



Cohen–Macaulayness for symbolic power ideals of edge ideals

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ABSTRACT

Let $S = K[x_1, \dots, x_n]$ be a polynomial ring over a field K . Let $I(G) \subseteq S$ denote the edge ideal of a graph G . We show that the ℓ th symbolic power $I(G)^{(\ell)}$ is a Cohen–Macaulay ideal (i.e., $S/I(G)^{(\ell)}$ is Cohen–Macaulay) for some integer $\ell \geq 3$ if and only if G is a disjoint union of finitely many complete graphs. When this is the case, all the symbolic powers $I(G)^{(\ell)}$ are Cohen–Macaulay ideals. Similarly, we characterize graphs G for which $S/I(G)^{(\ell)}$ has (FLC).

As an application, we show that an edge ideal $I(G)$ is complete intersection provided that $S/I(G)^\ell$ is Cohen–Macaulay for some integer $\ell \geq 3$. This strengthens the main theorem in Crupi et al. (2010) [3].

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0. Introduction

In this paper, we restrict our attention to the edge ideals of graphs. For a graph $G = (V(G), E(G))$, the edge ideal, denoted by $I(G)$, is defined by

$$I(G) = (x_i x_j : \{x_i, x_j\} \in E(G))S,$$

where $S = K[v : v \in V(G)] = K[x_1, \dots, x_n]$ is a polynomial ring over a field K . Then $E(G)$ is

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a squarefree monomial ideal which is generated by degree 2 elements, and thus it can be regarded as a Stanley–Reisner ideal and it is a radical ideal. Then the following theorem is well known.

Theorem. (See [1,2,12].) *Let S be a regular local ring (resp., a polynomial ring over a field K), and let I be a radical ideal (resp., a homogeneous radical ideal) of S . Then I is complete intersection if and only if S/I^ℓ is Cohen–Macaulay for infinitely many $\ell \geq 1$.*

In particular, for any edge ideal $I(G)$ of a graph G , $I(G)$ is a complete intersection ideal if and only if $S/I(G)^\ell$ is Cohen–Macaulay for infinitely many $\ell \geq 1$.

In what follows, let $G = (V(G), E(G))$ be a graph, and $I(G) \subseteq S = K[v : v \in V(G)]$ the edge ideal of G .

Recently, in [11], the last two authors gave a generalization of the theorem using a classification theorem for locally complete intersection Stanley–Reisner ideals; see [11, Theorem 1.15]. Note that the following theorem is also true for Stanley–Reisner ideals.

Theorem. (See [11, Theorem 2.1].) *If $S/I(G)^\ell$ is Buchsbaum for infinitely many $\ell \geq 1$, then $I(G)$ is complete intersection.*

Moreover, the authors [3] gave a refinement of the above theorem jointly with M. Crupi.

Theorem. (See [3, Theorem 2.1].) *$I(G)$ is complete intersection if and only if $S/I(G)^\ell$ is Cohen–Macaulay for some $\ell \geq \text{height } I$.*

The main purpose of this paper is to give another variation of the theorem in this context. Namely, we consider the following questions:

Questions. Let $\ell \geq 1$ be an integer. Let $I(G)^{(\ell)}$ denote the ℓ th symbolic power ideal of $I(G)$. Then:

- (1) When is $S/I(G)^{(\ell)}$ Cohen–Macaulay?
- (2) Is $I(G)$ complete intersection if $S/I(G)^\ell$ Cohen–Macaulay for a fixed $\ell \geq 1$?

The answers to these questions will give a generalization of the original theorem described as above. For instance, for each fixed $\ell \geq 1$, the Cohen–Macaulayness of $S/I(G)^\ell$ implies that of $S/I(G)^{(\ell)}$. Note that the converse is *not* true in general.

We first consider the above question. Let G be a graph on the vertex set $V = [n]$ such that $\dim S/I(G) = 1$. Such a graph G is isomorphic to the complete graph K_n . Then $S/I(G)^{(\ell)}$ is Cohen–Macaulay for every integer $\ell \geq 1$ because the symbolic power ideal has no embedded primes.

The following theorem characterizes graphs G for which all symbolic powers $S/I(G)^{(\ell)}$ are Cohen–Macaulay (or for $\ell \geq 3$).

Theorem. (See Theorem 3.6.) *The following conditions are equivalent:*

- (1) $S/I(G)^{(\ell)}$ is Cohen–Macaulay for every integer $\ell \geq 1$.
- (2) $S/I(G)^{(\ell)}$ is Cohen–Macaulay for some $\ell \geq 3$.
- (3) $S/I(G)^{(\ell)}$ satisfies Serre’s condition (S_2) for some $\ell \geq 3$.
- (4) G is a disjoint union of finitely many complete graphs.

As an application of the theorem, we can obtain some result for Cohen–Macaulayness of ordinary powers, which gives an improvement of the main theorem in [3].

Corollary. (See Theorem 3.8.) *If $S/I(G)^\ell$ is Cohen–Macaulay for some $\ell \geq 3$, then $I(G)$ is complete intersection.*

Next, we consider the following question. We need to assume that $I(G)$ is unmixed. Then if $\dim S/I(G) \leq 2$, for every integer $\ell \geq 1$, $S/I(G)^{(\ell)}$ is unmixed, and thus it has (FLC).

Question. Let $\ell \geq 1$ be an integer. When does $S/I(G)^{(\ell)}$ have (FLC)?

Let $\Delta = \Delta_{n_1, \dots, n_r}$ denote the simplicial complex whose Stanley–Reisner ideal is equal to the edge ideal of the disjoint union of complete graphs K_{n_1}, \dots, K_{n_r} . That is,

$$I_{\Delta_{n_1, \dots, n_r}} = I(K_{n_1} \sqcup \dots \sqcup K_{n_r}).$$

Then the following theorem gives an answer to the above question for $\ell \geq 3$:

Theorem. (See Theorem 4.7.) Let $\Delta(G)$ be the simplicial complex on $V(G)$ which satisfies $I_{\Delta(G)} = I(G)$. Suppose that $\Delta(G)$ is pure and $d = \dim S/I(G) \geq 3$. Let p denote the number of connected components of $\Delta(G)$. Then the following conditions are equivalent:

- (1) $S/I(G)^{(\ell)}$ has (FLC) for every integer $\ell \geq 1$.
- (2) $S/I(G)^{(\ell)}$ has (FLC) for some $\ell \geq 3$.
- (3) There exist $(n_{i1}, \dots, n_{id}) \in \mathbf{N}^d$ for every $i = 1, \dots, p$ such that Δ can be written as

$$\Delta = \Delta_{n_{11}, \dots, n_{1d}} \sqcup \Delta_{n_{21}, \dots, n_{2d}} \sqcup \dots \sqcup \Delta_{n_{p1}, \dots, n_{pd}}.$$

For $\ell = 2$, the problem is more complicated. For instance, if G is a pentagon, then $I(G)$ and $I(G)^2$ are Cohen–Macaulay although $I(G)^{(\ell)}$ (and hence $I(G)^\ell$) is not for any $\ell \geq 3$.

After finishing this work the authors have known that N.C. Minh obtained similar results independently.

1. Preliminaries

In this section we recall some definitions and properties that we will use later.

1.1. Edge ideals

Let G be a graph, which means a simple finite graph without loops and multiple edges. Let $V(G)$ (resp., $E(G)$) denote the set of vertices (resp., edges) of G . Put $V(G) = \{x_1, \dots, x_n\}$. Then the edge ideal of G , denoted by $I(G)$, is a squarefree monomial ideal of $S = K[x_1, \dots, x_n]$ defined by

$$I(G) = (x_i x_j : \{x_i, x_j\} \in E(G)).$$

A disjoint union of two graphs G_1 and G_2 , denoted by $G_1 \sqcup G_2$, is the graph G which satisfies $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. For a nonempty subset $W \subseteq V(G)$, $H = G|_W$ denotes the graph which satisfies $V(H) = W$ and $E(H) = \{\{x, y\} \in E(G) : x, y \in W\}$.

1.2. Stanley–Reisner ideals

Let $V = \{x_1, \dots, x_n\}$. A nonempty subset Δ of the power set 2^V is called a simplicial complex on V if $\{v\} \in \Delta$ for all $v \in V$, and $F \in \Delta$, $H \subseteq F$ imply $H \in \Delta$. An element $F \in \Delta$ is called a face of Δ . The dimension of Δ is defined by $\dim \Delta = \max\{\#(F) - 1 : F \text{ is a face of } \Delta\}$. A maximal face of Δ is called a facet of Δ . $\mathcal{F}(\Delta)$ denotes the set of all facets of Δ . The Stanley–Reisner ideal of Δ , denoted by I_Δ , is the squarefree monomial ideal generated by

$$\{x_{i_1} x_{i_2} \cdots x_{i_p} : 1 \leq i_1 < \dots < i_p \leq n, \{x_{i_1}, \dots, x_{i_p}\} \notin \Delta\},$$

and $K[\Delta] = K[x_1, \dots, x_n]/I_\Delta$ is called the Stanley–Reisner ring of Δ .

For an arbitrary graph G , the simplicial complex $\Delta(G)$ with $I(G) = I_{\Delta(G)}$ is called the *complementary simplicial complex* of G .

Put $d = \dim \Delta + 1$. A simplicial complex Δ is called *pure* if all the facets of Δ have the same cardinality d . A pure simplicial complex Δ is *connected in codimension 1* (or *strongly connected*) if for every two facets F and H of Δ , there is a sequence of facets $F = F_0, F_1, \dots, F_m = H$ such that $\sharp(F_i \cap F_{i+1}) = d - 1$ for each $i = 0, \dots, m - 1$. For every face $F \in \Delta$, the *star* and the *link* of F are defined by

$$\text{star}_\Delta F = \{H \in \Delta: H \cup F \in \Delta\},$$

$$\text{link}_\Delta F = \{H \in \Delta: H \cup F \in \Delta, H \cap F = \emptyset\}.$$

Note that these are also simplicial complexes.

1.3. Serre's condition

Let $S = K[x_1, \dots, x_n]$ and $\mathfrak{m} = (x_1, \dots, x_n)S$. Let I be a homogeneous ideal of S . For a positive integer k , S/I satisfies *Serre's condition* (S_k) if $\text{depth}(S/I)_P \geq \min\{\dim(S/I)_P, k\}$ for every $P \in \text{Spec } S/I$.

The ring S/I is called *Cohen–Macaulay* if $\text{depth } S/I = \dim S/I$. This is an equivalent condition that S/I satisfies Serre's condition (S_d) , where $d = \dim S/I$. Moreover, the ring S/I is called *(FLC)* if $H_{\mathfrak{m}}^i(S/I)$ has finite length for every $i \neq \dim S/I$. The ring S/I is called *Buchsbaum* if the natural map $\text{Ext}_S^i(S/\mathfrak{m}, S/I) \rightarrow H_{\mathfrak{m}}^i(S/I)$ is surjective for every $i \neq \dim S/I$. Note that any Cohen–Macaulay ring is Buchsbaum, and any Buchsbaum ring has (FLC).

A simplicial complex Δ is called *Cohen–Macaulay* (resp., *Buchsbaum*, *FLC*) if so is $K[\Delta]$. Note that Δ is Buchsbaum if and only if it satisfies (FLC). Moreover, if Δ is (FLC), then Δ is pure and $\text{link}_\Delta(F)$ is Cohen–Macaulay for every nonempty face $F \in \Delta$.

We notice that Δ is pure and connected in codimension 1 if $K[\Delta]$ satisfies (S_2) and $\dim \Delta \geq 1$.

1.4. Takayama's formula

Let I be an arbitrary monomial ideal in $S = K[x_1, \dots, x_n]$. Then the i th local cohomology module $H_{\mathfrak{m}}^i(S/I)$ can be regarded as a \mathbb{Z}^n -module over S/I . For every $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n$, we set $G_{\mathbf{a}} = \{i: a_i < 0\}$ and define

$$\Delta_{\mathbf{a}}(I) = \{F \subseteq [n]: F \text{ satisfies (C1) and (C2)}\},$$

where

(C1) $F \cap G_{\mathbf{a}} = \emptyset$;

(C2) for every minimal generator $u = x_1^{c_1} \cdots x_n^{c_n}$ of I there exists an index $i \notin F \cup G_{\mathbf{a}}$ with $c_i > a_i$.

Moreover, we define

$$\Delta(I) = \left\{ F \subseteq [n]: \prod_{i \in F} x_i \notin \sqrt{I} \right\}.$$

Then $\Delta(I)$ is a simplicial complex and $\Delta_{\mathbf{a}}(I)$ is a subcomplex of $\Delta(I)$ with $\dim \Delta_{\mathbf{a}}(I) = \dim \Delta(I) - \sharp(G_{\mathbf{a}})$ provided that $\Delta(I)$ is pure and $\Delta_{\mathbf{a}}(I) \neq \emptyset$ similarly as in [7, Lemma 1.3].

Now let us recall Takayama's formula, which is a generalization of well-known Hochster's formula.

Lemma 1.1 (Takayama's formula). (See e.g. [7, Theorem 1.1].) Let I be an arbitrary monomial ideal in $S = K[x_1, \dots, x_n]$. For every $\mathbf{a} \in \mathbb{Z}^n$, we have

$$\dim_K H_m^i(S/I)_{\mathbf{a}} = \begin{cases} \dim_K \tilde{H}_{i-\sharp(G_{\mathbf{a}})-1}(\Delta_{\mathbf{a}}(I)), & \text{if } G_{\mathbf{a}} \in \Delta(I), \\ 0, & \text{else.} \end{cases}$$

Using this lemma, we obtain the following criterion for Cohen–Macaulayness of S/I ; see also [7] in the case where $I = I_{\Delta}^{(\ell)}$ and $\dim S/I_{\Delta} = 1$.

Proposition 1.2. *The following conditions are equivalent:*

- (1) S/I is Cohen–Macaulay.
- (2) S/I has (FLC), and for any $\mathbf{a} \in \mathbb{N}^n$, we have that $\tilde{H}_i(\Delta_{\mathbf{a}}(I)) = 0$ for all $i < \dim \Delta_{\mathbf{a}}(I)$.

Proof. (1) \Rightarrow (2): Since S/I is Cohen–Macaulay, it has (FLC). For any $\mathbf{a} \in \mathbb{N}^n$, we have

$$\tilde{H}_i(\Delta_{\mathbf{a}}(I)) \cong H_m^{i+1}(S/I)_{\mathbf{a}} = 0$$

for all $i < \dim \Delta_{\mathbf{a}}(I) = \dim \Delta(I) = \dim S/I - 1$ by Lemma 1.1 since S/I is Cohen–Macaulay.

(2) \Rightarrow (1): Since S/I has (FLC), S/\sqrt{I} has also (FLC) and $\Delta(I)$ is pure; see [6].

Suppose that S/I is not Cohen–Macaulay. For any $\mathbf{a} \in \mathbb{N}^n$ we have

$$H_m^i(S/I)_{\mathbf{a}} \cong \tilde{H}_{i-1}(\Delta_{\mathbf{a}}(I)) = 0$$

for all $i \leq \dim \Delta_{\mathbf{a}}(I) = \dim \Delta(I)$. So there exist a vector $\mathbf{a} \in \mathbb{Z}^n \setminus \mathbb{N}^n$ and an index $i \leq \dim \Delta(I)$ such that

$$\tilde{H}_{i-\sharp(G_{\mathbf{a}})-1}(\Delta_{\mathbf{a}}(I)) \cong H_m^i(S/I)_{\mathbf{a}} \neq 0.$$

Set $\mathbf{a} = (a_1, \dots, a_n)$ and $a_j < 0$. Take any integer $k > 0$ and set $\mathbf{b} = \mathbf{a} - k\mathbf{e}_j$, where \mathbf{e}_j is the j th unit vector. Then we have $\Delta_{\mathbf{a}}(I) = \Delta_{\mathbf{b}}(I)$ because $G_{\mathbf{a}} = G_{\mathbf{b}}$. In particular, $H_m^i(S/I)_{\mathbf{b}} \neq 0$. But this contradicts the assumption that S/I has (FLC). \square

1.5. Symbolic power ideals

Let I be a radical ideal of S . Let $\text{Min}_S(S/I) = \{P_1, \dots, P_r\}$ be the set of the minimal prime ideals of I , and put $W = S \setminus \bigcup_{i=1}^r P_i$. Given an integer $\ell \geq 1$, the ℓ th symbolic power of I is defined to be the ideal

$$I^{(\ell)} = I^{\ell} S_W \cap S = \bigcap_{i=1}^r P_i^{\ell} S_{P_i} \cap S.$$

In particular, if I is a squarefree monomial ideal of S , then one has

$$I^{(\ell)} = P_1^{\ell} \cap \dots \cap P_r^{\ell}.$$

Let Δ be an arbitrary simplicial complex on $V = [n]$, and let $I_{\Delta} \subseteq S = K[x_1, \dots, x_n]$ denote the Stanley–Reisner ideal of Δ . For any integer $\ell \geq 1$ and $\mathbf{a} \in \mathbb{N}^n$, we set

$$\Delta_{\mathbf{a}}^{(\ell)} = \left\langle F \in \mathcal{F}(\Delta): \sum_{t \in V \setminus F} a_t \leq \ell - 1 \right\rangle.$$

We use the following remark and Proposition 1.2 in the proof of Theorem 2.1.

Remark 1.3. Under the notation above, for any $\mathbf{a} \in \mathbb{N}^n$, we have:

- (1) $\Delta_{\mathbf{a}}^{(\ell)} = \Delta_{\mathbf{a}}(I_{\Delta}^{(\ell)})$; see [7, Section 1].
- (2) If Δ is pure and $\Delta_{\mathbf{a}}^{(\ell)} \neq \emptyset$ then $\dim \Delta_{\mathbf{a}}^{(\ell)} = \dim \Delta$.

1.6. Polarizations

Now let $u = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ be a monomial in S . Then we can associate to it a squarefree monomial u^{pol} as follows: In the polarization process, each power of a variable $x_i^{a_i}$ is replaced by a product of a_i new variables $x_i^{(j)}$, $i \in \{1, \dots, n\}$, $j \in \{0, 1, \dots, a_i - 1\}$:

$$u^{\text{pol}} = x_1^{(0)} x_1^{(1)} \cdots x_1^{(a_1-1)} x_2^{(0)} x_2^{(1)} \cdots x_2^{(a_2-1)} \cdots x_n^{(0)} x_n^{(1)} \cdots x_n^{(a_n-1)},$$

where all $x_i^{(j)}$ are distinct variables and $x_i^{(0)} = x_i$ for each i . We call u^{pol} the *polarization* of u (see [9]). Let $I = (u_1, \dots, u_s)$ be a monomial ideal of S , where $\{u_1, \dots, u_s\}$ is the minimal set of monomial generators of I . If S^{pol} is a polynomial ring over K containing all monomials $u_1^{\text{pol}}, \dots, u_s^{\text{pol}}$, then we can consider the ideal $I^{\text{pol}} = (u_1^{\text{pol}}, \dots, u_s^{\text{pol}})$ of S^{pol} . It is known that, for monomial ideals I and J , one has [9]

$$(I \cap J)^{\text{pol}} = I^{\text{pol}} \cap J^{\text{pol}}. \quad (1.1)$$

It is well known that if S/I is Cohen–Macaulay then so is $S^{\text{pol}}/I^{\text{pol}}$. In the proof of the first main theorem, we need a stronger result: For a given positive integer k , if S/I satisfies Serre's condition (S_k) , then so does $S^{\text{pol}}/I^{\text{pol}}$; see [8]. Note that a similar statement for (FLC) does not hold in general.

1.7. Simplicial join

Let Γ (resp. Λ) be a nonempty simplicial complex on V_1 (resp. V_2) such that $V_1 \cap V_2 = \emptyset$. Then the *simplicial join* of Γ and Λ , denoted by $\Gamma * \Lambda$, is defined as follows

$$\Gamma * \Lambda = \{F_1 \cup F_2 : F_1 \in \Gamma, F_2 \in \Lambda\}.$$

Then $\Gamma * \Lambda$ is a simplicial complex on $V_1 \cup V_2$ and $\mathcal{F}(\Gamma * \Lambda) = \{F_1 \cup F_2 : F_1 \in \mathcal{F}(\Gamma), F_2 \in \mathcal{F}(\Lambda)\}$. In particular, $\dim \Gamma * \Lambda = \dim \Gamma + \dim \Lambda + 1$. Moreover, the i th reduced homology group $\tilde{H}_i(\Gamma * \Lambda)$ over a field K of $\Gamma * \Lambda$ is given by the so-called *Künneth formula*:

$$\tilde{H}_i(\Gamma * \Lambda) \cong \bigoplus_{p+q=i-1} \tilde{H}_p(\Gamma) \otimes \tilde{H}_q(\Lambda). \quad (1.2)$$

Notice that $K[\Gamma * \Lambda] \cong K[\Gamma] \otimes_K K[\Lambda]$ as K -algebras; see [4, Lemma 1].

For any disjoint union of two graphs G_1, G_2 , we have $\Delta(G_1 \sqcup G_2) = \Delta(G_1) * \Delta(G_2)$.

2. Symbolic powers of edge ideals of disjoint union of complete graphs

Let r, n_1, \dots, n_r be positive integers, and let

$$S = K[x_{ij} : 1 \leq i \leq r, 1 \leq j \leq n_i]$$

be a polynomial ring over a field K . For each integer i with $1 \leq i \leq r$, if we put

$$P_{ij} = (x_{i1}, \dots, \widehat{x_{ij}}, \dots, x_{in_i})S, \quad \text{and} \quad I_i = P_{i1} \cap P_{i2} \cap \dots \cap P_{in_i},$$

then I_i is equal to $I(K_{n_i})S$, where

$$I(K_{n_i}) = (x_{ij}x_{ik}: 1 \leq j < k \leq n_i)K[x_{i1}, \dots, x_{in_i}],$$

denotes the edge ideal of the complete n_i -graph K_{n_i} on the vertex set $V_i = \{x_{i1}, \dots, x_{in_i}\}$ for each $i = 1, \dots, r$.

Let G be the disjoint union of complete n_i -graphs for $i = 1, 2, \dots, r$:

$$G = K_{n_1} \sqcup K_{n_2} \sqcup \dots \sqcup K_{n_r}.$$

Then the edge ideal $I(G)$ of G is equal to $I_1 + I_2 + \dots + I_r$. Moreover, an irredundant primary decomposition of $I(G)$ is given by

$$I(G) = \bigcap_{j_1, \dots, j_r} (P_{1j_1} + \dots + P_{rj_r}), \quad (2.1)$$

where j_1, \dots, j_r move through the whole range $1 \leq j_1 \leq n_1, \dots, 1 \leq j_r \leq n_r$. In particular,

$$I(G)^{(\ell)} = \bigcap_{j_1, \dots, j_r} (P_{1j_1} + \dots + P_{rj_r})^\ell \quad (2.2)$$

for every integer $\ell \geq 1$. If we put

$$x_i = x_{i1} + x_{i2} + \dots + x_{in_i}$$

for $i = 1, 2, \dots, r$, then a sequence x_1, \dots, x_r forms a system of parameters of $S/I(G)$ (and hence $S/I(G)^{(\ell)}$ for every $\ell \geq 1$). Thus

$$\dim S/I(G)^{(\ell)} = \dim S/I(G) = r.$$

The main goal of this section is to prove the following theorem:

Theorem 2.1. *Let $S = K[x_{ij}: 1 \leq i \leq r, 1 \leq j \leq n_i]$ be a polynomial ring over a field K . Let G be a disjoint union of complete n_i -graphs: $G = K_{n_1} \sqcup K_{n_2} \sqcup \dots \sqcup K_{n_r}$. Then $S/I(G)^{(\ell)}$ is Cohen–Macaulay for every $\ell \geq 1$.*

Remark 2.2. In the above theorem, we do not need to assume that $\max\{n_1, \dots, n_d\} \geq 2$.

In order to prove Theorem 2.1, we need the following key lemma.

Lemma 2.3. *Let Γ (resp. Λ) be a simplicial complex on V_1 (resp. V_2) such that $V_1 \cap V_2 = \emptyset$. Put $\Delta = \Gamma * \Lambda$ and $V = V_1 \cup V_2$. Set $S_1 = K[V_1]$, $S_2 = K[V_2]$ and $S = S_1 \otimes_K S_2$. If $S_1/I_\Gamma^{(i)}$ and $S_2/I_\Lambda^{(i)}$ are Cohen–Macaulay for every $i \leq \ell$, then $S/I_\Delta^{(\ell)}$ is Cohen–Macaulay.*

Proof. We may assume that $V_1 = \{1, 2, \dots, m\}$, $V_2 = \{m+1, \dots, n\}$ and $V = [n]$. Note that Γ , Λ , and Δ are pure. By an inductive argument on $n = \sharp(V)$, we may assume that $S/I_\Delta^{(\ell)}$ has (FLC). Then we must show that $\tilde{H}_i(\Delta_{\mathbf{a}}^{(\ell)}) = 0$ for all $\mathbf{a} \in \mathbb{N}^n$ and $i < \dim \Delta_{\mathbf{a}}^{(\ell)} = \dim \Delta$ with $\Delta_{\mathbf{a}}^{(\ell)} \neq \emptyset$. We first prove the following claim.

Claim 1. For each $\mathbf{a} \in \mathbb{N}^n$, $\Delta_{\mathbf{a}}^{(\ell)} = \bigcup_{k=1}^{\ell} \Gamma_{\mathbf{a}_1}^{(\ell+1-k)} * \Lambda_{\mathbf{a}_2}^{(k)}$ holds, where $\mathbf{a}_1 = \mathbf{a}|_{V_1}$ and $\mathbf{a}_2 = \mathbf{a}|_{V_2}$.

Since $\mathcal{F}(\Delta) = \{F_1 \cup F_2 : F_1 \in \mathcal{F}(\Gamma), F_2 \in \mathcal{F}(\Lambda)\}$, we have

$$\begin{aligned} \Delta_{\mathbf{a}}^{(\ell)} &= \left\langle F_1 \cup F_2 : F_1 \in \mathcal{F}(\Gamma), F_2 \in \mathcal{F}(\Lambda), 0 \leq \sum_{t \in V \setminus F} a_t \leq \ell - 1 \right\rangle \\ &= \bigcup_{k=1}^{\ell} \left\langle F_1 \cup F_2 : F_1 \in \mathcal{F}(\Gamma), F_2 \in \mathcal{F}(\Lambda), \sum_{t \in V_1 \setminus F_1} a_t \leq \ell - k, \sum_{t' \in V_2 \setminus F_2} a_{t'} \leq k - 1 \right\rangle \\ &= \bigcup_{k=1}^{\ell} \Gamma_{\mathbf{a}_1}^{(\ell-k+1)} * \Lambda_{\mathbf{a}_2}^{(k)}, \end{aligned}$$

as required. We have proved Claim 1.

Put $d = \dim \Delta + 1 = \dim \Gamma + \dim \Lambda + 2$. For $j = 1, \dots, \ell$, we set

$$\Pi_j = \bigcup_{k=1}^j \Gamma_{\mathbf{a}_1}^{(\ell-k+1)} * \Lambda_{\mathbf{a}_2}^{(k)}.$$

We next prove the following claim.

Claim 2. $\tilde{H}_i(\Pi_j) = 0$ holds for every $i < d - 1$ and $j = 1, \dots, \ell$.

We use an induction on j . First consider the case where $j = 1$. Then $\Pi_1 = \Gamma_{\mathbf{a}_1}^{(\ell)} * \Lambda_{\mathbf{a}_2}$. As $I_{\Gamma}^{(\ell)}$ and I_{Λ} are Cohen–Macaulay by assumption, we get

$$p < \dim \Gamma = \dim \Gamma_{\mathbf{a}_1}^{(\ell)} \Rightarrow \tilde{H}_p(\Gamma_{\mathbf{a}_1}^{(\ell)}) = 0$$

and

$$q < \dim \Lambda = \dim \Lambda_{\mathbf{a}_2} \Rightarrow \tilde{H}_q(\Lambda_{\mathbf{a}_2}) = 0.$$

Now suppose that $i < \dim \Pi_1 = \dim \Gamma + \dim \Lambda + 1 = d - 1$. Then for any pair (p, q) with $p + q = i - 1$, either $p < \dim \Gamma$ or $q < \dim \Lambda$ holds. Hence the Künneth formula (see Section 1.6) yields that

$$\tilde{H}_i(\Pi_1) \cong \bigoplus_{p+q=i-1} \tilde{H}_p(\Gamma_{\mathbf{a}_1}^{(\ell)}) \otimes_K \tilde{H}_q(\Lambda_{\mathbf{a}_2}) = 0.$$

So we have proved the case where $j = 1$.

Now assume that $(\ell \geqslant) j \geqslant 2$ and $\tilde{H}_i(\Pi_{j-1}) = 0$ for all $i < d - 1$. Then we must show that $\tilde{H}_i(\Pi_j) = 0$ for all $i < d - 1$. In order to do that, we put $\Sigma = \Gamma_{\mathbf{a}_1}^{(\ell-j+1)} * \Lambda_{\mathbf{a}_2}^{(j)}$. Then $\Pi_{j-1} \cup \Sigma = \Pi_j$ and

$$\begin{aligned} \Pi_{j-1} \cap \Sigma &= \bigcup_{k=1}^{j-1} (\Gamma_{\mathbf{a}_1}^{(\ell-k+1)} * \Lambda_{\mathbf{a}_2}^{(k)}) \cap (\Gamma_{\mathbf{a}_1}^{(\ell-j+1)} * \Lambda_{\mathbf{a}_2}^{(j)}) \\ &= \bigcup_{k=1}^{j-1} \{(\Gamma_{\mathbf{a}_1}^{(\ell-k+1)} * \Lambda_{\mathbf{a}_2}^{(k)}) \cap (\Gamma_{\mathbf{a}_1}^{(\ell-j+1)} * \Lambda_{\mathbf{a}_2}^{(j)})\} \end{aligned}$$

$$\begin{aligned}
&= \bigcup_{k=1}^{j-1} (\Gamma_{\mathbf{a}_1}^{(\ell-j+1)} * \Lambda_{\mathbf{a}_2}^{(k)}) \\
&= \Gamma_{\mathbf{a}_1}^{(\ell-j+1)} * \Lambda_{\mathbf{a}_2}^{(j-1)}.
\end{aligned}$$

Thus the Mayer–Vietoris sequence yields the following exact sequence for each i :

$$\cdots \rightarrow \tilde{H}_i(\Pi_{j-1}) \oplus \tilde{H}_i(\Gamma_{\mathbf{a}_1}^{(\ell-j+1)} * \Lambda_{\mathbf{a}_2}^{(j)}) \rightarrow \tilde{H}_i(\Pi_j) \rightarrow \tilde{H}_{i-1}(\Gamma_{\mathbf{a}_1}^{(\ell-j+1)} * \Lambda_{\mathbf{a}_2}^{(j-1)}) \rightarrow \cdots.$$

By a similar argument as above, we have

$$\tilde{H}_i(\Gamma_{\mathbf{a}_1}^{(\ell-j+1)} * \Lambda_{\mathbf{a}_2}^{(j)}) = \tilde{H}_{i-1}(\Gamma_{\mathbf{a}_1}^{(\ell-j+1)} * \Lambda_{\mathbf{a}_2}^{(j-1)}) = 0$$

for all $i < d - 1$. Moreover, the induction hypothesis implies $\tilde{H}_i(\Pi_{j-1}) = 0$ for all $i < d - 1$. Hence $\tilde{H}_i(\Pi_j) = 0$ for all $i < d - 1$, as required. Therefore we obtain that $\tilde{H}_i(\Delta_{\mathbf{a}}^{(\ell)}) = 0$ for all $i < d - 1$. This completes the proof. \square

Proof of Theorem 2.1. We prove the assertion by an induction on r . If $r = 1$, then the assertion is clear because $\dim S/I(K_{n_1})^{(\ell)} = \dim S/I(K_{n_1}) = 1$.

Let $r \geq 2$. Put

$$S' = K[x_{ij} : 1 \leq i \leq r - 1, 1 \leq j \leq n_i] \quad \text{and} \quad G' = K_{n_1} \sqcup \cdots \sqcup K_{n_{r-1}}.$$

By the induction hypothesis, we may assume that $S'/I(G')^{(i)}$ is Cohen–Macaulay for all $i \geq 1$. As $\Delta(G) = \Delta(G' \sqcup K_{n_r}) = \Delta(G') * \Delta(K_{n_r})$, by virtue of Lemma 2.3, we can conclude that $I(G)^{(\ell)}$ is Cohen–Macaulay for all $\ell \geq 1$. \square

In order to discuss (FLC) properties of symbolic or ordinary powers, we generalize Theorem 2.1 to the following corollary.

Corollary 2.4. Let $G = K_{n_1} \sqcup \cdots \sqcup K_{n_r}$ be a disjoint union of finitely many complete graphs, and let y_1, \dots, y_s be variables which are not vertices of G . Put $S = K[v : v \in G]$, $T = S[y_1, \dots, y_s]$ and $I = I(G) + (y_1, \dots, y_s)$. Then $T/I^{(\ell)}$ is Cohen–Macaulay for all $\ell \geq 1$.

Proof. We may assume that $s = 1$, and put $y = y_1$ for simplicity. Let $I(G) = \bigcap_j P_j$ be an irredundant primary decomposition of $I(G)$. Since $I = \bigcap_j (P_j, y)$ gives an irredundant primary decomposition of I , we have

$$I^{(\ell)} = \bigcap_j (P_j, y)^\ell = \bigcap_j \sum_{k=0}^{\ell} P_j^k y^{\ell-k} = \sum_{k=0}^{\ell} \left(\bigcap_i P_j^k \right) y^{\ell-k} = \sum_{k=0}^{\ell} I(G)^{(k)} y^{\ell-k}.$$

Hence it follows that

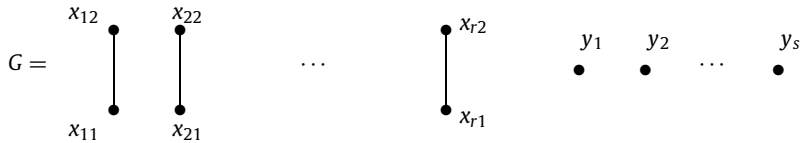
$$T/I^{(\ell)} \cong S/I(G)^{(\ell)} \oplus S/I(G)^{(\ell-1)} \oplus \cdots \oplus S/I(G)$$

as S -modules. Since all S -modules of the right-hand side are Cohen–Macaulay, so is $T/I^{(\ell)}$, as required. \square

Example 2.5. If G consists of r isolated edges and s isolated vertices, then

$$S = K[x_{11}, x_{12}, \dots, x_{r1}, x_{r2}, y_1, \dots, y_s], \quad I(G) = (x_{11}x_{12}, \dots, x_{r1}x_{r2}).$$

In particular, $S/I(G)^{(\ell)} = S/I(G)^\ell$ is Cohen–Macaulay for every $\ell \geq 1$.



This complete intersection complex is the boundary complex of a simplex or an iterated cone of a cross polytope. Namely, $I(G) = I_{\Delta(\mathcal{P})}$ holds, where \mathcal{P} is the s -iterated cone of the cross r -polytope.

The next example shows that our theorem cannot be generalized for mixed symbolic powers.

Example 2.6. Let G be a complete n -graph. Then $I(G) = P_1 \cap \dots \cap P_n$, where $P_i = (x_1, \dots, \widehat{x_i}, \dots, x_n)$ for each $i = 1, \dots, n$. Since $\dim S/I(G) = 1$ and P_i^a has no embedded primes for any integer $a \geq 1$, $S/P_1^{a_1} \cap \dots \cap P_n^{a_n}$ is a Cohen–Macaulay ring of dimension 1 for every positive integers a_1, \dots, a_n .

A similar assertion does *not* hold in general for two disjoint union of complete graphs. For example, let $I(G) = (x_1x_2, x_1x_3, x_2x_3, y_1y_2)$ be the edge ideal of $K_3 \sqcup K_2$ in $S = \mathbb{Q}[x_1, x_2, x_3, y_1, y_2]$. Then

$$I(G) = (x_1, x_2, y_1) \cap (x_1, x_2, y_2) \cap (x_1, x_3, y_1) \cap (x_1, x_3, y_2) \cap (x_2, x_3, y_1) \cap (x_2, x_3, y_2).$$

Our theorem says that

$$I(G)^{(2)} = (x_1, x_2, y_1)^2 \cap (x_1, x_2, y_2)^2 \cap (x_1, x_3, y_1)^2 \cap (x_1, x_3, y_2)^2 \cap (x_2, x_3, y_1)^2 \cap (x_2, x_3, y_2)^2$$

is a Cohen–Macaulay ring of dimension 2, that is, $\text{pd}_S S/I(G)^{(2)} = 3$. Indeed, by Macaulay 2, the minimal free resolution of $S/I(G)^{(2)}$ over S is given by

$$0 \rightarrow S^5 \rightarrow S^{12} \rightarrow S^8 \rightarrow S \rightarrow S/I(G)^{(2)} \rightarrow 0.$$

However, this is no longer true for mixed symbolic powers. For instance, put

$$J_a = (x_1, x_2, y_1)^2 \cap (x_1, x_2, y_2)^2 \cap (x_1, x_3, y_1)^2 \cap (x_1, x_3, y_2)^2 \cap (x_2, x_3, y_1)^2 \cap (x_2, x_3, y_2)^a$$

for every positive integer $a \geq 2$. When $a \leq 3$, S/J_a is Cohen–Macaulay. But S/J_4 is *not*.

The following question seems to be interesting.

Question 2.7. We use the same notation as in (2.1). Let ℓ_{j_1, \dots, j_r} be given integers. When is the following mixed symbolic power ideal

$$\bigcap_{j_1, \dots, j_r} (P_{1, j_1} + \dots + P_{r, j_r})^{\ell_{j_1, \dots, j_r}}$$

Cohen–Macaulay?

3. Non-Cohen–Macaulayness of symbolic powers

3.1. Cohen–Macaulay properties of symbolic powers

In the previous section, we proved that all symbolic powers of the edge ideal of a disjoint union of finitely many complete graphs are Cohen–Macaulay. In this section, we prove the converse. That is, the main purpose of this section is to prove Theorem 3.1. Using these results, we prove the first main theorem. Moreover, as an application, we also prove an improvement of the main theorem [3] with respect to Cohen–Macaulayness of ordinary powers.

Theorem 3.1. *Let G be a graph which is not a disjoint union of finitely many complete graphs. Then for any $\ell \geq 3$, $S/I(G)^{(\ell)}$ does not satisfy Serre’s condition (S_2) .*

Remark 3.2. The assumption that $\ell \geq 3$ is essential. For example, let G be a pentagon, and set $I(G) = (x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_1)$ in $S = K[x_1, x_2, x_3, x_4, x_5]$. Then $I(G)$ is not complete intersection, but $S/I(G)^{(2)} = S/I(G)^2$ is Cohen–Macaulay.

In order to study Cohen–Macaulayness of higher symbolic powers of edge ideals, we use the notion of polarization. Let I be a monomial ideal of S , and let $I^{\text{pol}} \subseteq S^{\text{pol}}$ denote the polarization of I .

Let G be a graph with vertex set $V(G) = \{x_1, x_2, \dots, x_n\}$. Let $\Delta = \Delta(G)$ be the complementary simplicial complex of G . For a positive integer ℓ , let $\Delta^{(\ell)}$ be the simplicial complex such that $I_{\Delta^{(\ell)}} = (I(G)^{(\ell)})^{\text{pol}}$.

For a positive integer ℓ and for any fixed i , we put $(x_i^\ell)^{\text{pol}} = x_i^{(0)} x_i^{(1)} \cdots x_i^{(\ell-1)}$, where $x_i^{(0)} = x_i$. Furthermore, we put $(x_1^{\ell_1} \cdots x_n^{\ell_n})^{\text{pol}} = (x_1^{\ell_1})^{\text{pol}} \cdots (x_n^{\ell_n})^{\text{pol}}$. See Section 2 for more details. In order to study facets of $\Delta^{(\ell)}$, we need the following lemma.

Lemma 3.3. *Under the above notation, we have*

$$((x_1, \dots, x_h)^\ell)^{\text{pol}} = \bigcap_{i_1 + \dots + i_h \leq \ell-1} (x_1^{(i_1)}, \dots, x_h^{(i_h)}).$$

Proof. By the definition of polarization, we have

$$\begin{aligned} ((x_1, \dots, x_h)^\ell)^{\text{pol}} &= (x_1^{j_1} \cdots x_h^{j_h} : j_1, \dots, j_h \geq 0, j_1 + \dots + j_h = \ell)^{\text{pol}} \\ &= \left(\prod_{k=1}^h x_k^{(0)} x_k^{(1)} \cdots x_k^{(j_k-1)} : j_1, \dots, j_h \geq 0, j_1 + \dots + j_h = \ell \right). \end{aligned}$$

So, in order to obtain the required primary decomposition, it suffices to show that

$$((x_1, \dots, x_h)^\ell)^{\text{pol}} \subseteq (x_1^{(i_1)}, \dots, x_h^{(i_h)}) \Leftrightarrow i_1 + \dots + i_h \leq \ell - 1.$$

Suppose $i_1 + \dots + i_h \geq \ell$. Take a monomial $M = \prod_{k=1}^h x_k^{(0)} x_k^{(1)} \cdots x_k^{(i_k-1)}$. Then it is clear that $M \notin (x_1^{(i_1)}, \dots, x_h^{(i_h)})$. On the other hand, M is contained in $((x_1, \dots, x_h)^\ell)^{\text{pol}}$ because there exists a sequence (j_1, \dots, j_h) such that $0 \leq j_k \leq i_k$ for each k and $j_1 + \dots + j_h = \ell$.

Next suppose that $i_1 + \dots + i_h \leq \ell - 1$. If $((x_1, \dots, x_h)^\ell)^{\text{pol}} \not\subseteq (x_1^{(i_1)}, \dots, x_h^{(i_h)})$, then there exists a monomial $M = \prod_{k=1}^h x_k^{(0)} \cdots x_k^{(j_k-1)}$ with $j_1 + \dots + j_h = \ell$ such that M is not contained in $(x_1^{(i_1)}, \dots, x_h^{(i_h)})$. Hence $j_k \leq i_k$ for each k . But $\ell = j_1 + \dots + j_h \leq i_1 + \dots + i_h \leq \ell - 1$. This is a contradiction. \square

By the above lemma, we get the following corollary.

Corollary 3.4. Under the above notation, we set $V^{(i)} = \{x_1^{(i)}, \dots, x_n^{(i)}\}$ for each $i = 1, 2, \dots, \ell - 1$. Then $\mathcal{F}(\Delta^{(\ell)})$ consists of the following subsets of $V \cup V^{(1)} \cup \dots \cup V^{(\ell-1)}$:

$$\begin{aligned} & (F \cup \{x_{i_{1,1}}, \dots, x_{i_{1,j_1}}, x_{i_{2,1}}, \dots, x_{i_{2,j_2}}, \dots, x_{i_{\ell-1,1}}, \dots, x_{i_{\ell-1,j_{\ell-1}}}\}) \\ & \cup (V^{(1)} \setminus \{x_{i_{1,1}}^{(1)}, \dots, x_{i_{1,j_1}}^{(1)}\}) \cup \dots \cup (V^{(\ell-1)} \setminus \{x_{i_{\ell-1,1}}^{(\ell-1)}, \dots, x_{i_{\ell-1,j_{\ell-1}}}^{(\ell-1)}\}), \end{aligned}$$

where F and x_i 's run through

- $F \in \mathcal{F}(\Delta)$;
- $0 \leq j_1, j_2, \dots, j_{\ell-1} \leq n$, $j_1 + 2j_2 + \dots + (\ell-1)j_{\ell-1} \leq \ell-1$;
- $\{x_{i_{1,1}}, \dots, x_{i_{\ell-1,j_{\ell-1}}}\} \cap F = \emptyset$, $\sharp\{x_{i_{1,1}}, \dots, x_{i_{\ell-1,j_{\ell-1}}}\} = j_1 + j_2 + \dots + j_{\ell-1}$.

In particular, if Δ is pure, then so is $\Delta^{(\ell)}$.

Proof. By definition, we have

$$I_{\Delta^{(\ell)}} = ((I(G))^{(\ell)})^{\text{pol}} = \left(\bigcap_{F \in \mathcal{F}(\Delta)} P_F^\ell \right)^{\text{pol}} = \bigcap_{F \in \mathcal{F}(\Delta)} (P_F^\ell)^{\text{pol}}.$$

If $P_F = (y_1, \dots, y_h)$, then

$$(P_F^\ell)^{\text{pol}} = \bigcap_{i_1 + \dots + i_h \leq \ell-1} (y_1^{(i_1)}, \dots, y_h^{(i_h)})$$

by the above lemma.

Let $G \in \mathcal{F}(\Delta^{(\ell)})$. Then there exist a facet $F \in \mathcal{F}(\Delta)$ and integers $0 \leq i_1 \leq \dots \leq i_h$ with $i_1 + \dots + i_h \leq \ell-1$ such that

$$\begin{aligned} V \cup V^{(1)} \cup \dots \cup V^{(\ell-1)} \setminus G &= \{y_1^{(i_1)}, \dots, y_h^{(i_h)}\}, \\ V \setminus F &= \{y_1, \dots, y_h\}. \end{aligned}$$

Putting

$$\{y_1^{(i_1)}, \dots, y_h^{(i_h)}\} = \{x_{i_{0,1}}^{(0)}, \dots, x_{i_{0,j_0}}^{(0)}, \dots, x_{i_{\ell-1,1}}^{(\ell-1)}, \dots, x_{i_{\ell-1,j_{\ell-1}}}^{(\ell-1)}\},$$

we get a required form of G . \square

Proof of Theorem 3.1. Assume that $S/I(G)^{(\ell)}$ satisfies (S_2) . As $I(G) = \sqrt{I(G)^{(\ell)}}$, $S/I(G)$ also satisfies (S_2) by [6]. In particular, $I(G)$ is pure. Since some connected component of G is not a complete graph by assumption, there exist $x_1, x_2, x_3 \in V(G)$ such that

$$\{x_1, x_2\}, \{x_1, x_3\} \in E(G), \quad \text{and} \quad \{x_2, x_3\} \notin E(G).$$

We may assume that $V(G) = \{x_1, x_2, x_3, \dots, x_n\}$, the vertex set of G by renumbering if necessary. Let $\Delta = \Delta(G)$ be the complementary simplicial complex of G , and let $\Delta^{(\ell)}$ be the simplicial complex defined as above. Set $\tilde{V} = V \cup V^{(1)} \cup \dots \cup V^{(\ell-1)}$. Note that $\Delta^{(\ell)}$ is a pure simplicial complex on \tilde{V} .

Now consider the following subset of \tilde{V} :

$$F_0 = \left\{ \begin{array}{ccccccc} x_1, & x_1^{(1)}, & x_1^{(2)}, & \dots & x_1^{(\ell-3)}, & x_1^{(\ell-2)}, & \square \\ x_2, & x_2^{(1)}, & x_2^{(2)}, & \dots & x_2^{(\ell-3)}, & \square & x_2^{(\ell-1)} \\ x_3, & \square & x_3^{(2)}, & \dots & x_3^{(\ell-3)}, & x_3^{(\ell-2)}, & x_3^{(\ell-1)} \\ \square & x_4^{(1)}, & x_4^{(2)}, & \dots & x_4^{(\ell-3)}, & x_4^{(\ell-2)}, & x_4^{(\ell-1)} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ \square & x_n^{(1)}, & x_n^{(2)}, & \dots & x_n^{(\ell-3)}, & x_n^{(\ell-2)}, & x_n^{(\ell-1)} \end{array} \right\}.$$

Then F_0 is a face of $\Delta^{(\ell)}$. Indeed, we can take a facet $F \in \mathcal{F}(\Delta)$ such that $\{x_2, x_3\} \subseteq F$. Since $x_1 \notin F$,

$$F' = (F \cup \{x_1\}) \cup V^{(1)} \cup V^{(2)} \cup \dots \cup V^{(\ell-2)} \cup (V^{(\ell-1)} \setminus \{x_1^{(\ell-1)}\})$$

is a facet of $\mathcal{F}(\Delta^{(\ell)})$ by Corollary 3.4. This implies that $F_0 \in \Delta^{(\ell)}$ because $F_0 \subseteq F'$.

We first prove the following claim:

Claim. Any facet of $\text{link}_{\Delta^{(\ell)}}(F_0)$ is given by

$$(F \setminus \{x_2, x_3\}) \cup \{x_3^{(1)}\} \cup \{x_2^{(\ell-2)}\}, \quad \text{where } F \in \mathcal{F}(\Delta) \text{ and } \{x_2, x_3\} \subseteq F;$$

or

$$(F \setminus \{x_1\}) \cup \{x_1^{(\ell-1)}\}, \quad \text{where } F \in \mathcal{F}(\Delta) \text{ and } x_1 \in F.$$

In order to prove the claim, it suffices to determine $\mathcal{F}(\text{star}_{\Delta^{(\ell)}}(F_0))$ because any facet G of $\text{link}_{\Delta^{(\ell)}}(F_0)$ can be written as $G = \tilde{F} \setminus F_0$ for some $\tilde{F} \in \mathcal{F}(\text{star}_{\Delta^{(\ell)}}(F_0))$.

Let $\tilde{F} \in \mathcal{F}(\text{star}_{\Delta^{(\ell)}}(F_0))$. Then $\tilde{F} \in \mathcal{F}(\Delta^{(\ell)})$ and $\tilde{F} \supseteq F_0$. In particular, $x_i^{(1)}, \dots, x_i^{(\ell-1)} \in \tilde{F}$ for each $i = 4, \dots, n$ and

$$W_1 = V^{(1)} \setminus \{x_3^{(1)}\}, \quad W_{\ell-2} = V^{(\ell-2)} \setminus \{x_2^{(\ell-2)}\}, \quad W_{\ell-1} = V^{(\ell-1)} \setminus \{x_1^{(\ell-1)}\} \subseteq \tilde{F}.$$

Hence \tilde{F} is given by one of the following complexes:

$$\begin{aligned} \tilde{F}_1 &= (F \cup \{x_1\}) \cup V^{(1)} \cup V^{(2)} \cup \dots \cup V^{(\ell-3)} \cup V^{(\ell-2)} \cup W_{\ell-1}, \\ \tilde{F}_2 &= (F \cup \{x_2\}) \cup V^{(1)} \cup V^{(2)} \cup \dots \cup V^{(\ell-3)} \cup W_{\ell-2} \cup V^{(\ell-1)}, \\ \tilde{F}_3 &= (F \cup \{x_3\}) \cup W_1 \cup V^{(2)} \cup \dots \cup V^{(\ell-3)} \cup V^{(\ell-2)} \cup V^{(\ell-1)}, \\ \tilde{F}_{12} &= (F \cup \{x_1, x_2\}) \cup V^{(1)} \cup V^{(2)} \cup \dots \cup V^{(\ell-3)} \cup W_{\ell-2} \cup W_{\ell-1}, \\ \tilde{F}_{13} &= (F \cup \{x_1, x_3\}) \cup W_1 \cup V^{(2)} \cup \dots \cup V^{(\ell-3)} \cup V^{(\ell-2)} \cup W_{\ell-1}, \\ \tilde{F}_{23} &= (F \cup \{x_2, x_3\}) \cup W_1 \cup V^{(2)} \cup \dots \cup V^{(\ell-3)} \cup W_{\ell-2} \cup V^{(\ell-1)}. \end{aligned}$$

Now suppose that $\tilde{F} = \tilde{F}_2$. Then we have $x_1, x_3 \in F$. This implies that $x_3 \notin P_F$. Hence, $x_1 x_3 \in I(G)$ yields $x_1 \in P_F$. This contradicts $x_1 \in F$. Therefore it does not occur that $\tilde{F} = \tilde{F}_2$. Similarly, we have $\tilde{F} \neq \tilde{F}_3$.

Next suppose that $\tilde{F} = \tilde{F}_{12}$. Then $(\ell - 2)j_{\ell-2} + (\ell - 1)j_{\ell-1} \geq 2\ell - 3 \geq \ell$ because $\ell \geq 3$. This is impossible. Hence $\tilde{F} \neq \tilde{F}_{12}$. Similarly, we have $\tilde{F} \neq \tilde{F}_{13}$. Consequently, either

$$\tilde{F} = \tilde{F}_1 \quad \text{and} \quad x_2, x_3 \in F, \quad x_1 \notin F$$

or

$$\tilde{F} = \tilde{F}_{23} \quad \text{and} \quad x_1 \in F, \quad x_2, x_3 \notin F$$

holds. In other words, any $G \in \mathcal{F}(\text{link}_{\Delta(\ell)}(F_0))$ can be written as

$$G' = (F \setminus \{x_2, x_3\}) \cup \{x_3^{(1)}\} \cup \{x_2^{(\ell-2)}\}$$

for some $F \in \mathcal{F}(\Delta)$ such that $x_1 \notin F$ and $x_2, x_3 \in F$; or

$$G'' = (F \setminus \{x_1\}) \cup \{x_1^{(\ell-1)}\}$$

for some $F \in \mathcal{F}(\Delta)$ such that $x_1 \in F$ and $x_2, x_3 \notin F$. So, we proved the claim.

Choose G' and G'' of the above type, respectively. Note that there exist those facets as $(x_1x_2, x_1x_3) \subseteq I_\Delta$. Then one can find no chain of facets in $\text{link}_{\Delta(\ell)}(F_0)$ such that

$$G' = G_0, G_1, \dots, G_r = G''$$

with $\sharp(G_i \cap G_{i-1}) = d - 1$, where $d = \dim K[\text{link}_{\Delta(\ell)}(F_0)]$ since both $x_3^{(1)}$ and $x_2^{(\ell-2)}$ are contained in G' but not in G'' . Thus $\text{link}_{\Delta(\ell)}(F_0)$ is *not* connected in codimension 1, and hence it does *not* satisfy (S_2) . By the lemma below, we can conclude that $S/I(G)^{(\ell)}$ does *not* satisfy (S_2) , as required. \square

The following lemma was used in the proof of [8, Theorem 4.1]. Moreover, it is clear that S/I is Cohen–Macaulay if and only if so is $S^{\text{pol}}/I^{\text{pol}}$ because S/I is isomorphic to a quotient of $S^{\text{pol}}/I^{\text{pol}}$ by a regular sequence.

Lemma 3.5. (See the proof of [8, Theorem 4.1].) Let $k \geq 1$ be any integer. Let $I \subseteq S$ be a monomial ideal, and let $I^{\text{pol}} \subseteq S^{\text{pol}}$ denote the polarization of I . If S/I satisfies (S_k) , then so does $S^{\text{pol}}/I^{\text{pol}}$.

We are now ready to prove the first main theorem in this paper.

Theorem 3.6. Let $I(G) \subseteq S$ be the edge ideal of a graph G . Then the following conditions are equivalent:

- (1) $S/I(G)^{(\ell)}$ is Cohen–Macaulay for every integer $\ell \geq 1$.
- (2) $S/I(G)^{(\ell)}$ is Cohen–Macaulay for some $\ell \geq 3$.
- (3) $S/I(G)^{(\ell)}$ satisfies Serre's condition (S_2) for some $\ell \geq 3$.
- (4) G is a disjoint union of finitely many complete graphs.

Proof. Let $I(G) \subseteq S$ be the edge ideal with $\dim S/I(G) \geq 2$.

(1) \Rightarrow (2): This is clear.

(2) \Rightarrow (3): Since any Cohen–Macaulay ring satisfies Serre's condition (S_2) , it is clear.

(3) \Rightarrow (4): Now suppose that G cannot be written as a disjoint union of finitely many complete graphs. Then, for any $\ell \geq 3$, $S/I(G)^{(\ell)}$ does not satisfy (S_2) by Theorem 3.1. This contradicts the assumption.

(4) \Rightarrow (1): By Theorem 2.1, if G is a disjoint union of finitely many complete graphs, then $S/I(G)^{(\ell)}$ is Cohen–Macaulay for every $\ell \geq 1$. \square

Remark 3.7. By a similar argument as in Corollary 2.4, we can generalize the above theorem to the case where I contains variables. Moreover, in this case, we can replace S with $S[t]$, where t is an indeterminate.

3.2. Cohen–Macaulay properties of ordinary powers

Using Theorem 3.6, we can give an improvement of the main theorem in [3].

Theorem 3.8. (Cf. [3, Theorem 2.1].) *Let $I(G)$ be the edge ideal of a graph G . If $S/I(G)^\ell$ is Cohen–Macaulay for some $\ell \geq 3$, then $I(G)$ is complete intersection.*

Remark 3.9. In [3], the authors proved an analogous theorem: $I(G)$ is complete intersection whenever $S/I(G)^\ell$ is Cohen–Macaulay for some $\ell \geq \text{height } I(G)$. Note that it is not difficult to derive this from Theorem 3.8.

In order to prove the theorem, we need the following lemma.

Lemma 3.10. (See also [10, Lemma 5.8, Theorem 5.9].) *Let $I(G)$ be the edge ideal of a graph G . Let $t \geq 2$ be an integer. Then the following conditions are equivalent.*

- (1) G contains no odd cycles of length $2s - 1$ for any $2 \leq s \leq t$.
- (2) $I(G)^{(t)} = I(G)^t$ holds.

Proof. Put $I = I(G)$ for simplicity.

(1) \Rightarrow (2): It follows from a similar argument as in the proof of [10, Lemma 5.8, Theorem 5.9]. But for the convenience of the readers, we give a sketch of the proof. It is enough to show that $\mathfrak{m} \notin \text{Ass}_S(S/I^t)$ if $\dim S/I \geq 1$. Now suppose *not*. Then we can take a monomial $M \notin I^t$ such that $I^t : M = \mathfrak{m}$. Since $\text{depth } S/I \geq 1$, we get $M \in I$. So we can write $M = x_1 x_2 L$ for some $x_1 x_2 \in G(I)$ and a monomial L . By definition, we have $x_2 M = x_1 x_2^2 L \in I^t$. It follows that $x_2^2 L \in I^{t-1}$ because I is generated by squarefree monomials. This yields $M \in x_1 I^{t-1} \cap (I^t : x_1)$.

On the other hand, by a similar argument as in the proof of [10, Lemma 5.8], we can show that $x I^m \cap (I^{m+1} : x) \subseteq I^{m+1}$ for any vertex x and for all $0 \leq m \leq t - 1$ using (1). (Notice that there exists a small gap in the final step of the proof of [10, Lemma 5.8]. That is, we obtain an odd cycle if only if i is even.) In particular, $M \in x_1 I^{t-1} \cap (I^t : x_1) \subseteq I^t$, which contradicts the choice of M .

(2) \Rightarrow (1): Suppose that G contains an odd cycle of length $2s - 1$ with $2 \leq s \leq t$; say, $x_1 x_2, x_2 x_3, \dots, x_{2s-2} x_{2s-1}, x_{2s-1} x_1$. Put $M = x_1 x_2 \cdots x_{2s-1}$. Then we show $M(x_1 x_2)^{t-s} \in I^{(t)} \setminus I^t$. Let P be any associated prime ideal of I . Then since P is prime and $x_1 x_2, x_2 x_3, \dots, x_{2s-2} x_{2s-1}, x_{2s-1} x_1 \in P$, we get $\sharp(P \cap \{x_1, x_2, \dots, x_{2s-1}\}) \geq s$. Hence $M \in P^s$ and thus $M(x_1 x_2)^{t-s} \in I^{(t)}$. On the other hand, $M(x_1 x_2)^{t-s} \notin I^t$ because $\deg M(x_1 x_2)^{t-s} = 2t - 1 < 2t = \text{indeg } I^t$, where $\text{indeg } I^t = \min\{m \in \mathbb{Z} : [I^t]_m \neq 0\}$. \square

Corollary 3.11. *Let G be a disjoint union of complete graphs K_{n_1}, \dots, K_{n_r} .*

If $\max\{n_1, \dots, n_r\} \geq 3$, then $I(G)^\ell \neq I(G)^\ell$ for every $\ell \geq 2$. In particular, $I(G)^\ell$ is not a Cohen–Macaulay ideal.

Proof. Under the assumption, G always contains a triangle (3-cycle). \square

Proof of Theorem 3.8. Now suppose that $S/I(G)^\ell$ is Cohen–Macaulay for some integer $\ell \geq 3$, and that $I(G)$ is *not* complete intersection.

By Theorem 3.6, G can be written as a disjoint union of finitely many complete graphs. However, this contradicts the above corollary. \square

The next example shows that the Cohen–Macaulayness of symbolic power ideals is different from that of ordinary power ideals.

Example 3.12. Let G be a disjoint union of d complete 3-graphs. Set

$$I = I(G) = (x_{11}x_{12}, x_{11}x_{13}, x_{12}x_{13}, \dots, x_{d1}x_{d2}, x_{d1}x_{d3}, x_{d2}x_{d3})$$

in a polynomial ring $S = K[x_{11}, x_{12}, x_{13}, \dots, x_{d1}, x_{d2}, x_{d3}]$. Then:

- (1) $S/I^{(\ell)}$ is Cohen–Macaulay of dimension d for every $\ell \geq 1$.
- (2) S/I^ℓ is not Cohen–Macaulay for any $\ell \geq 2$.
- (3) I is not complete intersection.

Proof. (1) follows from Theorem 3.6.

(2) If $\ell \geq 3$, then the assertion follows from Theorem 3.8. When $\ell = 2$, it follows from the fact $x_{11}x_{12}x_{13} \in I^{(2)} \setminus I^2$. \square

3.3. Some related results

In the final of this section, we comment a relationship between our results and the theorem by Minh and Trung [7]. Minh and Trung studied Cohen–Macaulay properties of the symbolic power ideals for one-dimensional simplicial complexes.

Theorem. (Minh–Trung; see [7].) Let $\ell \geq 3$ be an integer. Let $I = I_\Delta$ be the Stanley–Reisner ideal of a simplicial complex of dimension 1. Then $S/I^{(\ell)}$ is Cohen–Macaulay if and only if every pair of disjoint edges of Δ is contained in a cycle of length 4.

If I_Δ is generated by degree 2 monomials, the ideal I_Δ can be regarded as the edge ideal of a graph G . Then the required condition in the above theorem says that G is a disjoint union of two complete graphs. So, their theorem does not conflict our theorem.

4. Finite local cohomology and symbolic power

In [5], Goto and Takayama introduced the notion of generalized complete intersection complex. On the other hand, in [11], the last two authors defined the notion of locally complete intersection complex and gave a structure theorem for those complexes. Note that Δ is a generalized complete intersection complex if and only if Δ is a pure, locally complete intersection complex.

Definition 4.1. (Cf. [11].) Let Δ be a simplicial complex on the vertex set V . The complex Δ is called a *locally complete intersection complex* if $K[\text{link}_\Delta\{v\}]$ is complete intersection for every vertex $v \in V$.

The following result gives a structure theorem for locally complete intersection complexes.

Theorem 4.2. (Cf. [11].) Let Δ be a simplicial complex on V such that $V \neq \emptyset$. Then Δ is a locally complete intersection complex if and only if it is a finitely many disjoint union of the following connected complexes:

- (a) a complete intersection complex Γ with $\dim \Gamma \geq 2$;
- (b) m -gon ($m \geq 3$);
- (c) m' -pointed path ($m' \geq 2$);
- (d) a point.

When this is the case, $K[\Delta]$ is Cohen–Macaulay (resp., Buchsbaum) if and only if $\dim \Delta = 0$ or Δ is connected (resp., pure).

Moreover, for any pure simplicial complex Δ , it is a locally complete intersection complex if and only if S/I_Δ^ℓ has (FLC) for all $\ell \geq 1$ (or, more generally, for infinitely many $\ell \geq 1$). But, for a fixed $\ell \geq 1$, it is open when S/I^ℓ has (FLC).

4.1. FLC properties of symbolic powers

In this section, we consider the following question, which is closely related to the above question in the case of edge ideals.

Question 4.3. Let $I(G)$ be the edge ideal of a graph G . Let $\ell \geq 1$ be an integer. When does $S/I(G)^{(\ell)}$ have (FLC)?

As one of answers to this question, we prove the second main theorem (Theorem 4.7). We first prove the following proposition.

Proposition 4.4. Let Δ_{n_1, \dots, n_r} denote the simplicial complex whose Stanley–Reisner ideal is equal to the edge ideal of a disjoint union of complete graphs K_{n_1}, \dots, K_{n_r} . That is,

$$I_{\Delta_{n_1, \dots, n_r}} = I(K_{n_1} \sqcup \dots \sqcup K_{n_r}).$$

Let Δ be a simplicial complex defined by

$$\Delta = \Delta_{n_{11}, \dots, n_{1d}} \sqcup \Delta_{n_{21}, \dots, n_{2d}} \sqcup \dots \sqcup \Delta_{n_{p1}, \dots, n_{pd}},$$

where one can take all $n_{ij} = 1$ when $p \geq 2$. Put

$$S = K[x_{ij}^{(k)} : 1 \leq i \leq d; 1 \leq k \leq p; 1 \leq j \leq n_{ki}],$$

a polynomial ring over K , and

$$\begin{aligned} I_\Delta = & (x_{ij}^{(k)} x_{ij'}^{(k)} : 1 \leq i \leq d, 1 \leq j < j' \leq n_{ki}; 1 \leq k \leq p)S \\ & + (x_{ij}^{(k)} x_{i'j'}^{(m)} : 1 \leq i, i' \leq d, 1 \leq j \leq n_{ki}, 1 \leq j' \leq n_{mi'}, 1 \leq k < m \leq p)S. \end{aligned}$$

Then $S/I_\Delta^{(\ell)}$ has (FLC) for every $\ell \geq 1$.

Proof. Put $I = I_\Delta$. Since $\dim \Delta_{n_1, \dots, n_d} = d - 1$, Δ is a pure simplicial complex of dimension $d - 1$. Hence $S/I^{(\ell)}$ is an equidimensional ring of dimension d . So, it is enough to show that $(S/I^{(\ell)})_x$ is Cohen–Macaulay for any vertex x . Without loss of generality, we may assume that $x = x_{11}^{(1)}$. Then

$$\begin{aligned} I_x = & (x_{1j}^{(1)} : 2 \leq j \leq n_{11})S_x + (x_{ij}^{(k)} : 1 \leq i \leq d, 1 \leq j \leq n_{ki}, 2 \leq k \leq p)S_x \\ & + (x_{ij}^{(1)} x_{ij'}^{(1)} : 2 \leq i \leq d, 1 \leq j < j' \leq n_{1i})S_x. \end{aligned}$$

By Theorem 2.1 and Corollary 2.4, $(S/I^{(\ell)})_x$ is Cohen–Macaulay for all $\ell \geq 1$. \square

Example 4.5. Let $G = K_p$ be the complete p -graph. Then Δ_p is the complementary simplicial complex of K_p . Moreover, Δ_p has p connected component: $\Delta_p = \{x_1\} \sqcup \dots \sqcup \{x_p\}$. Then $K[\Delta_p] = K[x_1, \dots, x_p]/(x_i x_j : 1 \leq i < j \leq p)$.

On the other hand, $K[\Delta_1] = K[x_1, \dots, x_d]$, where $\mathbf{1} = \underbrace{1, \dots, 1}_d$.

Now suppose that $S/I(G)^{(\ell)}$ has (FLC) for some $\ell \geq 3$. As $I(G) = \sqrt{I(G)^{(\ell)}}$, $S/I(G)$ also has (FLC) (see e.g. [6]). Let $\Delta = \Delta(G)$ be the complementary simplicial complex of G : $I_\Delta = I(G)$. Then Δ is pure and $S_x/I_\Delta^{(\ell)}_x$ is Cohen–Macaulay for every vertex $x \in V$. Put $\Gamma = \text{link}_\Delta\{x\}$. This implies that $K[V \setminus \{x\}]/I_\Gamma^{(\ell)}$ is Cohen–Macaulay. Therefore, by Theorem 3.6, I_x can be written as

$$I_x = (y_1, \dots, y_m) + I(H_1)S_x + \dots + I(H_{d-1})S_x,$$

where H_1, \dots, H_{d-1} are disjoint complete subgraphs of G and $y_1, \dots, y_m \in V$ such that $\{x, y_j\} \in E(G)$ and no elements of $\{y_1, \dots, y_m\}$ are contained in $H_1 \cup \dots \cup H_{d-1}$.

In order to prove the second main theorem (Theorem 4.7), we need the following lemma.

Lemma 4.6. *Let G be a graph, and let Δ be the complementary simplicial complex of G : $I_\Delta = I(G)$. Suppose $d = \dim S/I(G) \geq 3$ and Δ is pure. Moreover, assume that for any vertex u , there exist vertices y_1, \dots, y_m and complete subgraphs H_1, \dots, H_{d-1} such that $I(G)_u$ can be written as*

$$I(G)_u = (y_1, \dots, y_m)S_u + I(H_1)S_u + \dots + I(H_{d-1})S_u,$$

where $V(G) = \{u\} \sqcup \{y_1, \dots, y_m\} \sqcup V(H_1) \sqcup \dots \sqcup V(H_{d-1})$.

Then for any vertex $x \in V(G)$, there exist subgraphs G_0, G_1, \dots, G_d which satisfies the following conditions:

- (1) $V(G) = V(G_0) \sqcup V(G_1) \sqcup \dots \sqcup V(G_{d-1}) \sqcup V(G_d)$ and $x \in G_d$.
- (2) $G|_{V(G_i)} = G_i$ for each $i = 0, 1, \dots, d-1, d$.
- (3) $G_1 \sqcup \dots \sqcup G_{d-1} \sqcup G_d$ is a disjoint union of complete graphs.
- (4) For every $y \in G_0$ and for every $z_i \in G_i$ ($i = 1, \dots, d$), we have $\{y, z_i\} \in E(G)$.

Proof. Fix $x \in V(G)$. Applying the assumption to the case of $u = x$, we can find disjoint complete subgraphs G_1, \dots, G_{d-1} of G and vertices y_1, \dots, y_m such that

$$I(G)_x = (y_1, \dots, y_m)S_x + I(G_1)S_x + \dots + I(G_{d-1})S_x$$

and $\{y_1, \dots, y_m\}$ are contained in $V(G) \setminus V(G_1 \cup \dots \cup G_{d-1})$. Then we prove the following claim.

Claim 1. *For any $y \in \{y_1, \dots, y_m\}$, if $\{y, z_1\} \in E(G)$ for some $z_1 \in V(G_1)$, then $\{y, z_i\} \in E(G)$ holds for all $i = 1, 2, \dots, d-1$ and for all $z_i \in V(G_i)$.*

Now suppose that $\{y, z_1\} \in E(G)$ for some $z_1 \in V(G_1)$. Then $yz_1 \in I(G)$.

For any $z_i \in V(G_i)$ ($i = 2, \dots, d-1$), if $\{y, z_i\} \notin E(G)$, then $yz_i \notin I(G)$. As $z_i \notin I(G)_x$, we have $xz_i \notin I(G)$. By the choice of G_i , $z_1z_i \notin I(G)$. Hence none of y, x, z_1 appears in $I(G)_{z_i}$. However, since $xy, yz_1 \in I(G)_{z_i}$, we have $xz_1 \in I(G)_{z_i}$ by assumption, and so $xz_1 \in I(G)$. This implies that $z_1 \in I(G)_x$. This contradicts the assumption. Thus we have $\{y, z_i\} \in E(G)$ for all $z_i \in V(G_i)$ ($i = 2, \dots, d-1$).

As $d \geq 3$, applying $\{y, z_2\} \in E(G)$ to the above argument, we obtain that $\{y, z'\} \in E(G)$ for all $z' \in V(G_1)$. Hence we proved the claim.

By the above claim, by renumbering if necessary, we may assume that there exists an integer k with $1 \leq k \leq m$ such that:

- (i) When $1 \leq j \leq k$, $\{y_j, z_i\} \in E(G)$ holds for every $1 \leq i \leq d-1$ and $z_i \in V(G_i)$.
- (ii) When $k+1 \leq j \leq m$, $\{y_j, z\} \notin E(G)$ holds for every $z \in V(G_1 \cup \dots \cup G_{d-1})$.

Then we put $V_0 = \{y_1, \dots, y_k\}$ and $V_d = \{y_{k+1}, \dots, y_m\}$ and $G_0 = G|_{V_0}$ and $G_d = G|_{V_d}$. In the following, we show that these G_j ($j = 0, \dots, d$) satisfy all conditions of the lemma. To show the condition (3), it is enough to show the following claim.

Claim 2. G_d is a complete graph, and G_i and G_d are disjoint for each $i = 1, \dots, d-1$.

To see that G_d is a complete graph, it is enough to show that $\{u, u'\} \in E(G)$ whenever $u, u' \in V(G_d) \setminus \{x\}$. Suppose $\{u, u'\} \notin E(G)$. Take $z_1 \in V(G_1)$. Then since $\{x, z_1\}, \{u, z_1\}, \{u', z_1\} \notin E(G)$ and $xu, xu' \in I(G)_{z_1}$, we have $uu' \in I(G)_{z_1}$, and thus $\{u, u'\} \in E(G)$. The latter assertion immediately follows from the definition of G_d .

To show the condition (4), it is enough to show the following claim.

Claim 3. For every $y \in G_0$, $\{y, u\} \in E(G)$ for every $u \in G_d$.

Suppose that $\{y, u\} \notin E(G)$. Take $z_1 \in V(G_1)$ and $z_2 \in V(G_2)$. Then, since $d \geq 3$, z_1, z_2, u are distinct vertices and $\{z_1, u\}, \{z_2, u\} \notin E(G)$ by Claim 2. By definition, $\{y, z_1\}, \{y, z_2\} \in E(G)$. By considering $yz_1, yz_2 \in I(G)_u$, we get $z_1z_2 \in I(G)_u$. Hence we have $\{z_1, z_2\} \in E(G)$. This is a contradiction. Therefore we conclude that $\{y, u\} \in E(G)$.

We have finished the proof of the lemma. \square

We are now ready to prove the second main theorem in this paper.

Theorem 4.7. Let G be a graph on $V = [n]$, and let $I(G) \subseteq S = K[v : v \in V]$ denote the edge ideal of G . Let $\Delta = \Delta(G)$ be the complementary simplicial complex of G , that is, $I_\Delta = I(G)$. Let p denote the number of connected components of Δ . Suppose that Δ is pure and $d = \dim S/I(G) \geq 3$. Then the following conditions are equivalent:

- (1) $S/I(G)^{(\ell)}$ has (FLC) for every $\ell \geq 1$.
- (2) $S/I(G)^{(\ell)}$ has (FLC) for some $\ell \geq 3$.
- (3) There exist $(n_{i1}, \dots, n_{id}) \in \mathbf{N}^d$ for every $i = 1, \dots, p$ such that Δ can be written as

$$\Delta = \Delta_{n_{11}, \dots, n_{1d}} \sqcup \Delta_{n_{21}, \dots, n_{2d}} \sqcup \dots \sqcup \Delta_{n_{p1}, \dots, n_{pd}}.$$

Proof. (3) \Rightarrow (1): It follows from Proposition 4.4.

(1) \Rightarrow (2) is clear.

(2) \Rightarrow (3): We may assume that $p \geq 2$ by Theorem 3.6. Then we note that Δ satisfies the assumption of Lemma 4.6. Fix $x \in V$. Let G_0, \dots, G_d be subgraphs of G determined by Lemma 4.6. Then it suffices to show that the connected component containing x (say, Δ') is the following form: $\Delta' = \Delta(G')$, where $G' = G_1 \cup \dots \cup G_d$, which is a disjoint union of complete graphs.

First we see that $V(\Delta') = V(G_1 \cup \dots \cup G_d)$. Let $z \in V(G_1 \cup \dots \cup G_d)$. If $z \in V(G_1 \cup \dots \cup G_{d-1})$, then as $\{x, z\} \notin E(G)$, $\{x, z\} \in \Delta'$, that is, $z \in V(\Delta')$. Otherwise, $z \in V(G_d)$. Then there exists a vertex $z' \in V(G_1 \cup \dots \cup G_{d-1})$ such that $\{z, z'\} \notin E(G)$. Moreover, as $\{x, z'\} \in \Delta'$, we have $z \in V(\Delta')$. Hence $V(G_1 \cup \dots \cup G_d) \subseteq V(\Delta')$. The converse follows from the condition (4) in Lemma 4.6.

Next we see that $I_{\Delta'} = I(G_1 \cup \dots \cup G_d)$. Since Δ' is a connected component of Δ , we get

$$\begin{aligned} I_{\Delta'} &= (I_\Delta \cap K[V(G_1 \cup \dots \cup G_d)])S \\ &= (I(G) \cap K[V(G_1 \cup \dots \cup G_d)])S \\ &= I(G_1 \cup \dots \cup G_d). \end{aligned}$$

This yields that $\Delta' = \Delta(G_1 \cup \dots \cup G_d)$, as required. \square

Remark 4.8. Let t be an indeterminate over R . If R has (FLC) but not Cohen–Macaulay, then $R[t]$ does not have (FLC). Hence, in the above theorem, we cannot replace S with $S[t]$, where t is an indeterminate over S .

Comparing Theorem 3.6 and Theorem 4.7, we obtain the following corollary.

Corollary 4.9. Suppose that $d = \dim S/I(G) \geq 3$. Let $\Delta(G)$ denote the complementary simplicial complex of G . Let $\ell \geq 3$ be an integer. Then the following conditions are equivalent.

- (1) $S/I(G)^{(\ell)}$ has (FLC) and $\Delta(G)$ is connected.
- (2) $S/I(G)^{(\ell)}$ is Cohen–Macaulay.

Then G is a disjoint union of finitely many complete graphs and $S/I(G)^{(k)}$ is Cohen–Macaulay for all $k \geq 1$.

Remark 4.10. In case of $\dim S/I(G) = 2$, the following conditions are equivalent:

- (1) $S/I(G)^{(\ell)}$ has (FLC) for every $\ell \geq 1$.
- (2) $S/I(G)^{(\ell)}$ has (FLC) for some $\ell \geq 1$.
- (3) $\Delta(G)$ is pure.

In particular, we cannot remove the condition $d = \dim S/I(G) \geq 3$ from the assumption in Theorem 4.7. For example, the pentagon cannot be expressed in the form as in Theorem 4.7(3).

4.2. FLC properties of ordinary powers

In the rest of this section, we consider (FLC) properties of ordinary powers. Fix a positive integer ℓ . Let $I = I_\Delta$ be a Stanley–Reisner ideal. If S/I^ℓ has (FLC), then $(S/I^\ell)_x$ is Cohen–Macaulay for all vertex x . Then $I^{(\ell)}/I^\ell$ has finite length, it is equal to $H_m^0(S/I^\ell)$. Then $S/I^{(\ell)}$ also has (FLC). Hence we have the following theorem, which gives an improvement of Goto–Takayama theorem in [5] in the case of edge ideals.

Theorem 4.11. Put $d = \dim S/I(G) \geq 1$. Let $\Delta(G)$ denote the complementary simplicial complex of G . Then the following conditions are equivalent:

- (1) $S/I(G)^\ell$ has (FLC) for every $\ell \geq 1$.
- (2) $S/I(G)^\ell$ has (FLC) for some $\ell \geq 3$.
- (3) $S/I(G)^{(\ell)}$ has (FLC) and $I(G)^{(\ell)}/I(G)^\ell$ has finite length for some $\ell \geq 3$.
- (4) $\Delta(G)$ is a pure, locally complete intersection complex.

Proof. (1) \Rightarrow (2) \Leftrightarrow (3) is clear. The equivalence of (1) and (4) follows from [5]. On the other hand, (2) \Rightarrow (4) follows from Theorem 3.8 by a similar argument as in [5]. \square

Remark 4.12. By Theorem 4.2, (4) can be rephrased as follows:

- (4)' When $d = 2$, $\Delta(G)$ is a disjoint union of finitely many paths and n -gons with $n \geq 4$. When $d \geq 3$, $\Delta(G)$ is a disjoint union of finitely many complete intersection complexes of dimension $d - 1$.

The next example shows that there exists a graph G for which $S/I(G)^\ell$ has (FLC) but not Cohen–Macaulay.

Example 4.13. Under the notation as in Theorem 4.7, Δ is locally complete intersection if and only if $\min\{n_{i1}, \dots, n_{id}\} \leq 2$.

For instance, for any positive integer d , the edge ideal of the complete bipartite graph $K_{d,d}$

$$I = (x_i y_j : 1 \leq i, j \leq d) \subseteq S = K[x_1, \dots, x_d, y_1, \dots, y_d]$$

satisfies the following statements:

- (1) S/I^ℓ has (FLC) of dimension d for every $\ell \geq 1$.
- (2) When $d \geq 2$, S/I^ℓ is not Cohen–Macaulay for all $\ell \geq 1$.

Proof. By [10], we know that $I^{(\ell)} = I^\ell$ for every $\ell \geq 1$; see also Lemma 3.10. Hence our theorem says that S/I^ℓ has (FLC) for all $\ell \geq 1$. On the other hand, as S/I is not Cohen–Macaulay, S/I^ℓ is not Cohen–Macaulay if $d \geq 2$ and $\ell \geq 1$. \square

Even if $S/I(G)$ is Cohen–Macaulay, one can find an example of G such that $S/I(G)^\ell$ has (FLC) but not Cohen–Macaulay.

Example 4.14. Let Δ be a 4-pointed path, and $I_\Delta = (x_1x_2, x_2x_3, x_3x_4)$. Then I_Δ is also the edge ideal of the 4-pointed path G . Then S/I_Δ is Cohen–Macaulay, and S/I_Δ^2 is Buchsbaum (thus (FLC)) but not Cohen–Macaulay.

Similarly, for the pentagon G , $S/I(G)$ is Cohen–Macaulay and $S/I(G)^3$ has (FLC) but not Cohen–Macaulay.

In general, even if $S/I(G)^{(\ell)}$ has (FLC), it is not necessarily $S/I(G)^\ell$ has (FLC) as the next example shows. Note that we can construct similar examples of graphs G with $\dim S/I(G) = d$ for every $d \geq 3$.

Example 4.15. Let $S = K[\{x_i\}_{1 \leq i \leq 9}, \{y_j\}_{1 \leq j \leq 9}]$, and let G be a graph such that $\Delta(G) = \Delta_{3,3,3} \sqcup \Delta_{3,3,3}$. Set

$$\begin{aligned} I(G) = & (x_1x_2, x_1x_3, x_2x_3, x_4x_5, x_4x_6, x_5x_6, x_7x_8, x_7x_9, x_8x_9) \\ & + (y_1y_2, y_1y_3, y_2y_3, y_4y_5, y_4y_6, y_5y_6, y_7y_8, y_7y_9, y_8y_9) \\ & + (x_iy_j: 1 \leq i, j \leq 9). \end{aligned}$$

Then

- (1) $\dim S/I(G) = 3$.
- (2) $S/I(G)^{(\ell)}$ has (FLC) for every $\ell \geq 1$.
- (3) $\Delta(G)$ is not a locally complete intersection complex.
- (4) $S/I(G)^\ell$ does not have (FLC) for every $\ell \geq 3$.

CI	Ex. 4.13 ($K_{d,d}$)	4-pointed path
$S/I^\ell : \mathbf{CM}$	$S/I^\ell : \mathbf{(FLC)}$	$I : \mathbf{pure, LCI}$
\Downarrow	\Downarrow	\Downarrow
$S/I^{(\ell)} : \mathbf{CM}$	$S/I^{(\ell)} : \mathbf{(FLC)}$	$S/I : \mathbf{Buchsbaum}$
Ex. 3.12 ($\Delta_{3,3}$)	Ex. 4.15 ($\Delta_{3,3,3} \sqcup \Delta_{3,3,3}$)	

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