



Generators for the mod 2 cohomology of the Steinberg summand of Thom spectra over $B(\mathbb{Z}/2)^n$

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ABSTRACT

Let $L_{n,k}$ be the Steinberg summand of the ideal generated by the k -th power of the top Dickson class of $H^*(B(\mathbb{Z}/2)^n; \mathbb{Z}/2)$. The module $L_{n,k}$ is the mod 2 cohomology of the Steinberg summand of a Thom spectrum over the classifying space $B(\mathbb{Z}/2)^n$. In this paper, using the Kameko map (Kameko, 1990 [6]) and short exact sequences induced by the cofibration sequences constructed by Takayasu (1999) [15], we construct a minimal generating set for $L_{n,k}$ as a module over the mod 2 Steenrod algebra. This generalises the result of Masateru Inoue (2002) [5] for the cases $k = 0, 1$.

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1. Introduction

Fix a natural number n and let ρ_n be the reduced real regular representation of a rank n elementary abelian 2-group $V_n := (\mathbb{Z}/2)^n$. Given $k \in \mathbb{N}$, let $BV_n^{k\rho_n}$ denote the Thom spectrum over the classifying space BV_n associated to the direct sum of k copies of the representation ρ_n .

The Steinberg idempotent [14], \mathbf{e}_n , of the general linear group $GL_n(\mathbb{F}_2)$ acts stably on BV_n and thus acts on the spectrum $BV_n^{k\rho_n}$. Denote by $L(n, k)$ the 2-completion of the spectrum

$$\mathrm{hocolim}(BV_n^{k\rho_n} \xrightarrow{\mathbf{e}_n} BV_n^{k\rho_n} \xrightarrow{\mathbf{e}_n} BV_n^{k\rho_n} \xrightarrow{\mathbf{e}_n} \dots),$$

and by $L_{n,k}$ its mod 2 cohomology. The Thom isomorphism for the bundle associated to $k\rho_n$ gives an isomorphism of modules over the mod 2 Steenrod algebra:

$$L_{n,k} \cong \mathbf{e}_n(P_n \cdot \omega_n^k).$$

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Here ω_n denotes the top Dickson class of $P_n := H^*(BV_n; \mathbb{Z}/2) \cong \mathbb{F}_2[x_1, \dots, x_n]$, which is identified with the Stiefel–Whitney class of the vector bundle over BV_n associated to the representation ρ_n .

The spectra $L(n, k)$, $n, k \in \mathbb{N}$, were studied by Kuhn, Mitchell, Priddy, Takayasu [8–10, 15] among others and seemed to play important roles in homotopy theory. For example, Mitchell and Priddy [10] used the spectra $L(n, 0)$ and $L(n, 1)$, which they denoted by $M(n)$ and $L(n)$ respectively, in the study of stable splitting of the classifying spaces of groups. They proved in particular that there was a decomposition $M(n) \simeq L(n) \vee L(n-1)$. The spectra $M(n)$ and $L(n)$ were also used by N. Kuhn [9] in his proof of the Whitehead conjecture on symmetric products of spheres. More recently, by work of Arone and Mahowald [2] and Arone and Dwyer [1], the spectra $L(n, k)$ have appeared again in the description of layers of the Goodwillie tower of the identity functor evaluated on spheres.

It is the purpose of this paper to study the mod 2 cohomology of the spectra $L(n, k)$ as a module over the mod 2 Steenrod algebra, \mathcal{A} . Specifically, for $(n, k) \in \mathbb{N} \times \mathbb{N}$, we would like to solve the *hit problem* [19] for the \mathcal{A} -module $L_{n,k}$, that is, to describe a minimal generating set for $L_{n,k}$ as an \mathcal{A} -module. This generalises work of M. Inoue [5] who constructed a minimal generating set for $L_{n,0}$ and $L_{n,1}$. We note however that our method is based on the Kameko map [6] and the Takayasu short exact sequences [15], which is completely different from the approach of Inoue who used a meticulous manipulation of the Adem relations.

Let us denote by Q the indecomposable functor which associates to each \mathcal{A} -module M the graded vector space $Q(M) := M/\mathcal{A}^+M$, where \mathcal{A}^+ is the augmentation ideal of the Steenrod algebra. Clearly any basis of $Q(M)$ lifts to a minimal generating set for M as an \mathcal{A} -module. Denote by Σ and Φ the suspension functor and the doubling functor of the category of \mathcal{A} -modules and \mathcal{A} -linear maps of degree zero (see for example [12]).

Here is the main result of this paper.

Theorem 1.1. (See Theorem 3.4 and Corollary 3.5.) *For $n \in \mathbb{N}^*$ and $m \in \mathbb{N}$, there are isomorphisms of graded vector spaces:*

- (1) $Q(\Sigma^n L_{n,2m+1}) \cong Q(\Phi \Sigma^n L_{n,m})$,
- (2) $Q(L_{n,2m}) \cong Q(L_{n,2m+1}) \oplus Q(\Sigma^{2m} L_{n-1,4m+1})$.

We note that in this paper, all maps of graded vector spaces or \mathcal{A} -modules are of degree zero. The first isomorphism in the theorem is induced by the Kameko map (see Section 3)

$$\psi_n : \Sigma^n P_n \rightarrow \Phi \Sigma^n P_n,$$

while the second is induced by the Takayasu short exact sequence (see Section 2)

$$0 \rightarrow L_{n,2m+1} \rightarrow L_{n,2m} \rightarrow \Sigma^{2m} L_{n-1,4m+1} \rightarrow 0.$$

It is clear from Theorem 1.1 that a minimal generating set for $L_{n,k}$ can be constructed by using a double induction on $(n, k) \in \mathbb{N} \times \mathbb{N}$. To state the result, we need the numerical function $\gamma : \mathbb{N} \rightarrow \mathbb{N}$ defined inductively by

$$\gamma(a) = \begin{cases} 2([a/2] + 1), & a \not\equiv 0 \pmod{2}, \\ 2\gamma(a/2), & a \equiv 0 \pmod{2}. \end{cases}$$

Explicitly, $\gamma(0) = 0$ and $\gamma(2^b(2c+1)) = 2^{b+1}(c+1)$ for $b, c \in \mathbb{N}$.

For $1 \leq \ell \leq n$, let ω_ℓ denote the top Dickson class of $\mathbb{F}_2[x_1, \dots, x_\ell]$, namely ω_ℓ is the product of all non-trivial one-dimensional classes of $\mathbb{F}_2[x_1, \dots, x_\ell]$. Given a sequence $I = (i_1, \dots, i_n)$ of integers with $i_1 > 2i_2 > \dots > 2^{n-1}i_n \geq 0$, put

$$\omega_I := \mathbf{e}_n(\omega_1^{i_1-2i_2} \dots \omega_{n-1}^{i_{n-1}-2i_n} \omega_n^{i_n}).$$

The formula of the Steinberg idempotent \mathbf{e}_n and the **left** action of GL_n on polynomials will be recalled in Section 2. We call ω_I an (n, k) -**spike** if there exist $\ell_1 \geq \dots \geq \ell_n \geq 0$ such that $i_j = 2^{n-j} \gamma^{\ell_j} (k+1) - 1$ for $1 \leq j \leq n$. Here for $\ell \geq 0$, we write γ^ℓ for the ℓ -fold composition of the function γ . For example, we have $\gamma^\ell(1) = 2^\ell$, which implies that an $(n, 0)$ -spike is a class of the form $\omega_{2^{\alpha_1-1}, \dots, 2^{\alpha_n-1}}$ with $\alpha_1 > \dots > \alpha_n \geq 0$.

The following theorem generalises the result¹ of Inoue [5] on minimal generating sets for the modules $L_{n,0}$ and $L_{n,1}$.

Theorem 1.2. *The (n, k) -spikes form a minimal generating set for $L_{n,k}$.*

The remainder of the paper is organised as follows. In Section 2, we recall the structure of $L_{n,k}$ as a graded vector space. We then study in Section 3 the behaviour of the Kameko map on $L_{n,k}$ and deduce Theorem 1.1. In Section 4 we prove by double induction Theorem 1.2. Finally in Section 5 we discuss a motivation for the study of the hit problem for $L_{n,k}$.

2. Linear structure of $L_{n,k}$

Recall that the polynomial algebra $P_n := \mathbb{F}_2[x_1, \dots, x_n]$, considered as the mod 2 cohomology of the classifying space of the elementary abelian group $(\mathbb{Z}/2)^n$, is an $\mathcal{A}[GL_n]$ -module. The general linear group GL_n acts on P_n on the **left** as follows. Given $g = (g_{i,j}) \in GL_n$ and $F(x_1, \dots, x_n) \in P_n$, gF is defined by

$$(gF)(x_1, \dots, x_n) := F(gx_1, \dots, gx_n),$$

where $gx_j := \sum_{i=1}^n g_{i,j} x_i$ for $1 \leq j \leq n$. This action commutes with that of the Steenrod algebra on P_n .

Let ω_n denote the top Dickson class in $\mathbb{F}_2[x_1, \dots, x_n]$, i.e. the product of all non-trivial one-dimensional classes of $\mathbb{F}_2[x_1, \dots, x_n]$. The localisation $P_n[\omega_n^{-1}]$ obtained by inverting ω_n has a unique \mathcal{A} -module structure extending that of P_n (see [17]). In particular, the action of the Steenrod squares on $x_1^{-1}, \dots, x_n^{-1}$ is given by

$$\mathrm{Sq}^i(x_j^{-1}) = x_j^{i-1}, \quad i \geq 0.$$

The Steinberg idempotent \mathbf{e}_n of $\mathbb{F}_2[GL_n]$ is defined as follows. Let B_n be the subgroup of upper triangular matrices in GL_n and let Σ_n be the subgroup of permutation matrices. Then

$$\mathbf{e}_n := \bar{B}_n \bar{\Sigma}_n$$

where $\bar{B}_n = \sum_{b \in B} b$ and $\bar{\Sigma}_n = \sum_{\sigma \in \Sigma_n} \sigma$.

Recall that a sequence $I = (i_1, \dots, i_n) \in \mathbb{N}^n$ is called *admissible* if $i_j \geq 2i_{j+1}$ for $1 \leq j \leq n-1$ and $i_n \geq 1$. Given $k \in \mathbb{N}$, we call I a *k-acceptable* sequence if $i_j > 2i_{j+1}$ for $1 \leq j \leq n-1$ and $i_n \geq k$. It is clear that I is *k-acceptable* if and only if $I+1 := (i_1+1, \dots, i_n+1)$ is admissible and $i_n \geq k$.

We put

$$\omega_{i_1, \dots, i_n} := \mathbf{e}_n(\omega_1^{i_1-2i_2} \dots \omega_{n-1}^{i_{n-1}-2i_n} \omega_n^{i_n}).$$

Here for $1 \leq \ell \leq n$, ω_ℓ is the top Dickson class in $\mathbb{F}_2[x_1, \dots, x_\ell]$.

¹ The hit problem for the Steinberg module $L_{n,0}$ was solved independently by Grant Walker and Reg Wood, using a similar approach to that of Inoue. The author is grateful to G. Walker for kindly sending him the unpublished preprint [16] of this result.

The linear structure of $L_{n,k} = \mathbf{e}_n(P_n \omega_n^k)$ is described as follows.

Proposition 2.1. *A basis of the graded vector space $L_{n,k}$ is given by*

$$\{\omega_{i_1, \dots, i_n} \mid (i_1, \dots, i_n) \text{ is } k\text{-acceptable}\}.$$

Proof. Mitchell and Priddy [10] used the right action of GL_n on P_n so that their model, M_n , for the summand of P_n associated to the Steinberg idempotent was $M_n = P_n \mathbf{e}_n$. In the context of the left action used in this paper, the module M_n is the same as the module $\mathbf{e}'_n P_n$ where \mathbf{e}'_n is the conjugate of \mathbf{e}_n defined by $\mathbf{e}'_n = \bar{\Sigma}_n \bar{B}_n$. As noted in [10, Proposition 2.6], the multiplication by \bar{B}_n gives an isomorphism of \mathcal{A} -modules $L_{n,0} \cong \bar{B}_n(M_n)$. Mitchell and Priddy also proved that a basis of M_n is given by the following set

$$\{\text{Sq}^{i_1+1} \dots \text{Sq}^{i_n+1} (x_1^{-1} \dots x_n^{-1}) \mid (i_1 + 1, \dots, i_n + 1) \text{ is admissible}\}.$$

We proved in [4] that the image under \bar{B}_n of each element of this set is given by

$$\bar{B}_n \text{Sq}^{i_1+1} \dots \text{Sq}^{i_n+1} (x_1^{-1} \dots x_n^{-1}) = \mathbf{e}_n(\omega_1^{i_1-2i_2} \dots \omega_{n-1}^{i_{n-1}-2i_n} \omega_n^{i_n}) =: \omega_{i_1, \dots, i_n}.$$

It follows that the classes ω_{i_1, \dots, i_n} with (i_1, \dots, i_n) 0-acceptable form a basis of the module $L_{n,0}$. As $L_{n,k} = \mathbf{e}_n(P_n \omega_n^k) = L_{n,0} \omega_n^k$, the set

$$\{\omega_{i_1, \dots, i_n} \omega_n^k \mid (i_1, \dots, i_n) \text{ is 0-acceptable}\}$$

is a basis of $L_{n,k}$. Now it is easy to check that $\omega_{i_1, \dots, i_n} \omega_n^k = \omega_{i_1+2^{n-1}k, \dots, i_{n-1}+2k, i_n+k}$ and that the map $(i_1, \dots, i_n) \mapsto (i_1+2^{n-1}k, \dots, i_{n-1}+2k, i_n+k)$ is a bijection between the set of 0-acceptable sequences and that of k -acceptable sequences. The proposition follows. \square

Generalising the Mitchell–Priddy decomposition $M(n) \simeq L(n) \vee L(n-1)$, Takayasu proved in [15] that given $n \in \mathbb{N}^*$ and $m \in \mathbb{N}$, there exists a cofibration sequence

$$\Sigma^m L(n-1, 2m+1) \rightarrow L(n, m) \rightarrow L(n, m+1).$$

This induces in cohomology a short exact sequence of \mathcal{A} -modules

$$0 \rightarrow L_{n,m+1} \xrightarrow{\alpha} L_{n,m} \xrightarrow{\beta} \Sigma^m L_{n-1,2m+1} \rightarrow 0. \quad (1)$$

By convention, for $k \in \mathbb{N}$, the spectrum $L(0, k)$ is identified with the sphere spectrum S^0 , thus $L_{0,k}$ is the trivial module $\mathbb{Z}/2$. In the short exact sequence (1), the map α is the natural inclusion and the map β is given by

$$\beta(\omega_{i_1, \dots, i_n}) = \begin{cases} 0, & i_n > m, \\ \Sigma^m \omega_{i_1, \dots, i_{n-1}}, & i_n = m. \end{cases}$$

The exactness of the sequence (1) can be verified using the following.

Lemma 2.2. (See [4, Proposition 1.2].) *If (i_1, \dots, i_n) is 0-acceptable, then*

$$\omega_{i_1, \dots, i_n} = \omega_{i_1, \dots, i_{n-1}} x_n^{i_n} + \text{terms } \omega_{j_1, \dots, j_{n-1}} x_n^{j_n} \text{ with } j > j_n.$$

By induction, we can write

$$\omega_{i_1, \dots, i_n} = x_1^{i_1} \cdots x_n^{i_n} + \text{monomials of higher power of } x_n.$$

It follows that for $\alpha_1 > \cdots > \alpha_n \geq 0$, $\omega_{2^{\alpha_1}-1, \dots, 2^{\alpha_n}-1}$ is indecomposable because the monomial $x_1^{2^{\alpha_1}-1} \cdots x_n^{2^{\alpha_n}-1}$ can never appear in any hit polynomial when it is expressed irredundantly as a sum of monomials.

3. Restriction of the Kameko map on $L_{n,k}$

An important map in the study of the space $Q(P_n)$ is the Kameko map

$$\psi_n : \Sigma^n P_n \rightarrow \Phi \Sigma^n P_n$$

which is defined by

$$\psi_n(\Sigma^n x_1^{i_1} \cdots x_n^{i_n}) = \begin{cases} \Phi \Sigma^n x_1^{\frac{i_1-1}{2}} \cdots x_n^{\frac{i_n-1}{2}} & i_1, \dots, i_n \text{ odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Note that the n -fold suspension functor Σ^n and the doubling functor Φ are used here to make ψ_n a map of degree zero. These functors are defined as follows (see for example [12, Chapter 1]). Given M and \mathcal{A} -module, the \mathcal{A} -module ΣM is defined by

$$(\Sigma M)^d = M^{d-1}, \quad \text{Sq}^i(\Sigma x) = \Sigma(\text{Sq}^i x), \quad x \in M,$$

and the \mathcal{A} -module ΦM is defined by

$$(\Phi M)^d = M^{d/2}, \quad \text{Sq}^i(\Phi x) = \Phi(\text{Sq}^{i/2} x), \quad x \in M,$$

with the convention that $M^{d/2} = 0$ if d is odd, and $\text{Sq}^{i/2} x = 0$ if i is odd.

The Kameko map ψ_n is an $\mathcal{A}[GL_n]$ -linear map. Furthermore, ψ_n is compatible with the Verschiebung of the Steenrod algebra, as follows.

Recall that the Laurent ring $\mathbb{F}_2[x_1, x_1^{-1}]$ admits a unique \mathcal{A} -module structure which extends that of P_1 . The subspace \hat{P}_1 consisting of the classes x_1^i with $i \geq -1$ is an \mathcal{A} -submodule of $\mathbb{F}_2[x_1, x_1^{-1}]$. The module \hat{P}_1 is generated by x_1^{-1} , following the formula $\text{Sq}^{d+1}(x_1^{-1}) = x_1^d$, $d \geq -1$. There are then \mathcal{A} -linear maps

$$\mathcal{A} \xrightarrow{\tau} \Sigma \hat{P}_1 \xrightarrow{\hat{\psi}_1} \Phi \Sigma \hat{P}_1$$

which are defined respectively on generators by $\tau(1) = \Sigma x_1^{-1}$ and $\hat{\psi}_1(\Sigma x_1^{-1}) = \Phi \Sigma x_1^{-1}$ (and so $\hat{\psi}_1(\Sigma x_1^d) = \Phi \Sigma x_1^{(d-1)/2}$ for all $d \geq -1$). It is straightforward to verify that $\hat{\psi}_1$ commutes with the Verschiebung morphism \mathbf{V} of the Steenrod algebra via τ :

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\tau} & \Sigma \hat{P}_1 \\ \mathbf{V} \downarrow & & \downarrow \hat{\psi}_1 \\ \Phi \mathcal{A} & \xrightarrow{\Phi \tau} & \Phi \Sigma \hat{P}_1, \end{array} \quad \begin{array}{ccc} 1 & \xrightarrow{\tau} & \Sigma x_1^{-1} \\ \mathbf{V} \downarrow & & \downarrow \hat{\psi}_1 \\ \Phi 1 & \xrightarrow{\Phi \tau} & \Phi \Sigma x_1^{-1}. \end{array}$$

By restriction, the tensor product $\hat{\psi}_1^{\otimes n} : (\Sigma \hat{P}_1)^{\otimes n} \rightarrow (\Phi \Sigma \hat{P}_1)^{\otimes n}$ defines an \mathcal{A} -linear map $\psi_n : \Sigma^n P_n \rightarrow \Phi \Sigma^n P_n$. By noting that $\hat{\psi}_1(\Sigma x_1^d) = \Phi \Sigma x_1^{(d-1)/2}$, we see that the map ψ_n is nothing other than the Kameko map defined above.² We obtain then a commutative diagram

$$\begin{array}{ccccccc}
 \mathcal{A} & \xrightarrow{\Delta} & \mathcal{A}^{\otimes n} & \xrightarrow{\tau^{\otimes n}} & (\Sigma \hat{P}_1)^{\otimes n} & \longleftarrow & \Sigma^n P_n \\
 \mathbf{V} \downarrow & & \mathbf{V}^{\otimes n} \downarrow & & \downarrow (\hat{\psi}_1)^{\otimes n} & & \downarrow \psi_n \\
 \Phi \mathcal{A} & \xrightarrow{\Phi \Delta} & (\Phi \mathcal{A})^{\otimes n} & \xrightarrow{(\Phi \tau)^{\otimes n}} & (\Phi \Sigma \hat{P}_1)^{\otimes n} & \longleftarrow & \Phi \Sigma^n P_n.
 \end{array} \quad (\text{K})$$

Here the Verschiebung morphism \mathbf{V} commutes with the diagonal Δ of the Steenrod algebra \mathcal{A} (which is a Hopf algebra) because \mathbf{V} is a map of coalgebras.

Proposition 3.1. *Let $I = (i_1, \dots, i_n)$ be a 0-acceptable sequence. Then*

$$\psi_n(\Sigma^n \omega_I) = \Phi \Sigma^n(\omega_{\frac{I-1}{2}}).$$

Proof. Using the commutative diagram (K) above, the image of Sq^{I+1} under the composition $\tau^{\otimes n} \circ \Delta$ is given by

$$\tau^{\otimes n} \circ \Delta(\text{Sq}^{I+1}) = \Sigma^n \text{Sq}^{I+1}(x_1^{-1} \cdots x_n^{-1}),$$

which is an element of $\Sigma^n P_n$ thanks to the admissibility of the sequence $I+1$. As the Verschiebung morphism \mathbf{V} sends Sq^{I+1} to $\Phi \text{Sq}^{\frac{I+1}{2}}$, we get

$$\psi_n[\Sigma^n \text{Sq}^{I+1}(x_1^{-1} \cdots x_n^{-1})] = \Phi \Sigma^n[\text{Sq}^{\frac{I+1}{2}}(x_1^{-1} \cdots x_n^{-1})].$$

The proposition follows from the formula $\omega_I = \bar{B}_n \text{Sq}^{i_1+1} \cdots \text{Sq}^{i_n+1}(x_1^{-1} \cdots x_n^{-1})$ and the GL_n -linearity of ψ_n . \square

Remark 3.2. One may give a direct proof of this result by using the definition of ψ_n and the formula $\omega_I = \mathbf{e}_n(\omega_1^{i_1-2i_2} \cdots \omega_{n-1}^{i_{n-1}-2i_n} \omega_n^{i_n})$.

Corollary 3.3. *The map ψ_n induces by restriction a commutative diagram of \mathcal{A} -modules*

$$\begin{array}{ccc}
 \Sigma^n L_{n,2m+1} & \hookrightarrow & \Sigma^n L_{n,2m} \\
 \psi_{n,2m+1} \downarrow & & \downarrow \psi_{n,2m} \\
 \Phi \Sigma^n L_{n,m} & \xlongequal{\quad} & \Phi \Sigma^n L_{n,m}.
 \end{array}$$

The following result and its consequence (Corollary 3.5) will play an essential role in constructing by double induction a minimal generating set for $L_{n,k}$.

² The presentation of the Kameko map given here is inspired by the note of G. Powell [11] in which he interpreted the Kameko map as a morphism $P_n \rightarrow \Sigma^n \Phi P_n$. In view of the compatibility with the Verschiebung morphism $\mathcal{A} \rightarrow \Phi \mathcal{A}$, we find it more natural to define the Kameko map as a morphism $\Sigma^n P_n \rightarrow \Phi \Sigma^n P_n$.

Theorem 3.4. The map $\psi_{n,2m+1}$ induces an isomorphism of graded vector spaces

$$Q(\Sigma^n L_{n,2m+1}) \cong Q(\Phi \Sigma^n L_{n,m}).$$

The proof of this is given below. Here is a consequence of the theorem.

Corollary 3.5. The functor Q preserves the exactness of the Takayasu short exact sequence

$$0 \rightarrow L_{n,2m+1} \rightarrow L_{n,2m} \rightarrow \Sigma^{2m} L_{n-1,4m+1} \rightarrow 0.$$

Proof. As the functor Q is right exact, it suffices to prove the injectivity of the induced map $Q(L_{n,2m+1}) \rightarrow Q(L_{n,2m})$. By Corollary 3.3, we have a commutative diagram of graded vector spaces

$$\begin{array}{ccc} Q(\Sigma^n L_{n,2m+1}) & \longrightarrow & Q(\Sigma^n L_{n,2m}) \\ \downarrow & & \downarrow \\ Q(\Phi \Sigma^n L_{n,m}) & \xlongequal{\quad} & Q(\Phi \Sigma^n L_{n,m}). \end{array}$$

By Theorem 3.4, the left vertical map is injective. It follows that the upper horizontal map is also injective. \square

As an application of Theorem 3.4, we give now a short proof for the result of M. Inoue on the linear structure of the space $Q(L_{n,0})$.

Corollary 3.6. (Cf. [5].) The classes $\omega_{2^{\alpha_1}-1, \dots, 2^{\alpha_n}-1}$ for which $\alpha_1 > \dots > \alpha_n \geq 0$ form a minimal generating set for $L_{n,0}$.

Proof. From the isomorphisms

$$Q(\Sigma^n L_{n,1}) \cong Q(\Phi \Sigma^n L_{n,0}), \quad Q(L_{n,0}) \cong Q(L_{n,1}) \oplus Q(L_{n-1,1}),$$

we get

$$Q(\Sigma^n L_{n,0}) \cong Q(\Phi \Sigma^n L_{n,0}) \oplus Q(\Sigma \Phi \Sigma^{n-1} L_{n-1,0}). \quad (2)$$

Let $F_n(t) = \sum_{d \geq 0} \dim_{\mathbb{F}_2} Q^d(L_{n,0}) \cdot t^d$ be the Poincaré series of the graded vector space $Q(L_{n,0}) = \bigoplus_{d \geq 0} Q^d(L_{n,0})$. It follows from the decomposition (2) that

$$F_n(t) = t^n F_n(t^2) + t^{n-1} F_{n-1}(t^2).$$

It is easy to verify that this equation determines uniquely $F_n(t)$ once we have obtained the formula for $F_{n-1}(t)$. Using induction, starting from the case $n=0$ where $F_0(t) = 1$, we conclude that

$$F_n(t) = \sum_{\alpha_1 > \dots > \alpha_n \geq 0} t^{(2^{\alpha_1}-1) + \dots + (2^{\alpha_n}-1)}.$$

Now for $\alpha_1 > \dots > \alpha_n \geq 0$, the class $\omega_{2^{\alpha_1}-1, \dots, 2^{\alpha_n}-1}$ is indecomposable because its expansion as a sum of monomials contains $x_1^{2^{\alpha_1}-1} \dots x_n^{2^{\alpha_n}-1}$. Moreover these classes occur in different degrees of $L_{n,0}$, thus they are linearly independent in $Q(L_{n,0})$. The corollary follows. \square

We prove now Theorem 3.4. For this we need to know the action of Steenrod squares on top Dickson classes. For $a_1, \dots, a_k \in \mathbb{N}$, put

$$[a_1, \dots, a_k] = \det \begin{pmatrix} x_1^{a_1} & \cdots & x_k^{a_1} \\ \vdots & \ddots & \vdots \\ x_1^{a_k} & \cdots & x_k^{a_k} \end{pmatrix} \in \mathbb{F}_2[x_1, \dots, x_k].$$

It is well known that $\omega_k = [0, 1, \dots, k-1]$. The following can be proved easily using the Cartan formula and the fact that, if $s > 0$ then $\text{Sq}^s(x_j^{2^a}) = x_j^{2^{a+1}}$ if $s = 2^a$ and zero otherwise.

Lemma 3.7. (See [18].) For $r \geq 0$ and $k \geq 1$,

$$\text{Sq}^r([0, \dots, k-1]) = \begin{cases} [0, \dots, i-1, i+1, \dots, k] & \text{if } r = 2^k - 2^i \text{ for } 0 \leq i \leq k, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $\text{Sq}^{2^k-1}(\omega_k^a) = a\omega_k^{a+1}$ and $\text{Sq}^r(\omega_k) = 0$ if $0 < r < 2^{k-1}$.

Corollary 3.8. If j_k is even and $j_{k+1}, \dots, j_n \in \mathbb{N}$, then

$$\text{Sq}^{2^k-1}(\omega_k^{j_k-1} \omega_{k+1}^{j_{k+1}} \cdots \omega_n^{j_n}) = \omega_k^{j_k} \omega_{k+1}^{j_{k+1}} \cdots \omega_n^{j_n}.$$

Proof of Theorem 3.4. We have a short exact sequence of \mathcal{A} -modules

$$0 \longrightarrow K \longrightarrow \Sigma^n L_{n,2m+1} \xrightarrow{\psi_{n,2m+1}} \Phi \Sigma^n L_{n,m} \longrightarrow 0$$

where $K := \ker \psi_{n,2m+1}$, by Proposition 3.1, is the subspace of $\Sigma^n L_{n,2m+1}$ generated by the classes $\Sigma^n \omega_{i_1, \dots, i_n}$ with at least one i_j even. As the functor Q is right exact, it induces an exact sequence

$$Q(K) \rightarrow Q(\Sigma^n L_{n,2m+1}) \rightarrow Q(\Phi \Sigma^n L_{n,m}) \rightarrow 0.$$

In order to prove the isomorphism $Q(\Sigma^n L_{n,2m+1}) \cong Q(\Phi \Sigma^n L_{n,m})$, it suffices to prove that the map $Q(K) \rightarrow Q(\Sigma^n L_{n,2m+1})$ is trivial, that is, each $\omega_I := \omega_{i_1, \dots, i_n}$ in $L_{n,2m+1}$ with at least one i_j even is hit in $L_{n,2m+1}$, namely $\omega_I \in \mathcal{A}^+ L_{n,2m+1}$.

If i_1 is even then $\text{Sq}^1(\omega_{i_1-1, i_2, \dots, i_n}) = \omega_{i_1, \dots, i_n}$, by Corollary 3.8. So ω_I is hit by Sq^1 . Suppose we have proved that ω_I is hit if at least one i_j is even for $1 \leq j \leq k-1$. We prove now that if i_1, \dots, i_{k-1} are odd and i_k is even, then ω_I is hit.

Put $u = \omega_1^{j_1} \cdots \omega_{k-1}^{j_{k-1}}$ and $v = \omega_k^{j_k-1} \omega_{k+1}^{j_{k+1}} \cdots \omega_n^{j_n}$ where $j_\ell = i_\ell - 2i_{\ell+1}$ for $1 \leq \ell \leq n-1$ and $j_n = i_n$. By Corollary 3.8, we have $\omega_1^{j_1} \cdots \omega_n^{j_n} = u \text{Sq}^{2^k-1} v$. Denote by $\widehat{\text{Sq}}^k$ the conjugate $\chi(\text{Sq}^k)$ where χ is the canonical anti-automorphism of \mathcal{A} . Using the χ -trick,³ we have

$$u \text{Sq}^{2^k-1}(v) = (\widehat{\text{Sq}}^{2^k-1} u) v + \sum_{i=1}^{2^k-1} \text{Sq}^i [(\widehat{\text{Sq}}^{2^k-1-i} u) v] = g v + \sum_{i=1}^{2^k-1} \text{Sq}^i f_i, \quad (3)$$

³ The χ -trick is an important tool in the study of the hit problem for the polynomial algebra P_n , which says that if $u, v \in P_n$, then $u\theta(v) \equiv \chi(\theta(u))v \pmod{\mathcal{A}^+ P_n}$. One should be careful in using this trick when studying the hit problem for a submodule M of P_n , because in general one only has a congruence modulo $\mathcal{A}^+ P_n$ but not $\mathcal{A}^+ M$. In Eq. (3) above, we have therefore used the formula $u \text{Sq}^m v = (\widehat{\text{Sq}}^m u) v + \sum_{i=1}^m \text{Sq}^i [(\widehat{\text{Sq}}^{m-i} u) v]$ instead of using directly the χ -trick congruence $u \text{Sq}^m(v) \equiv \widehat{\text{Sq}}^m(u)v \pmod{\mathcal{A}^+ P_n}$.

where $g = \widehat{\text{Sq}}^{2^k-1} u \in P_{k-1} \omega_{k-1}$ and $f_i = (\widehat{\text{Sq}}^{2^k-1-i} u) v \in P_n \omega_n^{2m+1}$. That g is an element of $P_{k-1} \omega_{k-1}$ is clear because $u = \omega_1^{j_1} \cdots \omega_{k-1}^{j_{k-1}}$ and $j_{k-1} \geq 1$. For the fact that f_i is an element of $P_n \omega_n^{2m+1}$, it is also clear if $k \leq n-1$ because $v = \omega_k^{j_k-1} \omega_{k+1}^{j_{k+1}} \cdots \omega_n^{j_n}$ and $j_n \geq 2m+1$. If $k = n$, then $v = \omega_n^{j_n-1}$ and since j_n is even and $j_n \geq 2m+1$, it follows that $j_n - 1 \geq 2m+1$.

Now we have $\mathbf{e}_n \mathbf{e}_{k-1} = \mathbf{e}_n$ [7], where the Steinberg idempotent \mathbf{e}_{k-1} is considered as an element of $\mathbb{F}_2[GL_n]$ by using the natural inclusion $(\mathbb{Z}/2)^{k-1} \hookrightarrow (\mathbb{Z}/2)^n$ defined by $z \mapsto (z, 0^{n-k+1})$. Letting \mathbf{e}_n act on (3), we get

$$\omega_I = \mathbf{e}_n(gv) + \sum_{i=1}^{2^k-1} \text{Sq}^i(\mathbf{e}_n f_i) \equiv \mathbf{e}_n[(\mathbf{e}_{k-1}g)v] \pmod{\mathcal{A}^+ L_{n,2m+1}}. \quad (4)$$

As $\mathbf{e}_{k-1}g$ is an element of $L_{k-1,1}$, we can write

$$\mathbf{e}_{k-1}g = \sum_{t_1, \dots, t_{k-1} > 0} \alpha_{t_1, \dots, t_{k-1}} \cdot \mathbf{e}_{k-1} \omega_1^{t_1} \cdots \omega_{k-1}^{t_{k-1}}, \quad \alpha_{t_1, \dots, t_{k-1}} \in \mathbb{Z}/2.$$

For each $T = (t_1, \dots, t_{k-1})$ in this sum, we have

$$\mathbf{e}_n(\mathbf{e}_{k-1} \omega_1^{t_1} \cdots \omega_{k-1}^{t_{k-1}} v) = \mathbf{e}_n(\omega_1^{t_1} \cdots \omega_{k-1}^{t_{k-1}} \omega_k^{j_k-1} \omega_{k+1}^{j_{k+1}} \cdots \omega_n^{j_n}) =: \omega_{T[k]},$$

where $T[k] = (t'_1, \dots, t'_n)$ is a sequence that satisfies the condition: $t'_j \equiv t_j \pmod{2}$ for $1 \leq j \leq k-1$. Since

$$\deg g = (2^k - 1) + \deg \omega_1^{j_1} \cdots \omega_{k-1}^{j_{k-1}} = \deg \omega_1^{t_1} \cdots \omega_{k-1}^{t_{k-1}},$$

it follows that

$$t_1 + \cdots + t_{k-1} \equiv i_1 + \cdots + i_{k-1} + 1 \pmod{2}.$$

As all i_1, \dots, i_{k-1} are odd, there is at least one t_j even for $1 \leq j \leq k-1$. Eq. (4) is then rewritten as follows

$$\omega_I \equiv \sum_T \alpha_T \omega_{T[k]} \pmod{\mathcal{A}^+ L_{n,2m+1}},$$

where for each $T[k] = (t'_1, \dots, t'_n)$, there is at least one t'_j even for $1 \leq j \leq k-1$. By inductive hypothesis, $\omega_{T[k]}$ is hit in $L_{n,2m+1}$. Thus ω_{i_1, \dots, i_n} is also hit. \square

4. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. Recall the definition of the function $\gamma : \mathbb{N} \rightarrow \mathbb{N}$ given in the introduction:

$$\gamma(a) = \begin{cases} 2([a/2] + 1), & a \not\equiv 0 \pmod{2}, \\ 2\gamma(a/2), & a \equiv 0 \pmod{2}. \end{cases}$$

Recall also that a class $\omega_I := \omega_{i_1, \dots, i_n}$ is an (n, k) -spike if there exist $\ell_1 \geq \cdots \geq \ell_n \geq 0$ such that $i_j = 2^{n-j} \gamma^{\ell_j}(k+1) - 1$ for $1 \leq j \leq n$. By abuse of terminology, we also refer to the sequence I as

being an (n, k) -spike if I verifies that condition. The non-trivial element of $\mathbb{Z}/2$ is, by convention, a $(0, k)$ -spike for all $k \in \mathbb{N}$.

Proof of Theorem 1.2. We prove by double induction on $(n, k) \in \mathbb{N} \times \mathbb{N}$ that the (n, k) -spikes form a minimal generating set for $L_{n,k}$. For $k = 0$, this is true by Corollary 3.6. For $n = 0$, $L_{0,k}$ is by convention identified with the trivial module $\mathbb{Z}/2$ and the non-trivial element of $\mathbb{Z}/2$ is a $(0, k)$ -spike. We can therefore start the double induction.

For $n, k > 0$, suppose that for all couples (n', k') which are less than (n, k) in the lexicographic order, the (n', k') -spikes form a minimal generating set for $L_{n',k'}$. We consider the following two cases:

Case $k = 2m + 1$. By Corollary 3.5, there is an isomorphism of graded vector spaces

$$\psi_{n,2m+1} : Q(\Sigma^n L_{n,2m+1}) \xrightarrow{\cong} Q(\Phi \Sigma^n L_{n,m}).$$

By inductive hypothesis for (n, m) , the set

$$\{\omega_I \mid \omega_I \text{ is an } (n, m)\text{-spike}\}$$

is a basis of $Q(L_{n,m})$. Since $\psi_{n,2m+1}(\Sigma^n \omega_{2l+1}) = \Phi \Sigma^n \omega_I$, it follows that the set

$$\{\omega_{2l+1} \mid I \text{ is an } (n, m)\text{-spike}\}$$

is a basis of $Q(L_{n,2m+1})$. This set is identical with that of all $(n, 2m + 1)$ -spikes because if $i_j = 2^{n-j} \gamma^{\ell_j}(m + 1) - 1$ then

$$2i_j + 1 = 2^{n-j+1} \gamma^{\ell_j}(m + 1) - 1 = 2^{n-j} \gamma^{\ell_j}(2m + 2) - 1.$$

Hence the set of all $(n, 2m + 1)$ -spikes is a basis of $Q(L_{n,2m+1})$.

Case $k = 2m$. By Theorem 3.4 and Corollary 3.5, we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & Q(\Sigma^n L_{n,2m+1}) & \xrightarrow{\alpha} & Q(\Sigma^n L_{n,2m}) & \xrightarrow{\beta} & Q(\Sigma^{2m+n} L_{n-1,4m+1}) \longrightarrow 0 \\ & & \downarrow \psi_{n,2m+1} \cong & & \parallel & & \downarrow \psi_{n-1,4m+1} \cong \\ 0 & \longrightarrow & Q(\Phi \Sigma^n L_{n,m}) & \longrightarrow & Q(\Sigma^n L_{n,2m}) & \longrightarrow & Q(\Sigma^{2m+1} \Phi \Sigma^{n-1} L_{n-1,2m}) \longrightarrow 0 \end{array}$$

where the rows are exact. By inductive hypothesis for $(n - 1, 2m)$, the set

$$\{\omega_I \mid I \text{ is an } (n - 1, 2m)\text{-spike}\}$$

is a basis of $Q(L_{n-1,2m})$. By inductive hypothesis for (n, m) , the set

$$\{\omega_I \mid I \text{ is an } (n, m)\text{-spike}\}$$

is a basis of $Q(L_{n,m})$. It follows from the above diagram that the set

$$\{\omega_{2l+1,2m} \mid I \text{ is an } (n - 1, 2m)\text{-spike}\} \cup \{\omega_{2l+1} \mid I \text{ is an } (n, m)\text{-spike}\}$$

is a basis of $Q(L_{n,2m})$. It is clear that the first set of this union consists of all $(n, 2m)$ -spikes ω_{i_1, \dots, i_n} for which there exist $\ell_1 \geq \dots \geq \ell_{n-1} \geq \ell_n = 0$ such that

$$i_j = 2^{n-j} \gamma^{\ell_j} (2m+1) - 1, \quad 1 \leq j \leq n.$$

For the second set, let I be an (n, m) -spike with

$$i_j = 2^{n-j} \gamma^{\ell_j-1} (m+1) - 1, \quad 1 \leq j \leq n,$$

for $\ell_1 \geq \dots \geq \ell_n \geq 1$. We have

$$\begin{aligned} 2i_j + 1 &= 2^{n-j+1} \gamma^{\ell_j-1} (m+1) - 1 \\ &= 2^{n-j} \gamma^{\ell_j-1} (2m+2) - 1 \quad (\text{as } \gamma(2m+2) = 2\gamma(m+1)) \\ &= 2^{n-j} \gamma^{\ell_j} (2m+1) - 1 \quad (\text{as } \gamma(2m+1) = 2m+2). \end{aligned}$$

Hence the second set of the union consists of all $(n, 2m)$ -spikes ω_{i_1, \dots, i_n} for which there exist $\ell_1 \geq \dots \geq \ell_n \geq 1$ such that

$$i_j = 2^{n-j} \gamma^{\ell_j} (2m+1) - 1, \quad 1 \leq j \leq n.$$

The set of $(n, 2m)$ -spikes is thus a basis of $Q(L_{n,2m})$. The theorem is proved. \square

Remark 4.1. It can be checked, again by a double induction on $(n, k) \in \mathbb{N} \times \mathbb{N}$, that (n, k) -spikes occur in different degrees of $L_{n,k}$, so the graded space $Q(L_{n,k})$ is at most one-dimensional in each degree (cf. [3]).

5. Final remark

A motivation for the study of the hit problem for $P_n = H^*BV_n$ is the existence of the Singer algebraic transfer [13] from the dual of $Q(P_n)$ to the cohomology of the Steenrod algebra. The Singer algebraic transfer

$$\text{Tr}_n : \text{Ext}_{\mathcal{A}}^0(P_n, \Sigma^d \mathbb{Z}/2) \rightarrow \text{Ext}_{\mathcal{A}}^n(\Sigma^{-n} \mathbb{Z}/2, \Sigma^d \mathbb{Z}/2)$$

is essentially constructed by taking the Yoneda product with a particular cohomology class $\text{Ext}_{\mathcal{A}}^n(\Sigma^{-n} \mathbb{Z}/2, P_n)$. One may use this idea to construct a similar algebraic transfer

$$\tau_{n,m+1} : \text{Ext}_{\mathcal{A}}^0(L_{n,m+1}, \Sigma^d \mathbb{Z}/2) \rightarrow \text{Ext}_{\mathcal{A}}^n(\Sigma^{(2^n-1)m} \mathbb{Z}/2, \Sigma^d \mathbb{Z}/2),$$

by taking the Yoneda product with a particular cohomology class

$$\gamma_{n,m+1} \in \text{Ext}_{\mathcal{A}}^n(\Sigma^{(2^n-1)m} \mathbb{Z}/2, L_{n,m+1})$$

defined as follows. The class $\gamma_{n,m+1}$ is represented by the exact sequence of unstable \mathcal{A} -modules

$$0 \rightarrow L_{n,m+1} \rightarrow L_{n,m} \rightarrow \dots \rightarrow \Sigma^{(2^k-1)m} L_{n-k, 2^k m} \rightarrow \dots \rightarrow \Sigma^{(2^n-1)m} \mathbb{Z}/2 \rightarrow 0$$

which is obtained by splicing the Takayasu short exact sequences

$$0 \rightarrow L_{n-k, 2^k m+1} \rightarrow L_{n-k, 2^k m} \rightarrow \Sigma^{2^k m} L_{n-k-1, 2^{k+1} m+1} \rightarrow 0$$

for $0 \leq k \leq n-1$. Note that since these are short exact sequences of unstable \mathcal{A} -modules, the source of the transfer $\tau_{n, m+1}$ is in fact an extension group taken in the category \mathcal{U} of unstable modules [12].

By Corollary 3.5, the transfer $\tau_{n, m+1}$ is trivial if $m+1$ is odd. The transfer $\tau_{n, m+1}$ for which $m+1$ is even (it is interesting enough to study $\tau_{n, 2^k}!$) will be studied elsewhere.

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