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Hyperrings and α^*-relations. A general approach

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ABSTRACT

The study of the equivalence relations of particular multialgebras for which the factor multialgebras are universal algebras satisfying certain identities is a very important and intensively studied topic in multialgebra theory and not only. Our paper provides a general multialgebraic approach for the construction of all these relations and also some important and interesting properties concerning the construction of the corresponding factor universal algebra. One of the purposes of our approach is to improve some results concerning α^*-relations of hyperrings and Krasner hyperrings and we will do this in the last sections of this paper.

1. Introduction

Multialgebras are particular relational systems which generalize universal algebras. They offer a wide range of possibilities of application to different areas of mathematics and computer science (see [7]). The first multialgebras (called hypergroups) emerged as factorizations of groups modulo some equivalence relations determined by subgroups. Later, G. Grätzer proved in [17] that any multialgebra can be obtained by an appropriate factorization of a universal algebra modulo an equivalence relation. But the importance of the construction of the factor multialgebra is not only related to the

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genesis of multialgebras. Such factorizations of rings proved to be useful tools in approaching algebraic geometry and number theory topics (see [6]). From the very beginning of multialgebra theory, special attention was paid to those equivalence relations which determine factor multialgebras which are universal algebras. The study of these equivalence relations and the investigation of the corresponding factor multialgebras have been thriving since the last decade of the 20th century, mostly for particular multialgebras. We mention that such factorizations are used in [34] to define invariants for some particular G-algebras (G group).

We initiated a general approach of these relations of a multialgebra $\mathbf{A}$ using the polynomial functions of the universal algebra $\mathbf{P}^*(\mathbf{A})$ of the nonempty subsets of $\mathbf{A}$ in [3, Theorem 13] and we gave a characterization for these relations in [27, Proposition 4.1]. By then, the term functions of $\mathbf{P}^*(\mathbf{A})$ had already proved to be important tools in multialgebra theory (see, for instance, [9,10,22,32] or [33]). In our opinion, a general multialgebraic approach may provide, sometimes, not only widely applicable results, but even stronger results than the existing ones. This statement is supported by some of our previous papers [24–27] and is also proved by the present paper.

The current paper is strongly related to [13] and [21]. An interesting situation appears in [13] for hyperrings. Even if the form of the smallest equivalence relation of a hyperring for which the factor is a commutative ring is not very simple, [13, Theorem 6] shows that we can avoid using it to determine (up to an isomorphism) the corresponding ring. Unfortunately, there are some missing steps in the proof of this theorem. While trying to fill in the blanks of [13, Theorem 6], we managed to give a "recipe" for such results to hold.

We will start by mentioning a few things we consider to be missing from the proof of [13, Theorem 6] because this will show how our paper came into being. It seems that the authors start their proof considering that $(R/\gamma^*, \uplus, \odot)$ is a hyperring. But the factorization of the hyperring $(R, +, \cdot)$ through γ^* (which is defined using only the multiplication $\cdot$) preserves the identities of the hypergroup $(R, +)$ in a weak manner. Thus the fact that they are preserved in a strong manner, if it is the case, should be proved. Take, for instance, associativity. If it does not hold in a strong manner, then one cannot use [13, Theorem 5] (which, by the way, is valid only if the multiplication is commutative) to show that $(R/\gamma^*)/\gamma^*_{\uplus}$ is a commutative ring, and, hence, the proof of the inclusion $\alpha^* \subseteq \theta$ is not complete. Of course, in a hypergroupoid $(R, +)$ with $+$ only weak associative, the (hyper)sum $\sum_{i=1}^{n} x_i$ is meaningless for $n \geqslant 3$, so it seems that these (hyper)sums should be replaced with images of singletons through term functions of the nonempty subsets of $(R/\gamma^*, \uplus)$.

But, if we work with term functions of a groupoid which we do not know to be a semigroup, is it necessary to consider hyperrings or the result is valid for a larger class of multialgebras? As we will see in Example 5, the identities of $+$ do not seem to be important. We will show that the important fact is that the multiplication is distributive with respect to addition and, somehow, the multialgebra $(R/\gamma^*, \uplus, \odot)$ inherits the essential part of this property. Actually, we will prove that [13, Theorem 6] is a consequence of a more general result which holds for any multialgebras satisfying certain subdistributivity conditions. Our results allow a different approach of α^*-relations for Krasner hyperrings which will improve some main results from [21].

The first part of the paper contains a general multialgebraic approach of those equivalence relations of a multialgebra which act like hyperrings' α^*-relations and some isomorphism theorems involving these relations. The last sections are dedicated to three particular multialgebras: (a more general type of) hyperrings, Krasner hyperrings, and (m,n)-hyperrings. We consider the third class of multialgebras the best choice for showing that our general results can be applied to other multialgebras than hyperrings, without losing touch with the problems which can occur in the case of hyperrings.

2. Preliminary notions and notations

Let $\mathcal{F}$ be a set of function symbols such that an arity $n \in \mathbb{N}$ is assigned to each symbol f from $\mathcal{F}$ and let $\mathcal{F}_n$ be the subset of the n-ary symbols from $\mathcal{F}$. Let A be a set, $n \in \mathbb{N}$ and let $P^*(A)$ denote the set of the nonempty subsets of A. An *n-ary multioperation* f on A is a mapping $f : A^n \to P^*(A)$. A *multialgebra* $\mathbf{A} = \langle A, F\rangle$ of type $\mathcal{F}$ (or an *$\mathcal{F}$-multialgebra*) consists of a set A and a family of multioperations F obtained by associating to each symbol f from $\mathcal{F}$ a multioperation $f^{\mathbf{A}}$ on A. When the notation is not ambiguous, we write f instead of $f^{\mathbf{A}}$. If the multialgebra $\mathbf{A}$ has no nullary operations, then we allow the underlying set A to be empty. If each image of each multioperation from F is a singleton, then $\mathbf{A}$ is a universal algebra.

If $f \in \mathcal{F}_n$ and $A_1, \dots, A_n \in P^*(A)$, by defining

$$f^{\mathbf{P}^*(\mathbf{A})}(A_1, \dots, A_n) = \bigcup \{ f^{\mathbf{A}}(a_1, \dots, a_n) \mid a_i \in A_i,\ i \in \{1, \dots, n\} \},$$

one obtains an operation on $P^*(A)$. Thus $P^*(A)$ can be organized as a universal algebra (see [30]). We denote this algebra by $\mathbf{P}^*(\mathbf{A})$ and we call it *the algebra of the nonempty subsets of the multialgebra* $\mathbf{A}$.

Let $m \in \mathbb{N}$. Let $\mathrm{Clo}_m(\mathbf{P}^*(\mathbf{A}))$ be the set of the m-ary term functions on $\mathbf{P}^*(\mathbf{A})$ and $\mathrm{Pol}_m(\mathbf{P}^*(\mathbf{A}))$ the set of the m-ary polynomial functions of the universal algebra $\mathbf{P}^*(\mathbf{A})$. We denote by p (or by $p^{\mathbf{P}^*(\mathbf{A})}$ when necessary) the term function from $\mathrm{Clo}_m(\mathbf{P}^*(\mathbf{A}))$ induced by the m-ary term p.

Let $a \in A$, $i \in \{1, \dots, m\}$ and the mappings

$$c_a^m, e_i^m : P^*(A)^m \to P^*(A), \quad c_a^m(A_1, \dots, A_m) = \{a\}, \qquad e_i^m(A_1, \dots, A_m) = A_i.$$

We denote by $\mathrm{Pol}_m^A(\mathbf{P}^*(\mathbf{A}))$ the subuniverse of $\langle \mathrm{Pol}_m(\mathbf{P}^*(\mathbf{A})), F \rangle$ generated by

$$\{ c_a^m \mid a \in A \} \cup \{ e_i^m \mid i \in \{1, \dots, m\} \}.$$

A mapping $h : A \to B$ between the multialgebras $\mathbf{A}$ and $\mathbf{B}$ of the same type $\mathcal{F}$ is a *homomorphism* if for any $n \in \mathbb{N}$, $f \in \mathcal{F}_n$ and any $a_1, \dots, a_n \in A$,

$$h\big(f^{\mathbf{A}}(a_1, \dots, a_n)\big) \subseteq f^{\mathbf{B}}\big(h(a_1), \dots, h(a_n)\big). \tag{1}$$

A *multialgebra isomorphism* is a bijective map h such that for any $n \in \mathbb{N}$, $f \in \mathcal{F}_n$ and any $a_1, \dots, a_n \in A$

$$h\big(f^{\mathbf{A}}(a_1, \dots, a_n)\big) = f^{\mathbf{B}}\big(h(a_1), \dots, h(a_n)\big). \tag{1'}$$

Let ρ be an equivalence relation on A and $A/\rho = \{x/\rho \mid x \in A\}$ (x/ρ denotes the class of x modulo ρ). If $f \in \mathcal{F}_n$, the equalities

$$f^{\mathbf{A}/\rho}(a_1/\rho, \dots, a_n/\rho) = \{ b/\rho \mid b \in f^{\mathbf{A}}(b_1, \dots, b_n),\ a_i \,\rho\, b_i,\ i \in \{1, \dots, n\} \}$$

define a multioperation $f^{\mathbf{A}/\rho}$ on A/ρ (see [17]). One obtains a multialgebra $\mathbf{A}/\rho$ on A/ρ which will be called *the factor multialgebra of* $\mathbf{A}$ *modulo* ρ. The canonical projection $\pi_\rho : A \to A/\rho$, $\pi_\rho(a) = a/\rho$ is a surjective homomorphism. Using [9, Proposition 2.6] one can deduce that if p is an m-ary term,

$$\{ a/\rho \mid a \in p^{\mathbf{P}^*(\mathbf{A})}(a_1, \dots, a_m) \} \subseteq p^{\mathbf{P}^*(\mathbf{A}/\rho)}(a_1/\rho, \dots, a_m/\rho), \quad \forall a_1, \dots, a_m \in A.$$

Lemma 1. *Let $\mathbf{A} = \langle A, F \rangle$ be an $\mathcal{F}$-multialgebra and let τ, ρ be equivalence relations on A such that $\tau \subseteq \rho$. For any m-ary term p and any $a_1, \dots, a_m \in A$,*

$$a/\tau \in p^{\mathbf{P}^*(\mathbf{A}/\tau)}(a_1/\tau, \dots, a_m/\tau) \quad \Rightarrow \quad a/\rho \in p^{\mathbf{P}^*(\mathbf{A}/\rho)}(a_1/\rho, \dots, a_m/\rho).$$

Proof. Let π_τ and π_ρ be the canonical projections of τ and ρ, respectively. Since $\tau \subseteq \rho$, there exists a unique mapping φ which makes the diagram

$$\begin{tikzcd} A \arrow[r, "\pi_\tau"] \arrow[dr, "\pi_\rho"'] & A/\tau \arrow[d, dotted, "\varphi"] \\ & A/\rho \end{tikzcd}$$

commutative (i.e. $\varphi(a/\tau) = a/\rho$, for any $a \in A$). Moreover, φ is a multialgebra homomorphism between $\mathbf{A}/\tau$ and $\mathbf{A}/\rho$. Indeed, if $f \in \mathcal{F}_n$, $a_1, \dots, a_n \in A$, and

$$a/\tau \in f^{\mathbf{A}/\tau}(a_1/\tau, \dots, a_n/\tau),$$

there exist $b_1, \dots, b_n \in A$, $a_1 \,\tau\, b_1, \dots, a_n \,\tau\, b_n$ such that $a \in f^{\mathbf{A}}(b_1, \dots, b_n)$. But $\tau \subseteq \rho$, thus $a_1 \,\rho\, b_1, \dots, a_n \,\rho\, b_n$, and, consequently,

$$\varphi(a/\tau) = a/\rho \in f^{\mathbf{A}/\rho}(a_1/\rho, \dots, a_n/\rho) = f^{\mathbf{A}/\rho}\big(\varphi(a_1/\tau), \dots, \varphi(a_n/\tau)\big).$$

Consequently, for any m-ary term p of type $\mathcal{F}$ and any $a_1, \dots, a_m \in A$,

$$\varphi\big(p^{\mathbf{P}^*(\mathbf{A}/\tau)}(a_1/\tau, \dots, a_m/\tau)\big) \subseteq p^{\mathbf{P}^*(\mathbf{A}/\rho)}\big(\varphi(a_1/\tau), \dots, \varphi(a_m/\tau)\big)$$

(see, for instance, [9, Proposition 2.6]), thus we obtain the implication from the statement. □

Let q, r be two m-ary terms and let $\mathbf{A}$ be a multialgebra of type $\mathcal{F}$. The m-ary (*strong*) *identity* $q = r$ is satisfied in the multialgebra $\mathbf{A}$ if

$$q^{\mathbf{P}^*(\mathbf{A})}(a_1, \dots, a_m) = r^{\mathbf{P}^*(\mathbf{A})}(a_1, \dots, a_m), \quad \forall a_1, \dots, a_m \in A.$$

The *weak identity* $q \cap r \neq \emptyset$ is satisfied in the multialgebra $\mathbf{A}$ if

$$q^{\mathbf{P}^*(\mathbf{A})}(a_1, \dots, a_m) \cap r^{\mathbf{P}^*(\mathbf{A})}(a_1, \dots, a_m) \neq \emptyset, \quad \forall a_1, \dots, a_m \in A.$$

Remark 1. For any multialgebra $\mathbf{A}$ satisfying an identity $q \cap r \neq \emptyset$ or $q = r$, the weak identity $q \cap r \neq \emptyset$ is satisfied in the factor multialgebra $\mathbf{A}/\rho$.

3. On some equivalence relations of a multialgebra determined by identities

Let ρ be an equivalence relation on the set A. We denote by $\overline{\overline{\rho}}$ the relation defined on $P^*(A)$ as follows: if $X, Y \in P^*(A)$, then

$$X \,\overline{\overline{\rho}}\, Y \quad \Leftrightarrow \quad x \,\rho\, y, \quad \forall x \in X,\ \forall y \in Y \quad (\Leftrightarrow X \times Y \subseteq \rho).$$

Denote by $E_{ua}(\mathbf{A})$ or $E_{ua}(\langle A, F\rangle)$ the set of all equivalence relations ρ of a multialgebra $\mathbf{A} = \langle A, F\rangle$ for which $\mathbf{A}/\rho$ is a universal algebra.

Proposition 2. *(See [27, Proposition 4.1].) Let $\mathbf{A} = \langle A, F\rangle$ be an $\mathcal{F}$-multialgebra and let ρ be an equivalence relation on A. The following conditions are equivalent*:

(a) $\rho \in E_{ua}(\mathbf{A})$;
(b) *if* $n \in \mathbb{N}$, $f \in \mathcal{F}_n$, $a, b, x_1, \dots, x_n \in A$ *and* $a \, \rho \, b$, *then, for any* $i \in \{1, \dots, n\}$,

$$f^{\mathbf{A}}(x_1, \dots, x_{i-1}, a, x_{i+1}, \dots, x_n) \, \overline{\overline{\rho}} \, f^{\mathbf{A}}(x_1, \dots, x_{i-1}, b, x_{i+1}, \dots, x_n);$$

(c) *if* $n \in \mathbb{N}$, $f \in \mathcal{F}_n$, $x_i, y_i \in A$ *and* $x_i \, \rho \, y_i$ *for any* $i \in \{1, \dots, n\}$, *then*

$$f^{\mathbf{A}}(x_1, \dots, x_n) \, \overline{\overline{\rho}} \, f^{\mathbf{A}}(y_1, \dots, y_n);$$

(d) *if* $m \in \mathbb{N}$, p *is a polynomial function from* $\mathrm{Pol}_m^A(\mathbf{P}^*(\mathbf{A}))$, $x_i, y_i \in A$ *and* $x_i \, \rho \, y_i$ *for any* $i \in \{1, \dots, m\}$, *then* $p(x_1, \dots, x_m) \, \overline{\overline{\rho}} \, p(y_1, \dots, y_m)$.

Remark 2. We can add to the above list of equivalent conditions the following two:

(e) *if* $m \in \mathbb{N}$, p *is a term function from* $\mathrm{Clo}_m(\mathbf{P}^*(\mathbf{A}))$, $x_i, y_i \in A$ *and* $x_i \, \rho \, y_i$ *for any* $i \in \{1, \dots, m\}$, *then*

$$p^{\mathbf{P}^*(\mathbf{A})}(x_1, \dots, x_m) \, \overline{\overline{\rho}} \, p^{\mathbf{P}^*(\mathbf{A})}(y_1, \dots, y_m);$$

(f) *if* $p \in \mathrm{Pol}_1^A(\mathbf{P}^*(\mathbf{A}))$ *is a unary polynomial function,* $x, y \in A$ *and* $x \, \rho \, y$ *then*

$$p(x) \, \overline{\overline{\rho}} \, p(y),$$

since (d) $\Rightarrow$ (e) $\Rightarrow$ (c) and (d) $\Rightarrow$ (f) $\Rightarrow$ (b).

If $\rho \in E_{ua}(\mathbf{A})$, then the operations in the factor multialgebra (which is a universal algebra) are defined as follows: if $f \in \mathcal{F}_n$ and $a_1, \dots, a_n \in A$ then

$$f^{\mathbf{A}/\rho}(a_1/\rho, \dots, a_n/\rho) = b/\rho, \quad \forall b \in f^{\mathbf{A}}(a_1, \dots, a_n). \tag{2}$$

Remark 3. If $\rho \in E_{ua}(\mathbf{A})$, $p \in \mathrm{Clo}_m(\mathbf{P}^*(\mathbf{A}))$ and $a_1, \dots, a_m \in A$, then

$$p^{\mathbf{P}^*(\mathbf{A}/\rho)}(a_1/\rho, \dots, a_m/\rho) = p^{\mathbf{A}/\rho}(a_1/\rho, \dots, a_m/\rho) = b/\rho, \quad \forall b \in p^{\mathbf{P}^*(\mathbf{A})}(a_1, \dots, a_m)$$

(see [28, Remark 13]), hence any (weak or strong) identity of $\mathbf{A}$ becomes an identity of the universal algebra $\mathbf{A}/\rho$.

Lemma 3. *Let* $\mathbf{A} = \langle A, F \rangle$ *be a multialgebra of type* $\mathcal{F}$ *and let* τ, ρ *be equivalence relations on* A *such that* $\tau \subseteq \rho$ *and* $\rho \in E_{ua}(\mathbf{A})$. *If* p *is an* m*-ary term of type* $\mathcal{F}$, *and* $a_1, \dots, a_m \in A$, *then*

$$b/\tau \in p^{\mathbf{P}^*(\mathbf{A}/\tau)}(a_1/\tau, \dots, a_m/\tau), \qquad b' \in p^{\mathbf{P}^*(\mathbf{A})}(a_1, \dots, a_m) \quad \Rightarrow \quad b \, \rho \, b'.$$

Proof. According to Lemma 1, from $\tau \subseteq \rho$ we obtain

$$b/\tau \in p^{\mathbf{P}^*(\mathbf{A}/\tau)}(a_1/\tau, \dots, a_m/\tau) \quad \Rightarrow \quad b/\rho \in p^{\mathbf{P}^*(\mathbf{A}/\rho)}(a_1/\rho, \dots, a_m/\rho),$$

and $\rho \in E_{ua}(\mathbf{A})$ leads us to

$$b/\rho = p^{\mathbf{P}^*(\mathbf{A}/\rho)}(a_1/\rho, \dots, a_m/\rho) = p^{\mathbf{A}/\rho}(a_1/\rho, \dots, a_m/\rho) = b'/\rho,$$

thus $b \, \rho \, b'$. $\square$

The poset $\langle E_{ua}(\mathbf{A}), \subseteq\rangle$ is an algebraic closure system on $A \times A$ (see [27, Lemma 4.2]). The smallest relation from $E_{ua}(\mathbf{A})$ is called *the fundamental relation of* $\mathbf{A}$ and it is denoted by $\alpha^*_{\langle A,F\rangle}$ or $\alpha^*_{\mathbf{A}}$. We denote by $\alpha^{\mathbf{A}}$ the closure operator corresponding to $\langle E_{ua}(\mathbf{A}), \subseteq\rangle$. The smallest relation from $E_{ua}(\mathbf{A})$ which contains a relation $\theta \subseteq A \times A$ is

$$\alpha^{\mathbf{A}}(\theta) = \bigcap \{\rho \in E_{ua}(\mathbf{A}) \mid \theta \subseteq \rho\}.$$

This relation can be characterized as the congruence of a universal algebra generated by a given relation (see [18, Theorem 10.4]).

Theorem 4. *Let $\mathbf{A} = \langle A, F\rangle$ be a multialgebra of type $\mathcal{F}$ and $\theta \subseteq A \times A$. The relation $\alpha^{\mathbf{A}}(\theta)$ is defined as follows: $\langle x, y\rangle \in \alpha^{\mathbf{A}}(\theta)$ if and only if there exist $k \in \mathbb{N}^*$, a sequence $x = t_0, t_1, \ldots, t_k = y$ of elements from A, some pairs $\langle b_1, c_1\rangle, \ldots, \langle b_k, c_k\rangle \in \theta$ and some unary polynomial functions $p_1, \ldots, p_k$ from $\mathrm{Pol}_1^A(\mathbf{P}^*(\mathbf{A}))$ such that for all $i \in \{1, \ldots, k\}$,*

$$\langle t_{i-1}, t_i\rangle \in p_i(b_i) \times p_i(c_i) \quad \text{or} \quad \langle t_i, t_{i-1}\rangle \in p_i(b_i) \times p_i(c_i).$$

Proof. Let η be the relation defined as follows

$$x\,\eta\,y \quad \Leftrightarrow \quad \exists \langle b, c\rangle \in \theta,\ \exists p \in \mathrm{Pol}_1^A\big(\mathbf{P}^*(\mathbf{A})\big): \quad \langle x, y\rangle \in \big(p(b) \times p(c)\big) \cup \big(p(c) \times p(b)\big).$$

Taking $p = c_a^1$ for some $a \in A$, we deduce that η is reflexive. Obviously, η is symmetric, hence its transitive closure η^* is an equivalence relation on A. The relation η^* is the relation in the statement and we will show that

$$\alpha^{\mathbf{A}}(\theta) = \eta^*. \tag{3}$$

In order to prove that $\eta^* \in E_{ua}(\mathbf{A})$, we will use the condition (f) from Remark 2. For this, let us take $p \in \mathrm{Pol}_1^A(\mathbf{P}^*(\mathbf{A}))$ and $x, y \in A$ such that $x\,\eta^*\,y$. It means that there exist $k \in \mathbb{N}^*$, a sequence $x = t_0, t_1, \ldots, t_k = y$, some pairs $\langle b_i, c_i\rangle \in \theta$ and some $p_i \in \mathrm{Pol}_1^A(\mathbf{P}^*(\mathbf{A}))$ ($i \in \{1, \ldots, k\}$) such that

$$\langle t_{i-1}, t_i\rangle \in \big(p_i(b_i) \times p_i(c_i)\big) \cup \big(p_i(c_i) \times p_i(b_i)\big), \quad \forall i \in \{1, \ldots, k\}.$$

We deduce that for all $i \in \{1, \ldots, k\}$,

$$p(t_{i-1}) \times p(t_i) \subseteq \big(p\big(p_i(b_i)\big) \times p\big(p_i(c_i)\big)\big) \cup \big(p\big(p_i(c_i)\big) \times p\big(p_i(b_i)\big)\big).$$

But $p'_i = p \circ p_i \in \mathrm{Pol}_1^A(\mathbf{P}^*(\mathbf{A}))$, hence for all $i \in \{1, \ldots, k\}$,

$$p(t_{i-1}) \times p(t_i) \subseteq \big(p'_i(b_i) \times p'_i(c_i)\big) \cup \big(p'_i(c_i) \times p'_i(b_i)\big) \subseteq \eta^*.$$

From the transitivity of η^* it results $p(x) \times p(y) = p(t_0) \times p(t_k) \subseteq \eta^*$ or, equivalently, $p(x)\,\overline{\overline{\eta^*}}\,p(y)$.

If we consider $p = e_1^1$, we obtain the inclusion $\theta \subseteq \eta$, therefore η^* is an equivalence from $E_{ua}(\mathbf{A})$ which contains θ. The relation η^* is the smallest relation from $E_{ua}(\mathbf{A})$ which contains θ since, if we consider $\rho \in E_{ua}(\mathbf{A})$ with $\theta \subseteq \rho$, we have $\eta \subseteq \rho$. This is a consequence of the following implication

$$\langle b, c\rangle \in \theta, \quad p \in \mathrm{Pol}_1^A\big(\mathbf{P}^*(\mathbf{A})\big) \quad \Rightarrow \quad p(b) \times p(c) \subseteq \rho \tag{4}$$

which can be verified by using the inductive construction of the polynomial functions from $\mathrm{Pol}_1^A(\mathbf{P}^*(\mathbf{A}))$. Indeed, if $a \in A$ and $p = c_a^1$ then $p(b) = p(c) = a$ and $\langle a, a\rangle \in \rho$. If $p = e_1^1$ then $p(b) = b$, $p(c) = c$ and $\langle b, c\rangle \in \theta \subseteq \rho$. Assume that the statement is true for $p_1, \ldots, p_n$, consider $f \in \mathcal{F}$ an n-ary symbol and $p = f^{\mathbf{A}}(p_1, \ldots, p_n)$. If

$$x \in p(b) = f^{\mathbf{A}}(p_1, \ldots, p_n)(b) = f^{\mathbf{A}}\big(p_1(b), \ldots, p_n(b)\big)$$

and

$$y \in p(c) = f^{\mathbf{A}}(p_1, \ldots, p_n)(c) = f^{\mathbf{A}}\big(p_1(c), \ldots, p_n(c)\big)$$

there exist $x_i \in p_i(b)$, $y_i \in p_i(c)$ $(i \in \{1, \ldots, n\})$ such that $x \in f^{\mathbf{A}}(x_1, \ldots, x_n)$ and $y \in f^{\mathbf{A}}(y_1, \ldots, y_n)$. But $p_i(b) \times p_i(c) \subseteq \rho$, hence $x_i \,\rho\, y_i$ for all $i = 1, \ldots, n$. Using Proposition 2, we obtain $x \,\rho\, y$. This completes the proof of (4), hence also the proof of (3). □

For some considerations regarding the behaviour of the homomorphisms with respect to term functions, in some cases it is more convenient to use images of term functions instead of the images of unary polynomial functions from the above characterization theorem. This can be done since the algebraic functions from [18] are the polynomial functions from [4], so we have:

Lemma 5. *Let $\mathbf{A} = \langle A, F\rangle$ be a multialgebra of type $\mathcal{F}$. For any unary polynomial function $p \in \mathrm{Pol}_1^A(\mathbf{P}^*(\mathbf{A}))$, there exist $m \in \mathbb{N}$, $m \geqslant 1$, $b_1, \ldots, b_m \in A$ and a term function $t \in \mathrm{Clo}_m(\mathbf{P}^*(\mathbf{A}))$ such that*

$$p^{\mathbf{P}^*(\mathbf{A})}(A_1) = t^{\mathbf{P}^*(\mathbf{A})}(A_1, b_2, \ldots, b_m), \quad \forall A_1 \in P^*(A).$$

If I is a set and, for any $i \in I$, q_i, r_i are m_i-ary terms of type $\mathcal{F}$ $(m_i \in \mathbb{N}^*)$, then the smallest equivalence relation of the multialgebra $\mathbf{A} = \langle A, F\rangle$ of type $\mathcal{F}$ for which the factor multialgebra is a universal algebra satisfying all the identities from $\mathcal{I} = \{q_i = r_i \mid i \in I\}$ is the relation $\alpha(R_{\mathcal{I}}^{\mathbf{A}})$, where

$$R_{\mathcal{I}}^{\mathbf{A}} = \bigcup \big\{ q_i^{\mathbf{P}^*(\mathbf{A})}(a_1, \ldots, a_{m_i}) \times r_i^{\mathbf{P}^*(\mathbf{A})}(a_1, \ldots, a_{m_i}) \,\big|\, a_1, \ldots, a_{m_i} \in A,\ i \in I \big\}.$$

Looking at the Introduction of [2], it is easy to notice that the relations $\alpha(R_{\mathcal{I}}^{\mathbf{A}})$ are generalizations of the verbal congruence relations of universal algebras. Thus one can see these relations as the smallest equivalence relations of the multialgebra $\mathbf{A}$ for which the factor multialgebras are in some given varieties of universal algebras of type $\mathcal{F}$.

We will write α instead of $\alpha^{\mathbf{A}}$ and $R_{\mathcal{I}}$ instead of $R_{\mathcal{I}}^{\mathbf{A}}$ when we do not have to emphasize the multialgebra we are dealing with. Also, if I is a one-element set and the considered identity is $q = r$, we will write R_{qr} instead of $R_{\mathcal{I}}$ and we will denote $\alpha(R_{qr})$ by α_{qr}^*.

Remark 4. If the identities from $\mathcal{I}$ are satisfied at least in a weak manner on $\mathbf{A}$ then $\alpha_{\mathbf{A}}^* = \alpha(R_{\mathcal{I}})$. In particular, for any terms q, r for which the weak identity $q \cap r \neq \emptyset$ holds on $\mathbf{A}$, we have $\alpha_{\mathbf{A}}^* = \alpha_{qr}^*$.

By applying Theorem 4 to $R = R_{\mathcal{I}}$, we obtain [29, Theorem 7] which can be restated according to Lemma 5 as follows:

Theorem 6. *(See [29, Theorem 7].) Let $I \neq \emptyset$ be a set and for any $i \in I$, let q_i, r_i be m_i-ary terms of type $\mathcal{F}$, let $\mathcal{I} = \{q_i = r_i \mid i \in I\}$, and let $\mathbf{A} = \langle A, F\rangle$ be a multialgebra of type $\mathcal{F}$. Let $\alpha_{\mathcal{I}} \subseteq A \times A$ be the relation defined as follows*:

$$\begin{aligned} x\,\alpha_{\mathcal{I}}\,y \quad \Leftrightarrow \quad & \exists i\in I,\ \exists n_i\in\mathbb{N}^*,\ \exists p_i\in \mathrm{Clo}_{n_i}\big(\mathbf{P}^*(\mathbf{A})\big),\ \exists a_1^i,\dots,a_{m_i}^i,b_2^i,\dots,b_{n_i}^i\in A\colon \\ & x\in p_i^{\mathbf{P}^*(\mathbf{A})}\big(q_i^{\mathbf{P}^*(\mathbf{A})}\big(a_1^i,\dots,a_{m_i}^i\big),b_2^i,\dots,b_{n_i}^i\big), \\ & y\in p_i^{\mathbf{P}^*(\mathbf{A})}\big(r_i^{\mathbf{P}^*(\mathbf{A})}\big(a_1^i,\dots,a_{m_i}^i\big),b_2^i,\dots,b_{n_i}^i\big), \\ & \quad \textit{or} \\ & y\in p_i^{\mathbf{P}^*(\mathbf{A})}\big(q_i^{\mathbf{P}^*(\mathbf{A})}\big(a_1^i,\dots,a_{m_i}^i\big),b_2^i,\dots,b_{n_i}^i\big), \\ & x\in p_i^{\mathbf{P}^*(\mathbf{A})}\big(r_i^{\mathbf{P}^*(\mathbf{A})}\big(a_1^i,\dots,a_{m_i}^i\big),b_2^i,\dots,b_{n_i}^i\big). \end{aligned}$$

The transitive closure $\alpha_{\mathcal{I}}^*$ of $\alpha_{\mathcal{I}}$ is the smallest equivalence relation on $\mathbf{A}$ for which the factor multialgebra is a universal algebra satisfying all the identities from $\mathcal{I}$, i.e. $\alpha_{\mathcal{I}}^*=\alpha(R_{\mathcal{I}})$.

Applying the above theorem to the case when $\mathcal{I}$ consists of only one identity $q=r$, we obtain another statement for [27, Theorem 4.4]. Taking $I\neq\emptyset$, we did not lose the fundamental relation from our study, since for any variable $\mathbf{x}$, $\alpha_{\mathbf{A}}^*=\alpha_{\mathbf{xx}}^*=\alpha(R_{\mathbf{xx}})$. Thus, we also have:

Corollary 7. *(See [28, Corollary 11].) Let $\mathbf{A}=\langle A,F\rangle$ be a multialgebra of type $\mathcal{F}$. The fundamental relation $\alpha_{\mathbf{A}}^*$ of the multialgebra $\mathbf{A}$ is the transitive closure of the relation $\alpha_{\mathbf{A}}$ defined as follows*

$$x\,\alpha_{\mathbf{A}}\,y \quad \Leftrightarrow \quad \exists n\in\mathbb{N},\ \exists p\in \mathrm{Clo}_n\big(\mathbf{P}^*(\mathbf{A})\big),\ \exists a_1,\dots,a_n\in A\colon\quad x,y\in p^{\mathbf{P}^*(\mathbf{A})}(a_1,\dots,a_n).$$

Example 1. A multialgebra $\langle H,\cdot\rangle$ with one binary associative multioperation is called *semihypergroup*. A semihypergroup $\langle H,\cdot\rangle$ satisfying the *reproducibility condition* (i.e. $a\cdot H=H\cdot a=H$, for any $a\in H$) is called *hypergroup*. Applying Corollary 7 to the (semi)hypergroup $\langle H,\cdot\rangle$ (and using Remark 3) one obtains the well-known characterization of the fundamental relation β^* as the transitive closure of the relation $\beta=\bigcup_{n\in\mathbb{N}^*}\beta_n$, where

$$x\,\beta_n\,y \quad \Leftrightarrow \quad \exists a_1,\dots,\ a_n\in H\colon\quad x,y\in a_1\cdots a_n$$

(see, for instance, [15, Section 2.2]). In particular, if $\langle H,\cdot\rangle$ is a hypergroup, then β is already transitive, hence $\beta=\beta^*$.

Remark 5. In the hypergroup $\langle H,\cdot\rangle$, the reproducibility condition defines two binary multioperations $/,\backslash : H\times H\to P^*(H)$ as follows

$$b/a=\{x\in H \mid b\in x\cdot a\},\qquad a\backslash b=\{x\in H \mid b\in a\cdot x\}.$$

Thus, the hypergroup $\langle H,\cdot\rangle$ can be seen as a multialgebra $\langle H,\cdot,/,\backslash\rangle$ with three binary multioperations. The fundamental relations of $\langle H,\cdot\rangle$ and $\langle H,\cdot,/,\backslash\rangle$ coincide since $E_{ua}(\langle H,\cdot\rangle)=E_{ua}(\langle H,\cdot,/,\backslash\rangle)$ (see [28, Remark 15]). For the same reason, $\langle H/\rho,\cdot\rangle$ is a group for any $\rho\in E_{ua}(\langle H,\cdot\rangle)$.

Example 2. (See [27, Example 5].) In Theorem 6, let us take the multialgebra $\mathbf{A}$ to be a semihypergroup $\langle A,\cdot\rangle$ and $\mathcal{I}=\{\mathbf{x}_1\cdot\mathbf{x}_2=\mathbf{x}_2\cdot\mathbf{x}_1\}$. One obtains $\alpha_{\mathcal{I}}^*$ to be the transitive closure of the union of all pairs (x,y) for which $x=y$ or

$$\begin{aligned} \exists n\in\mathbb{N},\ n\geqslant 2,\ \exists a_1,\dots,a_n\in A,\ \exists i\in\{1,\dots,n-1\}\colon\quad & x\in a_1\cdots a_{i-1}a_ia_{i+1}a_{i+2}\cdots a_n, \\ & y\in a_1\cdots a_{i-1}a_{i+1}a_ia_{i+2}\cdots a_n. \end{aligned}$$

Since the transpositions $(1,2),(2,3),\ldots,(n-1,n)$ generate the symmetric group S_n $(n \geqslant 2)$, $\alpha^*_{\mathcal{I}}$ is equal to the relation γ^* defined in [14, Section 1] as the transitive closure of the relation

$$x\gamma y \quad \Leftrightarrow \quad \exists n \in \mathbb{N}^*,\ \exists a_1,\ldots,a_n \in A,\ \exists \sigma \in S_n\colon\ \ x \in \prod_{i=1}^{n} a_i, \qquad y \in \prod_{i=1}^{n} a_{\sigma(i)}.$$

If $\langle A,\cdot\rangle$ is a hypergroup, γ is already transitive (see [14, Theorem 3.3]), so $\alpha^*_{\mathcal{I}} = \gamma$. Of course, if $\langle A,\cdot\rangle$ is a group, then $\alpha^*_{\mathcal{I}}$ is the congruence determined by its commutator subgroup, thus the abelianizations of groups are particular cases of the factorizations we will further investigate.

Example 3. The *hyperrings* from [13] are multialgebras $\langle R,+,\cdot\rangle$ for which $\langle R,+\rangle$ is a hypergroup, $\langle R,\cdot\rangle$ is a semihypergroup and the multioperation $\cdot$ is distributive with respect to the multioperation $+$. If $\mathbf{A}$ is a hyperring $\langle A,+,\cdot\rangle$ and $\mathcal{I} = \{\mathbf{x}_1+\mathbf{x}_2 = \mathbf{x}_2+\mathbf{x}_1,\ \mathbf{x}_1\cdot\mathbf{x}_2 = \mathbf{x}_2\cdot\mathbf{x}_1\}$, then $\alpha^*_{\mathcal{I}}$ is the smallest equivalence relation on A for which the factor multialgebra is a commutative ring and it coincides with the relation defined in [13, Definition 1] as the transitive closure of the relation consisting of all the pairs $\langle x,y\rangle$ for which there exist $n,k_1,\ldots,k_n \in \mathbb{N}^*$, a permutation $\tau \in S_n$, $x_{i1},\ldots,x_{ik_i} \in A$, and $\sigma_i \in S_{k_i}$ $(i=1,\ldots,n)$ such that

$$x \in \sum_{i=1}^{n}\left(\prod_{j=1}^{k_i} x_{ij}\right) \quad \text{and} \quad y \in \sum_{i=1}^{n}\left(\prod_{j=1}^{k_i} x_{\tau(i)\sigma_{\tau(i)}(j)}\right).$$

For details concerning the equality of these relations, see [29, Section 4]. Obviously, if $\langle A,+,\cdot\rangle$ is a ring, then $\alpha^*_{\mathcal{I}}$ is the congruence determined by its commutator ideal.

Example 4. Let $n \in \mathbb{N}$, $n \geqslant 2$. A multialgebra $\langle H,f\rangle$ with one n-ary associative multioperation is called *n-semihypergroup*. If f is an operation, then $\langle H,f\rangle$ is an *n-semigroup*. If we use the associativity of f, it is easy to notice that the relation from Corollary 7 will provide in this case the relation β^* from [20, Section 4]. If we take $\mathcal{I} = \{f(\mathbf{x}_1,\ldots,\mathbf{x}_n) = f(\mathbf{x}_{\sigma(1)},\ldots,\mathbf{x}_{\sigma(n)}) \mid \sigma \in S_n\}$, then $\alpha^*_{\mathcal{I}}$ from Theorem 6 is the smallest equivalence relation of H for which the factor multialgebra is a commutative n-semigroup. The equality of $\alpha^*_{\mathcal{I}}$ with the relation $\widehat{\gamma}$ from [12, Definition 4.4] follows as in the proof of [29, Theorem 14].

In the proof of [29, Theorem 18] we showed that the following result holds.

Lemma 8. *Let $\mathcal{I}$ be a set of identities and let $\mathbf{A}$ be an $\mathcal{F}$-multialgebra. For any universal algebra $\mathbf{B}$ of type $\mathcal{F}$ which satisfies the identities from $\mathcal{I}$ and any multialgebra homomorphism $h : A \to B$ there exists a unique universal algebra homomorphism $\bar{h}$ which makes the following diagram commutative*:

$$\begin{tikzcd} A \arrow[r, "\pi_{\alpha^*_{\mathcal{I}}}"] \arrow[dr, "h"'] & A/\alpha^*_{\mathcal{I}} \arrow[d, dotted, "\bar{h}"] \\ & B \end{tikzcd}$$

Corollary 9. *If in the above lemma $\mathbf{B} = \mathbf{A}/\alpha^*_{\mathcal{I}}$ and $h = \pi_{\alpha^*_{\mathcal{I}}}$, then $\bar{h}$ is the identity homomorphism $1_{A/\alpha^*_{\mathcal{I}}}$ of $\mathbf{A}/\alpha^*_{\mathcal{I}}$.*

Remark 6. The above property was used to prove that the variety $M(\mathcal{I})$ of the $\mathcal{F}$-algebras which satisfy all the identities from $\mathcal{I}$ is a reflective subcategory of the category of $\mathcal{F}$-multialgebras (see

[29, Theorem 18]). Consequently, the factorization of $\mathcal{F}$-multialgebras modulo $\alpha^*_{\mathcal{I}}$ provides a functor which is a reflector for $M(\mathcal{I})$.

Next, we will consider two disjoint nonempty sets I, J, the m_i-ary terms q_i, r_i of type $\mathcal{F}$ ($i \in I \cup J$), and two sets of identities

$$\mathcal{I} = \{q_i = r_i \mid i \in I\} \quad \text{and} \quad \mathcal{J} = \{q_i = r_i \mid i \in J\}.$$

For a multialgebra $\mathbf{A} = \langle A, F\rangle$ of type $\mathcal{F}$, we will denote by

$$\alpha^*_{\mathcal{I}\cup\mathcal{J}} = \alpha^{\mathbf{A}}\big(R^{\mathbf{A}}_{\mathcal{I}} \cup R^{\mathbf{A}}_{\mathcal{J}}\big), \qquad \alpha^*_{\mathcal{I}} = \alpha^{\mathbf{A}}\big(R^{\mathbf{A}}_{\mathcal{I}}\big), \qquad \alpha^*_{\mathcal{J}} = \alpha^{\mathbf{A}}\big(R^{\mathbf{A}}_{\mathcal{J}}\big),$$

the smallest equivalence relations on $\mathbf{A}$ for which the factor multialgebras are universal algebras satisfying all the identities from $\mathcal{I}\cup\mathcal{J}$, $\mathcal{I}$, and $\mathcal{J}$, respectively. Obviously, $\alpha^*_{\mathcal{I}} \subseteq \alpha^*_{\mathcal{I}\cup\mathcal{J}}$ and $\alpha^*_{\mathcal{J}} \subseteq \alpha^*_{\mathcal{I}\cup\mathcal{J}}$.

The relation $R^{\mathbf{A}/\alpha^*_{\mathcal{J}}}_{\mathcal{I}}$ consists of all the pairs

$$\big\langle q_i^{\mathbf{A}/\alpha^*_{\mathcal{J}}}\big(a_1/\alpha^*_{\mathcal{J}}, \ldots, a_{m_i}/\alpha^*_{\mathcal{J}}\big), r_i^{\mathbf{A}/\alpha^*_{\mathcal{J}}}\big(a_1/\alpha^*_{\mathcal{J}}, \ldots, a_{m_i}/\alpha^*_{\mathcal{J}}\big)\big\rangle$$

with $a_1, \ldots, a_{m_i} \in A$, $i \in I$, and $\alpha^{\mathbf{A}/\alpha^*_{\mathcal{J}}}(R^{\mathbf{A}/\alpha^*_{\mathcal{J}}}_{\mathcal{I}})$ is the congruence of $\mathbf{A}/\alpha^*_{\mathcal{J}}$ generated by $R^{\mathbf{A}/\alpha^*_{\mathcal{J}}}_{\mathcal{I}}$ or, equivalently, the smallest congruence on $\mathbf{A}/\alpha^*_{\mathcal{J}}$ for which the factor algebra satisfies the identities from $\mathcal{J}$. Let us denote

$$\underline{\alpha}^*_{\mathcal{I}} = \alpha^{\mathbf{A}/\alpha^*_{\mathcal{J}}}\big(R^{\mathbf{A}/\alpha^*_{\mathcal{J}}}_{\mathcal{I}}\big), \qquad \underline{\alpha}^*_{\mathcal{J}} = \alpha^{\mathbf{A}/\alpha^*_{\mathcal{I}}}\big(R^{\mathbf{A}/\alpha^*_{\mathcal{I}}}_{\mathcal{J}}\big),$$

and let us consider that the presence or the absence of the superscript * has the same meaning as in Theorem 6 (and its corollary).

Theorem 10. *If $\mathbf{A} = \langle A, F\rangle$ is a multialgebra of type $\mathcal{F}$ then*

$$\mathbf{A}/\alpha^*_{\mathcal{I}\cup\mathcal{J}} \cong \big(\mathbf{A}/\alpha^*_{\mathcal{J}}\big)/\underline{\alpha}^*_{\mathcal{I}} \cong \big(\mathbf{A}/\alpha^*_{\mathcal{I}}\big)/\underline{\alpha}^*_{\mathcal{J}}.$$

Proof. Let $\pi_{\alpha^*_{\mathcal{I}\cup\mathcal{J}}}$, $\pi_{\alpha^*_{\mathcal{J}}}$ be the canonical projections of $\mathbf{A}$ determined by $\alpha^*_{\mathcal{I}\cup\mathcal{J}}$ and $\alpha^*_{\mathcal{J}}$, respectively, and let $\pi_{\underline{\alpha}^*_{\mathcal{I}}}$ be the canonical projection of $\mathbf{A}/\alpha^*_{\mathcal{J}}$ determined by $\underline{\alpha}^*_{\mathcal{I}}$. Applying Lemma 8 for multialgebra $\mathbf{A}$, the set of identities $\mathcal{I}\cup\mathcal{J}$ and the universal algebra $(\mathbf{A}/\alpha^*_{\mathcal{J}})/\underline{\alpha}^*_{\mathcal{I}}$, one gets the existence of a unique universal algebra homomorphism φ which makes the left diagram below commutative (defined by the correspondence $a/\alpha^*_{\mathcal{I}\cup\mathcal{J}} \mapsto (a/\alpha^*_{\mathcal{J}})/\underline{\alpha}^*_{\mathcal{I}}$).

$$\begin{tikzcd}
A \arrow[r, "\pi_{\alpha^*_{\mathcal{I}\cup\mathcal{J}}}"] \arrow[dr, "\pi_{\underline{\alpha}^*_{\mathcal{I}}}\pi_{\alpha^*_{\mathcal{J}}}"'] & A/\alpha^*_{\mathcal{I}\cup\mathcal{J}} \arrow[d, dotted, "\varphi"] \\
& (A/\alpha^*_{\mathcal{J}})/\underline{\alpha}^*_{\mathcal{I}}
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\pi_{\alpha^*_{\mathcal{J}}}"] \arrow[drr, "\pi_{\alpha^*_{\mathcal{I}\cup\mathcal{J}}}"'] & A/\alpha^*_{\mathcal{J}} \arrow[r, "\pi_{\underline{\alpha}^*_{\mathcal{I}}}"] \arrow[dr, dotted] & (A/\alpha^*_{\mathcal{J}})/\underline{\alpha}^*_{\mathcal{I}} \arrow[d, dotted, "\psi"] \\
& & A/\alpha^*_{\mathcal{I}\cup\mathcal{J}}
\end{tikzcd}$$

Applying successively Lemma 8 for the multialgebra $\mathbf{A}$, the set of identities $\mathcal{J}$ and the universal algebra $\mathbf{A}/\alpha^*_{\mathcal{I}\cup\mathcal{J}}$, and then, for the (multi)algebra $\mathbf{A}/\alpha^*_{\mathcal{J}}$, the set of identities $\mathcal{I}$ and the algebra $\mathbf{A}/\alpha^*_{\mathcal{I}\cup\mathcal{J}}$, we obtain the existence of a unique algebra homomorphism ψ which makes the right diagram above commutative. From Corollary 9, we deduce that $\varphi \circ \psi = 1_{(A/\alpha^*_{\mathcal{J}})/\underline{\alpha}^*_{\mathcal{I}}}$ and $\psi \circ \varphi = 1_{A/\alpha^*_{\mathcal{I}\cup\mathcal{J}}}$, thus φ is a universal algebra isomorphisms (and ψ is its inverse).

The isomorphism $\mathbf{A}/\alpha^*_{\mathcal{I}\cup\mathcal{J}} \cong (\mathbf{A}/\alpha^*_{\mathcal{I}})/\underline{\alpha}^*_{\mathcal{J}}$ can be proved in a similar way. $\square$

If, in the above theorem, we take $\mathcal{J} = \{\mathbf{x} = \mathbf{x}\}$, then $\alpha^*_{\mathcal{J}} = \alpha^*_{\mathbf{A}}$ is the fundamental relation of the multialgebra $\mathbf{A}$. Clearly, $R_{\mathcal{J}}$ is the equality relation on A. This implies $R_{\mathcal{J}} \subseteq \alpha^*_{\mathcal{I}}$, hence $\alpha^*_{\mathcal{I} \cup \mathcal{J}} = \alpha^*_{\mathcal{I}}$, and we have:

Corollary 11. *If $\mathbf{A} = \langle A, F\rangle$ is an $\mathcal{F}$-multialgebra, then $\mathbf{A}/\alpha^*_{\mathcal{I}} \cong (\mathbf{A}/\alpha^*_{\mathbf{A}})/\underline{\alpha}^*_{\mathcal{I}}$.*

Considering that the first factor multialgebra is the fundamental algebra of $\mathbf{A}$ and the second one is the factor of a universal algebra modulo a verbal congruence relation, one can say that we already made the construction of the universal algebra $\mathbf{A}/\alpha^*_{\mathcal{I}}$ less difficult. Yet, the construction of the fundamental algebra of $\mathbf{A}$ is not necessarily easy. But, as we will see in the next section, it also can be done by successive factorizations by fundamental relations of multialgebras with less multioperations, the construction of which is, at least theoretically, easier.

4. Subdistributivity, factor multialgebras and universal algebras

Let $\mathcal{F}$ and $\mathcal{G}$ be two sets of function symbols such that a nonnegative arity is assigned to each symbol from $\mathcal{F} \cup \mathcal{G}$. In this section, we will prove some interesting results concerning multialgebras $\mathbf{A} = \langle A, F, G\rangle$ of type $\mathcal{F} \cup \mathcal{G}$ for which every multioperation $g \in G$ associated to any symbol from $\mathcal{G}$ is subdistributive with respect to every multioperation $f \in F$ associated to any symbol from $\mathcal{F}$.

For a multialgebra $\mathbf{A} = \langle A, F, G\rangle$ of type $\mathcal{F} \cup \mathcal{G}$, we will say that the n-ary multioperation $g \in G$ is *subdistributive* with respect to the m-ary multioperation $f \in F$ if for any $a_1, \dots, a_m, b_1, \dots, b_n \in A$, and any $i \in \{1, \dots, n\}$,

$$\begin{aligned} &g\bigl(b_1, \dots, b_{i-1}, f(a_1, \dots, a_m), b_{i+1}, \dots, b_n\bigr) \\ &\quad \subseteq f\bigl(g(b_1, \dots, b_{i-1}, a_1, b_{i+1}, \dots, b_n), \dots, g(b_1, \dots, b_{i-1}, a_m, b_{i+1}, \dots, b_n)\bigr). \end{aligned}$$

Thus, for an n-ary symbol $g \in \mathcal{G}$, an m-ary symbol $f \in \mathcal{F}$ and the variables $\mathbf{x}_1, \dots, \mathbf{x}_m, \mathbf{y}_1, \dots, \mathbf{y}_n$, *the subdistributivity* of g with respect to f consists of n particular weak identities which can be written formally

$$\begin{aligned} &g\bigl(\mathbf{y}_1, \dots, \mathbf{y}_{i-1}, f(\mathbf{x}_1, \dots, \mathbf{x}_m), \mathbf{y}_{i+1}, \dots, \mathbf{y}_n\bigr) \\ &\quad \subseteq f\bigl(g(\mathbf{y}_1, \dots, \mathbf{y}_{i-1}, \mathbf{x}_1, \mathbf{y}_{i+1}, \dots, \mathbf{y}_n), \dots, g(\mathbf{y}_1, \dots, \mathbf{y}_{i-1}, \mathbf{x}_m, \mathbf{y}_{i+1}, \dots, \mathbf{y}_n)\bigr), \end{aligned}$$

for all $i \in \{1, \dots, n\}$. Of course, if one replaces $\subseteq$ in the definition of subdistributivity by $=$, one gets the definition of the *distributivity* (of g with respect to f). The above defined subdistributivity of $g \in G$ with respect to $f \in F$ is inherited by $\mathbf{P}^*(\mathbf{A})$ and it is not difficult to prove the following result:

Lemma 12. *Let $\mathbf{A} = \langle A, F, G\rangle$ be a multialgebra of type $\mathcal{F} \cup \mathcal{G}$. If the n-ary multioperation $g \in G$ is subdistributive with respect to each multioperation $f \in F$ and p is an m-ary term function of type $\mathcal{F}$ on $P^*(A)$ $(m, n \in \mathbb{N})$, then*

$$\begin{aligned} &g\bigl(B_1, \dots, B_{i-1}, p(A_1, \dots, A_m), B_{i+1}, \dots, B_n\bigr) \\ &\quad \subseteq p\bigl(g(B_1, \dots, B_{i-1}, A_1, B_{i+1}, \dots, B_n), \dots, g(B_1, \dots, B_{i-1}, A_m, B_{i+1}, \dots, B_n)\bigr), \end{aligned}$$

for any $i \in \{1, \dots, n\}$, and any $A_1, \dots, A_m, B_1, \dots, B_n \in P^*(A)$.

Remark 7. If the n-ary multioperation $g \in G$ is distributive with respect to each multioperation $f \in F$, and p is an m-ary term function of type $\mathcal{F}$ on $\mathbf{P}^*(\mathbf{A})$ $(m, n \in \mathbb{N})$, $A_1, \dots, A_m \in P^*(A)$, $b_1, \dots, b_n \in A$, and $i \in \{1, \dots, n\}$, then

$$g\big(b_1,\dots,b_{i-1},p(A_1,\dots,A_m),b_{i+1},\dots,b_n\big)$$
$$=p\big(g(b_1,\dots,b_{i-1},A_1,b_{i+1},\dots,b_n),\dots,g(b_1,\dots,b_{i-1},A_m,b_{i+1},\dots,b_n)\big).$$

But, if g is distributive with respect to f, $A_1,\dots,A_m,B_1,\dots,B_n\in P^*(A)$, and $i\in\{1,\dots,n\}$, we do not necessarily have

$$g\big(B_1,\dots,B_{i-1},f(A_1,\dots,A_m),B_{i+1},\dots,B_n\big)$$
$$=f\big(g(B_1,\dots,B_{i-1},A_1,B_{i+1},\dots,B_n),\dots,g(B_1,\dots,B_{i-1},A_m,B_{i+1},\dots,B_n)\big).$$

This is obvious since in the power algebra of the ring of integers $(\mathbb{Z},+,\cdot)$,

$$\{1,2\}\cdot(1+1)=\{1,2\}\cdot 2=\{2,4\}\subsetneq\{2,3,4\}=\{1,2\}+\{1,2\}=\{1,2\}\cdot 1+\{1,2\}\cdot 1.$$

The identities satisfied in the variety determined by the globals of the universal algebras from a given variety are determined in [19] and [22] contains a multialgebra theoretical version of this problem. These results can be useful for multialgebra theorists, since dealing with subsets instead of elements is a frequent situation in multialgebra theory.

We organized the next part of the paper so as to prove that the situation in [13, Theorem 6] will occur for any hyperstructure with two multioperations such that one of them is subdistributive with respect to the other one. We do not claim that subdistributivity is a minimal condition for a hyperstructure with two multioperations to satisfy a [13, Theorem 6]-like property. Finding a minimal condition for this to happen can be an open problem, but we do not approach it here. As we mention in Introduction, one of the first hints which challenged us to approach the problem as we do was provided by the following example.

Example 5. Let $R=\{e,v,w,x,y,z\}$ be a 6-elements set endowed with two binary operations $+$ and $\cdot$ such that $\langle R,+\rangle$ is isomorphic to the group $\langle S_3,\circ\rangle$ of the permutations of a 3-elements set and $\langle R,\cdot\rangle$ is isomorphic to the 6-elements cyclic group $\langle\mathbb{Z}_6,+\rangle$ as follows:

$+$	e	v	w	x	y	z
e	e	v	w	x	y	z
v	v	w	e	y	z	x
w	w	e	v	z	x	y
x	x	z	y	e	w	v
y	y	x	z	v	e	w
z	z	y	x	w	v	e

$\cdot$	e	v	w	x	y	z
e	e	v	w	x	y	z
v	v	w	x	y	z	e
w	w	x	y	z	e	v
x	x	y	z	e	v	w
y	y	z	e	v	w	x
z	z	e	v	w	x	y

The universal algebra $\langle R,+,\cdot\rangle$ satisfies all the identities from the hyperring definition from [13], except for (sub)distributivity, because

$$x(y+z)=x\cdot w=z\neq e=v+w=xy+xz.$$

Using the notations from [13, Theorem 6], the relation γ^* is the equality relation on R, since $\langle R,\cdot\rangle$ is already an Abelian group, hence $\gamma^*_{\uplus}$ is the equivalence relation $\rho_{\{e,v,w\}}$ determined by the derived subgroup of the group $\langle R,+\rangle$, so

$$\big\langle\big(R/\gamma^*\big)/\gamma^*_{\uplus},+,\cdot\big\rangle=\langle R/\rho_{\{e,v,w\}},+,\cdot\rangle.$$

It is known that the congruence relations of a group are the equivalence relations determined by its normal subgroups, thus $\langle (R/\gamma^*)/\gamma^*_{\uplus}, +, \cdot\rangle$ is not a universal algebra, since $\{e, v, w\}$ is not even a subgroup of $\langle R/\gamma^*, \cdot\rangle = \langle R, \cdot\rangle$.

Lemma 13. *Let $\mathbf{A} = \langle A, F, G\rangle$ be a multialgebra of type $\mathcal{F} \cup \mathcal{G}$. If the multioperation $g \in G$ is subdistributive with respect to the multioperation $f \in F$ and $\rho \in E_{ua}(\langle A, g\rangle)$, then, in the factor multialgebra $\mathbf{A}/\rho$, the operation $g^{\mathbf{A}/\rho}$ is subdistributive with respect to the multioperation $f^{\mathbf{A}/\rho}$.*

Proof. If the arity of f is m and the arity of g is n ($m, n \in \mathbb{N}$), we have to show that for any $i \in \{1, \dots, n\}$ and any $a_1, \dots, a_m, b_1, \dots, b_n \in A$ we have

$$\begin{aligned} &g^{\mathbf{A}/\rho}\big(b_1/\rho, \dots, b_{i-1}/\rho, f^{\mathbf{A}/\rho}(a_1/\rho, \dots, a_m/\rho), b_{i+1}/\rho, \dots, b_n/\rho\big) \\ &\quad\subseteq f^{\mathbf{A}/\rho}\big(g^{\mathbf{A}/\rho}(b_1/\rho, \dots, b_{i-1}/\rho, a_1/\rho, b_{i+1}/\rho, \dots, b_n/\rho), \dots, \\ &\qquad g^{\mathbf{A}/\rho}(b_1/\rho, \dots, b_{i-1}/\rho, a_m/\rho, b_{i+1}/\rho, \dots, b_n/\rho)\big). \end{aligned}$$

If $x/\rho \in g^{\mathbf{A}/\rho}(b_1/\rho, \dots, b_{i-1}/\rho, f^{\mathbf{A}/\rho}(a_1/\rho, \dots, a_m/\rho), b_{i+1}/\rho, \dots, b_n/\rho)$, there exists $a/\rho \in f^{\mathbf{A}/\rho}(a_1/\rho, \dots, a_m/\rho)$ such that

$$x/\rho = g^{\mathbf{A}/\rho}(b_1/\rho, \dots, b_{i-1}/\rho, a/\rho, b_{i+1}/\rho, \dots, b_n/\rho).$$

Thus, there exist $a'_1, \dots, a'_m \in A$ with $a_j\, \rho\, a'_j$ for all $j \in \{1, \dots, m\}$ such that

$$a \in f^{\mathbf{A}}\big(a'_1, \dots, a'_m\big) \tag{5}$$

and there exists $x' \in A$, $x\, \rho\, x'$ such that

$$x' \in g^{\mathbf{A}}(b_1, \dots, b_{i-1}, a, b_{i+1}, \dots, b_n). \tag{6}$$

From (5) and (6), it follows

$$\begin{aligned} x' &\in g^{\mathbf{A}}\big(b_1, \dots, b_{i-1}, f^{\mathbf{A}}\big(a'_1, \dots, a'_m\big), b_{i+1}, \dots, b_n\big) \\ &\subseteq f^{\mathbf{A}}\big(g^{\mathbf{A}}\big(b_1, \dots, b_{i-1}, a'_1, b_{i+1}, \dots, b_n\big), \dots, g^{\mathbf{A}}\big(b_1, \dots, b_{i-1}, a'_m, b_{i+1}, \dots, b_n\big)\big). \end{aligned}$$

Thus, for each $j \in \{1, \dots, m\}$, there exists

$$x_j \in g^{\mathbf{A}}\big(b_1, \dots, b_{i-1}, a'_j, b_{i+1}, \dots, b_n\big) \tag{7}$$

such that

$$x' \in f^{\mathbf{A}}(x_1, \dots, x_m). \tag{8}$$

From (7) we deduce that for each $j \in \{1, \dots, m\}$

$$\begin{aligned} x_j/\rho &= g^{\mathbf{A}/\rho}\big(b_1/\rho, \dots, b_{i-1}/\rho, a'_j/\rho, b_{i+1}/\rho, \dots, b_n/\rho\big) \\ &= g^{\mathbf{A}/\rho}(b_1/\rho, \dots, b_{i-1}/\rho, a_j/\rho, b_{i+1}/\rho, \dots, b_n/\rho), \end{aligned}$$

hence, according to (8), we have

$$x/\rho = x'/\rho \in f^{\mathbf{A}/\rho}(x_1/\rho, \dots, x_m/\rho)$$
$$= f^{\mathbf{A}/\rho}\big(g^{\mathbf{A}/\rho}(b_1/\rho, \dots, b_{i-1}/\rho, a_1/\rho, b_{i+1}/\rho, \dots, b_n/\rho), \dots,$$
$$g^{\mathbf{A}/\rho}(b_1/\rho, \dots, b_{i-1}/\rho, a_m/\rho, b_{i+1}/\rho, \dots, b_n/\rho)\big),$$

which ends the proof of the lemma. □

Let $I \neq \emptyset$, let q_i, r_i be m_i-ary terms of type $\mathcal{F}$ $(i \in I)$ and

$$\mathcal{I} = \{q_i = r_i \mid i \in I\}.$$

Let $\mathbf{A} = \langle A, F, G\rangle$ be a multialgebra of type $\mathcal{F} \cup \mathcal{G}$. We will denote by $\alpha_{F,\mathcal{I}}$ the relation defined as in Theorem 6 for the multialgebra $\langle A, F\rangle$ and for the set of identities $\mathcal{I}$, and by $\alpha^*_{F,\mathcal{I}}$ its transitive closure. Thus,

$$\alpha^*_{F,\mathcal{I}} = \alpha^{\langle A,F\rangle}\big(R_{\mathcal{I}}^{\langle A,F\rangle}\big)$$

is the smallest equivalence relation on A providing a factor multialgebra of $\mathbf{A}$ for which all the multioperations from F are operations and which satisfies all the identities from $\mathcal{I}$.

Lemma 14. *Let $\mathbf{A} = \langle A, F, G\rangle$ be a multialgebra of type $\mathcal{F} \cup \mathcal{G}$ such that each multioperation $g \in G$ is an operation. If each operation $g \in G$ is subdistributive with respect to each multioperation $f \in F$, then the factor multialgebra*

$$\mathbf{A}/\alpha^*_{F,\mathcal{I}} = \big\langle A/\alpha^*_{F,\mathcal{I}}, F, G\big\rangle$$

is a universal algebra.

Proof. The factor multialgebra $\mathbf{A}/\alpha^*_{F,\mathcal{I}}$ is a universal algebra if and only if $\alpha^*_{F,\mathcal{I}}$ is a congruence relation on the universal algebra $\langle A, G\rangle$. So, we have to show that for any n-ary $g \in \mathcal{G}$, any $x_1, \dots, x_n, x, y \in A$, and any $j \in \{1, \dots, n\}$, $\langle x, y\rangle \in \alpha^*_{F,\mathcal{I}}$ implies that in $\mathbf{A}$ we have

$$g(x_1, \dots, x_{j-1}, x, x_{j+1}, \dots, x_n)\, \alpha^*_{F,\mathcal{I}}\, g(x_1, \dots, x_{j-1}, y, x_{j+1}, \dots, x_n)$$

(see [5, Proposition II.6.1]). If we use the definition of the transitive closure of $\alpha_{F,\mathcal{I}}$, it is not difficult to observe that it is enough to prove the property for $\langle x, y\rangle \in \alpha_{F,\mathcal{I}}$. According to Theorem 6, $\langle x, y\rangle \in \alpha_{F,\mathcal{I}}$ means that there exist $i \in I$, $n_i \in \mathbb{N}^*$, some n_i-ary term functions p_i of type $\mathcal{F}$ on $\mathbf{P}^*(\mathbf{A})$, and some elements $a^i_1, \dots, a^i_{m_i}, b^i_2, \dots, b^i_{n_i} \in A$ such that

$$x \in p_i\big(q_i\big(a^i_1, \dots, a^i_{m_i}\big), b^i_2, \dots, b^i_{n_i}\big), \qquad y \in p_i\big(r_i\big(a^i_1, \dots, a^i_{m_i}\big), b^i_2, \dots, b^i_{n_i}\big),$$

or

$$y \in p_i\big(q_i\big(a^i_1, \dots, a^i_{m_i}\big), b^i_2, \dots, b^i_{n_i}\big), \qquad x \in p_i\big(r_i\big(a^i_1, \dots, a^i_{m_i}\big), b^i_2, \dots, b^i_{n_i}\big).$$

We assume that

$$x \in p_i\big(q_i\big(a^i_1, \dots, a^i_{m_i}\big), b^i_2, \dots, b^i_{n_i}\big), \qquad y \in p_i\big(r_i\big(a^i_1, \dots, a^i_{m_i}\big), b^i_2, \dots, b^i_{n_i}\big)$$

because the other case can be treated the same way. Then

$$\begin{aligned}
&g(x_1,\dots,x_{j-1},x,x_{j+1},\dots,x_n)\\
&\quad\in g\bigl(x_1,\dots,x_{j-1},p_i\bigl(q_i\bigl(a_1^i,\dots,a_{m_i}^i\bigr),b_2^i,\dots,b_{n_i}^i\bigr),x_{j+1},\dots,x_n\bigr),\\
&g(x_1,\dots,x_{j-1},y,x_{j+1},\dots,x_n)\\
&\quad\in g\bigl(x_1,\dots,x_{j-1},p_i\bigl(r_i\bigl(a_1^i,\dots,a_{m_i}^i\bigr),b_2^i,\dots,b_{n_i}^i\bigr),x_{j+1},\dots,x_n\bigr).
\end{aligned}$$

Since p_i, q_i, r_i are term functions of type $\mathcal{F}$, we can successively apply Lemma 12 and thus obtain

$$\begin{aligned}
&g(x_1,\dots,x_{j-1},x,x_{j+1},\dots,x_n)\\
&\quad\in p_i\bigl(g\bigl(x_1,\dots,x_{j-1},q_i\bigl(a_1^i,\dots,a_{m_i}^i\bigr),x_{j+1},\dots,x_n\bigr),\\
&\qquad g\bigl(x_1,\dots,x_{j-1},b_2^i,x_{j+1},\dots,x_n\bigr),\dots,g\bigl(x_1,\dots,x_{j-1},b_{n_i}^i,x_{j+1},\dots,x_n\bigr)\bigr)\\
&\quad\subseteq p_i\bigl(q_i\bigl(g\bigl(x_1,\dots,x_{j-1},a_1^i,x_{j+1},\dots,x_n\bigr),\dots,g\bigl(x_1,\dots,x_{j-1},a_{m_i}^i,x_{j+1},\dots,x_n\bigr)\bigr),\\
&\qquad g\bigl(x_1,\dots,x_{j-1},b_2^i,x_{j+1},\dots,x_n\bigr),\dots,g\bigl(x_1,\dots,x_{j-1},b_{n_i}^i,x_{j+1},\dots,x_n\bigr)\bigr),
\end{aligned}$$

and, similarly,

$$\begin{aligned}
&g(x_1,\dots,x_{j-1},y,x_{j+1},\dots,x_n)\\
&\quad\in p_i\bigl(r_i\bigl(g\bigl(x_1,\dots,x_{j-1},a_1^i,x_{j+1},\dots,x_n\bigr),\dots,g\bigl(x_1,\dots,x_{j-1},a_{m_i}^i,x_{j+1},\dots,x_n\bigr)\bigr),\\
&\qquad g\bigl(x_1,\dots,x_{j-1},b_2^i,x_{j+1},\dots,x_n\bigr),\dots,g\bigl(x_1,\dots,x_{j-1},b_{n_i}^i,x_{j+1},\dots,x_n\bigr)\bigr).
\end{aligned}$$

Since $g(x_1,\dots,x_{j-1},a_1^i,x_{j+1},\dots,x_n),\dots,g(x_1,\dots,x_{j-1},a_{m_i}^i,x_{j+1},\dots,x_n)$ and $g(x_1,\dots,x_{j-1},b_2^i,x_{j+1},\dots,x_n),\dots,g(x_1,\dots,x_{j-1},b_{n_i}^i,x_{j+1},\dots,x_n)$ are elements from A, it follows that

$$g(x_1,\dots,x_{j-1},x,x_{j+1},\dots,x_n)\,\alpha_{F,\mathcal{I}}\,g(x_1,\dots,x_{j-1},y,x_{j+1},\dots,x_n),$$

which ends the proof of the lemma. □

From the above theorem, one deduces that $\alpha^*_{F,\mathcal{I}}\in E_{ua}(\langle A,F,G\rangle)$. Of course, the algebra $\langle A/\alpha^*_{F,\mathcal{I}},F,G\rangle$ satisfies all the identities from $\mathcal{I}$. From Theorem 6, we easily deduce that $\alpha^*_{F,\mathcal{I}}\subseteq\alpha^*_{\mathcal{I}}$. But $\alpha^*_{\mathcal{I}}$ is the smallest relation from $E_{ua}(\langle A,F,G\rangle)$ for which the factor of $\langle A,F,G\rangle$ satisfies the identities from $\mathcal{I}$. Thus, we have:

Corollary 15. *If $\mathbf{A}=\langle A,F,G\rangle$ is a multialgebra of type $\mathcal{F}\cup\mathcal{G}$ and each multioperation $g\in G$ is an operation that is subdistributive with respect to each multioperation $f\in F$, then $\alpha^*_{\mathcal{I}}=\alpha^*_{F,\mathcal{I}}$.*

As a matter of fact, Lemma 14 and Corollary 15 are equivalent statements. If $\mathcal{I}=\{\mathbf{x}=\mathbf{x}\}$ for some variable $\mathbf{x}$, then $\alpha^*_{F,\mathcal{I}}=\alpha^*_{\langle A,F\rangle}$ and $\alpha^*_{\mathcal{I}}=\alpha^*_{\langle A,F,G\rangle}$. Hence, we also have:

Corollary 16. *If $\mathbf{A}=\langle A,F,G\rangle$ is a multialgebra of type $\mathcal{F}\cup\mathcal{G}$ and each $g\in G$ is an operation subdistributive with respect to each multioperation $f\in F$, then the fundamental relation of $\mathbf{A}$ is the fundamental relation of $\langle A,F\rangle$.*

Theorem 17. *Let $\mathbf{A}=\langle A,F,G\rangle$ be a multialgebra of type $\mathcal{F}\cup\mathcal{G}$. If each multioperation $g\in G$ is subdistributive with respect to each multioperation $f\in F$ and $\rho\in E_{ua}(\langle A,G\rangle)$ then the factor multialgebra*

$$(\mathbf{A}/\rho)/\alpha^*_{F,\mathcal{I}} = \langle (A/\rho)/\alpha^*_{F,\mathcal{I}}, F, G\rangle$$

is a universal algebra.

Proof. According to Lemma 13, in the factor multialgebra $\mathbf{A}/\rho = \langle A/\rho, F, G\rangle$, each multioperation $g^{\mathbf{A}/\rho} \in G$ is an operation which is subdistributive with respect to each multioperation $f^{\mathbf{A}/\rho} \in F$. By applying Lemma 14 to $\mathbf{A}/\rho$ we reach the conclusion that $(\mathbf{A}/\rho)/\alpha^*_{F,\mathcal{I}}$ is a universal algebra. □

Remark 8. Since the factor of any multialgebra preserves the multialgebra's identities, at least in their weak form (see Remark 1), all the (weak or strong) identities of the multialgebras $\mathbf{A}$ and $\mathbf{A}/\rho$ will turn into identities of the universal algebra $(\mathbf{A}/\rho)/\alpha^*_{F,\mathcal{I}}$. In particular, each operation $g^{(\mathbf{A}/\rho)/\alpha^*_{F,\mathcal{I}}}$ is distributive with respect to each operation $f^{(\mathbf{A}/\rho)/\alpha^*_{F,\mathcal{I}}}$.

Consider two disjoint nonempty sets I, J, let q_i, r_i be m_i-ary terms of type $\mathcal{F}$ ($i \in I$), q_j, r_j m_j-ary terms of type $\mathcal{G}$ ($j \in J$) and

$$\mathcal{I} = \{q_i = r_i \mid i \in I\}, \qquad \mathcal{J} = \{q_j = r_j \mid j \in J\}.$$

In the next part of the paper, for a multialgebra $\mathbf{A} = \langle A, F, G\rangle$ of type $\mathcal{F} \cup \mathcal{G}$, we will denote

$$\alpha^*_{\mathcal{I}\cup\mathcal{J}} = \alpha^{\mathbf{A}}\big(R^{\mathbf{A}}_{\mathcal{I}} \cup R^{\mathbf{A}}_{\mathcal{J}}\big), \qquad \alpha^*_{G,\mathcal{J}} = \alpha^{\langle A,G\rangle}\big(R^{\mathbf{A}}_{\mathcal{J}}\big), \qquad \underline{\alpha}^*_{F,\mathcal{I}} = \alpha^{\langle A/\alpha^*_{G,\mathcal{J}},F\rangle}\big(R_{\mathcal{I}}^{\langle A/\alpha^*_{G,\mathcal{J}},F\rangle}\big),$$

and we will consider that the presence or the absence of the superscript * has the same meaning as in Theorem 6.

Thus, $\alpha^*_{\mathcal{I}\cup\mathcal{J}}$ is the smallest equivalence relation on $\mathbf{A}$ for which the factor multialgebra is a universal algebra satisfying all the identities from $\mathcal{I} \cup \mathcal{J}$,

$$\alpha^*_{G,\mathcal{J}} = \alpha^{\langle A,G\rangle}\big(R^{\mathbf{A}}_{\mathcal{J}}\big) = \alpha^{\langle A,G\rangle}\big(R^{\langle A,G\rangle}_{\mathcal{J}}\big)$$

is the smallest equivalence relation on A providing a factor multialgebra on $\mathbf{A}$ for which all the multioperations from G are operations and which satisfies all the identities from $\mathcal{J}$, and $\underline{\alpha}^*_{F,\mathcal{I}}$ is the smallest equivalence relation on $A/\alpha^*_{G,\mathcal{J}}$ providing a factor multialgebra of $\mathbf{A}/\alpha^*_{G,\mathcal{J}} = \langle A/\alpha^*_{G,\mathcal{J}}, F, G\rangle$ for which all the multioperations from F are operations and which satisfies all the identities from $\mathcal{I}$. From Theorem 6, one can easily deduce the inclusion $\alpha^*_{G,\mathcal{J}} \subseteq \alpha^*_{\mathcal{I}\cup\mathcal{J}}$.

Theorem 18. *Let $\mathbf{A} = \langle A, F, G\rangle$ be a multialgebra of type $\mathcal{F} \cup \mathcal{G}$. If each multioperation $g \in G$ is subdistributive with respect to each multioperation $f \in F$, then*

$$\mathbf{A}/\alpha^*_{\mathcal{I}\cup\mathcal{J}} \cong \big(\mathbf{A}/\alpha^*_{G,\mathcal{J}}\big)/\underline{\alpha}^*_{F,\mathcal{I}}.$$

Proof. Our intention is to show that the correspondence

$$a/\alpha^*_{\mathcal{I}\cup\mathcal{J}} \mapsto \big(a/\alpha^*_{G,\mathcal{J}}\big)/\underline{\alpha}^*_{F,\mathcal{I}} \tag{9}$$

defines the required universal algebra isomorphism. Since $(\mathbf{A}/\alpha^*_{G,\mathcal{J}})/\underline{\alpha}^*_{F,\mathcal{I}}$ is a universal algebra (see Theorem 17), this will follow by applying Pickett's isomorphism theorem to the composition of the projections

$$\mathbf{A} \to \mathbf{A}/\alpha^*_{G,\mathcal{J}} \to \big(\mathbf{A}/\alpha^*_{G,\mathcal{J}}\big)/\underline{\alpha}^*_{F,\mathcal{I}}. \tag{10}$$

More precisely, we have to show that the kernel of the above composition is $\alpha^*_{\mathcal{I}\cup\mathcal{J}}$, hence to show that for any $a, b \in A$,

$$a\alpha^*_{\mathcal{I}\cup\mathcal{J}}b \quad \Leftrightarrow \quad \langle a/\alpha^*_{G,\mathcal{J}}, b/\alpha^*_{G,\mathcal{J}}\rangle \in \underline{\alpha}^*_{F,\mathcal{I}}. \tag{11}$$

From Remark 8 it follows that $(\mathbf{A}/\alpha^*_{G,\mathcal{J}})/\underline{\alpha}^*_{F,\mathcal{I}}$ is a universal algebra satisfying all the identities from $\mathcal{I}\cup\mathcal{J}$. Since the factor of the multialgebra $\mathbf{A}$ modulo the kernel of the composition (10) is isomorphic to the universal algebra $(\mathbf{A}/\alpha^*_{G,\mathcal{J}})/\underline{\alpha}^*_{F,\mathcal{I}}$, we deduce that the kernel of the composition (10) is a relation from $E_{ua}(\mathbf{A})$ for which the corresponding factor algebra satisfies all the identities from $\mathcal{I}\cup\mathcal{J}$, thus it contains the relation $\alpha^*_{\mathcal{I}\cup\mathcal{J}}$. This proves the left–right implication from (11).

Since $\underline{\alpha}^*_{F,\mathcal{I}}$ is the transitive closure of $\underline{\alpha}_{F,\mathcal{I}}$, in order to prove the right–left implication from (11) it is enough to show that

$$\langle a/\alpha^*_{G,\mathcal{J}}, b/\alpha^*_{G,\mathcal{J}}\rangle \in \underline{\alpha}_{F,\mathcal{I}} \quad \Rightarrow \quad a\alpha^*_{\mathcal{I}\cup\mathcal{J}}b. \tag{12}$$

According to Theorem 6, $\langle a/\alpha^*_{G,\mathcal{J}}, b/\alpha^*_{G,\mathcal{J}}\rangle \in \underline{\alpha}_{F,\mathcal{I}}$ if and only if there exist $i \in I$, $n_i \in \mathbb{N}^*$, some n_i-ary terms p_i of type $\mathcal{F}$, and $a^i_1, \ldots, a^i_{m_i}, b^i_2, \ldots, b^i_{n_i} \in A$ such that

$$\begin{aligned}
a/\alpha^*_{G,\mathcal{J}} &\in p_i^{\mathbf{P}^*(\mathbf{A}/\alpha^*_{G,\mathcal{J}})}\big(q_i^{\mathbf{P}^*(\mathbf{A}/\alpha^*_{G,\mathcal{J}})}\big(a^i_1/\alpha^*_{G,\mathcal{J}}, \ldots, a^i_{m_i}/\alpha^*_{G,\mathcal{J}}\big), b^i_2/\alpha^*_{G,\mathcal{J}}, \ldots, b^i_{n_i}/\alpha^*_{G,\mathcal{J}}\big),\\
b/\alpha^*_{G,\mathcal{J}} &\in p_i^{\mathbf{P}^*(\mathbf{A}/\alpha^*_{G,\mathcal{J}})}\big(r_i^{\mathbf{P}^*(\mathbf{A}/\alpha^*_{G,\mathcal{J}})}\big(a^i_1/\alpha^*_{G,\mathcal{J}}, \ldots, a^i_{m_i}/\alpha^*_{G,\mathcal{J}}\big), b^i_2/\alpha^*_{G,\mathcal{J}}, \ldots, b^i_{n_i}/\alpha^*_{G,\mathcal{J}}\big),
\end{aligned}$$

or

$$\begin{aligned}
b/\alpha^*_{G,\mathcal{J}} &\in p_i^{\mathbf{P}^*(\mathbf{A}/\alpha^*_{G,\mathcal{J}})}\big(q_i^{\mathbf{P}^*(\mathbf{A}/\alpha^*_{G,\mathcal{J}})}\big(a^i_1/\alpha^*_{G,\mathcal{J}}, \ldots, a^i_{m_i}/\alpha^*_{G,\mathcal{J}}\big), b^i_2/\alpha^*_{G,\mathcal{J}}, \ldots, b^i_{n_i}/\alpha^*_{G,\mathcal{J}}\big),\\
a/\alpha^*_{G,\mathcal{J}} &\in p_i^{\mathbf{P}^*(\mathbf{A}/\alpha^*_{G,\mathcal{J}})}\big(r_i^{\mathbf{P}^*(\mathbf{A}/\alpha^*_{G,\mathcal{J}})}\big(a^i_1/\alpha^*_{G,\mathcal{J}}, \ldots, a^i_{m_i}/\alpha^*_{G,\mathcal{J}}\big), b^i_2/\alpha^*_{G,\mathcal{J}}, \ldots, b^i_{n_i}/\alpha^*_{G,\mathcal{J}}\big).
\end{aligned}$$

Let us consider that we are in the first of the two cases above (the latter can be solved similarly). If we take

$$\begin{aligned}
a' &\in p_i^{\mathbf{P}^*(\mathbf{A})}\big(q_i^{\mathbf{P}^*(\mathbf{A})}\big(a^i_1, \ldots, a^i_{m_i}\big), b^i_2, \ldots, b^i_{n_i}\big),\\
b' &\in p_i^{\mathbf{P}^*(\mathbf{A})}\big(r_i^{\mathbf{P}^*(\mathbf{A})}\big(a^i_1, \ldots, a^i_{m_i}\big), b^i_2, \ldots, b^i_{n_i}\big),
\end{aligned} \tag{13}$$

and we apply Lemma 3 to $\mathbf{A}$, $\alpha^*_{G,\mathcal{J}}$, $\alpha^*_{\mathcal{I}\cup\mathcal{J}}$, we obtain $a\alpha^*_{\mathcal{I}\cup\mathcal{J}}a'$, $b'\alpha^*_{\mathcal{I}\cup\mathcal{J}}b$. By using again Theorem 6, from (13) we get $a'\,\alpha^*_{\mathcal{I}\cup\mathcal{J}}\,b'$. Thus, $a\,\alpha^*_{\mathcal{I}\cup\mathcal{J}}\,b$, and the proof of (12) is now complete. $\square$

Remark 9. Taking $\mathcal{J} = \{\mathbf{x} = \mathbf{x}\}$ or $\mathcal{I} = \{\mathbf{x} = \mathbf{x}\}$ for some variable $\mathbf{x}$, one can replace the corresponding equivalence from the right side of the isomorphism in the above theorem by the corresponding fundamental relation, while the index $\mathcal{J}$ or $\mathcal{I}$, respectively, disappears from the left side of the isomorphism.

If we take both $\mathcal{J}$ and $\mathcal{I}$ to be $\{\mathbf{x} = \mathbf{x}\}$, we have:

Corollary 19. *For any multialgebra $\mathbf{A} = \langle A, F, G\rangle$ of type $\mathcal{F}\cup\mathcal{G}$ for which each $g \in G$ is subdistributive with respect to each $f \in F$ we have*

$$\mathbf{A}/\alpha^*_{\langle A,F,G\rangle} \cong \big(\mathbf{A}/\alpha^*_{\langle A,G\rangle}\big)/\alpha^*_{\langle A/\alpha^*_{\langle A,G\rangle},F\rangle}.$$

Remark 10. There are situations when Theorem 10 and Theorem 18 provide the same isomorphism for **A** multialgebras satisfying the required subdistributivity conditions. But, in general, for the isomorphic image of $\mathbf{A}/\alpha^*_{\mathcal{I}\cup\mathcal{J}}$ in Theorem 10, the first factor multialgebra is a universal algebra which is then factorized through a congruence, while for the isomorphic image of $\mathbf{A}/\alpha^*_{\mathcal{I}\cup\mathcal{J}}$ in Theorem 18 both factorizations could be factors of multialgebras. The advantage of Theorem 18 consists in the fact that the construction of the equivalence relations involved could be easier. As for the construction of the isomorphic image of $\mathbf{A}/\alpha^*_{\mathcal{I}\cup\mathcal{J}}$, this can be a difficult task in both cases. A combination of the two theorems can be useful sometimes in this respect. One can apply first Corollary 11 and then Corollary 19 to determine the required factor (multi)algebra. Such a situation will appear in Section 5.

Next, we will apply our results to some particular classes of multialgebras. We will emphasize the importance of our general results by showing how they can be used to improve some of the results from [13] and [21].

5. Applications to hyperrings

There are many types of hyperrings known in the literature (to get a partial picture of this, it is enough to read the remarks that follow Definition 1.2 from [23]). Most of them require a set R endowed with two binary multioperations $+$ and $\cdot$ such that $\langle R,+\rangle$ is a hypergroup, $\langle R,\cdot\rangle$ is a semihypergroup and $\cdot$ is subdistributive with respect to $+$, i.e.

$$a(b+c)\subseteq ab+ac,\qquad (b+c)a\subseteq ba+ca,\quad \forall a,b,c\in R.$$

We will call a multialgebra $\langle R,+,\cdot\rangle$ defined by the above "minimal" conditions a *hyperring-like structure* in order to avoid a possible terminological confusion. We insist that this should be seen rather like a description of the multialgebra $\langle R,+,\cdot\rangle$ than the definition of a new type of hyperring.

A hyperring-like structure $\langle R,+,\cdot\rangle$ is a *Krasner hyperring* if $\langle R,+\rangle$ is a canonical hypergroup (i.e. a commutative hypergroup with an element 0 such that $0+a=a$, for any $a\in R$, which satisfies the property that for each $a\in R$ there exists an element $-a\in R$ such that for any $b,c\in R$, $c\in a+b$ implies $b\in(-a)+c$), $\langle R,\cdot\rangle$ is a semigroup,

$$0\cdot a=a\cdot 0=0,\quad \forall a\in R,$$

and the operation $\cdot$ is distributive with respect to the multioperation $+$.

Remark 11. The element 0 above is uniquely determined, and for each $a\in R$, there exists only one element $-a\in R$ satisfying the above condition. Moreover, for each $a\in R$, $-a$ is the only element such that $0\in a+(-a)$.

Thus, a Krasner hyperring can be seen as a multialgebra $\langle R,+,/,\backslash,0,-,\cdot\rangle$ with $/,\backslash$ defined as in Remark 5, 0 nullary operation, $-$ unary operation and $\cdot$ binary operation, satisfying some identities (see [26, Example 13]). Important particular nonzero Krasner hyperrings are those which have a multiplicative identity 1 (see, for instance, [6]). We can see this identity as an additional nullary operation. Krasner hyperfields are such particular hyperrings.

Remark 12. If $\langle R,+,\cdot\rangle$ be a Krasner hyperring and $\rho\in E_{ua}(\langle R,+,\cdot\rangle)$, then $\langle R/\rho,+\rangle$ is an Abelian group, $0/\rho$ is its zero element, the symmetric $-(a/\rho)$ of a/ρ is $(-a)/\rho$, $\langle R/\rho,\cdot\rangle$ is a semigroup, and $\cdot$ is distributive with respect to $+$. Thus, $\langle R/\rho,+,0/\rho,-,\cdot\rangle$ is a ring. If the Krasner hyperring $\langle R,+,\cdot\rangle$ has a multiplicative identity 1, then $1/\rho$ is a multiplicative identity for $\langle R/\rho,\cdot\rangle$, hence $\langle R/\rho,+,0/\rho,-,\cdot,1/\rho\rangle$ is a ring with identity.

If we want to factorize a hyperring-like structure $\langle R, +, \cdot\rangle$ in order to obtain a ring, we must make $+$ commutative in the factor multialgebra. For this ring to be commutative, $\cdot$ must become a commutative operation, too. Thus, taking the binary symbols $+, \cdot$, and the variables $\mathbf{x}_1, \mathbf{x}_2$, our sets of identities will be

$$\mathcal{I} = \{\mathbf{x}_1 + \mathbf{x}_2 = \mathbf{x}_2 + \mathbf{x}_1\} \quad \text{and} \quad \mathcal{J} = \{\mathbf{x}_1 \cdot \mathbf{x}_2 = \mathbf{x}_2 \cdot \mathbf{x}_1\},$$

and (using Remark 5 and Remark 3) one deduces that $\alpha^*_{\mathcal{I}\cup\mathcal{J}}$ is the smallest equivalence of $\langle R, +, \cdot\rangle$ for which the factor multialgebra is a commutative ring.

Remark 13. According to Theorem 10, the commutative ring $\langle R/\alpha^*_{\mathcal{I}\cup\mathcal{J}}, +, \cdot\rangle$ can be obtained as follows: we factorize $\langle R, +, \cdot\rangle$ over the smallest equivalence relation from $E_{ua}(\langle R, +, \cdot\rangle)$ for which the operation $+$ is commutative, we obtain a ring, then we take the factor of this ring over its commutator ideal.

Using again Remark 3, we have:

Proposition 20. *For a hyperring-like structure $\langle R, +, \cdot\rangle$ with $+$ at least weak commutative, the fundamental relation of $\langle R, +, \cdot\rangle$ and $\alpha^*_{\mathcal{I}}$ coincide, $\alpha^*_{\mathcal{I}\cup\mathcal{J}} = \alpha^*_{\mathcal{J}}$ and the ring $\langle R/\alpha^*_{\mathcal{I}\cup\mathcal{J}}, +, \cdot\rangle$ is isomorphic to the factor of the fundamental ring of $\langle R, +, \cdot\rangle$ over its commutator ideal.*

A hyperring-like structure $\langle R, +, \cdot\rangle$ with $+$ and $\cdot$ operations is a distributive nearring. Since the distributive nearrings which have multiplicative identity are (associative) rings (see, for instance, [13, p. 3309]), we have:

Lemma 21. *If $\langle R, +, \cdot\rangle$ is a hyperring-like structure with a multiplicative identity and $\rho \in E_{ua}(\langle R, +, \cdot\rangle)$, then $\langle R/\rho, +, \cdot\rangle$ is an associative ring.*

An immediate consequence of Proposition 20 and Lemma 21 is the following:

Corollary 22. *For a hyperring-like structure $\langle R, +, \cdot\rangle$ with a multiplicative identity the fundamental relation of $\langle R, +, \cdot\rangle$ and $\alpha^*_{\mathcal{I}}$ coincide, $\alpha^*_{\mathcal{I}\cup\mathcal{J}} = \alpha^*_{\mathcal{J}}$ and the ring $\langle R/\alpha^*_{\mathcal{I}\cup\mathcal{J}}, +, \cdot\rangle$ is isomorphic to the factor of the fundamental ring of $\langle R, +, \cdot\rangle$ over its commutator ideal.*

Let $\langle R, +, \cdot\rangle$ be a hyperring-like structure. We denote by β_+ and $\beta_\times$ the relations defined as in Example 1 for $\langle R, +\rangle$ and $\langle R, \cdot\rangle$ respectively, by γ_+ and $\gamma_\times$ the relations defined as in Example 2 for $\langle R, +\rangle$ and $\langle R, \cdot\rangle$ respectively, and by β^*_+, $\beta^*_\times$, γ^*_+, $\gamma^*_\times$ their transitive closures. Thus

$$\beta^*_+ = \alpha^*_{\langle R, +\rangle}, \qquad \beta^*_\times = \alpha^*_{\langle R, \cdot\rangle}, \qquad \gamma^*_+ = \alpha^*_{\{+\},\mathcal{I}}, \qquad \gamma^*_\times = \alpha^*_{\{\cdot\},\mathcal{J}}.$$

We us also denote

$$\underline{\beta}^*_+ = \alpha^*_{\langle R/\beta^*_\times, +\rangle}, \qquad \underline{\gamma}^*_+ = \underline{\alpha}^*_{\{+\},\mathcal{I}}.$$

As we mentioned in Examples 1 and 2, the relations β_+ and γ_+ are transitive, hence $\beta^*_+ = \beta_+$ and $\gamma^*_+ = \gamma_+$. From Corollary 15 and Corollary 16 we deduce:

Proposition 23. *If $\langle R, +, \cdot\rangle$ is a hyperring-like structure with $\cdot$ operation, then*:

(i) *the fundamental relation of $\langle R, +, \cdot\rangle$ and β_+ coincide*;
(ii) *the multialgebra $\langle R/\gamma_+, +, \cdot\rangle$ is a ring or, equivalently, $\alpha^*_{\mathcal{I}} = \gamma_+$*;

(iii) *if the operation $\cdot$ is commutative, then* $\alpha^*_{\mathcal{I}\cup\mathcal{J}} = \alpha^*_{\mathcal{I}} = \gamma_+$;
(iv) *if the operation $\cdot$ is commutative and it has an identity element, then*

$$\alpha^*_{\mathcal{I}\cup\mathcal{J}} = \alpha^*_{\mathcal{J}} = \alpha^*_{\mathcal{I}} = \gamma_+ = \beta_+$$

and they are all equal to the fundamental relation of $\langle R,+,\cdot\rangle$;
(v) *if $+$ and $\cdot$ are* (*at least weak*) *commutative, then*

$$\alpha^*_{\mathcal{I}\cup\mathcal{J}} = \alpha^*_{\mathcal{J}} = \alpha^*_{\mathcal{I}} = \gamma_+ = \beta_+.$$

Remark 14. If $+$ and $\cdot$ are operations, one can recognize in (ii) from the above proposition an elementary property of nearrings: *the derived subgroup of the additive group of a distributive nearring R is an ideal of R* (see, for instance, [1, p. 355]).

Let $\langle R,+,\cdot\rangle$ be a Krasner hyperring and let us denote, as in [21], the fundamental relation $\alpha^*_{\langle R,+,\cdot\rangle}$ of $\langle R,+,\cdot\rangle$ by Γ^*. From Proposition 23 and the commutativity of the addition of a Krasner hyperring we have:

Corollary 24. *If $\langle R,+,\cdot\rangle$ is a Krasner hyperring, then*

$$\Gamma^* = \alpha^*_{\mathcal{I}} = \gamma^*_+ = \gamma_+ = \beta^*_+ = \beta_+ \quad \textit{and} \quad \alpha^*_{\mathcal{I}\cup\mathcal{J}} = \alpha^*_{\mathcal{J}}.$$

If the multiplication of $\langle R,+,\cdot\rangle$ is commutative, then

$$\alpha^*_{\mathcal{I}\cup\mathcal{J}} = \alpha^*_{\mathcal{J}} = \alpha^*_{\mathcal{I}} = \Gamma^* = \gamma_+ = \beta_+.$$

From Theorem 18, it follows:

Proposition 25. *For a hyperring-like structure $\langle R,+,\cdot\rangle$ we have*

$$R/\alpha^*_{\mathcal{I}\cup\mathcal{J}} \cong \big(R/\gamma^*_\times\big)/\underline{\gamma}^*_+.$$

Remark 15. If $\cdot$ is distributive with respect to $+$, the hyperring-like structures $\langle R,+,\cdot\rangle$ are the hyperrings from [13], the relation $\alpha^*_{\mathcal{I}\cup\mathcal{J}}$ is the α^*-relation from [13], Proposition 23(iii) becomes [13, Theorem 5] and Proposition 25 becomes [13, Theorem 6]. If $\langle R,+,\cdot\rangle$ is a Krasner hyperring, then $\alpha^*_{\mathcal{I}\cup\mathcal{J}}$ is the α^*-relation from [21], [21, Theorem 2.1] is an immediate consequence of Proposition 23, and Corollary 24 is a stronger version of [21, Theorem 2.1].

It is easy to notice that the isomorphism from Proposition 25 is independent from the axioms of $+$ and from the axioms of $\cdot$, so Proposition 25 could be stated as follows: *for a multialgebra $\langle R,+,\cdot\rangle$ with two binary multioperations, if $\cdot$ is subdistributive with respect to $+$, then $R/\alpha^*_{\mathcal{I}\cup\mathcal{J}} \cong (R/\gamma^*_\times)/\underline{\gamma}^*_+$.* Yet, the axioms of $+$ and $\cdot$ improve the form of the relations involved in this isomorphism. The associativity of $\cdot$ gives $\gamma^*_\times$ the friendly form in Example 2. A close look at the proof of [13, Lemma 2] shows that only the subdistributivity of $\cdot$ with respect to $+$ is needed for the lemma to hold. Using the approach from [29, Section 4], one easily deduces that, for the hyperring-like structure $\langle R,+,\cdot\rangle$, the relation $\alpha^*_{\mathcal{I}\cup\mathcal{J}}$ is the relation from Example 3.

Remark 16. Since we do not know anything about the associativity of $+$ in $\langle R/\gamma^*_\times,+\rangle$, we cannot characterize $\underline{\gamma}^*_+$ as in Example 2. Yet, it is not very difficult to use Theorem 6 (or [27, Theorem 4.4]) to determine it.

Remark 17. As we anticipated in Remark 10, for obtaining the ring $\langle R/\alpha^*_{\mathcal{I}}, +, \cdot\rangle$ we can first factorize $\langle R, +, \cdot\rangle$ modulo the fundamental relation $\beta^*_\times$ of $\langle R, \cdot\rangle$, then we factorize $\langle R/\beta^*_\times, +, \cdot\rangle$ over the fundamental relation $\underline{\beta}^*_+$ of $\langle R/\beta^*_\times, +\rangle$ and, finally, we factorize the distributive nearring $\langle (R/\beta^*_\times)/\underline{\beta}^*_+, +, \cdot\rangle$ by the derived subgroup of its additive group.

Proposition 23 also makes us see some of the isomorphism theorems from [21, Section 3] in a different way. For instance, using [25, Proposition 7], one can get a stronger result than [21, Theorem 3.5] without the exhausting computations from [21, Lemma 3.2, Theorem 3.3 and Theorem 3.4]. Also, [21, Theorem 3.10] is only a rewritten version of the well-known ring homomorphism theorem.

Since the fundamental algebra preserves multialgebra's identities, [25, Theorem 1] can be rewritten for hyperring-like structures:

Proposition 26. *Let $\langle R_1, +, \cdot\rangle$, $\langle R_2, +, \cdot\rangle$ be two hyperring-like structures with $+$ at least weak commutative (or with a multiplicative identity), let Γ^*_1, Γ^*_2 be their fundamental relations, and let π_1, π_2, respectively, be the corresponding canonical projections. For any homomorphism $f : R_1 \to R_2$ there exists a unique ring homomorphism $\overline{f} : R_1/\Gamma^*_1 \to R_2/\Gamma^*_2$ which makes the following diagram commutative:*

$$\begin{tikzcd} R_1 \arrow[r, "f"] \arrow[d, "\pi_1"'] & R_2 \arrow[d, "\pi_2"] \\ R_1/\Gamma^*_1 \arrow[r, "\overline{f}"] & R_2/\Gamma^*_2 \end{tikzcd}$$

Remark 18. For two hyperring-like structures with both $+$ and $\cdot$ commutative, their α^*-relations coincide with their fundamental relations. Even if we are not dealing with strong homomorphisms, the ideal defined in [21, Definition 3.8 and Lemma 3.9] is the kernel of the ring homomorphism $\overline{f}$ introduced by Proposition 26, and [21, Theorem 3.10] needs no proof, since it is a particular case of the ring homomorphism theorem.

Theorem [25, Theorem 1] shows that the construction of the fundamental algebra determines a functor from the category of $\mathcal{F}$-multialgebras into the category of the universal algebras of type $\mathcal{F}$ (which is the functor from Remark 6 in the case $\mathcal{I} = \{\mathbf{x} = \mathbf{x}\}$ for some variable $\mathbf{x}$). This functor does not preserve the finite (direct) products. Yet, we found in [25, Corollary 5] a sufficient condition for the fundamental algebra of a finite direct product of multialgebras to be isomorphic to the direct product of their fundamental algebras. In [25, Proposition 7], we showed that hypergroups fulfill this condition, so we have:

Lemma 27. *If $\langle H_1, +\rangle$, $\langle H_2, +\rangle$ are two hypergroups and β_1, β_2, respectively, are their fundamental relations, then the fundamental group of the direct product $\langle H_1 \times H_2, +\rangle$ is isomorphic to $\langle H_1/\beta_1 \times H_2/\beta_2, +\rangle$.*

Remark 19. As a matter of fact, if $\mathbf{A}_1$, $\mathbf{A}_2$ are multialgebras, e_1, e_2 are the canonical projections of the direct product $\mathbf{A}_1 \times \mathbf{A}_2$, $\overline{e_1}$, $\overline{e_2}$ are the induced homomorphisms between the corresponding fundamental algebras (see [25, Theorem 1]), and p_1, p_2 are the projections of the product $\mathbf{A}_1/\alpha^*_{\mathbf{A}_1} \times \mathbf{A}_2/\alpha^*_{\mathbf{A}_2}$, from the universal property of the direct product one deduces that the correspondence $\langle a_1, a_2\rangle/\alpha^*_{\mathbf{A}_1\times\mathbf{A}_2} \mapsto \langle a_1/\alpha^*_{\mathbf{A}_1}, a_2/\alpha^*_{\mathbf{A}_2}\rangle$ defines the unique universal algebra homomorphism $\varphi : (A_1 \times A_2)/\alpha^*_{\mathbf{A}_1\times\mathbf{A}_2} \to A_1/\alpha^*_{\mathbf{A}_1} \times A_2/\alpha^*_{\mathbf{A}_2}$ for which $p_1 \circ \varphi = \overline{e_1}$ and $p_2 \circ \varphi = \overline{e_2}$. The homomorphism φ is surjective, and, if the condition from [25, Corollary 5] holds (as it happens for hypergroups in Lemma 27), φ is an isomorphism.

In the next part of this section, we consider $\langle R_1, +, \cdot\rangle$ and $\langle R_2, +, \cdot\rangle$ to be hyperring-like structures for which $\cdot$ is an operation and $+$ is at least weak commutative – in particular, they can be Krasner hyperrings – or hyperring-like structures for which $\cdot$ is an operation which has an identity element. It is not difficult to verify that the condition from [25, Corollary 5] holds for such pairs of hyperring-like structures, and it is even easier to show that φ is an isomorphism in their case using Proposition 23.

Theorem 28. *If Γ_1^* and Γ_2^* are the fundamental relations of $\langle R_1, +, \cdot\rangle$ and $\langle R_2, +, \cdot\rangle$, respectively, then the fundamental ring of $\langle R_1 \times R_2, +, \cdot\rangle$ is isomorphic to $\langle R_1/\Gamma_1^* \times R_2/\Gamma_2^*, +, \cdot\rangle$.*

Proof. Let Γ_{12}^*, β_1, β_2, and β_{12} be the fundamental relations of $\langle R_1 \times R_2, +, \cdot\rangle$, $\langle R_1, +\rangle$, $\langle R_2, +\rangle$, and $\langle R_1 \times R_2, +\rangle$, respectively. In our case, φ from Remark 19 is a surjective ring homomorphism from $(R_1 \times R_2)/\Gamma_{12}^*$ into $R_1/\Gamma_1^* \times R_2/\Gamma_2^*$. According to (i) from Proposition 23,

$$R_1/\Gamma_1^* = R_1/\beta_1, \qquad R_2/\Gamma_2^* = R_2/\beta_2, \qquad (R_1 \times R_2)/\Gamma_{12}^* = (R_1 \times R_2)/\beta_{12},$$

so $\varphi : (R_1 \times R_2)/\beta_{12} \to R_1/\beta_1 \times R_2/\beta_2$ is the mapping from Lemma 27. Thus φ it is also injective, which means it is a ring isomorphism. □

It is easy to show that the commutator ideal of a direct product of two rings is the direct product of their commutator ideals and, consequently, the factor of the direct product modulo its commutator ideal is isomorphic to the product of the factors of the rings by their commutator ideals – as a matter of fact, for unitary rings, this is the ring theoretical version of [21, Theorem 3.4 and Theorem 3.5]. Based on this fact, the above theorem, Proposition 20 and Corollary 22, one easily deduces the following:

Corollary 29. *If α_1^*, α_2^* and α_{12}^* are the α^*-relations of $\langle R_1, +, \cdot\rangle$, $\langle R_2, +, \cdot\rangle$ and $\langle R_1 \times R_2, +, \cdot\rangle$, respectively, then we have the following ring isomorphism*

$$(R_1 \times R_2)/\alpha_{12}^* \cong R_1/\alpha_1^* \times R_2/\alpha_2^*.$$

Remark 20. Since Krasner's hyperrings are a particular case of the hyperring-like structures involved in Theorem 28 and Corollary 29, [21, Theorem 3.5] follows immediately from the above corollary. Corollary 29 also shows that [21, Theorem 3.5] can be restated for any Krasner hyperrings, not only for those which have multiplicative identity.

6. Applications to (m, n)-hyperrings

Let $m, n \in \mathbb{N}$, $m, n \geqslant 2$. An m-semihypergroup $\langle R, f\rangle$ satisfying condition

$$R = f(a_1, \ldots, a_{i-1}, R, a_{i+1}, \ldots, a_m), \quad \forall a_1, \ldots, a_m \in R,\ \forall i \in \{1, \ldots, m\} \tag{14}$$

is called an *m-hypergroup*. The condition (14) can also be written as follows:

$$\forall b, a_1, \ldots, a_m \in R,\ \forall i \in \{1, \ldots, m\},\ \exists x \in R\colon\quad b \in f(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_m). \tag{$*$}$$

If f is an operation and the element x from $(*)$ is unique, then $\langle R, f\rangle$ is an *m-group*. A multialgebra $\langle R, f, g\rangle$ is an *(m, n)-hyperring* if $\langle R, f\rangle$ is an m-hypergroup, $\langle R, g\rangle$ is an n-semihypergroup and the multioperation g is distributive with respect to f (see [11, Section 5]). The $(2, 2)$-hyperrings are the hyperrings from [13], so the results from this section generalize some results from the previous section. $\langle R, f, g\rangle$ is an *(m, n)-ring* if $\langle R, f\rangle$ is a commutative m-group, $\langle R, g\rangle$ is an n-semigroup and the operation g is distributive with respect to f (see [8]).

Remark 21. Using (14) we can organize the m-hypergroup $\langle R, f\rangle$ as a multialgebra $\langle R, f, f_1, \dots, f_m\rangle$ where for all $i \in \{1, \dots, m\}$, f_i is the m-ary multioperation which assigns to each m-tuple $\langle a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_m\rangle$ the nonempty subset

$$f_i(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_m) = \bigl\{x \in R \bigm| b \in f(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_m)\bigr\}. \tag{15}$$

The similar configuration of m-groups can be found in [16, Section 1] and the corresponding configuration of 2-hypergroups (i.e. hypergroups) is presented (in great detail) in [28].

Lemma 30. *Let us configure the m-hypergroup $\langle R, f\rangle$ like in the above remark as a multialgebra $\langle R, f, f_1, \dots, f_m\rangle$ and let $\rho \in E_{ua}(\langle R, f\rangle)$. In the factor multialgebra of $\langle R, f, f_1, \dots, f_m\rangle$ over ρ, each f_i determines an operation which associates to each m-tuple $\langle a_1/\rho, \dots, a_{i-1}/\rho, b/\rho, a_{i+1}/\rho, \dots, a_m/\rho\rangle$ the unique element $\underline{x}$ for which*

$$b/\rho = f(a_1/\rho, \dots, a_{i-1}/\rho, \underline{x}, a_{i+1}/\rho, \dots, a_m/\rho).$$

Proof. Let us denote by $f_1', \dots, f_m'$ the multioperations which correspond to $f_1, \dots, f_m$ in the factor multialgebra of $\langle R, f, f_1, \dots, f_m\rangle$. Since the associativity of f is determined by a set of identities of $\langle R, f\rangle$, f is an associative operation in the factor multialgebra $\langle R/\rho, f, f_1', \dots, f_m'\rangle$. For any $i \in \{1, \dots, m\}$,

$$\underline{x} \in f_i'(a_1/\rho, \dots, a_{i-1}/\rho, b/\rho, a_{i+1}/\rho, \dots, a_m/\rho)$$

if and only if there exist $a_1', \dots, a_m', b', c \in R$ such that $a_1 \rho a_1', \dots, a_m \rho a_m'$, $b \rho b'$, $\underline{x} = c/\rho$ and

$$c \in f_i\bigl(a_1', \dots, a_{i-1}', b', a_{i+1}', \dots, a_m'\bigr). \tag{16}$$

But (16) is equivalent to $b' \in f_i(a_1', \dots, a_{i-1}', c, a_{i+1}', \dots, a_m')$, hence

$$b'/\rho = f\bigl(a_1'/\rho, \dots, a_{i-1}'/\rho, c/\rho, a_{i+1}'/\rho, \dots, a_m'/\rho\bigr)$$

in the factor multialgebra $\langle R/\rho, f, f_1', \dots, f_m'\rangle$, or, equivalently,

$$b/\rho = f\bigl(a_1/\rho, \dots, a_{i-1}/\rho, \underline{x}, a_{i+1}/\rho, \dots, a_m/\rho\bigr). \tag{17}$$

According to a remark from [31, p. 213], if f is associative, the existence of a solution $\underline{x}$ for (each of) the equations (17) implies its uniqueness, which completes the proof of the lemma. □

Using the notations from the above lemma, we have:

Corollary 31. $E_{ua}(\langle R, f\rangle) = E_{ua}(\langle R, f, f_1, \dots, f_n\rangle)$.

But $E_{ua}(\langle R, f, g\rangle) = E_{ua}(\langle R, f\rangle) \cap E_{ua}(\langle R, g\rangle)$ and

$$E_{ua}\bigl(\langle R, f, f_1, \dots, f_n, g\rangle\bigr) = E_{ua}\bigl(\langle R, f, f_1, \dots, f_n\rangle\bigr) \cap E_{ua}\bigl(\langle R, g\rangle\bigr).$$

Thus, we also have:

Corollary 32. $E_{ua}(\langle R, f, f_1, \dots, f_n, g\rangle) = E_{ua}(\langle R, f, g\rangle)$.

Remark 22. Lemma 30 and its corollaries are very important for the next part of this section. They allow us not to consider the multioperations $f_1, \dots, f_n$ in the characterization of the smallest equivalence relations of $\langle R, f\rangle$ (or $\langle R, f, g\rangle$) for which the factor multialgebra is an m-group (or an (m,n)-ring, respectively) satisfying certain identities. Moreover, they show that the factor (multi)algebras determined by such relations do not lose the information provided by the equalities (14), even if these equalities do not look (in this form) as multialgebra identities. In particular, the fundamental relation of the m-hypergroup $\langle R, f\rangle$ coincides with the fundamental relation of the m-semihypergroup $\langle R, f\rangle$ (and it can be found in [20, Section 4]).

Since we do not have to add fundamental operations to the multialgebra $\langle R, f, g\rangle$ and g is distributive with respect to f, we can apply the results from Section 4 to (m,n)-hyperrings. The following results are valid because of the subdistributivity of g with respect to f. From Corollary 15, we deduce:

Proposition 33. *If g is an operation in the (m,n)-hyperring $\langle R, f, g\rangle$, the fundamental relations $\alpha^*_{\langle R,f,g\rangle}$ of $\langle R, f, g\rangle$ and $\alpha^*_{\langle R,f\rangle}$ of $\langle R, f\rangle$ coincide.*

As for the construction of the fundamental algebra of an (m,n)-hyperring, from Corollary 19, we have:

Proposition 34. *Let $\langle R, f, g\rangle$ be an (m,n)-hyperring. Consider the factor multialgebra of $\langle R, f, g\rangle$ modulo the fundamental relation of $\langle R, g\rangle$ and then the factor of this multialgebra over the fundamental relation of $\langle R/\alpha^*_{\langle R,g\rangle}, f\rangle$. The resulting multialgebra is an algebra isomorphic to the fundamental algebra of the (m,n)-hyperring of $\langle R, f, g\rangle$.*

Even if we do not know if $\langle R/\alpha^*_{\langle R,g\rangle}, f\rangle$ is an m-semihypergroup, the characterization of its fundamental relation follows immediately from Corollary 7. The resulting fundamental algebra might not be an (m,n)-ring since the operation f might not be commutative. But if we take $\mathcal{I}$ to be the set of the identities which characterize the commutativity of f and we call the algebra $\langle R/\alpha^*_{\mathcal{I}}, f, g\rangle$ *fundamental (m,n)-ring*, then we have:

Proposition 35. *Let $\langle R, f, g\rangle$ be an (m,n)-hyperring. Consider the factor multialgebra of $\langle R, f, g\rangle$ modulo the fundamental relation of $\langle R, g\rangle$ and then the factor of this multialgebra over $\underline{\alpha}^*_{f,\mathcal{I}}$. The resulting multialgebra is isomorphic to the fundamental (m,n)-ring of $\langle R, f, g\rangle$.*

More generally, if $\mathcal{I}$ is the set of identities for which the terms are constructed using only f and $\mathcal{J}$ the set of identities for which the terms are constructed using only g (in particular, $\mathcal{I}$ can be the set of the identities which characterize the commutativity of f and $\mathcal{J}$ the set of the identities which characterize the commutativity of g), then, from Corollary 15 and Theorem 18, we deduce:

Proposition 36. *If g is operation in the (m,n)-hyperring $\langle R, f, g\rangle$, then*

$$\alpha^*_{\mathcal{I}} = \alpha^*_{f,\mathcal{I}}.$$

Proposition 37. *If $\langle R, f, g\rangle$ is an (m,n)-hyperring then*

$$\left\langle R/\alpha^*_{\mathcal{I}\cup\mathcal{J}}, f, g\right\rangle \cong \left\langle \left(R/\alpha^*_{g,\mathcal{J}}\right)/\underline{\alpha}^*_{f,\mathcal{I}}, f, g\right\rangle.$$

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