q^{(1/2 - \beta_i)\langle \alpha, x \rangle} \frac{(b_i^{-1} q^{2+\langle \alpha, x \rangle}; q^2)_\infty}{(b_i q^{\langle \alpha, x \rangle}; q^2)_\infty} \\ &\quad \times \prod_{j=1}^\ell \prod_{\substack{\alpha > 0 \\ \alpha: \text{short} \\ \text{in } F_4^\vee}} q^{(1/2 - \gamma_j)\langle \alpha, x \rangle} \frac{(c_j^{-1} q^{1+\langle \alpha, x \rangle}; q)_\infty}{(c_j q^{\langle \alpha, x \rangle}; q)_\infty}, \\ \Delta_{F_4}(x) &= \prod_{\substack{\alpha > 0 \\ \text{in } F_4^\vee}} (q^{-\langle \alpha, x \rangle/2} - q^{\langle \alpha, x \rangle/2}) \end{aligned} \quad (3)$$

where

$$b_i = q^{2\beta_i} \quad (i = 1, 2, \dots, s) \quad \text{and} \quad c_j = q^{\gamma_j} \quad (j = 1, 2, \dots, \ell).$$

We abbreviate $\Phi_{F_4}^{(s,\ell)}(\{b_i\}, \{c_j\}; x)$ and $\Delta_{F_4}(x)$ to $\Phi^{(s,\ell)}(x)$ and $\Delta(x)$, respectively. For $z \in E_\mathbb{C}$ the sum

$$J^{(s,\ell)}(\{b_i\}, \{c_j\}; z; Q) := \sum_{\omega \in Q} \Phi^{(s,\ell)}(z + \omega) \Delta(z + \omega) \quad (4)$$

over the lattice Q converges if

$$\left| \prod_{\substack{\alpha > 0 \\ \alpha: \text{long} \\ \text{in } F_4^\vee}} q^{((s-1)/2 - \beta_1 - \dots - \beta_s)\langle \alpha, \omega_i \rangle} \prod_{\substack{\alpha > 0 \\ \alpha: \text{short} \\ \text{in } F_4^\vee}} q^{((\ell-1)/2 - \gamma_1 - \dots - \gamma_\ell)\langle \alpha, \omega_i \rangle} \right| < 1$$

for $i = 1, 2, 3$ and 4, i.e., b_i and c_j satisfy the following:

$$\begin{aligned} |(b_1 \dots b_s)^6 (c_1 \dots c_\ell)^{10}| &> q^{6(s-1)+5(\ell-1)}, \quad |(b_1 \dots b_s)^4 (c_1 \dots c_\ell)^6| > q^{4(s-1)+3(\ell-1)}, \\ |(b_1 \dots b_s)^3 (c_1 \dots c_\ell)^4| &> q^{3(s-1)+2(\ell-1)}, \quad |(b_1 \dots b_s)^5 (c_1 \dots c_\ell)^6| > q^{5(s-1)+3(\ell-1)}. \end{aligned}$$

(See [15, p. 158, Theorem 4] for the convergence condition.) We call the sum $J^{(s,\ell)}(\{b_i\}, \{c_j\}; z; Q)$ the *Jackson integral of type (F_4^\vee, F_4)* , to distinguish it from that of type F_4 under the condition $q_\alpha = q$ for all roots α . The following lemma is to be used in Section 6:

Lemma 3.1. Let $\varphi(x)$ be a function such that $\sum_{\omega \in L} \Phi^{(s,\ell)}(z+\omega)\varphi(z+\omega)$ converges. For $\eta \in L$ we define the operator ∇_η by

$$\nabla_\eta \varphi(x) := \varphi(x) - \frac{\Phi^{(s,\ell)}(x+\eta)}{\Phi^{(s,\ell)}(x)} \varphi(x+\eta). \quad (5)$$

Then

$$\sum_{\omega \in L} \Phi^{(s,\ell)}(z+\omega) \nabla_\eta \varphi(z+\omega) = 0. \quad (6)$$

Moreover,

$$\sum_{\omega \in L} \Phi^{(s,\ell)}(z+\omega) \mathcal{A} \nabla_\eta \varphi(z+\omega) = 0. \quad (7)$$

Proof. See Appendix A. \square

4. Main results

We set

$$\begin{aligned} & \Theta^{(s,\ell)}(\{b_i\}, \{c_j\}; z) \\ &= \prod_{\substack{\alpha > 0 \\ \alpha: \text{long} \\ \text{in } F_4^\vee}} \frac{q^{(s-1)\langle \alpha, x \rangle / 2} \theta(q^{\langle \alpha, x \rangle}; q^2)}{\prod_{i=1}^s q^{\beta_i \langle \alpha, x \rangle} \theta(b_i q^{\langle \alpha, x \rangle}; q^2)} \prod_{\substack{\alpha > 0 \\ \alpha: \text{short} \\ \text{in } F_4^\vee}} \frac{q^{(\ell-1)\langle \alpha, x \rangle / 2} \theta(q^{\langle \alpha, x \rangle}; q)}{\prod_{j=1}^\ell q^{\gamma_j \langle \alpha, x \rangle} \theta(c_j q^{\langle \alpha, x \rangle}; q)}, \end{aligned} \quad (8)$$

where $\theta(x; q)$ denotes the function $(x; q)_\infty (q/x; q)_\infty$, which satisfies

$$\theta(qx; q) = -\theta(x; q)/x. \quad (9)$$

Then

Theorem 4.1. The ratio $J^{(s,\ell)}(\{b_i\}, \{c_j\}; z; Q) / \Theta^{(s,\ell)}(\{b_i\}, \{c_j\}; z)$ does not depend on $z \in E_{\mathbb{C}}$ if and only if $(s, \ell) = (0, 2), (1, 1)$ or $(2, 0)$.

Proof. See Appendix B. \square

From Theorem 4.1, if $(s, \ell) = (0, 2), (1, 1)$ or $(2, 0)$, we set the constant

$$C^{(s,\ell)}(\{b_i\}, \{c_j\}) := \frac{J^{(s,\ell)}(\{b_i\}, \{c_j\}; z; Q)}{\Theta^{(s,\ell)}(\{b_i\}, \{c_j\}; z)}, \quad (10)$$

whose explicit expression is the subject of interest in this paper.

Remark. Since $\Theta^{(s,\ell)}(\{b_i\}, \{c_j\}; z)$ is invariant under the shift $z \rightarrow z + \omega$ ($\omega \in Q$) if $(s, \ell) = (0, 2), (1, 1)$ or $(2, 0)$, for these values of (s, ℓ) the right-hand side of (10) can be rewritten

$$\frac{J^{(s,\ell)}(\{b_i\}, \{c_j\}; z; Q)}{\Theta^{(s,\ell)}(\{b_i\}, \{c_j\}; z)} = \sum_{\omega \in Q} \Psi^{(s,\ell)}(z + \omega),$$

where

$$\psi^{(s,\ell)}(x) = \prod_{\substack{\alpha \in F_4^\vee \\ \alpha: \text{short}}} \frac{\prod_{j=1}^{\ell} (c_j^{-1} q^{1+\langle \alpha, x \rangle}; q)_{\infty}}{(q^{1+\langle \alpha, x \rangle}; q)_{\infty}} \prod_{\substack{\alpha \in F_4^\vee \\ \alpha: \text{long}}} \frac{\prod_{i=1}^s (b_i^{-1} q^{2+\langle \alpha, x \rangle}; q^2)_{\infty}}{(q^{2+\langle \alpha, x \rangle}; q^2)_{\infty}}, \quad (11)$$

which is invariant under the Weyl group action. Consequently

Corollary 4.2. *If $(s, \ell) = (0, 2), (1, 1)$ or $(2, 0)$, then*

$$\sum_{\omega \in Q} \psi^{(s,\ell)}(z + \omega) = C^{(s,\ell)}(\{b_i\}, \{c_j\}). \quad (12)$$

Following [5,19], we call the left-hand side of (12) the *Macdonald type sum* of type (F_4^\vee, F_4) .

The aim of this section is to evaluate the constants $C^{(s,\ell)}(\{b_i\}, \{c_j\})$.

Theorem 4.3. *If $(s, \ell) = (0, 2), (1, 1)$ or $(2, 0)$, then the constant $C^{(s,\ell)}(\{b_i\}, \{c_j\})$ for each case is expressed as follows:*

$$\begin{aligned} C^{(0,2)}(c_1, c_2) &= (q; q)_{\infty}^2 (qc_1^{-3}; q)_{\infty} (qc_2^{-3}; q)_{\infty} (qc_1^{-1}c_2^{-1}; q)_{\infty} \\ &\quad \times (qc_1^{-1}c_2^{-2}; q)_{\infty} (qc_2^{-2}c_1^{-1}; q)_{\infty} (c_1^{-1}c_2^{-3}; q)_{\infty} (c_1^{-3}c_2^{-1}; q)_{\infty} \\ &\quad \times (c_1^{-2}c_2^{-2}; q)_{\infty} (c_1^{-2}c_2^{-3}; q)_{\infty} (c_1^{-3}c_2^{-2}; q)_{\infty} (q^{-1}c_1^{-3}c_2^{-3}; q)_{\infty} \\ &\quad \times (q^2; q^2)_{\infty}^2 (q^2c_1^{-2}; q^2)_{\infty} (q^2c_2^{-2}; q^2)_{\infty} (c_1^{-4}; q^2)_{\infty} (c_2^{-4}; q^2)_{\infty} (q^2c_1^{-2}c_2^{-2}; q^2)_{\infty} \\ &\quad \times \frac{(c_1^{-2}c_2^{-4}; q^2)_{\infty} (c_1^{-4}c_2^{-2}; q^2)_{\infty} (c_1^{-4}c_2^{-4}; q^2)_{\infty} (q^{-1}c_1^{-6}c_2^{-6}; q^2)_{\infty}}{(c_1^{-1}; q)_{\infty} (qc_1^{-1}; q)_{\infty}^2 (c_2^{-1}; q)_{\infty} (qc_2^{-1}; q)_{\infty}^2 (q^{-4}c_1^{-12}c_2^{-12}; q^2)_{\infty}}, \end{aligned} \quad (13)$$

$$\begin{aligned} C^{(1,1)}(b_1, c_1) &= \frac{(q; q)_{\infty}^2 (b_1^{-1}; q)_{\infty} (b_1^{-2}; q)_{\infty} (b_1^{-3}c_1^{-2}; q)_{\infty}}{(b_1^{-1}c_1^{-3}; q)_{\infty} (b_1^{-2}c_1^{-3}; q)_{\infty} (b_1^{-3}c_1^{-5}; q)_{\infty} (c_1^{-1}; q)_{\infty} (c_1^{-2}; q)_{\infty}} \\ &\quad \times \frac{(qb_1^{-3}c_1^{-6}; q)_{\infty} (qb_1^{-2}c_1^{-4}; q)_{\infty} (qb_1^{-1}c_1^{-4}; q)_{\infty} (qc_1^{-2}; q)_{\infty} (qc_1^{-3}; q)_{\infty}}{(qb_1^{-3}c_1^{-3}; q)_{\infty} (qb_1^{-2}c_1^{-1}; q)_{\infty} (qb_1^{-1}c_1^{-1}; q)_{\infty} (qc_1^{-1}; q)_{\infty}^2} \\ &\quad \times \frac{(q^2; q^2)_{\infty}^2 (b_1^{-2}c_1^{-6}; q^2)_{\infty} (c_1^{-2}; q^2)_{\infty} (c_1^{-4}; q^2)_{\infty} (q^2b_1^{-2}; q^2)_{\infty} (q^2b_1^{-3}; q^2)_{\infty}}{(b_1^{-1}; q^2)_{\infty} (b_1^{-2}; q^2)_{\infty} (b_1^{-3}c_1^{-2}; q^2)_{\infty} (b_1^{-3}c_1^{-4}; q^2)_{\infty} (b_1^{-5}c_1^{-6}; q^2)_{\infty}} \\ &\quad \times \frac{(q^2b_1^{-4}c_1^{-2}; q^2)_{\infty} (q^2b_1^{-4}c_1^{-4}; q^2)_{\infty} (q^2b_1^{-6}c_1^{-6}; q^2)_{\infty}}{(q^2b_1^{-1}; q^2)_{\infty}^2 (q^2b_1^{-1}c_1^{-2}; q^2)_{\infty} (q^2b_1^{-1}c_1^{-4}; q^2)_{\infty} (q^2b_1^{-3}c_1^{-6}; q^2)_{\infty}}, \end{aligned} \quad (14)$$

$$\begin{aligned} C^{(2,0)}(b_1, b_2) &= (q; q)_{\infty}^2 (qb_1^{-2}; q)_{\infty} (qb_2^{-2}; q)_{\infty} \\ &\quad \times (qb_1^{-1}b_2^{-1}; q)_{\infty} (qb_1^{-1}b_2^{-2}; q)_{\infty} (qb_1^{-2}b_2^{-1}; q)_{\infty} (qb_1^{-2}b_2^{-2}; q)_{\infty} \\ &\quad \times (q^2; q^2)_{\infty}^2 (qb_1^{-1}; q^2)_{\infty} (qb_2^{-1}; q^2)_{\infty} (q^2b_1^{-3}; q^2)_{\infty} (q^2b_2^{-3}; q^2)_{\infty} \\ &\quad \times (q^2b_1^{-1}b_2^{-1}; q^2)_{\infty} (q^2b_1^{-1}b_2^{-2}; q^2)_{\infty} (q^2b_1^{-2}b_2^{-1}; q^2)_{\infty} (q^2b_1^{-2}b_2^{-2}; q^2)_{\infty} \\ &\quad \times \frac{(q^2b_1^{-1}b_2^{-3}; q^2)_{\infty} (q^2b_1^{-3}b_2^{-1}; q^2)_{\infty} (q^2b_1^{-2}b_2^{-3}; q^2)_{\infty} (q^2b_1^{-3}b_2^{-2}; q^2)_{\infty}}{(q^2b_1^{-1}; q^2)_{\infty}^2 (q^2b_2^{-1}; q^2)_{\infty}^2 (qb_1^{-3}b_2^{-3}; q^2)_{\infty} (qb_1^{-6}b_2^{-6}; q^2)_{\infty}}. \end{aligned} \quad (15)$$

The rest of this paper is devoted to proving Theorem 4.3.

5. The case $(s, \ell) = (1, 1)$

According to Macdonald's theory [19], only the $(s, \ell) = (1, 1)$ case of the (F_4^\vee, F_4) type Jackson integral is known. It is expressed as in the general formula for any irreducible reduced root system R .

When $(s, \ell) = (1, 1)$, we set

$$k_\alpha = \begin{cases} \gamma_1 & \text{if } \alpha \text{ is short in } F_4^\vee, \\ \beta_1 & \text{if } \alpha \text{ is long in } F_4^\vee, \end{cases} \quad \text{and} \quad q_\alpha = \begin{cases} q & \text{if } \alpha \text{ is short in } F_4^\vee, \\ q^2 & \text{if } \alpha \text{ is long in } F_4^\vee. \end{cases}$$

For $(s, \ell) = (1, 1)$, the Macdonald type sum has already been evaluated explicitly in [19, p. 206, Eq. (6.5)]. It reads

$$\sum_{\omega \in Q} \Psi^{(1,1)}(z + \omega) = \prod_{\alpha > 0} \frac{(q_\alpha^{1 - \langle \rho_k, \alpha^\vee \rangle - k_\alpha}; q_\alpha)_\infty (q_\alpha^{\delta_\alpha - \langle \rho_k, \alpha^\vee \rangle + k_\alpha}; q_\alpha)_\infty}{(q_\alpha^{1 - \langle \rho_k, \alpha^\vee \rangle}; q_\alpha)_\infty (q_\alpha^{-\langle \rho_k, \alpha^\vee \rangle}; q_\alpha)_\infty} \quad (16)$$

where

$$\rho_k := \frac{1}{2} \sum_{\alpha > 0} k_\alpha \alpha \quad \text{and} \quad \delta_\alpha := \begin{cases} 1 & \text{if } \langle \rho_k, \alpha^\vee \rangle = k_\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Eq. (14) is nothing but the right-hand side of (16). In particular

$$\lim_{N \rightarrow \infty} C^{(1,1)}(q^{-2N} b_1, q^{-N} c_1) = (q; q)_\infty^2 (q^2; q^2)_\infty^2. \quad (17)$$

From (11), (12) and [15, p. 173, Eq. (56) in Theorem 6], we have

$$\lim_{N \rightarrow \infty} C^{(1,1)}(q^{-2N} b_1, q^{-N} c_1) = \sum_{\omega \in Q} \Psi(z + \omega) \quad (18)$$

where

$$\Psi(x) = \prod_{\substack{\alpha \in F_4^\vee \\ \alpha: \text{short}}} \frac{1}{(q^{1 + \langle \alpha, x \rangle}; q)_\infty} \prod_{\substack{\alpha \in F_4^\vee \\ \alpha: \text{long}}} \frac{1}{(q^{2 + \langle \alpha, x \rangle}; q^2)_\infty}.$$

Similarly, for the cases $(s, \ell) = (0, 2)$ and $(2, 0)$ we have

$$\lim_{N \rightarrow \infty} C^{(0,2)}(q^{-N} c_1, q^{-N} c_2) = \lim_{N \rightarrow \infty} C^{(2,0)}(q^{-2N} b_1, q^{-2N} b_2) = \sum_{\omega \in Q} \Psi(z + \omega). \quad (19)$$

From (17), (18) and (19), we obtain

Proposition 5.1.

$$\begin{aligned} \lim_{N \rightarrow \infty} C^{(0,2)}(q^{-N} c_1, q^{-N} c_2) &= \lim_{N \rightarrow \infty} C^{(2,0)}(q^{-2N} b_1, q^{-2N} b_2) \\ &= \lim_{N \rightarrow \infty} C^{(1,1)}(q^{-2N} b_1, q^{-N} c_1) = (q; q)_\infty^2 (q^2; q^2)_\infty^2. \end{aligned}$$

This proposition will be used in Section 6.

6. The cases $(s, \ell) = (0, 2)$ and $(2, 0)$

In this section we will consider the cases $(s, \ell) = (0, 2)$ and $(2, 0)$, and will prove (13) and (15) in Theorem 4.3. Before that we will explain the dominance ordering and the reverse lexicographic ordering for dominance weights of F_4^\vee . Next, using the 86 dominance weights, we will show that the symmetric polynomials which occurs from the q -shifts of the Jackson integrals with respect to their parameters can be written as a linear combination of Weyl characters of F_4^\vee . Finally we will give the proofs for (13) and (15) individually.

6.1. Orderings on P^+ of F_4^\vee

The dominance ordering $<$ on P of F_4^\vee is defined by the relations

$$\mu \preceq \lambda \quad \text{whenever } \lambda - \mu \in Q^+, \quad (20)$$

i.e., $\lambda - \mu$ is a nonnegative integral sum of positive roots. In this paper, the dominant weights $\lambda \in P^+$ satisfying $\lambda \preceq 2\omega_3 + 2\omega_4$ are needed. As we see in Fig. 1, there are 86 dominant weights in the set $P^+(2\omega_3 + 2\omega_4)$, where

$$P^+(\lambda) := \{\mu \in P^+; \mu \preceq \lambda\}.$$

In the figure, the vertex labels indicate the coordinates of the weights; i.e., the weight $m_1\omega_1 + \cdots + m_4\omega_4$ is labeled $m_1 \dots m_4$. If a line links the upper vertex which represents the weight λ with the lower vertex which represents the weight μ , it means that $\mu < \lambda$, but there is no $\nu \in P^+$ such that $\mu < \nu < \lambda$. (This poset expression is introduced by Stembridge in [20,21].) On the other hand, for $\lambda = \lambda_1\varepsilon_1 + \lambda_2\varepsilon_2 + \lambda_3\varepsilon_3 + \lambda_4\varepsilon_4$ and $\mu = \mu_1\varepsilon_1 + \mu_2\varepsilon_2 + \mu_3\varepsilon_3 + \mu_4\varepsilon_4$ in P^+ , the reverse lexicographic ordering $\lambda < \mu$ is defined if the following holds for some $k \in \{1, 2, 3, 4\}$:

$$\lambda_1 = \mu_1, \dots, \lambda_{k-1} = \mu_{k-1} \quad \text{and} \quad \lambda_k < \mu_k. \quad (21)$$

By definition, if $\lambda < \mu$ in the dominance ordering, then $\lambda < \mu$ in the reverse lexicographic ordering, because the condition

$$\begin{aligned} \lambda_1 &\leq \mu_1, \\ \lambda_1 + \lambda_2 &\leq \mu_1 + \mu_2, \\ 2\lambda_1 + \lambda_2 + \lambda_3 &\leq 2\mu_1 + \mu_2 + \mu_3, \\ 3\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 &\leq 3\mu_1 + \mu_2 + \mu_3 + \mu_4, \end{aligned}$$

which is equivalent to (20), leads the condition (21) or $\lambda = \mu$. Table 1 is the list of 86 weights η_i ($0 \leq i \leq 85$) in the set $P^+(2\omega_3 + 2\omega_4)$ ordered by the reverse lexicographic ordering $<$, using the notation

$$\eta_i = \lambda_{1i}\varepsilon_1 + \cdots + \lambda_{4i}\varepsilon_4 \quad (0 \leq i \leq 85). \quad (22)$$

6.2. Symmetric polynomials arising from the q -shifts of the Jackson integrals

We denote by $e_i(\{X_1, \dots, X_{12}\})$ the i th elementary symmetric polynomial of $\{X_1, \dots, X_{12}\}$, i.e.,

$$e_i(\{X_1, \dots, X_{12}\}) := \sum_{1 \leq k_1 < \cdots < k_i \leq 12} X_{k_1} \cdots X_{k_i}.$$

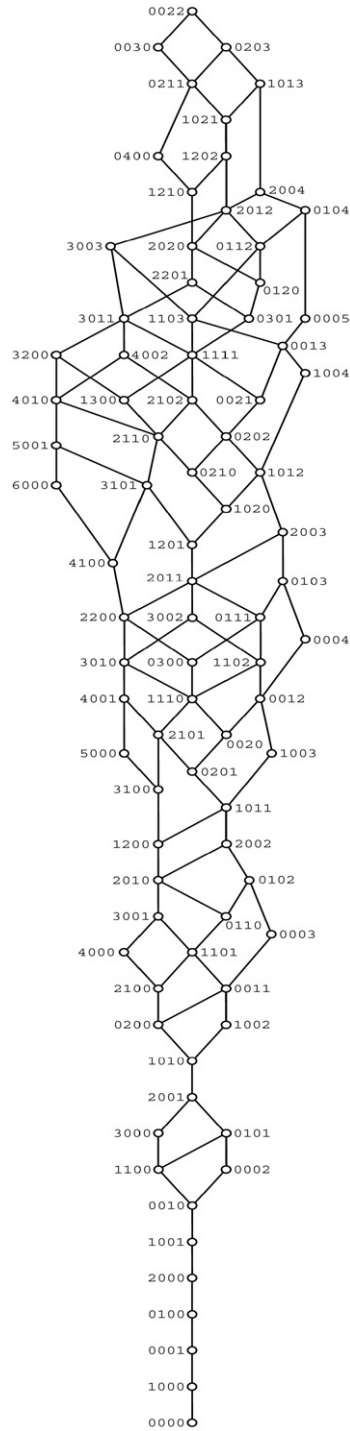


Fig. 1. The partial ordering of $P^+(2\omega_3 + 2\omega_4)$ in F_4^\vee .

Table 1The reverse lexicographic ordering of $P^+(2\omega_3 + 2\omega_4)$ in F_4^\vee

i	(λ_{1i})	λ_{2i}	λ_{3i}	λ_{4i}	Weight η_i	i	(λ_{1i})	λ_{2i}	λ_{3i}	λ_{4i}	Weight η_i
0	0	0	0	0	0	43	7	3	0	0	$3\omega_1 + 2\omega_4$
1	1	1	0	0	ω_1	44	7	3	1	1	$2\omega_1 + \omega_3 + \omega_4$
2	2	0	0	0	ω_4	45	7	3	2	0	$\omega_1 + 2\omega_2 + \omega_4$
3	2	1	1	0	ω_2	46	7	3	2	2	$\omega_1 + 2\omega_3$
4	2	2	0	0	$2\omega_1$	47	7	3	3	1	$2\omega_2 + \omega_3$
5	3	1	0	0	$\omega_1 + \omega_4$	48	7	4	1	0	$3\omega_1 + \omega_2 + \omega_4$
6	3	1	1	1	ω_3	49	7	4	2	1	$2\omega_1 + \omega_2 + \omega_3$
7	3	2	1	0	$\omega_1 + \omega_2$	50	7	4	3	0	$\omega_1 + 3\omega_2$
8	3	3	0	0	$3\omega_1$	51	7	5	0	0	$5\omega_1 + \omega_4$
9	4	0	0	0	$2\omega_4$	52	7	5	1	1	$4\omega_1 + \omega_3$
10	4	1	1	0	$\omega_2 + \omega_4$	53	7	5	2	0	$3\omega_1 + 2\omega_2$
11	4	2	0	0	$2\omega_1 + \omega_4$	54	8	0	0	0	$4\omega_4$
12	4	2	1	1	$\omega_1 + \omega_3$	55	8	1	1	0	$\omega_2 + 3\omega_4$
13	4	2	2	0	$2\omega_2$	56	8	2	0	0	$2\omega_1 + 3\omega_4$
14	4	3	1	0	$2\omega_1 + \omega_2$	57	8	2	1	1	$\omega_1 + \omega_3 + 2\omega_4$
15	4	4	0	0	$4\omega_1$	58	8	2	2	0	$2\omega_2 + 2\omega_4$
16	5	1	0	0	$\omega_1 + 2\omega_4$	59	8	2	2	2	$2\omega_3 + \omega_4$
17	5	1	1	1	$\omega_3 + \omega_4$	60	8	3	1	0	$2\omega_1 + \omega_2 + 2\omega_4$
18	5	2	1	0	$\omega_1 + \omega_2 + \omega_4$	61	8	3	2	1	$\omega_1 + \omega_2 + \omega_3 + \omega_4$
19	5	2	2	1	$\omega_2 + \omega_3$	62	8	3	3	0	$3\omega_2 + \omega_4$
20	5	3	0	0	$3\omega_1 + \omega_4$	63	8	3	3	2	$\omega_2 + 2\omega_3$
21	5	3	1	1	$2\omega_1 + \omega_3$	64	8	4	0	0	$4\omega_1 + 2\omega_4$
22	5	3	2	0	$\omega_1 + 2\omega_2$	65	8	4	1	1	$3\omega_1 + \omega_3 + \omega_4$
23	5	4	1	0	$3\omega_1 + \omega_2$	66	8	4	2	0	$2\omega_1 + 2\omega_2 + \omega_4$
24	5	5	0	0	$5\omega_1$	67	8	4	2	2	$2\omega_1 + 2\omega_3$
25	6	0	0	0	$3\omega_4$	68	8	4	3	1	$\omega_1 + 2\omega_2 + \omega_3$
26	6	1	1	0	$\omega_2 + 2\omega_4$	69	8	4	4	0	$4\omega_2$
27	6	2	0	0	$2\omega_1 + 2\omega_4$	70	9	1	0	0	$\omega_1 + 4\omega_4$
28	6	2	1	1	$\omega_1 + \omega_3 + \omega_4$	71	9	1	1	1	$\omega_3 + 3\omega_4$
29	6	2	2	0	$2\omega_2 + \omega_4$	72	9	2	1	0	$\omega_1 + \omega_2 + 3\omega_4$
30	6	2	2	2	$2\omega_3$	73	9	2	2	1	$\omega_2 + \omega_3 + 2\omega_4$
31	6	3	1	0	$2\omega_1 + \omega_2 + \omega_4$	74	9	3	0	0	$3\omega_1 + 3\omega_4$
32	6	3	2	1	$\omega_1 + \omega_2 + \omega_3$	75	9	3	1	1	$2\omega_1 + \omega_3 + 2\omega_4$
33	6	3	3	0	$3\omega_2$	76	9	3	2	0	$\omega_1 + 2\omega_2 + 2\omega_4$
34	6	4	0	0	$4\omega_1 + \omega_4$	77	9	3	2	2	$\omega_1 + 2\omega_3 + \omega_4$
35	6	4	1	1	$3\omega_1 + \omega_3$	78	9	3	3	1	$2\omega_2 + \omega_3 + \omega_4$
36	6	4	2	0	$2\omega_1 + 2\omega_2$	79	9	3	3	3	$3\omega_3$
37	6	5	1	0	$4\omega_1 + \omega_2$	80	10	0	0	0	$5\omega_4$
38	6	6	0	0	$6\omega_1$	81	10	1	1	0	$\omega_2 + 4\omega_4$
39	7	1	0	0	$\omega_1 + 3\omega_4$	82	10	2	0	0	$2\omega_1 + 4\omega_4$
40	7	1	1	1	$\omega_3 + 2\omega_4$	83	10	2	1	1	$\omega_1 + \omega_3 + 3\omega_4$
41	7	2	1	0	$\omega_1 + \omega_2 + 2\omega_4$	84	10	2	2	0	$2\omega_2 + 3\omega_4$
42	7	2	2	1	$\omega_2 + \omega_3 + \omega_4$	85	10	2	2	2	$2\omega_3 + 2\omega_4$

We denote by T_u the q -shift operator with respect to $u \rightarrow qu$. From the definition (3), it follows that

$$\begin{aligned}
 \frac{T_{c_1} \Phi^{(s, \ell)}(x)}{\Phi^{(s, \ell)}(x)} &= \prod_{\substack{\alpha > 0 \\ \alpha: \text{short}}} q^{-(\alpha, x)} (1 - c_1 q^{(\alpha, x)}) (1 - c_1^{-1} q^{(\alpha, x)}) \\
 &= \prod_{\substack{\alpha > 0 \\ \alpha: \text{short}}} ((q^{(\alpha, x)} + q^{-(\alpha, x)}) - (c_1 + c_1^{-1})) \\
 &= \sum_{i=0}^{12} (-1)^i (c_1 + c_1^{-1})^{12-i} S_i(x)
 \end{aligned} \tag{23}$$

where

$$\begin{aligned} S_i(x) &:= e_i(\{q^{(\alpha, x)} + q^{-(\alpha, x)}; \alpha \text{ is short positive in } F_4^\vee\}) \\ &= e_i(\{q^{(\varepsilon_i \pm \varepsilon_j, x)} + q^{-(\varepsilon_i \pm \varepsilon_j, x)}; 1 \leq i < j \leq 4\}). \end{aligned}$$

Set

$$\mathcal{A}_\eta(x) := \mathcal{A}(q^{(\eta, x)}) = \sum_{w \in W} (\text{sgn } w) q^{(w\eta, x)}.$$

If $\rho = \sum_{\alpha > 0} \alpha/2 = 8\varepsilon_1 + 3\varepsilon_2 + 2\varepsilon_3 + \varepsilon_4$ and η_i 's are defined in (22), then a direct calculation shows

$$S_i(x) = \sum_{j=0}^{36} c_{ij} \mathcal{A}_{\rho+\eta_j}(x) / \mathcal{A}_\rho(x) \quad (24)$$

where the coefficients c_{ij} are listed in Table 2. For example,

$$\begin{aligned} S_1(x) \mathcal{A}_\rho(x) &= \mathcal{A}_{\rho+\eta_1}(x) - 2\mathcal{A}_\rho(x), \\ S_2(x) \mathcal{A}_\rho(x) &= \mathcal{A}_{\rho+\eta_3}(x) + \mathcal{A}_{\rho+\eta_2}(x) - 2\mathcal{A}_{\rho+\eta_1}(x) - 9\mathcal{A}_\rho(x), \end{aligned}$$

and so on.

In the same way, we have

$$\frac{T_{b_1}^2 \Phi^{(s, \ell)}(x)}{\Phi^{(s, \ell)}(x)} = \sum_{i=0}^{12} (-1)^i (b_1 + b_1^{-1})^{12-i} L_i(x) \quad (25)$$

where

$$L_i(x) := e_i(\{q^{(\alpha, x)} + q^{-(\alpha, x)}; \alpha \text{ is long positive in } F_4^\vee\}).$$

From direct calculations we also have

$$L_i(x) = \sum_{j=0}^{85} d_{ij} \mathcal{A}_{\rho+\eta_j}(x) / \mathcal{A}_\rho(x), \quad (26)$$

where the coefficients d_{ij} are listed in Table 3. For example,

$$\begin{aligned} L_1(x) \mathcal{A}_\rho(x) &= \mathcal{A}_{\rho+\eta_2}(x) - \mathcal{A}_{\rho+\eta_1}(x) - 2\mathcal{A}_\rho(x), \\ L_2(x) \mathcal{A}_\rho(x) &= \mathcal{A}_{\rho+\eta_6}(x) - \mathcal{A}_{\rho+\eta_5}(x) + \mathcal{A}_{\rho+\eta_4}(x) - \mathcal{A}_{\rho+\eta_3}(x) \\ &\quad - \mathcal{A}_{\rho+\eta_2}(x) + 2\mathcal{A}_{\rho+\eta_1}(x) - 8\mathcal{A}_\rho(x), \end{aligned}$$

and so on.

For $0 \leq i \leq 85$ we define

$$J_i := \sum \Phi^{(s, \ell)}(z + \omega) \mathcal{A}_{\rho+\eta_i}(z + \omega). \quad (27)$$

Table 2The coefficients of c_{ij} of $\mathcal{A}_{\rho+\eta_j}(x)/\mathcal{A}_{\rho}(x)$ in $S_i(x)$

	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}	S_{12}
c_{i0}	-2	-9	18	29	-58	-40	80	22	-43	-5	6	1
c_{i1}	1	-2	-8	16	22	-44	-24	49	8	-20	0	2
c_{i2}		1	-2	-7	14	16	-31	-15	24	8	-6	-2
c_{i3}		1	-1	-9	8	29	-22	-39	23	19	-6	-3
c_{i4}			0	1	-1	-7	6	16	-10	-14	4	4
c_{i5}			1	-1	-7	6	16	-11	-13	7	2	-1
c_{i6}			1	-2	-6	13	10	-26	-3	16	0	-3
c_{i7}				1	-1	-6	6	10	-10	-3	3	0
c_{i8}				0	1	-1	-6	6	10	-10	-3	3
c_{i9}				1	-1	-7	6	16	-11	-13	6	3
c_{i10}				1	-1	-6	5	12	-7	-10	3	3
c_{i11}					1	0	-6	-1	11	4	-6	-3
c_{i12}					0	1	0	-5	0	7	0	-3
c_{i13}					1	-2	-4	10	3	-14	1	5
c_{i14}					0	1	0	-5	0	7	0	-2
c_{i15}					0	0	0	1	0	-4	0	3
c_{i16}					1	-1	-5	4	7	-4	-2	1
c_{i17}						0	1	-1	-3	3	0	0
c_{i18}						1	-1	-3	3	1	-1	0
c_{i19}						0	0	1	0	-3	0	1
c_{i20}						0	1	-1	-3	3	1	-1
c_{i21}						0	1	-1	-3	3	2	-2
c_{i22}						0	0	0	1	0	-2	1
c_{i23}						0	0	1	-1	-2	2	0
c_{i24}						0	0	0	1	-1	-2	1
c_{i25}						1	-2	-3	6	2	-3	-1
c_{i26}							1	-2	-2	5	0	-2
c_{i27}								1	-1	-3	2	2
c_{i28}								1	-1	-2	1	1
c_{i29}									0	1	0	-2
c_{i30}									1	-2	0	1
c_{i31}									1	-1	-1	1
c_{i32}										1	-1	0
c_{i33}										0	1	-1
c_{i34}										1	-1	-1
c_{i35}											1	-1
c_{i36}												1

In particular, from (4) we have

$$J_0 = J^{(s,\ell)}(\{b_i\}, \{c_j\}; z; Q)$$

because $\mathcal{A}_{\rho}(x) = \Delta_{F_4^{\vee}}(x)$ by Weyl's denominator formula. From (23), (24) and (27), the sum $T_{c_1} J^{(s,\ell)}(\{b_i\}, \{c_j\}; z; Q)$ is expressed as a linear combination of J_i ($0 \leq i \leq 36$). From (25), (26) and (27), $T_{b_1}^2 J^{(s,\ell)}(\{b_i\}, \{c_j\}; z; Q)$ is also expressed as a linear combination of J_i ($0 \leq i \leq 85$):

$$T_{c_1} J^{(s,\ell)}(\{b_i\}, \{c_j\}; z; Q) = \sum_{i=0}^{12} \sum_{j=0}^{36} (-1)^i (c_1 + c_1^{-1})^{12-i} c_{ij} J_j, \quad (28)$$

$$T_{b_1}^2 J^{(s,\ell)}(\{b_i\}, \{c_j\}; z; Q) = \sum_{i=0}^{12} \sum_{j=0}^{85} (-1)^i (b_1 + b_1^{-1})^{12-i} d_{ij} J_j. \quad (29)$$

But it will be shown in the remaining section that each J_i is proportional to $J_0 = J^{(s,\ell)}(\{b_i\}, \{c_j\}; z; Q)$. Consequently we obtain recurrence relations for $J^{(s,\ell)}(\{b_i\}, \{c_j\}; z; Q)$, with respect to $c_1 \rightarrow qc_1$

Table 3The coefficients d_{ij} of $\mathcal{A}_{\rho+\eta_j}(x)/\mathcal{A}_{\rho}(x)$ in $L_i(x)$

	L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}	L_{11}	L_{12}
d_{i0}	-2	-8	16	23	-45	-30	53	20	-24	-8	3	1
d_{i1}	-1	2	7	-15	-16	39	13	-42	-2	17	0	-2
d_{i2}	1	-1	-8	7	23	-16	-29	13	16	-3	-3	0
d_{i3}		-1	1	8	-7	-23	16	29	-13	-16	3	3
d_{i4}		1	-1	-7	7	16	-16	-13	13	3	-3	0
d_{i5}		-1	1	6	-6	-11	11	6	-7	1	1	-1
d_{i6}		1	0	-8	0	24	-6	-33	0	20	0	-4
d_{i7}			0	0	0	0	0	0	0	0	0	0
d_{i8}			-1	1	5	-5	-6	6	0	-1	1	0
d_{i9}			1	0	-6	-1	12	6	-10	-9	3	3
d_{i10}			-1	0	7	0	-17	0	16	0	-4	0
d_{i11}			1	1	-7	-5	17	7	-15	-3	3	1
d_{i12}			-1	1	5	-7	-7	16	2	-14	0	4
d_{i13}			1	-1	-5	5	7	-6	-3	0	2	0
d_{i14}				-1	1	6	-5	-12	7	10	-3	-3
d_{i15}				1	0	-6	0	12	0	-9	0	2
d_{i16}				-1	-1	6	4	-13	-4	11	1	-3
d_{i17}				2	0	-10	1	15	-2	-6	0	0
d_{i18}				0	0	0	0	0	0	0	0	0
d_{i19}				-1	0	6	0	-13	0	12	0	-4
d_{i20}				-1	0	4	0	-4	-1	1	1	0
d_{i21}				1	0	-4	1	4	-3	0	2	0
d_{i22}					0	-1	-1	5	3	-8	-1	3
d_{i23}					0	0	0	0	0	0	0	0
d_{i24}					-1	1	4	-4	-4	4	1	-1
d_{i25}					1	1	-5	-3	8	1	-3	0
d_{i26}					-1	-1	5	5	-8	-8	4	4
d_{i27}					2	0	-8	1	9	-1	-3	0
d_{i28}					-1	-1	3	2	-2	0	-1	0
d_{i29}					1	1	-4	-4	5	5	-2	-2
d_{i30}					1	-1	-3	6	2	-10	0	5
d_{i31}					-1	1	4	-3	-4	1	1	0
d_{i32}					0	0	0	0	0	0	0	0
d_{i33}					0	-1	1	3	-3	-1	1	0
d_{i34}					1	0	-4	0	5	0	-2	0
d_{i35}						-1	-1	3	3	-2	-2	0
d_{i36}						1	0	-4	0	5	0	-2
d_{i37}						-1	2	3	-6	-2	3	1
d_{i38}						1	-2	-2	5	0	-2	0
d_{i39}						-1	-1	3	2	-2	0	-1
d_{i40}						2	1	-7	-2	8	1	-3
d_{i41}						0	0	0	0	0	0	0
d_{i42}						-1	0	3	0	-2	0	0
d_{i43}						-1	0	2	-1	0	1	0
d_{i44}						1	1	-2	-2	1	1	0
d_{i45}							-1	0	2	-1	0	1
d_{i46}							0	-2	-1	5	1	-3
d_{i47}							0	0	0	0	0	0
d_{i48}							1	0	-2	0	0	0
d_{i49}							-1	1	2	-2	0	0
d_{i50}							-1	2	2	-5	0	2
d_{i51}							0	0	0	0	0	0
d_{i52}							1	-1	-2	2	1	-1
d_{i53}							0	-1	1	2	-2	0
d_{i54}							1	1	-3	-3	2	2
d_{i55}							-1	0	3	0	-2	0
d_{i56}							1	1	-2	-2	1	1
d_{i57}							-1	0	1	-2	0	2
d_{i58}							1	0	-2	0	1	0
d_{i59}								1	1	-1	-1	0

(continued on next page)

Table 3 (continued)

	L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}	L_{11}	L_{12}
d_{i60}								-1	1	3	-2	-2
d_{i61}								0	0	0	0	0
d_{i62}								-1	1	2	-1	-1
d_{i63}								0	0	-1	0	2
d_{i64}								1	0	-2	0	1
d_{i65}								-1	0	1	0	0
d_{i66}								1	-1	-1	1	0
d_{i67}								0	1	0	-1	0
d_{i68}								0	-1	1	1	-1
d_{i69}								0	1	-2	0	1
d_{i70}								-1	0	2	0	-1
d_{i71}								1	0	-1	0	0
d_{i72}									0	0	0	0
d_{i73}									-1	1	1	-1
d_{i74}									-1	0	1	0
d_{i75}									1	0	0	0
d_{i76}									0	-1	0	1
d_{i77}									0	-1	0	0
d_{i78}									0	1	-1	0
d_{i79}									0	0	1	-1
d_{i80}									1	-1	-1	0
d_{i81}										-1	1	1
d_{i82}										1	-1	0
d_{i83}											-1	0
d_{i84}											1	-1
d_{i85}												1

and $b_1 \rightarrow q^2 b_1$ from (28) and (29) respectively. We present a key tool for deducing the relations among the J_i 's, required to establish the proportionality.

For an arbitrary function $\varphi(x)$ on $E_{\mathbb{C}}$ and $\eta \in P$, let $\bar{\varphi}(z; \eta)$ be the sum defined by

$$\bar{\varphi}(z; \eta) := \sum_{\omega \in Q} \Phi^{(s, \ell)}(z + \omega) \mathcal{A} \nabla_{\eta} \varphi(z + \omega). \quad (30)$$

Eq. (7) in Lemma 3.1 indicates

$$\bar{\varphi}(z; \eta) = 0, \quad (31)$$

which plays a key role in deducing the relations among the J_i 's.

We will now discuss the cases $(s, \ell) = (0, 2)$ and $(2, 0)$ individually.

6.3. The case $(s, \ell) = (2, 0)$

Lemma 6.1. *The sum $J^{(2,0)}(b_1, b_2; z; Q)$ satisfies the recurrence relation*

$$J^{(2,0)}(q^2 b_1, b_2; z; Q) = R(b_1, b_2) J^{(2,0)}(b_1, b_2; z; Q), \quad (32)$$

where

$$\begin{aligned} R(b_1, b_2) = & (b_1 b_2; q)_2 (b_1 b_2^2; q)_2 (b_1^2; q)_4 (b_1^2 b_2; q)_4 (b_1^2 b_2^2; q)_4 \\ & \times (q b_1; q^2)_1 (b_1 b_2; q^2)_1 (b_1 b_2^2; q^2)_1 (b_1 b_2^3; q^2)_1 \\ & \times \frac{(b_1^2 b_2; q^2)_2 (b_1^2 b_2^2; q^2)_2 (b_1^2 b_2^3; q^2)_2 (b_1^3; q^2)_3 (b_1^3 b_2; q^2)_3 (b_1^3 b_2^2; q^2)_3}{b_1^{12} (b_1; q^2)_1^2 (q b_1^3 b_2^3; q^2)_3 (q b_1^6 b_2^6; q^2)_6}. \end{aligned}$$

Remark. The parameters b_1 and b_2 can be interchanged in the above equation.

Proof. Let $f_1(x)$ and $g_1(x)$ be functions defined by

$$\begin{aligned} f_1(x) &:= \prod_{i=1}^2 (1 - b_i q^{(2\varepsilon_1, x)}) (1 - b_i q^{(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4, x)}) (1 - b_i q^{(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4, x)}) \\ &\quad \times (1 - b_i q^{(2\varepsilon_2, x)}) (1 - b_i q^{(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + \varepsilon_4, x)}) (1 - b_i q^{(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4, x)}), \\ g_1(x) &:= \prod_{i=1}^2 (b_i - q^{(2\varepsilon_1, x)}) (b_i - q^{(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4, x)}) (b_i - q^{(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4, x)}) \\ &\quad \times (b_i - q^{(2\varepsilon_2, x)}) (b_i - q^{(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + \varepsilon_4, x)}) (b_i - q^{(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4, x)}). \end{aligned}$$

By the definition (3) of $\Phi^{(2,0)}(x)$, we have

$$\frac{\Phi^{(2,0)}(x + \omega_1)}{\Phi^{(2,0)}(x)} = q^{12} \frac{f_1(x)}{g_1(x - \omega_1)}.$$

For $\eta_i \in P^+(2\omega_3 + 2\omega_4)$, which is defined in (22) and Table 1, we define

$$\varphi_{\eta_i}(x) := q^{-(\rho + \eta_i, x)} g_1(x),$$

where $\rho = \sum_{\alpha > 0} \alpha/2 = 8\varepsilon_1 + 3\varepsilon_2 + 2\varepsilon_3 + \varepsilon_4$. Then, from (5), we have

$$\nabla_{\omega_1} \varphi_{\eta_i}(x) = q^{-(\rho + \eta_i, x)} [g_1(x) - q^{12 - (\rho + \eta_i, \omega_1)} f_1(x)].$$

Using *MATHEMATICA* [22] (or any other computer algebra program) for calculation of the alternating sum $\mathcal{A}_{\nabla_{\omega_1} \varphi_{\eta_i}(x)}$ over W , from (30), we see that $\bar{\varphi}_{\eta_i}(z; \omega_1)$, $i = 1, 2, \dots, 85$, can be expanded as a linear combination of J_0, J_1, \dots, J_i , i.e.,

$$\bar{\varphi}_{\eta_i}(z; \omega_1) = \sum_{j=0}^i a_{i,j} J_j \quad (i = 1, 2, \dots, 85).$$

For instance, some of the coefficients $a_{i,j}$ are expressed as

$$\begin{aligned} a_{1,1} &= a_{2,2} = -(1 - qb_1^6 b_2^6)/q, \quad a_{2,1} = 0, \\ a_{3,3} &= -(1 - q^2 b_1^6 b_2^6)/q^2, \quad a_{3,2} = -(b_1 + b_2)(1 - q^2 b_1^5 b_2^5)/q^2, \\ a_{4,4} &= -(1 - q^3 b_1^6 b_2^6)/q^3, \quad a_{4,3} = 0, \quad a_{4,2} = -a_{4,1} = (b_1 + b_2)(1 - q^3 b_1^5 b_2^5)/q^3, \quad \dots \end{aligned}$$

(Most of the coefficients are too long to write down conveniently.) Since $\bar{\varphi}_{\eta_i}(z; \omega_1) = 0$ from (31), we have the linear equations

$$\begin{pmatrix} a_{85,85} & \cdots & a_{85,2} & a_{85,1} \\ & \ddots & \vdots & \vdots \\ & & a_{2,2} & a_{2,1} \\ & & & a_{1,1} \end{pmatrix} \begin{pmatrix} J_{85} \\ \vdots \\ J_2 \\ J_1 \end{pmatrix} = -J_0 \begin{pmatrix} a_{85,0} \\ \vdots \\ a_{2,0} \\ a_{1,0} \end{pmatrix}. \quad (33)$$

By virtue of the triangularity of (33), we can easily solve the system and obtain explicit expressions for J_1, \dots, J_{85} . For example, we obtain

$$J_1 = -\frac{a_{1,0}}{a_{1,1}} J_0 = \left[\frac{(1-q)(1-b_1^9 b_2^9)}{(1-q b_1^6 b_2^6)(1-b_1^3 b_2^3)} - \frac{(1-b_1^3)(1-b_2^3)(1-q b_1^4 b_2^4)}{(1-b_1)(1-b_2)(1-q b_1^6 b_2^6)} \right] J_0, \\ J_7 = J_{18} = J_{23} = J_{32} = J_{41} = J_{47} = J_{51} = J_{61} = J_{72} = 0.$$

(Most of the other expressions are too long to write down here.) We can eliminate J_1, \dots, J_{85} from Eq. (29) by putting their explicit forms into the equation. Then, factorizing the right-hand side of (29) using *MATHEMATICA*, we eventually obtain the recurrence relation (32). \square

Proof of (15) in Theorem 4.3. Using (9), Eq. (8) implies

$$\Theta^{(2,0)}(q^2 b_1, b_2; z) = b_1^{12} \Theta^{(2,0)}(b_1, b_2; z). \quad (34)$$

From (10), (34) and Lemma 6.1, we have

$$C^{(2,0)}(q^2 b_1, b_2) = C^{(2,0)}(b_1, b_2) \times (q^{-1} b_1^{-1} b_2^{-1}; q)_2 (q^{-1} b_1^{-1} b_2^{-2}; q)_2 \\ \times (q^{-3} b_1^{-2}; q)_4 (q^{-3} b_1^{-2} b_2^{-1}; q)_4 (q^{-3} b_1^{-2} b_2^{-2}; q)_4 \\ \times (q^{-1} b_1^{-1}; q^2)_1 (b_1^{-1} b_2^{-1}; q^2)_1 (b_1^{-1} b_2^{-2}; q^2)_1 (b_1^{-1} b_2^{-3}; q^2)_1 \\ \times (q^{-2} b_1^{-2} b_2^{-1}; q^2)_2 (q^{-2} b_1^{-2} b_2^{-2}; q^2)_2 (q^{-2} b_1^{-2} b_2^{-3}; q^2)_2 \\ \times \frac{(q^{-4} b_1^{-3}; q^2)_3 (q^{-4} b_1^{-3} b_2^{-1}; q^2)_3 (q^{-4} b_1^{-3} b_2^{-2}; q^2)_3}{(b_1^{-1}; q^2)_1^2 (q^{-5} b_1^{-3} b_2^{-3}; q^2)_3 (q^{-11} b_1^{-6} b_2^{-6}; q^2)_6}.$$

By repeated use of this recurrence relation, we obtain

$$C^{(2,0)}(b_1, b_2) = C^{(2,0)}(q^{-2N} b_1, q^{-2N} b_2) \times (q b_1^{-2}; q)_{4N} (q b_2^{-2}; q)_{4N} \\ \times (q b_1^{-1} b_2^{-1}; q)_{4N} (q b_1^{-1} b_2^{-2}; q)_{6N} (q b_1^{-2} b_2^{-1}; q)_{6N} (q b_1^{-2} b_2^{-2}; q)_{8N} \\ \times (q b_1^{-1}; q^2)_N (q b_2^{-1}; q^2)_N (q^2 b_1^{-3}; q^2)_{3N} (q^2 b_2^{-3}; q^2)_{3N} \\ \times (q^2 b_1^{-1} b_2^{-1}; q^2)_{2N} (q^2 b_1^{-1} b_2^{-2}; q^2)_{3N} (q^2 b_1^{-2} b_2^{-1}; q^2)_{3N} (q^2 b_1^{-2} b_2^{-2}; q^2)_{4N} \\ \times \frac{(q^2 b_1^{-1} b_2^{-3}; q^2)_{4N} (q^2 b_1^{-3} b_2^{-1}; q^2)_{4N} (q^2 b_1^{-2} b_2^{-3}; q^2)_{5N} (q^2 b_1^{-3} b_2^{-2}; q^2)_{5N}}{(q^2 b_1^{-1}; q^2)_N^2 (q^2 b_2^{-1}; q^2)_N^2 (q b_1^{-3} b_2^{-3}; q^2)_{6N} (q b_1^{-6} b_2^{-6}; q^2)_{12N}} \\ = \lim_{N \rightarrow \infty} C^{(2,0)}(q^{-2N} b_1, q^{-2N} b_2) \times (q b_1^{-2}; q)_\infty (q b_2^{-2}; q)_\infty \\ \times (q b_1^{-1} b_2^{-1}; q)_\infty (q b_1^{-1} b_2^{-2}; q)_\infty (q b_1^{-2} b_2^{-1}; q)_\infty (q b_1^{-2} b_2^{-2}; q)_\infty \\ \times (q b_1^{-1}; q^2)_\infty (q b_2^{-1}; q^2)_\infty (q^2 b_1^{-3}; q^2)_\infty (q^2 b_2^{-3}; q^2)_\infty \\ \times (q^2 b_1^{-1} b_2^{-1}; q^2)_\infty (q^2 b_1^{-1} b_2^{-2}; q^2)_\infty (q^2 b_1^{-2} b_2^{-1}; q^2)_\infty (q^2 b_1^{-2} b_2^{-2}; q^2)_\infty \\ \times \frac{(q^2 b_1^{-1} b_2^{-3}; q^2)_\infty (q^2 b_1^{-3} b_2^{-1}; q^2)_\infty (q^2 b_1^{-2} b_2^{-3}; q^2)_\infty (q^2 b_1^{-3} b_2^{-2}; q^2)_\infty}{(q^2 b_1^{-1}; q^2)_\infty^2 (q^2 b_2^{-1}; q^2)_\infty^2 (q b_1^{-3} b_2^{-3}; q^2)_\infty (q b_1^{-6} b_2^{-6}; q^2)_\infty}. \quad (35)$$

Combining (35) and Proposition 5.1, we obtain (15). The proof is now complete. \square

6.4. The case $(s, \ell) = (0, 2)$

Lemma 6.2. The sum $J^{(0,2)}(c_1, c_2; z; Q)$ satisfies the recurrence relation

$$J^{(0,2)}(qc_1, c_2; z; Q) = R(c_1, c_2)J^{(0,2)}(c_1, c_2; z; Q), \quad (36)$$

where

$$\begin{aligned} R(c_1, c_2) = & (c_1c_2; q)_1 (c_1c_2^2; q)_1 (qc_1c_2^3; q)_1 \\ & \times (c_1^2c_2; q)_2 (qc_1^2c_2^2; q)_2 (qc_1^3c_2^3; q)_3 (qc_1^3c_2^2; q)_3 (q^2c_1^3c_2^3; q)_3 \\ & \times \frac{(c_1^2; q^2)_1 (c_1^2c_2^2; q^2)_1 (q^2c_1^2c_2^4; q^2)_1 (q^2c_1^4; q^2)_2 (q^2c_1^4c_2^2; q^2)_2 (q^2c_1^4c_2^4; q^2)_2 (q^3c_1^6c_2^6; q^2)_3}{c_1^{12}(c_1; q)_1^2(qc_1; q)_1(q^6c_1^{12}c_2^{12}; q^2)_6}. \end{aligned}$$

Remark. The parameters c_1 and c_2 can be interchanged in the above equation.

Proof. Let $f_1(x)$ and $g_1(x)$ be functions defined by

$$\begin{aligned} f_1(x) := & \prod_{j=1}^2 (1 - c_j q^{(\varepsilon_1 + \varepsilon_3, x)}) (1 - c_j q^{(\varepsilon_1 - \varepsilon_3, x)}) (1 - c_j q^{(\varepsilon_1 + \varepsilon_4, x)}) (1 - c_j q^{(\varepsilon_1 - \varepsilon_4, x)}) \\ & \times (1 - c_j q^{(\varepsilon_2 + \varepsilon_3, x)}) (1 - c_j q^{(\varepsilon_2 - \varepsilon_3, x)}) (1 - c_j q^{(\varepsilon_2 + \varepsilon_4, x)}) (1 - c_j q^{(\varepsilon_2 - \varepsilon_4, x)}) \\ & \times (1 - c_j q^{(\varepsilon_1 + \varepsilon_2, x)}) (1 - c_j q^{1 + (\varepsilon_1 + \varepsilon_2, x)}), \\ g_1(x) := & \prod_{j=1}^2 (c_j - q^{(\varepsilon_1 + \varepsilon_3, x)}) (c_j - q^{(\varepsilon_1 - \varepsilon_3, x)}) (c_j - q^{(\varepsilon_1 + \varepsilon_4, x)}) (c_j - q^{(\varepsilon_1 - \varepsilon_4, x)}) \\ & \times (c_j - q^{(\varepsilon_2 + \varepsilon_3, x)}) (c_j - q^{(\varepsilon_2 - \varepsilon_3, x)}) (c_j - q^{(\varepsilon_2 + \varepsilon_4, x)}) (c_j - q^{(\varepsilon_2 - \varepsilon_4, x)}) \\ & \times (c_j - q^{(\varepsilon_1 + \varepsilon_2, x)}) (c_j - q^{1 + (\varepsilon_1 + \varepsilon_2, x)}). \end{aligned}$$

By the definition (3) of $\Phi^{(0,2)}(x)$, we have

$$\frac{\Phi^{(0,2)}(x + \omega_1)}{\Phi^{(0,2)}(x)} = q^{10} \frac{f_1(x)}{g_1(x - \omega_1)}.$$

For $\eta_i \in P^+(2\omega_3 + 2\omega_4)$, which is specified in (22) and Table 1, we define

$$\varphi_{\eta_i}(x) := q^{-\langle \rho + \eta_i, x \rangle} g_1(x)$$

where $\rho = \sum_{\alpha > 0} \alpha / 2 = 8\varepsilon_1 + 3\varepsilon_2 + 2\varepsilon_3 + \varepsilon_4$. Then, from (5), we have

$$\nabla_{\omega_1} \varphi_{\eta_i}(x) = q^{-\langle \rho + \eta_i, x \rangle} [g_1(x) - q^{10 - \langle \rho + \eta_i, \omega_1 \rangle} f_1(x)].$$

Using *MATHEMATICA* for calculation of the alternating sum $\mathcal{A} \nabla_{\omega_1} \varphi_{\eta_i}(x)$ over W , from (30), we see that $\bar{\varphi}_{\eta_i}(z; \omega_1)$, $i = 1, \dots, 8$, can be expanded as a linear combination of J_0, \dots, J_8 , i.e.,

$$\bar{\varphi}_{\eta_i}(z; \omega_1) = \sum_{j=0}^8 a_{i,j} J_j \quad (1 \leq i \leq 8).$$

For instance, some of the coefficients a_{ij} are expressed as

$$\begin{aligned} a_{1,1} &= -(1 - c_1 c_2)(1 - c_1^2 c_2)(1 - c_1 c_2^2)(1 + q^3 c_1^6 c_2^6)/q^3, \\ a_{2,4} &= c_1^4 c_2^4 (1 - q^3 c_1^2 c_2^2)/q^3, \\ a_{3,4} &= -c_1 c_2 (1 + c_1)(1 + c_2)(1 - c_1 c_2)(1 + q^4 c_1^6 c_2^6)/q^4, \\ a_{4,4} &= -(1 - q^5 c_1^{10} c_2^{10})/q^5, \quad \dots \end{aligned}$$

(Most of the coefficients are too long to write down conveniently.) Since $\bar{\varphi}_{\eta_i}(z; \omega_1) = 0$ from (31), we have the linear equations

$$\begin{pmatrix} a_{8,8} & a_{8,7} & a_{8,6} & a_{8,5} & a_{8,4} & a_{8,3} & a_{8,2} & a_{8,1} \\ & a_{7,7} & a_{7,6} & a_{7,5} & a_{7,4} & a_{7,3} & a_{7,2} & a_{7,1} \\ & & a_{6,7} & a_{6,6} & a_{6,5} & a_{6,4} & a_{6,3} & a_{6,2} & a_{6,1} \\ & & & a_{5,5} & a_{5,4} & a_{5,3} & a_{5,2} & a_{5,1} \\ & & & & a_{4,4} & a_{4,3} & a_{4,2} & a_{4,1} \\ & & & & & a_{3,4} & a_{3,3} & a_{3,2} & a_{3,1} \\ & & & & & & a_{2,4} & a_{2,3} & a_{2,2} & a_{2,1} \\ & & & & & & & & a_{1,1} \end{pmatrix} \begin{pmatrix} J_8 \\ J_7 \\ J_6 \\ J_5 \\ J_4 \\ J_3 \\ J_2 \\ J_1 \end{pmatrix} = -J_0 \begin{pmatrix} a_{8,0} \\ a_{7,0} \\ a_{6,0} \\ a_{5,0} \\ a_{4,0} \\ a_{3,0} \\ a_{2,0} \\ a_{1,0} \end{pmatrix}.$$

Solving the above linear equations, we obtain J_1, \dots, J_8 in terms of J_0 . For example,

$$\begin{aligned} J_1 &= -\frac{a_{1,0}}{a_{1,1}} J_0 \\ &= \left[\frac{(1 - c_1^3)(1 - c_2^3)}{1 - qc_1^2 c_2^2 + q^2 c_1^4 c_2^4} \left(\frac{1 + qc_1 c_2 + q^2 c_1^2 c_2^2}{(1 - c_1)(1 - c_2)} - q(1 + c_1 c_2) \right) - 1 \right] J_0. \end{aligned}$$

The explicit forms of J_2, \dots, J_8 are too long to write down here.

Next we compute the explicit forms of J_9, \dots, J_{36} . Let $f_2(x)$, $g_2(x)$ be functions defined by

$$\begin{aligned} f_2(x) &:= \prod_{j=1}^2 (1 - c_j q^{(\varepsilon_1 + \varepsilon_2, x)})(1 - c_j q^{1 + (\varepsilon_1 + \varepsilon_2, x)})(1 - c_j q^{(\varepsilon_1 - \varepsilon_2, x)})(1 - c_j q^{1 + (\varepsilon_1 - \varepsilon_2, x)}) \\ &\quad \times (1 - c_j q^{(\varepsilon_1 + \varepsilon_3, x)})(1 - c_j q^{1 + (\varepsilon_1 + \varepsilon_3, x)})(1 - c_j q^{(\varepsilon_1 - \varepsilon_3, x)})(1 - c_j q^{1 + (\varepsilon_1 - \varepsilon_3, x)}) \\ &\quad \times (1 - c_j q^{(\varepsilon_1 + \varepsilon_4, x)})(1 - c_j q^{1 + (\varepsilon_1 + \varepsilon_4, x)})(1 - c_j q^{(\varepsilon_1 - \varepsilon_4, x)})(1 - c_j q^{1 + (\varepsilon_1 - \varepsilon_4, x)}), \\ g_2(x) &:= \prod_{j=1}^2 (c_j - q^{(\varepsilon_1 + \varepsilon_2, x)})(c_j - q^{1 + (\varepsilon_1 + \varepsilon_2, x)})(c_j - q^{(\varepsilon_1 - \varepsilon_2, x)})(c_j - q^{1 + (\varepsilon_1 - \varepsilon_2, x)}) \\ &\quad \times (c_j - q^{(\varepsilon_1 + \varepsilon_3, x)})(c_j - q^{1 + (\varepsilon_1 + \varepsilon_3, x)})(c_j - q^{(\varepsilon_1 - \varepsilon_3, x)})(c_j - q^{1 + (\varepsilon_1 - \varepsilon_3, x)}) \\ &\quad \times (c_j - q^{(\varepsilon_1 + \varepsilon_4, x)})(c_j - q^{1 + (\varepsilon_1 + \varepsilon_4, x)})(c_j - q^{(\varepsilon_1 - \varepsilon_4, x)})(c_j - q^{1 + (\varepsilon_1 - \varepsilon_4, x)}). \end{aligned}$$

By the definition (3) of $\Phi^{(0,2)}(x)$, we have

$$\frac{\Phi^{(0,2)}(x + \omega_4)}{\Phi^{(0,2)}(x)} = q^{12} \frac{f_2(x)}{g_2(x - \omega_4)}.$$

For $\eta_i \in P^+(2\omega_3 + 2\omega_4)$, if we define

$$\psi_{\eta_i}(x) := q^{-\langle \rho + \eta_i, x \rangle} g_2(x),$$

then, we have

$$\nabla_{\omega_4} \psi_{\eta_i}(x) = q^{-\langle \rho + \eta_i, x \rangle} [g_2(x) - q^{12 - \langle \rho + \eta_i, \omega_4 \rangle} f_2(x)].$$

From (30), using *MATHEMATICA*, we see that $\bar{\psi}_{\eta_i}(z; \omega_4)$, $i = 9, \dots, 36$, can be expanded as a linear combination of J_0, \dots, J_i , i.e.,

$$\bar{\psi}_{\eta_i}(z; \omega_4) = \sum_{j=0}^i a_{i,j} J_j \quad (9 \leq i \leq 36).$$

Since $\bar{\psi}_{\eta_i}(z; \omega_4) = 0$ from (31), we have the following linear equations:

$$\begin{pmatrix} a_{36,36} & \cdots & a_{36,10} & a_{36,9} \\ & \ddots & \vdots & \vdots \\ & & a_{10,10} & a_{10,9} \\ & & & a_{9,9} \end{pmatrix} \begin{pmatrix} J_{36} \\ \vdots \\ J_{10} \\ J_9 \end{pmatrix} = - \begin{pmatrix} a_{36,8} & \cdots & a_{36,1} & a_{36,0} \\ \vdots & \ddots & \vdots & \vdots \\ a_{10,8} & \cdots & a_{10,1} & a_{10,0} \\ a_{9,8} & \cdots & a_{9,1} & a_{9,0} \end{pmatrix} \begin{pmatrix} J_8 \\ \vdots \\ J_1 \\ J_0 \end{pmatrix}. \quad (37)$$

Since we have already had J_1, \dots, J_8 , we obtain J_9, \dots, J_{36} solving the above linear equations.

We can eliminate J_1, \dots, J_{36} from Eq. (28) by putting their explicit forms that have been obtained above into the equation. Then, factorizing the right-hand side of (28) using *MATHEMATICA*, we eventually obtain the recurrence relation (36). \square

Remark. Logically speaking, J_j/J_0 ($j = 1, 2, \dots, 36$) can be obtained only from $\bar{\varphi}_{\eta_i}(z; \omega_1)$ considering $i = 1, \dots, 36$. However, it is much easier to use $\bar{\psi}_{\eta_i}(z; \omega_4)$ for $i = 9, \dots, 36$ because the matrix in the left-hand side of (37) is already upper triangular, which is the reason why we used $\bar{\psi}_{\eta_i}(z; \omega_4)$.

Proof of (13) in Theorem 4.3. Using (9), Eq. (8) implies

$$\Theta^{(0,2)}(qc_1, c_2; z) = c_1^{12} \Theta^{(0,2)}(c_1, c_2; z). \quad (38)$$

From (10), (38) and Lemma 6.2, we have

$$\begin{aligned} C^{(0,2)}(qc_1, c_2) &= C^{(0,2)}(c_1, c_2) \times (c_1^{-1}c_2^{-1}; q)_1 (c_1^{-1}c_2^{-2}; q)_1 (q^{-1}c_1^{-1}c_2^{-3}; q)_1 \\ &\quad \times (q^{-1}c_1^2c_2^{-1}; q)_2 (q^{-2}c_1^{-2}c_2^{-2}; q)_2 (q^{-2}c_1^{-2}c_2^{-3}; q)_2 \\ &\quad \times (q^{-2}c_1^{-3}; q)_3 (q^{-3}c_1^{-3}c_2^{-1}; q)_3 (q^{-3}c_1^{-3}c_2^{-2}; q)_3 (q^{-4}c_1^{-3}c_2^{-3}; q)_3 \\ &\quad \times (c_1^{-2}; q^2)_1 (c_1^{-2}c_2^{-2}; q^2)_1 (q^{-2}c_1^{-2}c_2^{-4}; q^2)_1 \\ &\quad \times \frac{(q^{-4}c_1^{-4}; q^2)_2 (q^{-4}c_1^{-4}c_2^{-2}; q^2)_2 (q^{-4}c_1^{-4}c_2^{-4}; q^2)_2 (q^{-7}c_1^{-6}c_2^{-6}; q^2)_3}{(c_1^{-1}; q)_1^2 (q^{-1}c_1^{-1}; q)_1 (q^{-16}c_1^{-12}c_2^{-12}; q^2)_6}. \end{aligned}$$

By repeated use of this recurrence relation, we obtain

$$\begin{aligned}
C^{(0,2)}(c_1, c_2) &= C^{(0,2)}(q^{-N}c_1, q^{-N}c_2) \times (qc_1^{-3}; q)_{3N} (qc_2^{-3}; q)_{3N} (qc_1^{-1}c_2^{-1}; q)_{2N} \\
&\quad \times (qc_1^{-1}c_2^{-2}; q)_{3N} (qc_1^{-2}c_2^{-1}; q)_{3N} (c_1^{-1}c_2^{-3}; q)_{4N} (c_1^{-3}c_2^{-1}; q)_{4N} \\
&\quad \times (c_1^{-2}c_2^{-2}; q)_{4N} (c_1^{-2}c_2^{-3}; q)_{5N} (c_1^{-3}c_2^{-2}; q)_{5N} (q^{-1}c_1^{-3}c_2^{-3}; q)_{6N} \\
&\quad \times (q^2c_1^{-2}; q^2)_N (q^2c_2^{-2}; q^2)_N (c_1^{-4}; q^2)_{2N} (c_2^{-4}; q^2)_{2N} (q^2c_1^{-2}c_2^{-2}; q^2)_{2N} \\
&\quad \times \frac{(c_1^{-2}c_2^{-4}; q^2)_{3N} (c_1^{-4}c_2^{-2}; q^2)_{3N} (c_1^{-4}c_2^{-4}; q^2)_{4N} (q^{-1}c_1^{-6}c_2^{-6}; q^2)_{6N}}{(c_1^{-1}; q)_N (qc_1^{-1}; q)_N^2 (c_2^{-1}; q)_N (qc_2^{-1}; q)_N^2 (q^{-4}c_1^{-12}c_2^{-12}; q^2)_{12N}} \\
&= \lim_{N \rightarrow \infty} C^{(0,2)}(q^{-N}c_1, q^{-N}c_2) \times (qc_1^{-3}; q)_\infty (qc_2^{-3}; q)_\infty (qc_1^{-1}c_2^{-1}; q)_\infty \\
&\quad \times (qc_1^{-1}c_2^{-2}; q)_\infty (qc_1^{-2}c_2^{-1}; q)_\infty (c_1^{-1}c_2^{-3}; q)_\infty (c_1^{-3}c_2^{-1}; q)_\infty \\
&\quad \times (c_1^{-2}c_2^{-2}; q)_\infty (c_1^{-2}c_2^{-3}; q)_\infty (c_1^{-3}c_2^{-2}; q)_\infty (q^{-1}c_1^{-3}c_2^{-3}; q)_\infty \\
&\quad \times (q^2c_1^{-2}; q^2)_\infty (q^2c_2^{-2}; q^2)_\infty (c_1^{-4}; q^2)_\infty (c_2^{-4}; q^2)_\infty (q^2c_1^{-2}c_2^{-2}; q^2)_\infty \\
&\quad \times \frac{(c_1^{-2}c_2^{-4}; q^2)_\infty (c_1^{-4}c_2^{-2}; q^2)_\infty (c_1^{-4}c_2^{-4}; q^2)_\infty (q^{-1}c_1^{-6}c_2^{-6}; q^2)_\infty}{(c_1^{-1}; q)_\infty (qc_1^{-1}; q)_\infty^2 (c_2^{-1}; q)_\infty (qc_2^{-1}; q)_\infty^2 (q^{-4}c_1^{-12}c_2^{-12}; q^2)_\infty}. \tag{39}
\end{aligned}$$

Combining (39) and Proposition 5.1, we obtain (13). This completes the proof. \square

Appendix A. Proof of Lemma 3.1

Proof of Lemma 3.1. We abbreviate $\Phi^{(s, \ell)}(x)$ as $\Phi(x)$. By definition, if $\eta \in L$, then

$$\sum_{\omega \in L} \Phi(z + \omega) \varphi(z + \omega) = \sum_{\omega \in L} \Phi(z + \omega + \eta) \varphi(z + \omega + \eta)$$

so that

$$\sum_{\omega \in L} \Phi(z + \omega) \left[\varphi(z + \omega) - \frac{\Phi(z + \omega + \eta)}{\Phi(z + \omega)} \varphi(z + \omega + \eta) \right] = 0,$$

which implies (6).

Next we consider (7). For $w \in W$, if we set

$$\begin{aligned}
U_w(x) &:= \frac{w\Phi(x)}{\Phi(x)} = \prod_{j=1}^{\ell} \prod_{\substack{\alpha > 0 \\ -w^{-1}\alpha > 0 \\ \alpha: \text{short in } F_4^\vee}} q^{(2\gamma_j - 1)\langle \alpha, x \rangle} \frac{\theta(c_j q^{\langle \alpha, x \rangle}; q)}{\theta(c_j^{-1} q^{1 + \langle \alpha, x \rangle}; q)} \\
&\quad \times \prod_{i=1}^s \prod_{\substack{\alpha > 0 \\ -w^{-1}\alpha > 0 \\ \alpha: \text{long in } F_4^\vee}} q^{(2\beta_i - 1)\langle \alpha, x \rangle} \frac{\theta(b_i q^{\langle \alpha, x \rangle}; q^2)}{\theta(b_i^{-1} q^{2 + \langle \alpha, x \rangle}; q^2)},
\end{aligned}$$

then $U_w(x)$ is invariant under the shift $x \rightarrow x + \omega$ ($\omega \in P$), because $t^{a-b}\theta(q^a t; q)/\theta(q^b t; q)$ is invariant under the shift $t \rightarrow qt$. Thus we obtain

$$w \left[\sum_{\omega \in L} \Phi(z + \omega) \nabla_\eta \varphi(z + \omega) \right] = U_w(z) \sum_{\omega \in L} \Phi(z + \omega) w \nabla_\eta \varphi(z + \omega), \tag{A.1}$$

for $w \in W$. From (6) and (A.1) it follows that

$$\sum_{\omega \in L} \Phi(z + \omega) w \nabla_{\eta} \varphi(z + \omega) = 0. \quad (\text{A.2})$$

Since $\mathcal{A} \nabla_{\eta} \varphi(x) = \sum_{w \in W} (\text{sgn } w) w \nabla_{\eta} \varphi(x)$, we have the expression

$$\sum_{\omega \in L} \Phi(z + \omega) \mathcal{A} \nabla_{\eta} \varphi(z + \omega) = \sum_{w \in W} (\text{sgn } w) \sum_{\omega \in L} \Phi(z + \omega) w \nabla_{\eta} \varphi(z + \omega).$$

This is equal to zero because of (A.2), and hence (7) is obtained. This completes the proof. \square

Appendix B. Proof of Theorem 4.1

Proof of Theorem 4.1. We set

$$g(z) := \frac{J^{(s, \ell)}(\{b_i\}, \{c_j\}; z; Q)}{\Theta^{(s, \ell)}(\{b_i\}, \{c_j\}; z)}.$$

Since $\Theta^{(s, \ell)}(\{b_i\}, \{c_j\}; z)$ is written as

$$\prod_{\substack{\alpha > 0 \\ \alpha: \text{short} \\ \text{in } F_4}} \frac{q^{(s-1)\langle \alpha, z \rangle} \theta(q^{2\langle \alpha, z \rangle}; q^2)}{\prod_{i=1}^s q^{2\beta_i \langle \alpha, z \rangle} \theta(q^{2(\beta_i + \langle \alpha, z \rangle)}; q^2)} \prod_{\substack{\alpha > 0 \\ \alpha: \text{long} \\ \text{in } F_4}} \frac{q^{(\ell-1)\langle \alpha, z \rangle/2} \theta(q^{\langle \alpha, z \rangle}; q)}{\prod_{j=1}^{\ell} q^{\gamma_j \langle \alpha, z \rangle} \theta(q^{\gamma_j + \langle \alpha, z \rangle}; q)},$$

using (9), we have

$$\Theta^{(s, \ell)}(\{b_i\}, \{c_j\}; z + \omega) = V_{\omega}(z) \Theta^{(s, \ell)}(\{b_i\}, \{c_j\}; z),$$

where

$$\begin{aligned} V_{\omega}(z) &= \prod_{\substack{\alpha > 0 \\ \alpha: \text{short} \\ \text{in } F_4}} (-1)^{(s-1)\langle \alpha, \omega \rangle} q^{2(s-1)[\langle \alpha, \omega \rangle^2/2 + \langle \alpha, z \rangle \langle \alpha, \omega \rangle]} \\ &\quad \times \prod_{\substack{\alpha > 0 \\ \alpha: \text{long} \\ \text{in } F_4}} (-1)^{(\ell-1)\langle \alpha, \omega \rangle} q^{(\ell-1)[\langle \alpha, \omega \rangle^2/2 + \langle \alpha, z \rangle \langle \alpha, \omega \rangle]}. \end{aligned}$$

Since $J^{(s, \ell)}(\{b_i\}, \{c_j\}; z; Q)$ is invariant under the shift $z \rightarrow z + \omega$, $\omega \in P$, the function $g(z)$ satisfies

$$g(z + \omega) = g(z)/V_{\omega}(z).$$

First we suppose $g(z)$ is a constant, i.e., $V_{\omega}(z) \equiv 1$ for $\omega \in P$. Then it is necessary that

$$(s-1) \sum_{\substack{\alpha > 0 \\ \alpha: \text{short} \\ \text{in } F_4}} \langle \alpha, \omega_i \rangle + (\ell-1) \sum_{\substack{\alpha > 0 \\ \alpha: \text{long} \\ \text{in } F_4}} \langle \alpha, \omega_i \rangle \equiv 0 \pmod{2} \quad (\text{B.1})$$

for $i = 1, 2, 3, 4$, and

$$2(s-1)A + (\ell-1)B = O \quad (\text{B.2})$$

where $A = (a_{ij})$ and $B = (b_{ij})$ are 4×4 matrices given by

$$a_{ij} = \sum_{\substack{\alpha > 0 \\ \alpha: \text{short} \\ \text{in } F_4}} \langle \alpha, \omega_i \rangle \langle \alpha, \omega_j \rangle, \quad b_{ij} = \sum_{\substack{\alpha > 0 \\ \alpha: \text{long} \\ \text{in } F_4}} \langle \alpha, \omega_i \rangle \langle \alpha, \omega_j \rangle.$$

An explicit computation gives

$$B = 2A \quad \text{where} \quad A = \begin{pmatrix} 6 & 9 & 12 & 6 \\ 9 & 18 & 24 & 12 \\ 12 & 24 & 36 & 18 \\ 6 & 12 & 18 & 12 \end{pmatrix}. \quad (\text{B.3})$$

From (B.2) and (B.3), we obtain

$$s + \ell = 2, \quad s \geq 0, \quad \ell \geq 0.$$

This implies that $(s, \ell) = (0, 2)$, $(1, 1)$ or $(2, 0)$, which also satisfies

$$\langle (s-1)(5\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4) + (\ell-1)(6\varepsilon_1 + 4\varepsilon_2 + 2\varepsilon_3), \omega_i \rangle \equiv 0 \pmod{2}$$

for $i = 1, 2, 3, 4$, which is equal to (B.1).

Conversely if $(s, \ell) = (0, 2)$, $(1, 1)$ or $(2, 0)$, then $g(z)$ is invariant under the shift $z \rightarrow z + \omega$. Moreover, since Proposition 1.1 also holds for F_4 under condition (1), $g(z)$ is holomorphic, and hence $g(z)$ is constant. \square

References

- [1] K. Aomoto, On elliptic product formulas for Jackson integrals associated with reduced root systems, *J. Algebraic Combin.* 8 (1998) 115–126.
- [2] W.N. Bailey, Series of hypergeometric type which are infinite in both directions, *Q. J. Math.* 7 (1936) 105–115, doi:10.1093/qmath/os-7.1.105.
- [3] N. Bourbaki, *Groupe et algèbres de Lie*, Chapitres 4, 5 et 6, Hermann, Paris, 1969.
- [4] I. Cherednik, Double affine Hecke algebras and Macdonald's conjectures, *Ann. of Math.* (2) 141 (1995) 191–216.
- [5] J.F. van Diejen, On certain multiple Bailey, Rogers and Dougall type summation formulas, *Publ. RIMS Kyoto Univ.* 33 (1997) 483–508.
- [6] F.G. Garvan, A proof of the Macdonald–Morris root system conjecture for F_4 , *SIAM J. Math. Anal.* 21 (1990) 803–821.
- [7] F.G. Garvan, G. Gonnet, Macdonald's constant term conjectures for exceptional root systems, *Bull. Amer. Math. Soc.* 24 (1991) 343–347.
- [8] F.G. Garvan, G. Gonnet, A proof of the two parameter q -cases of the Macdonald–Morris root system conjecture for $S(F_4)$ and $S(F_4)^\vee$ via Zeilberger's method, *J. Symbolic Comput.* 14 (1992) 141–177.
- [9] G. Gasper, M. Rahman, *Basic Hypergeometric Series*, 2nd ed., *Encyclopedia Math. Appl.*, vol. 96, Cambridge University Press, 2004.
- [10] R.A. Gustafson, The Macdonald identities for affine root systems of classical type and hypergeometric series very-well-poised on semisimple Lie algebras, in: *Ramanujan International Symposium on Analysis*, Pune, 1987, Macmillan of India, New Delhi, 1989, pp. 185–224.
- [11] M. Ito, On a theta product formula for Jackson integrals associated with root system of rank two, *J. Math. Anal. Appl.* 216 (1997) 122–163.
- [12] M. Ito, Symmetry classification for Jackson integral associated with irreducible reduced root systems, *Compos. Math.* 129 (2001) 325–340.
- [13] M. Ito, A product formula for Jackson integral associated with the root system F_4 , *Ramanujan J.* 6 (2002) 279–293.
- [14] M. Ito, Symmetry classification for Jackson integral associated with the root system BC_n , *Compos. Math.* 136 (2003) 209–216.

- [15] M. Ito, Convergence and asymptotic behavior of Jackson integrals associated with irreducible reduced root systems, *J. Approx. Theory* 124 (2003) 154–180.
- [16] M. Ito, Askey–Wilson type integrals associated with root systems, *Ramanujan J.* 12 (2006) 131–151.
- [17] M. Ito, A. Tsubouchi, Bailey type summation formulas associated with the root system G_2^\vee , preprint, 2006.
- [18] I.G. Macdonald, Some conjectures for root systems, *SIAM J. Math. Anal.* 13 (1982) 998–1007.
- [19] I.G. Macdonald, A formal identity for affine root systems, in: *Lie Groups and Symmetric Spaces*, in: *Amer. Math. Soc. Transl. Ser. 2*, vol. 210, Amer. Math. Soc., Providence, RI, 2003, pp. 195–211.
- [20] J.R. Stembridge, The partial order of dominant weights, *Adv. Math.* 136 (1998) 340–364.
- [21] J.R. Stembridge, Computational aspects of root systems, Coxeter groups, and Weyl characters, in: *Interaction of Combinatorics and Representation Theory*, in: *MSJ Mem.*, vol. 11, Math. Soc. Japan, Tokyo, 2001, pp. 1–38.
- [22] Wolfram Research, Inc., *Mathematica*, Version 5.2, Wolfram Research, Inc., Champaign, IL, 2005.